THE HEAT SEMIGROUP ON CONFIGURATION SPACES

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Abstract

In this paper, we study properties of the heat semigroup of configuration space analysis. Using a natural "Riemannian-like" structure of the configuration space Γ_X over a complete, connected, oriented, and stochastically complete Riemannian manifold X of infinite volume, the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$ was introduced and studied in [*J. Func. Anal.* **154** (1998), 444–500]. Here, H^Γ is the Dirichlet operator of the Dirichlet form $\mathcal{E}^\Gamma$ over the space $L^2(\Gamma_X,\pi_m)$, where π_m is the Poisson measure on Γ_X with intensity m—the volume measure on X. We construct a metric space Γ_∞ that is continuously embedded into Γ_X. Under some conditions on the manifold X, we prove that Γ_∞ is a set of full π_m measure and derive an explicit formula for the heat semigroup: $(e^{-tH^\Gamma}F)(\gamma)=\int_{\Gamma_\infty}F(\xi)\,\mathbf{P}_{t,\gamma}(d\xi)$, where $\mathbf{P}_{t,\gamma}$ is a probability measure on Γ_∞ for all $t>0$, $\gamma\in\Gamma_\infty$. The central results of the paper are two types of Feller properties for the heat semigroup. The first one is a kind of strong Feller property with respect to the metric on the space Γ_∞. The second one, obtained in the case $X=\mathbb{R}^d$, is the Feller property with respect to the intrinsic metric of the Dirichlet form $\mathcal{E}^\Gamma$. Next, we give a direct construction of the independent infinite particle process on the manifold X, which is a realization of the Brownian motion on the configuration space. The main point here is that we prove that this process can start in every $\gamma\in\Gamma_\infty$, will never leave Γ_∞, and has continuous sample path in Γ_∞, provided $\dim X\geq 2$. In this case, we also prove that this process is a strong Markov process whose transition probabilities are given by the $\mathbf{P}_{t,\gamma}(\cdot)$ above. Furthermore, we discuss the necessary changes to be done for constructing the process in the case $\dim X=1$. Finally, as an easy consequence we get a "path-wise" construction of the independent particle process on Γ_∞ from the underlying Brownian motion.

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Contents

1 Introduction

In [3, 4, 5, 6], stochastic analysis and differential geometry on configuration spaces were considerably developed by using the so-called "lifting procedure," see also [33, 28, 34, 35, 1, 2] for further results and reviews.

Let us recall that the configuration space Γ_X over a complete, connected, oriented, and stochastically complete Riemannian manifold X of infinite volume is defined as the set of all infinite subsets of X which are locally finite. Each configuration $\gamma \in \Gamma_X$ can be identified with the Radon measure $\sum_{x\in\gamma} \varepsilon_x$. The tangent space to Γ_X at a point $\gamma \in \Gamma_X$, denoted by $T_\gamma(\Gamma_X)$, is defined as the direct sum of the tangent spaces to X at x, where x runs over the points of the configuration γ; that is, $T_\gamma(\Gamma_X) := \bigoplus_{x\in\gamma} T_x(X)$. The gradient $\nabla^\Gamma F(\gamma)$ of a differentiable function $F\colon \Gamma_X \to \mathbb{R}$ at a point $\gamma \in \Gamma_X$ is defined as an element of the tangent space $T_\gamma(\Gamma_X)$ through a natural lifting of the gradient on X. Analogously, one introduces also the notion of divergence of a vector field over Γ_X.

Let π_m denote the Poisson measure on Γ_X with intensity m—the volume measure on X. By using the integration by parts formula for the Poisson measure, it was shown in [5] that π_m is a volume measure on Γ_X, in the sense that the gradient and the divergence become dual operators on $L^2(\pi_m) := L^2(\Gamma_X, \pi_m)$.

Thus, having identified differentiation and a volume measure on the configuration space, the next step in [5] was to consider the Dirichlet form over $L^2(\pi_m)$, which is defined by

$$\mathcal{E}^\Gamma(F_1, F_2) := \frac{1}{2} \int_{\Gamma_X} \langle \nabla^\Gamma F_1(\gamma), \nabla^\Gamma F_2(\gamma)\rangle_{T_\gamma(\Gamma_X)}\, \pi_m(d\gamma)$$

on an appropriate set of smooth cylinder functions on Γ_X. Using again the integration by parts formula, one obtains the associated Dirichlet operator, i.e., the operator H^Γ in $L^2(\pi_m)$ satisfying $\mathcal{E}^\Gamma(F_1, F_2) = (H^\Gamma F_1, F_2)_{L^2(\pi_m)}$. This yields, in particular, that the bilinear form $\mathcal{E}^\Gamma$ is closable. Moreover, the operator H^Γ was shown to be essentially selfadjoint. We will preserve the notation H^Γ for its closure.

The present paper is devoted to the study of properties of the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$.

By using the general theory of Dirichlet forms, it has been already proved in [5, 28] (see also [35]) that there exists a diffusion process (i.e. a strong Markov process with continuous sample paths) on the configuration space that is canonically associated with the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$, i.e., for each $F \in L^2(\pi_m)$,

$$(e^{-tH^\Gamma} F)(\gamma) = \int_{\mathbf{\Omega}} F(\mathbf{X}_t)\, d\mathbf{P}_\gamma$$

for π_m-a.a. (or even quasi-every) $\gamma \in \Gamma_X$. This process is then the Brownian motion on Γ_X. Moreover, this process is, in fact, the well-known independent infinite particle process (cf. [5]). The latter is obtained by taking countably many independent Brownian motions on X, see [11].

The first part of this paper is devoted to deriving an explicit formula for the heat semigroup. We introduce functionals B_n, $n \in \mathbb{N}$, on Γ_X by

$$B_n(\gamma) := \sum_{x\in\gamma} \exp\big[-\tfrac{1}{n} \operatorname{dist}(x_0, x) \big],$$

where x_0 is a fixed point of the manifold X. We define a subset Γ_∞ of Γ_X consisting of those configurations γ for which $B_n(\gamma) < \infty$ for all $n \in \mathbb{N}$, and equip Γ_∞ with a metric in such a way that the convergence in Γ_∞ means vague convergence together with convergence of all the functionals B_n (see also [21]). Under some conditions on the geometry of the manifold X, we prove that Γ_∞ is of full π_m measure and that, for each $\gamma \in \Gamma_\infty$, $t > 0$, there exists a probability measure $\mathbf{P}_{t,\gamma}$ on Γ_∞ such that for each $F \in L^2(\Gamma_\infty, \pi_m)$

$$(e^{-tH^\Gamma} F)(\gamma) = \int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi), \qquad \pi_m\text{-a.a. } \gamma \in \Gamma_\infty. \tag{1.1}$$

To this end, we apply the method for constructing probability measures on Γ_X described in [39], and define $\mathbf{P}_{t,\gamma}$ via a product measure $\bigotimes_{k=1}^\infty p_{t,x_k}$ on $X^{\mathbb{N}}$. Here, $p_{t,x}(dy) := p(t, x, y)\, m(dy)$, $p(t, x, y)$ is the heat kernel of the manifold X, and $\gamma = \{x_k\}_{k=1}^\infty$ (the resulting measure $\mathbf{P}_{t,\gamma}$ will, however, be independent of the chosen ordering of the points of γ).

The second part of the paper is devoted to our main results which concern two types of Feller properties of the heat semigroup.

We introduce a class $\mathbf{D}$ of measurable functions on Γ_∞, which particularly contains all bounded local functions, and show that $\mathbf{D}$ is invariant under the action of the semigroup e^{-tH^Γ}. Moreover, we prove that, for each $F \in \mathbf{D}$, the map

$$\Gamma_\infty \ni \gamma \mapsto (\mathbf{P}_t F)(\gamma) := \int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) \in \mathbb{R}$$

is a continuous function on the space Γ_∞ ($\mathbf{P}_t F$ is even continuous with respect to some weaker metric). Thus, we obtain a kind of strong Feller property of the heat semigroup. Here, we use results on harmonic analysis over configuration spaces from [18, 19, 22] (see also [23, 24, 25, 26, 7]),

Next, we consider a metric space $\ddot{\Gamma}_\infty$ which is an appropriate extension of Γ_∞ to multiple configurations in X, i.e., to $\mathbb{Z}_+$-valued Radon measures on X. Restricting ourselves to the case $X = \mathbb{R}^d$, we prove that the operators $(\mathbf{P}_t)_{t>0}$ defined by

$$\ddot{\Gamma}_\infty \ni \gamma \mapsto (\mathbf{P}_t F)(\gamma) := \int_{\ddot{\Gamma}_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) \in \mathbb{R}$$

preserve the class of all bounded functions on $\ddot{\Gamma}_\infty$ which are continuous with respect to the intrinsic metric of the Dirichlet form $\mathcal{E}^\Gamma$ (see [34]). Thus, for this metric, we have the usual Feller property of the heat semigroup.

In the third part of the paper, which is more probabilistic, we present a direct construction of the independent infinite particle process on the manifold X with the state space $\ddot{\Gamma}_\infty$, which will be therefore a realization of the Brownian motion on the configuration space mentioned above. We show that, if the dimension of X is ≥ 2, the constructed process is the unique continuous strong Markov process on Γ_∞ whose transition probabilities are given by $\mathbf{P}_t(\gamma, \cdot) := \mathbf{P}_{t,\gamma}(\cdot)$. In particular, it starts at any configuration in Γ_∞ and never leaves Γ_∞. If $\dim X = 1$, one cannot exclude collisions of the particles, but it is still possible to realize the Brownian motion on the configuration space as a continuous Markov process on $\ddot{\Gamma}_\infty$. Finally, we describe a path-wise construction of the infinite particle process starting from any point in Γ_∞ (respectively $\ddot{\Gamma}_\infty$). More precisely, we show that the obvious heuristic construction can be performed rigorously.

We should mention that independent infinite particle processes have been studied by many authors, see e.g. [36], but neither in this paper, nor in any other reference we are aware of, it was proved that the process takes values in the configuration space for all values of $t > 0$.

2 Intrinsic Dirichlet operator on the Poisson space

In this section, we will briefly recall the definition and some properties of the intrinsic Dirichlet operator on the Poisson space. We refer the reader to [5, 3, 4] for details and proofs.

Let X be a complete, connected, oriented C^∞ Riemannian manifold. Let m denote the volume measure on X, and we suppose that $m(X) = \infty$. Let ∇^X and $H^X := -\frac{1}{2}\Delta^X$ be the gradient and Laplace–Beltrami operator on X, respectively. We denote by $\mathcal{D} := C_0^\infty(X)$ the space of all C^∞ functions on X with compact support. It is well-known that $(H^X, \mathcal{D})$ is essentially selfadjoint on $L^2(m) := L^2(X, \mathcal{B}(X), m)$, where $\mathcal{B}(X)$ is the Borel σ-algebra on X. In what follows, we will always suppose that H^X is conservative (cf. e.g. [38]).

Let $p(t, x, y)$, $t \in (0, \infty)$, $x, y \in X$, denote the heat kernel of the operator H^X:

$$(e^{-tH^X}\varphi)(x) = \int_X \varphi(y) p(t, x, y)\, m(dy), \qquad m\text{-a.e. } x \in X, \tag{2.1}$$

where φ is a bounded measurable function on X. We recall that $p(t, x, y)$ is a strictly positive C^∞ function on $(0, \infty) \times X \times X$ (cf. e.g. [10]).

The conservativity condition yields, in particular, that

$$\int_X p(t,x,y)\,m(dy) = 1, \qquad t \in (0,\infty),\ x \in X, \tag{2.2}$$

i.e., for each $t > 0$ and $x \in X$ the heat kernel determines a probability measure

$$p_{t,x}(dy) := p_t(x,dy) = p(t,x,y)\,m(dy) \tag{2.3}$$

on X. Thus, the manifold X is stochastically complete.

Next, we consider the configuration space Γ_X over X—the set of all infinite subsets in X which are locally finite:

$$\Gamma_X := \{\gamma \subset X \mid |\gamma| = \infty \text{ and } |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here, $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma_X$ with the positive Radon measure

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(X),$$

where $\mathcal{M}(X)$ stands for the set of all positive Radon measures on $\mathcal{B}(X)$. The space Γ_X can be endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on Γ_X with respect to which all maps

$$\Gamma_X \ni \gamma \mapsto \langle \varphi, \gamma\rangle := \int_X \varphi(x)\,\gamma(dx) = \sum_{x\in\gamma} \varphi(x), \qquad \varphi \in \mathcal{D},$$

are continuous. We shall denote the Borel σ-algebra on Γ_X by $\mathcal{B}(\Gamma_X)$.

Let π_m denote the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity m. This measure can be characterized by its Laplace transform

$$\ell_{\pi_m}(\varphi) := \int_{\Gamma_X} e^{\langle\varphi,\gamma\rangle}\,\pi_m(d\gamma) = \exp\bigg(\int_X (e^{\varphi(x)} - 1)\,m(dx)\bigg), \qquad \varphi \in \mathcal{D}. \tag{2.4}$$

We refer to e.g. [39, 37, 5] for a detailed discussion of the construction of the Poisson measure on the configuration space. Now, we recall how to define the intrinsic Dirichlet operator H^Γ in the space $L^2(\pi_m) := L^2(\Gamma_X, \mathcal{B}(\Gamma_X), \pi_m)$.

Let $T_x(X)$ denote the tangent space to X at a point $x \in X$. The tangent space to Γ_X at a point $\gamma \in \Gamma_X$ is defined as the Hilbert space

$$T_\gamma(\Gamma_X) := \bigoplus_{x\in\gamma} T_x(X).$$

Thus, each $V(\gamma) \in T_\gamma(\Gamma_X)$ has the form $V(\gamma) = (V(\gamma, x))_{x\in\gamma}$, where $V(\gamma, x) \in T_x(X)$, and

$$\|V(\gamma)\|^2_{T_\gamma(\Gamma_X)} = \sum_{x\in\gamma} \|V(\gamma, x)\|^2_{T_x(X)}.$$

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. We denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \varnothing$. Now, for a function $F\colon \Gamma_X \to \mathbb{R}$, $\gamma \in \Gamma_X$, and $x \in \gamma$, we define a function $F_x(\gamma, \cdot)\colon \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by

$$\mathcal{O}_{\gamma,x} \ni y \mapsto F_x(\gamma, y) := F((\gamma \setminus \{x\}) \cup \{y\}) \in \mathbb{R}.$$

We say that a function $F\colon \Gamma_X \to \mathbb{R}$ is differentiable at $\gamma \in \Gamma_X$ if for each $x \in \gamma$ the function $F_x(\gamma, \cdot)$ is differentiable at x and

$$\nabla^\Gamma F(\gamma) := (\nabla^X F_x(\gamma, x))_{x\in\gamma} \in T_\gamma(\Gamma_X),$$

where

$$\nabla^X F_x(\gamma, x) := \nabla^X_y F_x(\gamma, y)\big|_{y=x}$$

(cf. [1, 2]). Evidently, this definition is independent of the choice of the set $\mathcal{O}_{\gamma,x}$. We will call $\nabla^\Gamma F(\gamma)$ the gradient of F at γ.

We introduce the set $\mathcal{F}C^\infty_{\mathrm{b}}\big(\mathcal{D}, \Gamma_X\big)$ consisting of all smooth cylinder functions on Γ_X, i.e., all functions of the form

$$F(\gamma) = g_F(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_N, \gamma\rangle), \qquad \gamma \in \Gamma_X, \tag{2.5}$$

where $N \in \mathbb{N}$, $\varphi_1, \dots, \varphi_N \in \mathcal{D}$, and $g_F \in C^\infty_{\mathrm{b}}(\mathbb{R}^N)$. Any function F of the form (2.5) is differentiable at each point $\gamma \in \Gamma_X$, and its gradient is given by

$$(\nabla^\Gamma F)(\gamma) = \sum_{j=1}^N \partial_j\, g_F(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_N, \gamma\rangle)\nabla^X \varphi_j, \tag{2.6}$$

where $\partial_j\, g_F$ means derivative with respect to the j-th coordinate.

Then, the corresponding pre-Dirichlet form is

$$\mathcal{E}^\Gamma(F, G) := \frac{1}{2}\int_{\Gamma_X} \langle\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma_X)}\, \pi_m(d\gamma), \qquad F, G \in \mathcal{F}C^\infty_{\mathrm{b}}\big(\mathcal{D}, \Gamma_X\big). \tag{2.7}$$

By using the integration by parts formula on the Poisson space, one shows that the associated Dirichlet operator H^Γ, i.e., the operator satisfying

$$\mathcal{E}^\Gamma(F, G) = (H^\Gamma F, G)_{L^2(\pi_m)}, \qquad F, G \in \mathcal{F}C^\infty_{\mathrm{b}}\big(\mathcal{D}, \Gamma_X\big),$$

is of the form

$$(H^\Gamma F)(\gamma) = -\sum_{i,j=1}^{N} \partial_i\,\partial_j\, g_F(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle) \int_X \frac{1}{2}\,\langle\nabla^X\varphi_i(x),\nabla^X\varphi_j(x)\rangle_{T_x(X)}\,\gamma(dx)$$
$$+\sum_{j=1}^{N}\partial_j\, g_F(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\int_X (H^X\varphi_j)(x)\,\gamma(dx),$$

where F is given by (2.5). Therefore, the bilinear form $(\mathcal{E}^\Gamma, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma_X))$ is closable on $L^2(\pi_m)$, and with its closure we can associate a positive definite selfadjoint operator, the Friedrichs extension of H^Γ, which will be also denoted by H^Γ. (In fact, $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma_X)$ is a domain of essential selfadjointness of H^Γ, see [5, Theorem 5.3].)

Consider the corresponding heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$ in $L^2(\pi_m)$, where as usual $\mathbb{R}_+ := [0,\infty)$. We set

$$E(\mathcal{D}_1,\Gamma) := \mathrm{l.\,h.}\left\{\ \exp\left[\,\langle\log(1+\varphi),\cdot\rangle\,\right] \mid \varphi\in\mathcal{D}_1\,\right\}.$$

Here, l.h. means linear hull and

$$\mathcal{D}_1 := \left\{\,\varphi\in D(H^X)\cap L^1(m)\mid H^X\varphi\in L^1(m) \text{ and } -\delta\le\varphi\le 0 \text{ for some } \delta\in(0,1)\,\right\}. \tag{2.8}$$

Proposition 2.1 *We have*

$$e^{-tH^\Gamma}\exp\left[\,\langle\log(1+\varphi),\cdot\rangle\,\right] = \exp\left[\,\langle\log(1+e^{-tH^X}\varphi),\cdot\rangle\,\right] \qquad \pi_m\text{-a.e. for all } \varphi\in\mathcal{D}_1.$$

Proof. See [5, Proposition 4.1]. ∎

As a direct consequence of this proposition, in particular, one obtains that $(H^\Gamma, E(\mathcal{D}_1,\Gamma))$ is essentially selfadjoint on $L^2(\pi_m)$.

Finally, the diffusion process that is properly associated with the Dirichlet form $(\mathcal{E}^\Gamma, \mathrm{Dom}(\mathcal{E}^\Gamma))$ is the usual independent infinite particle process, or in other terms, Brownian motion on Γ_X (cf. [5, Subsection 6.2]).

3 Correlation measures in configuration space analysis

In this section, we shall recall some facts on K-transforms and correlation measures. We shall follow [18, 22] (see also [23, 24, 25, 26, 19, 20, 7]; in [18, 22, 7] the reader can also find many further references and historical comments).

Denote by $\Gamma_{X,0}$ the space of all finite configurations over X:

$$\Gamma_{X,0} := \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}, \qquad \Gamma_X^{(0)} = \{\varnothing\}, \qquad \Gamma_X^{(n)} = \{\eta \subset X \mid |\eta| = n\}, \qquad n \in \mathbb{N}.$$

Let

$$\widetilde{X}^n = \left\{ (x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ when } i \neq j \right\}$$

and let S_n denote the group of permutations of $\{1, \dots, n\}$, which acts on $\widetilde{X}^n$ by

$$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}), \qquad \sigma \in S_n.$$

Through the natural bijection

$$\widetilde{X}^n / S_n \mapsto \Gamma_X^{(n)} \tag{3.1}$$

one defines a topology on $\Gamma_X^{(n)}$. The space $\Gamma_{X,0}$ is then equipped with the topology of disjoint unions. Let $\mathcal{B}(\Gamma_{X,0})$ denote the corresponding Borel σ-algebra. A set $K \subset \Gamma_{X,0}$ is compact iff there exists $N \in \mathbb{N}$ with $K \cap \Gamma_X^{(n)} = \varnothing$ for all $n > N$ and $K \cap \Gamma_X^{(n)}$ is compact for all $n \leq N$. The set of all Borel sets in $\Gamma_{X,0}$ with compact closure is denoted by $\mathcal{B}_{\mathrm{c}}(\Gamma_{X,0})$.

A $\mathcal{B}(\Gamma_{X,0})$-measurable function $G \colon \Gamma_{X,0} \to \mathbb{R}$ is said to have bounded support if there exist a relatively compact open set $\Lambda \subset X$ and $N \in \mathbb{N}$ such that $\{G \neq 0\} \subset \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}$. The space of bounded functions on $\Gamma_{X,0}$ with bounded support is denoted by $B_{\mathrm{bs}}(\Gamma_{X,0})$.

In what follows, for any $\gamma \in \Gamma_X$, we shall use the notation $\sum_{\eta \Subset \gamma}$ for the summation over all $\eta \subset \gamma$ such that $|\eta| < \infty$. For a function $G \colon \Gamma_{X,0} \to \mathbb{R}$, the K-transform of G is then defined by

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta) \tag{3.2}$$

for each $\gamma \in \Gamma_X$ such that at least one of the series $\sum_{\eta \Subset \gamma} G^+(\eta)$ or $\sum_{\eta \Subset \gamma} G^-(\eta)$ converges in $\mathbb{R}_+$, where $G^+(\gamma) := \max\{0, G(\gamma)\}$, $G^-(\gamma) = -\min\{0, G(\gamma)\}$. For each $G \in B_{\mathrm{bs}}(\Gamma_{X,0})$ and each $\gamma \in \Gamma_X$, the series $\sum_{\eta \Subset \gamma} G(\eta)$ is always finite, and moreover, $(KG)(\cdot)$ is a $\mathcal{B}(\Gamma_X)$-measurable function on Γ_X (cf. [18, Proposition 3.5]).

Let μ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. The correlation measure corresponding to μ is defined by

$$\rho_\mu(A) := \int_{\Gamma_X} (K\mathbf{1}_A)(\gamma)\, \mu(d\gamma), \qquad A \in \mathcal{B}(\Gamma_{X,0}).$$

ρ_μ is obviously a measure on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$.

Proposition 3.1 *Let μ be a probability measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$. Then, the measure ρ_μ is locally finite, i.e.,*

$$\rho_\mu(A) < \infty \quad \text{for all } A \in \mathcal{B}_{\mathrm{c}}(\Gamma_{X,0}), \tag{3.3}$$

if and only if

$$\int_{\Gamma_X} |\gamma_\Lambda|^n \, \mu(d\gamma) < \infty \quad \text{for all } n \in \mathbb{N} \text{ and } \Lambda \in \mathcal{B}_{\mathrm{c}}(X). \tag{3.4}$$

Proof. See [18, Proposition 4.2]. ■

We say that a measure μ satisfying (3.4) has finite local moments and denote the set of all such measures on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ by $\mathcal{M}_{\mathrm{fm}}(\Gamma_X)$. The set of all locally finite measures on $\Gamma_{X,0}$ will be denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_{X,0})$.

Proposition 3.2 *Let $\mu \in \mathcal{M}_{\mathrm{fm}}(\Gamma_X)$ and let $G \colon \Gamma_{X,0} \to \mathbb{R}$ be a measurable function which is integrable with respect to the measure ρ_μ. Then, KG is well-defined and finite μ-a.e., and integrable with respect to the measure μ. If for some $G' \colon \Gamma_{X,0} \to \mathbb{R}$, $G = G'$ ρ_μ-a.e., then $KG = KG'$ μ-a.e., and hence the K-transform defines a linear mapping*

$$K \colon L^1(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}), \rho_\mu) \to L^1(\Gamma_X, \mathcal{B}(\Gamma_X), \mu).$$

Furthermore, we have

$$\|KG\|_{L^1(\mu)} \le \|K|G|\,\|_{L^1(\mu)} = \|G\|_{L^1(\rho_\mu)}$$

and

$$\int_{\Gamma_{X,0}} G(\eta)\, \rho_\mu(d\eta) = \int_{\Gamma_X} (KG)(\gamma)\, \mu(d\gamma).$$

Proof. See [18, Theorem 4.11]. ■

For two functions $G_1, G_2 \colon \Gamma_{X,0} \to \mathbb{R}$, the $\star$-convolution of G_1 and G_2 is defined as the mapping $G_1 \star G_2 \colon \Gamma_{X,0} \to \mathbb{R}$ given by

$$(G_1 \star G_2)(\eta) := \sum_{(\eta_1, \eta_2, \eta_3) \in \mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3), \tag{3.5}$$

where $\mathcal{P}_3(\eta)$ denotes the set of all ordered partitions (η_1, η_2, η_3) of η into 3 parts. Clearly, if G_1, G_2 are $\mathcal{B}(\Gamma_{X,0})$-measurable, then so is $G_1 \star G_2$. The main property of the $\star$-convolution is given by the following formula (see [18, Proposition 3.11]):

$$(K(G_1 \star G_2))(\gamma) = (KG_1)(\gamma) \cdot (KG_2)(\gamma), \tag{3.6}$$

provided $(KG_1)(\gamma)$ and $(KG_2)(\gamma)$ exist.

Let σ be a non-atomic Radon measure. The Lebesgue–Poisson measure λ_σ on $(\Gamma_{X,0}, \mathcal{B}(\Gamma_{X,0}))$ with intensity σ is defined by

$$\lambda_\sigma := \varepsilon_\varnothing + \sum_{n=1}^{\infty} \frac{1}{n!}\, \sigma^{\otimes n},$$

where the measure $\sigma^{\otimes n}$ is defined on $\Gamma_X^{(n)}$ via the bijection (3.1).

Finally, let us introduce the notion of correlation functions. Suppose that a measure $\rho \in \mathcal{M}_{\mathrm{lf}}(\Gamma_{X,0})$ is absolutely continuous with respect to the Lebesgue–Poisson measure λ_m with intensity m, and define the functions $k^{(n)} \colon \Gamma_X^{(n)} \to \mathbb{R}$ as the restrictions of the Radon–Nikodym derivative $k := \dfrac{d\rho}{d\lambda_m}$ to $\Gamma_X^{(n)}$. In the case where $\rho = \rho_\mu$ is a correlation measure, the functions $(k_\mu^{(n)})_{n=1}^\infty$ are called correlation functions of the measure μ.

4 Heat kernel measures $\mathbf{P}_{t,\gamma}$

In this section, we shall construct a family of probability measures $\mathbf{P}_{t,\gamma}$ on the configuration space so that $\mathbf{P}_{t,\bullet}(\cdot)$ is the kernel of the integral operator e^{-tH^Γ}.

First, we recall the construction of probability measures on the configuration space Γ_X proposed by A. M. Vershik et al. [39], see also [16].

Let us consider the infinite product $X^{\mathbb{N}} = \mathop{\times}\limits_{k=1}^{\infty} X_k$, $X_k = X$, furnished with the product topology, and let $\mathcal{B}(X^{\mathbb{N}})$ denote the Borel σ-algebra on $X^{\mathbb{N}}$. We define $\widetilde{X}^{\mathbb{N}}$ as the set of all elements $(x_1, x_2, \dots) \in X^{\mathbb{N}}$ such that 1) $x_i \neq x_j$ when $i \neq j$, and 2) the sequence $\{x_k\}_{k=1}^\infty$ has no accumulation points in X. Evidently,

$$\widetilde{X}^{\mathbb{N}} = \left[\bigcap_{i\neq j} \{\, (x_1, x_2, \dots) \in X^{\mathbb{N}} : x_i \neq x_j \,\}\right] \cap \\ \cap \left[\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{\, (x_1, x_2, \dots) \in X^{\mathbb{N}} : \forall l \geq k \ d(x_0, x_l) \geq n \,\}\right], \quad (4.1)$$

where x_0 is a fixed point of X and $d(\cdot,\cdot)$ denotes the distance on X. Hence, $\widetilde{X}^{\mathbb{N}} \in \mathcal{B}(X^{\mathbb{N}})$.

Let ν_k, $k \in \mathbb{N}$, be nonatomic probability measures on $(X, \mathcal{B}(X))$ and consider the product measure $\nu := \bigotimes_{k=1}^\infty \nu_k$ on $(X^{\mathbb{N}}, \mathcal{B}(X^{\mathbb{N}}))$. (4.1) and the Borel–Cantelli lemma imply the following:

Lemma 4.1 [39] *We have $\nu(\widetilde{X}^{\mathbb{N}}) = 0$ or 1, and $\nu(\widetilde{X}^{\mathbb{N}}) = 1$ if and only if*

$$\sum_{k=1}^{\infty} \nu_k(\Lambda) < \infty \qquad \text{for each compact } \Lambda \subset X. \quad (4.2)$$

Let S_∞ denote the group of all permutations of the sequence of natural numbers, which acts on $X^{\mathbb{N}}$:

$$\sigma(y_1, y_2, \dots) = (y_{\sigma(1)}, y_{\sigma(2)}, \dots), \qquad \sigma \in S_\infty.$$

The space $\widetilde{X}^{\mathbb{N}}$ is invariant under the action of S_∞. Through the natural bijection $\widetilde{X}^{\mathbb{N}}/S_\infty \mapsto \Gamma_X$, we shall identify these two spaces. Let $I\colon \widetilde{X}^{\mathbb{N}} \to \Gamma_X$ be given by

$$\widetilde{X}^{\mathbb{N}} \ni \mathbf{x} = (x_1, x_2, \dots) \mapsto I\mathbf{x} = \{x_1, x_2, \dots\} \in \Gamma_X. \tag{4.3}$$

Thus, I maps an element $\mathbf{x} \in \widetilde{X}^{\mathbb{N}}$ into the corresponding equivalence class $[\mathbf{x}] \in \widetilde{X}^{\mathbb{N}}/S_\infty$.

The mapping $I\colon \widetilde{X}^{\mathbb{N}} \to \Gamma_X$ defined by (4.3) is $\mathcal{B}(\widetilde{X}^{\mathbb{N}})$-$\mathcal{B}(\Gamma_X)$-measurable (here $\mathcal{B}(\widetilde{X}^{\mathbb{N}})$ denotes the trace σ-algebra of $\mathcal{B}(X^{\mathbb{N}})$ on $\widetilde{X}^{\mathbb{N}}$). Indeed, the σ-algebra $\mathcal{B}(\Gamma_X)$ is generated by the sets of the form

$$A_{\Lambda,n} = \big\{\, \gamma \in \Gamma_X\colon \langle \mathbf{1}_\Lambda, \gamma\rangle = n \,\big\},$$

where $n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, Λ is a compactum in X, and $\mathbf{1}_\Lambda$ is the indicator of Λ (see e.g. [17, 29]). Then,

$$I^{-1}(A_{\Lambda,n}) = \bigcup_{\sigma \in S_\infty^{\mathrm{fin}}} \big\{\, (x_1, x_2, \dots) \in \widetilde{X}^{\mathbb{N}}\colon x_{\sigma(i)} \in \Lambda,\ i = 1, \dots, n,\ x_{\sigma(i)} \in \Lambda^{\mathrm{c}},\ i \geq n+1 \,\big\}, \tag{4.4}$$

where S_∞^{fin} denotes the group of all finite permutations of the sequence of natural numbers, and $\Lambda^{\mathrm{c}} := X \setminus \Lambda$. Since S_∞^{fin} has a countable number of elements, we conclude from (4.4) that $I^{-1}(A_{\Lambda,n}) \in \mathcal{B}(\widetilde{X}^{\mathbb{N}})$, which implies the measurability of I.

Hence, if the measures ν_k, $k \in \mathbb{N}$, satisfy condition (4.2), we can consider the image of the probability measure ν on $\widetilde{X}^{\mathbb{N}}$ under the mapping I, which is a probability measure on Γ_X. Evidently, this image-measure is independent of the order of the ν_k's, that is, it coincides with the measure on Γ_X constructed through the product-measure $\bigotimes_{k=1}^\infty \nu_{\sigma(k)}$ for each $\sigma \in S_\infty$.

Let now $t > 0$ and let γ be a fixed point of Γ_X such that

$$\sum_{x \in \gamma} p_{t,x}(\Lambda) < \infty \quad \text{for each compact } \Lambda \subset X, \tag{4.5}$$

where $p_{t,x}$ is as in (2.3). Define

$$\mathbf{P}_{t,\gamma} := \mathbb{P}_{t,\mathbf{x}} \circ I^{-1}, \text{ where } \mathbb{P}_{t,\mathbf{x}} := \bigotimes_{k=1}^{\infty} p_{t,x_k} \tag{4.6}$$

and where $\mathbf{x} = (x_k)_{k=1}^\infty \in \widetilde{X}^{\mathbb{N}}$ is an arbitrary element of the set $I^{-1}\{\gamma\}$ (the resulting measure $\mathbf{P}_{t,\gamma}$ being independent of the choice of $\mathbf{x}$).

Below, we shall need the correlation measure $\rho_{t,\gamma}$ of $\mathbf{P}_{t,\gamma}$.

Proposition 4.1 *Let $t>0$ and let $\gamma\in\Gamma_X$ satisfy* (4.5). *Then, the correlation measure $\rho_{t,\gamma}$ of $\mathbf{P}_{t,\gamma}$ is given by*

$$\begin{aligned}\rho_{t,\gamma}\restriction\Gamma_X^{(0)}&:=\rho_{t,\gamma}^{(0)}:=\varepsilon_\varnothing,\\ \rho_{t,\gamma}\restriction\Gamma_X^{(n)}&:=\rho_{t,\gamma}^{(n)}:=\sum_{\theta\Subset\gamma:\,|\theta|=n}\left(\bigotimes_{x\in\theta}p_{t,x}\right)\circ T_n^{-1},\qquad n\in\mathbb{N},\end{aligned}\tag{4.7}$$

where $T_n\colon\widetilde{X}^n\to\Gamma_X^{(n)}$ is the composition of the natural quotient map $\widetilde{X}^n\to\widetilde{X}^n/S_n$ and the bijection (3.1) (*the measure $(\otimes_{x\in\theta}p_{t,x})\circ T_n^{-1}$ is independent of a chosen order of the product of the measures $p_{t,x}$*). *Moreover, we have*

$$\rho_{t,\gamma}(\Gamma_\Lambda^{(n)})\le\frac{1}{n!}\left(\sum_{x\in\gamma}p_{t,x}(\Lambda)\right)^n<\infty\qquad\text{for each compact }\Lambda\subset X,$$

in particular, $\rho_{t,\gamma}\in\mathcal{M}_{\mathrm{lf}}(\Gamma_{X,0})$ and $\mathbf{P}_{t,\gamma}\in\mathcal{M}_{\mathrm{fm}}(\Gamma_X)$.

Proof. Let the measure $\rho_{t,\gamma}$ on $(\Gamma_{X,0},\mathcal{B}(\Gamma_{X,0}))$ be defined by (4.7). For a measurable function $G\colon\Gamma_{X,0}\to\mathbb{R}$, we have $G=(G^{(n)})_{n=0}^\infty$, where $G^{(n)}:=G\restriction\Gamma_X^{(n)}$. Then, by using the definition of $\mathbf{P}_{t,\gamma}$ and the monotone convergence theorem, we have, for any $\mathcal{B}(\Gamma_{X,0})$-measurable function $G\colon\Gamma_{X,0}\to\mathbb{R}_+$, that

$$\begin{aligned}\int_{\Gamma_X}(KG)(\xi)\,\mathbf{P}_{t,\gamma}(d\xi)&=\int_{\widetilde{X}^{\mathbb{N}}}(KG)(I\mathbf{y})\,\mathbb{P}_{t,\mathbf{x}}(d\mathbf{y})\\ &=G^{(0)}(\varnothing)+\sum_{n=1}^\infty\sum_{\{i_1,\dots,i_n\}\subset\{1,2,\dots\}}\int_{\widetilde{X}^{\mathbb{N}}}G^{(n)}\circ T_n(y_{i_1},\dots,y_{i_n})\bigotimes_{k=1}^\infty p_{t,x_k}(dy_k)\\ &=G^{(0)}(\varnothing)+\sum_{n=1}^\infty\sum_{\{i_1,\dots,i_n\}\subset\{1,2,\dots\}}\int_{\widetilde{X}^n}G^{(n)}\circ T_n(y_1,\dots,y_n)\,p_{t,x_{i_1}}\otimes\dots\otimes p_{t,x_{i_n}}(dy_1,\dots,dy_n)\\ &=G^{(0)}(\varnothing)+\sum_{n=1}^\infty\int_{\widetilde{X}^n}G^{(n)}\circ T_n(y_1,\dots,y_n)\sum_{\{i_1,\dots,i_n\}\subset\{1,2,\dots\}}p_{t,x_{i_1}}\otimes\dots\otimes p_{t,x_{i_n}}(dy_1,\dots,dy_n)\\ &=\int_{\Gamma_{X,0}}G(\eta)\,\rho_{t,\gamma}(d\eta),\end{aligned}\tag{4.8}$$

where $\mathbf{x}=(x_k)_{k=1}^\infty\in I^{-1}\{\gamma\}$. The final inequality in the assertion immediately follows from (4.5) and (4.7). Hence, the measure $\rho_{t,\gamma}$ is from $\mathcal{M}_{\mathrm{lf}}(\Gamma_{X,0})$, and therefore, by Proposition 3.1, $\mathbf{P}_{t,\gamma}\in\mathcal{M}_{\mathrm{fm}}(\Gamma_X)$. ■

Remark 4.1 One could also start with a measure $\rho_{t,\gamma}$ on $\Gamma_{X,0}$ that is given *a priori* by formula (4.7) for each $t>0$ and each $\gamma\in\Gamma_X$ satisfying (4.5), and then, using [18, Theorem 6.5], identify $\mathbf{P}_{t,\gamma}$ as the unique probability measure on Γ_X whose correlation measure is $\rho_{t,\gamma}$.

Our next aim is to show that condition (4.5) is satisfied for π_m-a.a. $\gamma \in \Gamma_X$, at least under some additional conditions on the manifold X.

Let us assume that the manifold X satisfies the following two conditions:

(C1) For each $t > 0$, there exist constants $C_t > 0$ and $\varepsilon_t > 0$ such that

$$p(t, x, y) \le C_t \exp\big[- d(x, y)^{1+\varepsilon_t} \big], \qquad t > 0,\ x, y \in X.$$

(C2) For some fixed $x_0 \in X$,

$$m\big(B(x_0, r)\big) \le c_{x_0}\, r^N, \qquad r > 0,$$

where $c_{x_0} > 0$, $N \in \mathbb{N}$, and $B(x_0, r)$ denotes the geodesic ball with center at x and radius r.

Concerning these conditions, in particular, the upper estimate of the heat kernel, we refer the reader e.g. to [10, 14, 15] and the references therein. For example, in the case of a manifold X of nonnegative Ricci curvature, one has

$$p(t, x, y) \le \frac{C}{m\big(B(y, \sqrt{t})\big)} \exp\left(- \frac{d(x, y)^2}{(4 + \varepsilon)t} \right), \qquad \varepsilon > 0, \tag{4.9}$$

$$m\big(B(x, r)\big) \le \mathrm{const}_d\, r^d \tag{4.10}$$

(d being the dimension of X). Thus, conditions (C1) and (C2) are satisfied if the manifold X possesses the following additional property:

$$\forall r > 0: \qquad \inf_{x \in X} m\big(B(x, r)\big) > 0, \tag{4.11}$$

which is true, for example, in case of a manifold having bounded geometry (see [10]).

Now, we shall follow the idea of [21] to consider subsets of the configuration space on which some special functionals take finite values. So, for each $n \in \mathbb{N}$, we introduce the functional

$$B_n(\gamma) := \big\langle \exp\big[- \tfrac{1}{n}\, d(x_0, \cdot) \big], \gamma \big\rangle = \sum_{x \in \gamma} \exp\big[- \tfrac{1}{n}\, d(x_0, x) \big], \qquad \gamma \in \Gamma_X, \tag{4.12}$$

and define $\Gamma_n \in \mathcal{B}(\Gamma_X)$ by

$$\Gamma_n := \big\{ \gamma \in \Gamma_X : B_n(\gamma) < \infty \big\}. \tag{4.13}$$

Here, x_0 is as in (C2). Evidently, we have, for each $n \in \mathbb{N}$, $\Gamma_{n+1} \subset \Gamma_n$, and let

$$\Gamma_\infty := \bigcap_{n=1}^{\infty} \Gamma_n.$$

Let d_{V} be any metric on $\mathcal{M}(X)$ determining the vague topology. For example, we can take as d_{V} the metric d_{K} that was introduced in [32]:

$$d_{\mathrm{K}}(\nu_1,\nu_2) := \sum_{i=1}^{\infty} 2^{-i}\, d_{\mathrm{K},i}(\nu_1,\nu_2)/[1+d_{\mathrm{K},i}(\nu_1,\nu_2)], \qquad \nu_1,\nu_2\in\mathcal{M}(X),$$

where

$$d_{\mathrm{K},i}(\nu_1,\nu_2) := \sup\bigg\{ \Big| \int_X f\, d(\nu_1-\nu_2)\Big| :\ f: X\to\mathbb{R},\\ \sup_{x,y\in X} \frac{d(f(x),f(y))}{d(x,y)} \le 1,\ f(x)=0 \text{ if } d(x_0,x)\ge i \bigg\}.$$

The metric d_{K} is a generalization of the Kantorovich metric, and on any set of measures from $\mathcal{M}(X)$ which have uniformly bounded support, d_{K} is just equivalent to the Kantorovich metric.

Then, we can metrize the set Γ_∞ as follows: for $\gamma_1,\gamma_2\in\Gamma_\infty$

$$d_\infty(\gamma_1,\gamma_2) := d_{\mathrm{V}}(\gamma_1,\gamma_2) + \sum_{n=1}^{\infty} 2^{-n} |B_n(\gamma_1)-B_n(\gamma_2)| \big/ \big[1+|B_n(\gamma_1)-B_n(\gamma_2)|\big]. \quad (4.14)$$

Let $\mathcal{B}(\Gamma_\infty)$ denote the trace σ-algebra of $\mathcal{B}(\Gamma_X)$ on Γ_∞. It can be shown that this σ-algebra coincides with the Borel σ-algebra on Γ_∞ that corresponds to the topology generated by the d_∞ metric.

Proposition 4.2 *Let* (C1) *and* (C2) *be satisfied. Then,* Γ_∞ *is a set of full* π_m *measure. Furthermore, for each* $\gamma\in\Gamma_\infty$ *condition* (4.5) *is satisfied and* Γ_∞ *is a set of full* $\mathbf{P}_{t,\gamma}$ *measure for each* $t>0$.

Proof. We have by (C2) that

$$\begin{aligned}\int_{\Gamma_X} B_n(\gamma)\,\pi_m(d\gamma) &= \int_X \exp\big[-\tfrac{1}{n}\, d(x_0,x)\big]\, m(dx)\\ &= \sum_{k=1}^{\infty} \int_{B(x_0,k)\setminus B(x_0,k-1)} \exp\big[-\tfrac{1}{n}\, d(x_0,x)\big]\, m(dx)\\ &\le \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}(k-1)\big] m\big(B(x_0,k)\big)\\ &\le \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}(k-1)\big] c_{x_0} k^N < \infty. \qquad (4.15)\end{aligned}$$

Therefore, B_n is π_m-a.e. finite, i.e., $\pi_m(\Gamma_n) = 1$ for all $n \in \mathbb{N}$, which yields that $\pi_m(\Gamma_\infty) = 1$.

Next, from (C1) we get, for each $r > 0$, $t > 0$, and $\gamma \in \Gamma_\infty$,

$$\begin{aligned}
\sum_{x\in\gamma} p_{t,x}(B(x_0,r)) &= \sum_{x\in\gamma} \int_{B(x_0,r)} p(t,x,y)\, m(dy) \\
&\le \widetilde{C}_t \sum_{x\in\gamma} \int_{B(x_0,r)} \exp\big[-d(x,y)\big]\, m(dy) \\
&\le \widetilde{C}_t \sum_{x\in\gamma} \exp\big[-d(x_0,x)\big] \int_{B(x_0,r)} \exp\big[d(x_0,y)\big]\, m(dy) < \infty,
\end{aligned}$$

so that (4.5) is satisfied.

Finally, for each $\gamma \in \Gamma_\infty$, $t > 0$, and $n \in \mathbb{N}$, we get from (C1), (C2) (cf. also (4.15)), and the monotone convergence theorem that

$$\begin{aligned}
\int_{\Gamma_X} B_n(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) &= \sum_{x\in\gamma} \int_X \exp\big[-\tfrac{1}{n}\, d(x_0,y)\big] p(t,x,y)\, m(dy) \\
&\le \sum_{x\in\gamma} \int_X \exp\big[-\tfrac{1}{n}\, d(x_0,y)\big] C_t \exp\big[-d(x,y)^{1+\varepsilon_t}\big]\, m(dy) \\
&\le C_{t,n} \sum_{x\in\gamma} \int_X \exp\big[-\tfrac{1}{n}\, d(x_0,y)\big] \exp\big[-\tfrac{1}{2n}\, d(x,y)\big]\, m(dy) \\
&\le C_{t,n} \sum_{x\in\gamma} \exp\big[-\tfrac{1}{2n}\, d(x_0,x)\big] \int_X \exp\big[-\tfrac{1}{2n}\, d(x_0,y)\big]\, m(dy) < \infty,
\end{aligned} \tag{4.16}$$

which yields that $\mathbf{P}_{t,\gamma}(\Gamma_\infty) = 1$. ■

Remark 4.2 Let π_{zm} denote the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity zm, where $z > 0$. Since the correlation measure of π_{zm} is the Lebesgue–Poisson measure λ_{zm}, it follows from the proof of Proposition 4.2 that $\pi_{zm}(\Gamma_\infty) = 1$ for all $z > 0$. Furthermore, let $\mu_{\nu,m} := \int_0^\infty \pi_{zm}\, \nu(dz)$ be a mixed Poisson measure such that ν is a probability measure on $(0,\infty)$. Then, Γ_∞ is a set of full $\mu_{\nu,m}$ measure.

5 Explicit formula for the heat semigroup

Due to Proposition 4.2, we can consider π_m as a probability measure on $(\Gamma_\infty, \mathcal{B}(\Gamma_\infty))$. In this section, we shall derive an explicit formula for the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$.

Theorem 5.1 *Let the conditions* (C1) *and* (C2) *be satisfied. Then, for each* $F \in L^2(\Gamma_\infty, \pi_m)$*, we have*

$$(e^{-tH^\Gamma} F)(\gamma) = \int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) \tag{5.1}$$

for π_m*-a.a.* $\gamma \in \Gamma_\infty$.

Proof. We start with the following

Lemma 5.1 *Let* $\widetilde{\mathcal{D}}_1$ *denote the subset of* $\mathcal{D}_1$ *(see* (2.8)*) given by*

$$\widetilde{\mathcal{D}}_1 := \bigl\{\, \varphi \in \mathcal{D} \mid \exists \delta \in (0,1) : -\delta \le \varphi \le 0 \,\bigr\}.$$

Then, for any $\varphi \in \widetilde{\mathcal{D}}_1$, $\gamma \in \Gamma_\infty$*, and* $t > 0$*, we have*

$$\int_{\Gamma_\infty} \exp\bigl[\, \langle \log(1+\varphi), \xi\rangle \,\bigr]\, \mathbf{P}_{t,\gamma}(d\xi) = \exp\left[\left\langle \log\left(1 + \int \varphi\, dp_{t,\bullet}\right), \gamma \right\rangle \right].$$

Proof. First, we observe that by Proposition 4.2

$$\sum_{x\in\gamma} \int_X |\varphi(y)|\, p_{t,x}(dy) < \infty \qquad \text{for each } \gamma \in \Gamma_\infty \text{ and } \varphi \in \mathcal{D}.$$

But for each $\gamma \in \Gamma_\infty$, $\mathbf{x} = (x_k)_{k=1}^\infty \in I^{-1}\{\gamma\}$, and $\varphi \in \widetilde{\mathcal{D}}_1$,

$$\begin{aligned}\int_{\Gamma_\infty} \exp\bigl[\langle \log(1+\varphi), \xi\rangle\bigr]\, \mathbf{P}_{t,\gamma}(d\xi) &= \prod_{k=1}^\infty \int_X (1+\varphi(y))\, p_{t,x_k}(dy) \\ &= \exp\left[\left\langle \log\left(1 + \int \varphi\, dp_{t,\bullet}\right), \gamma \right\rangle \right]. \quad \blacksquare\end{aligned}$$

Lemma 5.2 *For any measurable function* $F\colon \Gamma_\infty \to \mathbb{R}_+$*, we have*

$$\int_{\Gamma_\infty}\int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi)\, \pi_m(d\gamma) = \int_{\Gamma_\infty} F(\gamma)\, \pi_m(d\gamma). \tag{5.2}$$

Proof. It is easy to check that $\bigl\{\, \exp\bigl[\langle \log(1+\varphi), \cdot\rangle\bigr] \mid \varphi \in \widetilde{\mathcal{D}}_1 \,\bigr\}$ is stable under multiplication and that it contains a countable subset separating the points of Γ_∞, so it generates $\mathcal{B}(\Gamma_\infty)$. Therefore, we only have to check (5.2) for $F := \exp\bigl[\langle \log(1+\varphi), \cdot\rangle\bigr]$, $\varphi \in \widetilde{\mathcal{D}}_1$. But for such functions (5.2) immediately follows from Lemma 5.1. Indeed, (2.4) extends to all functions $\varphi\colon X \to \mathbb{R}_+$ which are increasing limits of functions $\varphi_n \in \mathcal{D}$, $n \in \mathbb{N}$, such as $\log(1 + \int \varphi\, dp_{t,\bullet})$. Furthermore, $\iint \varphi\, dp_{t,\bullet}\, dm = \int \varphi\, dm$, since H^X is assumed to be conservative. $\blacksquare$

Now, we can easily finish the proof of the theorem. It follows from Lemma 5.2 that, if $A \in \mathcal{B}(\Gamma_\infty)$ is of zero π_m measure, then $\mathbf{P}_{t,\gamma}(A) = 0$ for π_m-a.e. $\gamma \in \Gamma_\infty$. Moreover, using the Cauchy-Schwarz inequality and Lemma 5.2, we get

$$\begin{aligned}\int_{\Gamma_\infty} \left(\int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) \right)^2 \pi_m(d\gamma) &\le \int_{\Gamma_\infty} \int_{\Gamma_\infty} |F(\xi)|^2\, \mathbf{P}_{t,\gamma}(d\xi)\, \pi_m(d\gamma) \\ &= \int_{\Gamma_\infty} |F(\gamma)|^2\, \pi_m(d\gamma).\end{aligned}$$

Thus, for each $t > 0$, we can define a linear continuous operator

$$\mathbf{P}_t : L^2(\Gamma_\infty, \pi_m) \to L^2(\Gamma_\infty, \pi_m)$$

by setting

$$(\mathbf{P}_t F)(\gamma) := \int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi).$$

By Proposition 2.1 and Lemma 5.1, the action of the operator $\mathbf{P}_t$ coincides with the action of the operator e^{-tH^Γ} on the set $\left\{ \exp\left[\langle \log(1+\varphi), \cdot\rangle\right] \mid \varphi \in \widetilde{\mathcal{D}}_1 \right\}$, which is total in $L^2(\Gamma_\infty, \pi_m)$ (i.e., its linear hull is a dense set in $L^2(\Gamma_\infty, \pi_m)$). Hence, we get the equality $e^{-tH^\Gamma} = \mathbf{P}_t$, which proves the theorem. $\blacksquare$

In what follows, for a measurable function F on Γ_∞, we set

$$(\mathbf{P}_t F)(\gamma) := \int_{\Gamma_\infty} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi), \qquad t > 0,\ \gamma \in \Gamma_\infty, \tag{5.3}$$

provided the integral on the right hand side exists. Hence, by virtue of Theorem 5.1, $\mathbf{P}_t F$ is a π_m-version of $e^{-tH^\Gamma} F$ for each $F \in L^2(\Gamma_\infty, \pi_m)$.

Remark 5.1 One can easily prove an explicit formula for the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$ in the weak sense. More specifically, we define for each $t > 0$ a function $R_t \colon \Gamma_{X,0} \times \Gamma_{X,0} \to \mathbb{R}$ setting: $R_t(\eta, \theta) := 0$ if $|\eta| \ne |\theta|$, $R_t(\{\varnothing\}, \{\varnothing\}) = 1$, and for $\eta = \{x_1, \dots, x_n\}$, $\theta = \{y_1, \dots, y_n\}$, $n \in \mathbb{N}$,

$$R_t(\eta, \theta) := \sum_{\sigma \in S_n} \prod_{k=1}^n p_t(x_k, y_{\sigma(k)}),$$

where $p_t(x, y) := p(t, x, y)$. Suppose that conditions (C1) and (C2) are satisfied. Then, for arbitrary measurable functions $G_1, G_2 \colon \Gamma_{X,0} \to \mathbb{R}$ such that

$$\int_{\Gamma_{X,0}} \int_{\Gamma_{X,0}} |G_1(\theta)| \cdot \left[R_t(\eta, \theta) \star_\eta |G_2(\eta)| \right] \lambda_m(d\eta)\, \lambda_m(d\theta) < \infty \qquad \text{for all } t > 0$$

($\star_\eta$ denoting the $\star$-convolution with respect to the η variable), we have

$$\int_{\Gamma_\infty} |\mathbf{P}_t F_1(\gamma)|\, |F_2(\gamma)|\, \pi_m(d\gamma) < \infty,$$

where $F_1(\gamma) = (KG_1)(\gamma)$, $F_2(\gamma) = (KG_2)(\gamma)$, and

$$\int_{\Gamma_\infty} (\mathbf{P}_t F_1)(\gamma) F_2(\gamma)\, \pi_m(d\gamma) = \int_{\Gamma_{X,0}} \int_{\Gamma_{X,0}} G_1(\theta) \cdot \big[R_t(\eta,\theta) \star_\eta G_2(\eta)\big]\, \lambda_m(d\eta)\, \lambda_m(d\theta).$$

Now, we define a family of probability kernels $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$ on the space $(\Gamma_\infty, \mathcal{B}(\Gamma_\infty))$ setting

$$\mathbf{P}_t(\gamma, A) := \mathbf{P}_{t,\gamma}(A), \qquad \gamma \in \Gamma_\infty,\ A \in \mathcal{B}(\Gamma_\infty),\ t \in \mathbb{R}_+, \tag{5.4}$$

where

$$\mathbf{P}_{0,\gamma} := \varepsilon_\gamma. \tag{5.5}$$

Since $\gamma \mapsto \mathbf{P}_t F(\gamma)$ is measurable for F in the linear span of $\big\{ \exp\big[\langle \log(1+\varphi), \cdot\rangle\big] \mid \varphi \in \widetilde{\mathcal{D}}_1 \big\}$ by Lemma 5.1, a monotone class argument shows that, indeed, $\gamma \mapsto \mathbf{P}_t(\gamma, A)$ is $\mathcal{B}(\Gamma_\infty)$-measurable for all $A \in \mathcal{B}(\Gamma_\infty)$.

We finish this section with the following proposition.

Proposition 5.1 *Let* (C1) *and* (C2) *be satisfied. Then,* $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$ *is a Markov semigroup of kernels on* $(\Gamma_\infty, \mathcal{B}(\Gamma_\infty))$.

Proof. The Markov property of the kernels $\mathbf{P}_t$, i.e., $\mathbf{P}_t(\gamma, \Gamma_\infty) = 1$, $\gamma \in \Gamma_\infty$, follows from Proposition 4.2.

Let us show the semigroup property: $\mathbf{P}_t\mathbf{P}_s = \mathbf{P}_{t+s}$, $t, s \in \mathbb{R}_+$. To this end, we fix $t, s > 0$, $\gamma \in \Gamma_\infty$, and $A \in \mathcal{B}(\Gamma_\infty)$. Then, by the construction of the measure $\mathbf{P}_{t,\gamma}$ and the semigroup property of the heat kernel on X, we get

$$\begin{aligned}(\mathbf{P}_t\mathbf{P}_s)(\gamma, A) &= \int_{\Gamma_\infty} \mathbf{P}_{s,\xi}(A)\, \mathbf{P}_{t,\gamma}(d\xi) = \int_{\widetilde{X}^{\mathbb{N}}} \mathbf{P}_{s,I\mathbf{y}}(A)\, \mathbb{P}_{t,\mathbf{x}}(d\mathbf{y}) \\ &= \int_{\widetilde{X}^{\mathbb{N}}} \mathbb{P}_{s,\mathbf{y}}(I^{-1}A)\, \mathbb{P}_{t,\mathbf{x}}(d\mathbf{y}) = \int_{X^{\mathbb{N}}} \mathbb{P}_{s,\mathbf{y}}(I^{-1}A)\, \mathbb{P}_{t,\mathbf{x}}(d\mathbf{y}) \\ &= \mathbb{P}_{t+s,\mathbf{x}}(I^{-1}A) = \mathbf{P}_{t+s,\gamma}(A) = \mathbf{P}_{t+s}(\gamma, A),\end{aligned}$$

where $\mathbf{x} \in I^{-1}\{\gamma\}$. $\blacksquare$

6 A strong Feller property of the heat semigroup

Let us introduce a new metric d_1 on the set Γ_∞ as follows:

$$d_1(\gamma_1,\gamma_2) := d_{\mathrm{V}}(\gamma_1,\gamma_2) + |B_1(\gamma_1) - B_1(\gamma_2)|, \qquad \gamma_1,\gamma_2 \in \Gamma_\infty.$$

Evidently, convergence with respect to the d_∞ metric implies convergence with respect to the d_1 metric.

In this section, we shall show that the "concrete version" $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$ of the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$ constructed in the previous section possesses a kind of strong Feller property with respect to the metric d_1, and therefore also with respect to d_∞.

Theorem 6.1 *Let* (C1) *and* (C2) *hold. Let* $G\colon \Gamma_{X,0} \to \mathbb{R}$ *be a measurable function satisfying the following condition*:

$$\forall c > 0: \quad \int_{\Gamma_{X,0}} |G(\eta)|\, \lambda_{m_c}(d\eta) < \infty, \tag{6.1}$$

where, for each $c > 0$, λ_{m_c} *is the Lebesgue–Poisson measure on* $\Gamma_{X,0}$ *with intensity*

$$m_c(dx) := c\, e^{d(x_0,x)}\, m(dx). \tag{6.2}$$

Then, for each $t > 0$, *the function*

$$\Gamma_\infty \ni \gamma \mapsto (\mathbf{P}_t(KG))(\gamma) = \int_{\Gamma_\infty} (KG)(\xi)\, \mathbf{P}_{t,\gamma}(d\xi) \in \mathbb{R} \tag{6.3}$$

is continuous with respect to the metric d_1.

Proof. Let $\gamma\in\Gamma_\infty$. By Propositions 4.1, 4.2 and the definition of correlation functions, we see that for each $t > 0$ the measure $\rho_{t,\gamma}$ is absolutely continuous with respect to the Lebesgue–Poisson measure λ_m, and the correlation functions $k^{(n)}_{t,\gamma}$ of $\mathbf{P}_{t,\gamma}$ are given by

$$k^{(n)}_{t,\gamma}(\theta) = \sum_{(i_1,\dots,i_n)\in\widetilde{\mathbb{N}}^n} \prod_{k=1}^n p_t(x_{i_k}, y_k) \quad \text{for } \gamma = \{x_i\}_{i=1}^\infty \text{ and } \theta = \{y_1,\dots,y_n\}, \tag{6.4}$$

where

$$\widetilde{\mathbb{N}}^n := \big\{\, (i_i,\dots,i_n)\in\mathbb{N}^n : i_k \ne i_l \text{ if } k\ne l \,\big\}.$$

Denote

$$k_t(\gamma,\theta) := k_{t,\gamma}(\theta) := \frac{d\rho_{t,\gamma}}{d\lambda_m}(\theta), \tag{6.5}$$

so that $k_{t,\gamma}(\theta) = k^{(n)}_{t,\gamma}(\theta)$ for $|\theta| = n$.

By using (C1), we get

$$
\begin{aligned}
|k_t(\gamma,\{y_1,\dots,y_n\})| &\le \prod_{k=1}^n \Bigg(\sum_{x\in\gamma} p_t(x,y_k)\Bigg)\\
&\le \prod_{k=1}^n \Bigg(\sum_{x\in\gamma} C_t' \exp\big[-d(x,y_k)\big]\Bigg)\\
&\le \Bigg(C_t'\sum_{x\in\gamma}\exp\big[-d(x_0,x)\big]\Bigg)^n \exp\big[\,d(x_0,y_1)+\dots+d(x_0,y_n)\big].
\end{aligned}
\tag{6.6}
$$

Hence, (6.1) and (6.2) imply that $G\in L^1(\Gamma_{X,0},\mathcal{B}(\Gamma_{X,0}),\rho_{t,\gamma})$. Therefore, if $\gamma_j\to\gamma$ in Γ_∞ with respect to d_1, by Proposition 3.2 and (6.5) we have to prove that

$$\int_{\Gamma_{X,0}} G(\eta)k_t(\gamma^j,\eta)\,\lambda_m(d\eta)\to\int_{\Gamma_{X,0}} G(\eta)k_t(\gamma,\eta)\,\lambda_m(d\eta)\qquad\text{as } j\to\infty. \tag{6.7}$$

First, we show that

$$k_t(\gamma^j,\eta)\to k_t(\gamma,\eta)\quad\text{as } j\to\infty \text{ for each fixed } \eta\in\Gamma_{X,0}. \tag{6.8}$$

Since $\gamma^j\to\gamma$ in the d_1 metric, we have, particularly, that $\gamma^j\to\gamma$ in the d_{V} metric. We claim that that there exists a numeration of the points of the configurations γ^j, $j\in\mathbb{N}$, and of γ such that

$$\gamma^j=\{x_k^j\}_{k=1}^\infty,\quad \gamma=\{x_k\}_{k=1}^\infty,\qquad \forall k\in\mathbb{N}:\quad d(x_k^j,x_k)\to 0 \text{ as } j\to\infty. \tag{6.9}$$

Indeed, let us fix any numeration of points of γ such that

$$\gamma=\{x_k\}_{k=1}^\infty,\qquad d(x_0,x_{k+1})\ge d(x_0,x_k),\quad k\in\mathbb{N}.$$

Next, we fix positive numbers r_n, $n\in\mathbb{N}$, so that

$$
\begin{gathered}
r_{n+1}>r_n,\quad n\in\mathbb{N},\qquad r_n\to\infty \text{ as } n\to\infty,\\
\forall x\in\Bigg(\bigcup_{j=1}^\infty\gamma^j\Bigg)\cup\gamma:\quad d(x_0,x)\ne r_n,\\
k_1:=|\gamma\cap B(x_0,r_1)|>0,\qquad k_n:=|\gamma\cap(B(x_0,r_n)\setminus B(x_0,r_{n-1}))|>0,\qquad n\ge 2.
\end{gathered}
$$

Since $\gamma_j\to\gamma$ vaguely, we then conclude that there exist $j_1\in\mathbb{N}$ such that

$$|\gamma^j\cap B(x_0,r_1)|=k_1\qquad\text{for all } j\ge j_1,$$

and a numeration of the points of $\gamma^j \cap B(x_0, r_1)$, $j \geq j_1$, such that

$$\gamma^j \cap B(x_0, r_1) = \{x_k^j\}_{k=1}^{k_1}, \qquad x_k^j \to x_k \text{ as } j \to \infty \text{ for all } k = 1, \dots, k_1.$$

Next, there exist $j_2 \in \mathbb{N}$, $j_2 > j_1$, such that

$$|\gamma^j \cap (B(x_0, r_2) \setminus B(x_0, r_1))| = k_2 \qquad \text{for all } j \geq j_2,$$

and a numeration of the points of $\gamma^j \cap (B(x_0, r_2) \setminus B(x_0, r_1))$, $j \geq j_2$, such that

$$\gamma^j \cap (B(x_0, r_2) \setminus B(x_0, r_1)) = \{x_k^j\}_{k=k_1+1}^{k_1+k_2},$$
$$x_k^j \to x_k \text{ as } j \to \infty \text{ for all } k = k_1 + 1, \dots, k_1 + k_2.$$

Continuing this procedure by induction, we get for each $j_n \leq j < j_{n+1}$ a numeration $\{x_k^j\}_{k=1}^{k_1+\cdots+k_n}$ of the points of $\gamma^j \cap B(x_0, r_n)$. For such j we choose an arbitrary numeration $\{x_k^j\}_{k=k_1+\cdots+k_n+1}^{\infty}$ of $\gamma^j \cap B(x_0, r_n)^{\mathrm{c}}$, and therefore obtain a numeration $\{x_k^j\}_{k=1}^{\infty}$ of γ_j. Since $j_n \to \infty$, we thus have a numeration of all γ^j with $j \geq j_1$ (for the first $j_1 - 1$ configurations, we again take an arbitrary numeration). Now, for any fixed $l \in \mathbb{N}$, take the minimal $n(l) \in \mathbb{N}$ satisfying $x_l \in B(x_0, r_{n(l)})$ (this $n(l)$ always exists since $r_n \to \infty$). Then, $k_1 + \cdots + k_{n(l)-1} + 1 \leq l \leq k_1 + \cdots + k_{n(l)}$ (where $k_1 + \cdots + k_{n(l)-1} := 0$ if $n(l) = 1$). By induction, the sequence $(x_k^j)_{j=j_{n(l)}}^{\infty}$ converges to x_k as $j \to \infty$ for each k satisfying $k_1 + \cdots + k_{n(l)-1} + 1 \leq k \leq k_1 + \cdots + k_{n(l)}$, which immediately yields that the sequence $(x_l^j)_{j=1}^{\infty}$ converges to x_l as $j \to \infty$.

According to (6.4), (6.8) is, therefore, equivalent to the convergence

$$\sum_{(i_1,\dots,i_n)\in\widetilde{\mathbb{N}}^n} \prod_{k=1}^{n} p_t(x_{i_k}^j, y_k) \to \sum_{(i_1,\dots,i_n)\in\widetilde{\mathbb{N}}^n} \prod_{k=1}^{n} p_t(x_{i_k}, y_k) \qquad \text{as } j \to \infty \tag{6.10}$$

for each fixed $(y_1, \dots, y_n) \in \widetilde{X}^n$, $n \in \mathbb{N}$.

We now claim that, for any fixed $\varepsilon > 0$, there exist $J, K \in \mathbb{N}$ such that

$$\forall j > J: \quad \sum_{k=K+1}^{\infty} \exp\big[-d(x_0, x_k^j)\big] < \varepsilon, \qquad \sum_{k=K+1}^{\infty} \exp\big[-d(x_0, x_k)\big] < \varepsilon. \tag{6.11}$$

Indeed, choose any $K \in \mathbb{N}$ such that

$$\sum_{k=K+1}^{\infty} \exp\big[-d(x_0, x_k)\big] < \frac{\varepsilon}{3}, \tag{6.12}$$

then choose any $J_1 \in \mathbb{N}$ such that

$$\forall j > J_1: \quad \left| \sum_{k=1}^{K} \exp\big[-d(x_0, x_k^j)\big] - \sum_{k=1}^{K} \exp\big[-d(x_0, x_k)\big] \right| < \frac{\varepsilon}{3}, \tag{6.13}$$

and finally take any $J_2 \in \mathbb{N}$ such that

$$\forall j > J_2 : \quad \left| \sum_{k=1}^{\infty} \exp\big[-d(x_0, x_k^j)\big] - \sum_{k=1}^{\infty} \exp\big[-d(x_0, x_k)\big] \right| < \frac{\varepsilon}{3}. \tag{6.14}$$

Then, it follows from (6.12)–(6.14) that (6.11) holds with K as above and $J := \max\{J_1, J_2\}$.

Now, we conclude from (C1) that, for each fixed $t > 0$ and $y \in X$, there exists $\mathrm{const}_{t,y} > 0$ such that

$$p_t(x, y) \le \mathrm{const}_{t,y} \exp\big[-d(x_0, x)\big]. \tag{6.15}$$

Thus, from (6.9), (6.11), and (6.15), we easily derive that

$$\sum_{k=1}^{\infty} \big|p_t(x_k^j, y) - p_t(x_k, y)\big| \to 0 \qquad \text{as } j \to \infty. \tag{6.16}$$

Thus, (6.10) holds for $n = 1$.

Next, we show that (6.10) holds for $n = 2$, i.e.,

$$\sum_{i_1=1}^{\infty} \left| p_t(x_{i_1}^j, y_1) \sum_{i_2 \in \mathbb{N},\, i_2 \ne i_1} p_t(x_{i_2}^j, y_2) - p_t(x_{i_1}, y_1) \sum_{i_2 \in \mathbb{N},\, i_2 \ne i_1} p_t(x_{i_2}, y_2) \right| \to 0 \qquad \text{as } j \to \infty. \tag{6.17}$$

It follows from (6.9) and (6.16) that, for each $i_1 \in \mathbb{N}$,

$$p_t(x_{i_1}^j, y_1) \sum_{i_2 \in \mathbb{N},\, i_2 \ne i_1} p_t(x_{i_2}^j, y_2) \to p_t(x_{i_1}, y_1) \sum_{i_2 \in \mathbb{N},\, i_2 \ne i_1} p_t(x_{i_2}, y_2) \qquad \text{as } j \to \infty. \tag{6.18}$$

Moreover, we get from (6.16) that

$$\sum_{i_2 \in \mathbb{N},\, i_2 \ne i_1} p_t(x_{i_2}^j, y_2) \le \sum_{i_2=1}^{\infty} p_t(x_{i_2}^j, y_2) \le \mathrm{const} \qquad \forall j \in \mathbb{N}. \tag{6.19}$$

Thus, we obtain (6.17) from (6.11), (6.15), (6.18), and (6.19).

Continuing this way, by induction we prove (6.10) for each $n \in \mathbb{N}$.

By virtue of the majorized convergence theorem, it still remains to verify that all the functions $G(\cdot)k_t(\gamma^j, \cdot)$ are majorized by a function from $L^1(\lambda_m)$. But it follows from (6.6) that

$$|k_t(\gamma^j, \{y_1, \dots, y_n\})| \le \mathrm{const}_t^n \exp\big[d(x_0, y_1) + \dots + d(x_0, y_n)\big].$$

Thus, (6.1) implies the assertion of the theorem. ■

We have also the following theorem.

Theorem 6.2 *Let $\mathbb{D}$ denote the set of all measurable functions $G = (G^{(n)})_{n=0}^{\infty}$ on $\Gamma_{X,0}$ such that there exist $\varepsilon = \varepsilon(G)$, $C = C(G) > 0$ such that*

$$\big|G^{(n)} \circ T_n(x_1, \dots, x_n)\big| \le C^n \exp\big[-(1+\varepsilon)(d(x_0,x_1) + \cdots + d(x_0,x_n)) \big], \qquad (6.20)$$
$$(x_1, \dots, x_n) \in \widetilde{X}^n, \ n \in \mathbb{N}.$$

We define

$$\mathbf{D} := \big\{ (KG) \restriction \Gamma_\infty \mid G \in \mathbb{D} \big\}.$$

Then,

$$\forall p \ge 1: \quad \mathbf{D} \subset L^p(\Gamma_\infty, \pi_m) \qquad (6.21)$$

and

$$\forall t > 0: \quad \mathbf{P}_t \mathbf{D} \subset \mathbf{D}. \qquad (6.22)$$

Furthermore, for each $F \in \mathbf{D}$, $\mathbf{P}_t F$ is a continuous function on Γ_∞ with respect to the metric d_1.

Proof. By virtue of (4.15), (6.20) implies (6.1), and hence by Theorem 6.1, $\mathbf{P}_t F$ is a continuous function on Γ_∞ with respect to d_1 for each $F \in \mathbf{D}$.

Next, $\mathbb{D} \subset L^1(\Gamma_{X,0}, \lambda_m)$, and therefore, by Proposition 3.2,

$$\mathbf{D} \subset L^1(\Gamma_\infty, \pi_m). \qquad (6.23)$$

For each $\eta \in \Gamma_{X,0}$, $\sum_{\theta \subset \eta} 1 = 2^{|\eta|}$, which yields that

$$\sum_{(\theta_1,\theta_2,\theta_3) \in \mathcal{P}_3(\eta)} 1 \le 6 \cdot 4^{|\eta|}. \qquad (6.24)$$

By (6.24) and definition (3.5), we get that $G_1 \star G_2 \in \mathbb{D}$ for arbitrary $G_1, G_2 \in \mathbb{D}$. Consequently, by (3.6),

$$F_1 \cdot F_2 \in \mathbf{D}, \qquad F_1, F_2 \in \mathbf{D}. \qquad (6.25)$$

From (6.23) and (6.25), we get (6.21).

Finally, let us show that the set $\mathbf{D}$ is invariant under the action of $\mathbf{P}_t$. By (4.7) and (4.8), we have

$$(\mathbf{P}_t(KG))(\gamma) = (K(\widetilde{\mathbf{P}}_t G))(\gamma), \qquad \gamma \in \Gamma_\infty, \ G \in \mathbb{D}, \qquad (6.26)$$

where $\widetilde{\mathbf{P}}_t G = ((\widetilde{\mathbf{P}}_t G)^{(n)})_{n=0}^{\infty}$ is given by

$$(\widetilde{\mathbf{P}}_t G)^{(0)} = G^{(0)},$$

$$(\widetilde{\mathbf{P}}_t G)^{(n)} \circ T_n(x_1, \dots, x_n) = \int_{\widetilde{X}^n} G^{(n)} \circ T_n(y_1, \dots, y_n)\, p_{t,x_1}(dy_1) \cdots p_{t,x_n}(dy_n),$$
$$(x_1, \dots, x_n) \in \widetilde{X}^n,\ n \in \mathbb{N}.$$

Now, by using (C1), we derive, for $G = (G^{(n)})_{n=0}^{\infty}$ satisfying (6.20) with some $\varepsilon > 0$ and $C > 0$,

$$\begin{aligned}
&\big|(\widetilde{\mathbf{P}}_t G)^{(n)} \circ T_n(x_1, \dots, x_n)\big| \le \\
&\le \int_{X^n} C^n \exp\big[-(1+\varepsilon)(d(x_0, y_1) + \dots + d(x_0, y_n)) \big] \\
&\times (C'_t)^n \exp\big[-(1+\tfrac{\varepsilon}{2})(d(x_1, y_1) + \dots + d(x_n, y_n)) \big]\, m(dy_1) \cdots m(dy_n) \\
&\le (CC'_t)^n \exp\big[-(1+\tfrac{\varepsilon}{2})(d(x_0, x_1) + \dots + d(x_0, x_n)) \big] \bigg(\int_X \exp\big[-\tfrac{\varepsilon}{2}\, d(x_0, y) \big]\, m(dy) \bigg)^n .
\end{aligned}$$

Hence, because of (4.15), $\widetilde{\mathbf{P}}_t G \in \mathbb{D}$, and (6.22) follows from (6.26). ■

For an arbitrary measurable, bounded, symmetric function $G^{(n)}(x_1, \dots, x_n)$ on X^n, $n \in \mathbb{N}$, with bounded support, one can introduce the following monomial on Γ_∞ of n-th order with kernel $G^{(n)}$:

$$\Gamma_\infty \ni \gamma \mapsto \langle G^{(n)}, :\gamma^{\otimes n}: \rangle := \sum_{\{x_1, \dots, x_n\} \subset \gamma} G^{(n)}(x_1, \dots, x_n) = (KG^{(n)})(\gamma).$$

It is natural to call a finite sum of functions of such type and a constant a cylindrical polynomial on Γ_∞ with bounded coefficients. We denote by $\mathcal{FP}_{\mathrm{bc}}(\Gamma_\infty)$ the set of all such polynomials on Γ_∞. Thus, $\mathcal{FP}_{\mathrm{bc}}(\Gamma_X)$ is nothing but the image of the set $B_{\mathrm{bs}}(\Gamma_{X,0})$ under the K-transform.

Since every function $G \in B_{\mathrm{bs}}(\Gamma_{X,0})$ satisfies (6.20), we get the following consequence of Theorem 6.2:

Corollary 6.1 *We have the inclusion $\mathcal{FP}_{\mathrm{bc}}(\Gamma_\infty) \subset \mathbf{D}$. In particular, for each polynomial $F \in \mathcal{FP}_{\mathrm{bc}}(\Gamma_\infty)$, the function $\mathbf{P}_t F$ is continuous on Γ_∞ with respect to the metric d_1.*

A measurable function $F \colon \Gamma_\infty \to \mathbb{R}$ is called local if there exist an open, relatively compact set $\Lambda \subset X$ and a measurable function $\widetilde{F} : \Gamma_{\Lambda,0} \to \mathbb{R}$ such that $F(\gamma) = \widetilde{F}(\gamma_\Lambda)$ for all $\gamma \in \Gamma_\infty$. The following corollary is an analog of the classical strong Feller property for the heat semigroup on the configuration space.

Corollary 6.2 *Each measurable bounded local function $F \colon \Gamma_\infty \to \mathbb{R}$ belongs to $\mathbf{D}$, and hence the function $\mathbf{P}_t F$ is continuous on Γ_∞ with respect to the metric d_1.*

Proof. Let $F(\gamma)=\widetilde{F}(\gamma_\Lambda)$ with Λ and $\widetilde{F}$ as above. By [18, Proposition 3.5], one can explicitly calculate the inverse K-transform of F:

$$(K^{-1}F)(\eta)=\begin{cases}\sum\limits_{\theta\subset\eta}(-1)^{|\eta\setminus\theta|}\,\widetilde{F}(\theta), & \text{if } \eta\in\Gamma_{\Lambda,0},\\ 0, & \text{otherwise.}\end{cases}\tag{6.27}$$

Set $G=(G^{(n)})_{n=0}^\infty:=K^{-1}F$. Let $C:=\sup|F|$. Since $\sum_{\theta\subset\eta}1=2^{|\eta|}$, we conclude from (6.27) that, for each $n\in\mathbb{N}$, $\{G^{(n)}\neq 0\}\subset\Gamma_\Lambda^{(n)}$ and $|G^{(n)}|$ is bounded by the constant $C2^n$. Therefore, $G\in\mathbb{D}$ and $F=KG\in\mathbf{D}$. ■

Remark 6.1 Let $t>0$. As a consequence of Corollary 6.2 we have that any Markovian kernel $\widetilde{\mathbf{P}}_t$ on $(\Gamma_\infty,\mathcal{B}(\Gamma_\infty))$ such that $\widetilde{\mathbf{P}}_tF$ is continuous with respect to d_1 and $\widetilde{\mathbf{P}}_tF$ is a π_m-version of $e^{-tH^\Gamma}F$ for each measurable bounded local function $F:\Gamma_\infty\to\mathbb{R}$ must coincide with $\mathbf{P}_t$. This follows from the fact that $\pi_m(U)>0$ for any nonempty set $U\in\mathcal{B}(\Gamma_\infty)$ which is open in the topology generated by the metric d_1. The latter can be proved as follows. For any fixed $\hat\gamma\in\Gamma_\infty$ and $\varepsilon>0$, there exist $R>0$ and $\delta>0$ such that for each $r\geq R$

$$\begin{aligned}&\pi_m\big(d_1(\gamma,\hat\gamma)<\varepsilon\big)\\ &\quad\geq\pi_m\big(d_{\mathrm{K}}(\gamma_{\Lambda_r},\hat\gamma_{\Lambda_r})<\delta,\ |B_1(\gamma_{\Lambda_r})-B_1(\hat\gamma_{\Lambda_r})|<\delta,\ B_1(\gamma_{\Lambda_r^c})<\delta\big)\\ &\quad=\pi_m\big(d_{\mathrm{K}}(\gamma_{\Lambda_r},\hat\gamma_{\Lambda_r})<\delta,\ |B_1(\gamma_{\Lambda_r})-B_1(\hat\gamma_{\Lambda_r})|<\delta\big)\,\pi_m\big(B_1(\gamma_{\Lambda_r^c})<\delta\big).\end{aligned}\tag{6.28}$$

Here, $\Lambda_r:=\{x\in X: d(x_0,x)<r\}$ and the functional B_1 is defined on the space $\Gamma_{X,0}$ by the same formula (4.12). The first factor in (6.28) is obviously positive, while the positivity of the second factor for sufficiently big $r>0$ is implied by

$$1=\pi_m\big(B_1(\gamma)<\infty\big)=\pi_m\bigg(\bigcup_{r=1}^\infty\{B_1(\gamma_{\Lambda_r^c})<\delta\}\bigg)=\lim_{r\to\infty}\pi_m\big(B_1(\gamma_{\Lambda_r^c})<\delta\big).$$

7 Feller property of the heat semigroup with respect to the intrinsic metric of the Dirichlet form

For presenting another type of Feller property of the heat semigroup $(e^{-tH^\Gamma})_{t\in\mathbb{R}_+}$, we shall need the space $\ddot\Gamma_X$ of all $\mathbb{Z}_+$-valued Radon measures γ on X such that $\gamma(X)=\infty$. This space is the closure of Γ_X in the d_{K} metric. The space $\ddot\Gamma_X$ is equipped with the topology induced by the vague topology on $\mathcal{M}(X)$, and let $\mathcal{B}(\ddot\Gamma_X)$ denote the corresponding Borel σ-algebra.

Furthermore, let

$$\ddot\Gamma_n:=\big\{\,\gamma\in\ddot\Gamma_X: B_n(\gamma)<\infty\,\big\},$$

where B_n is as in (4.12), but defined on all of $\ddot{\Gamma}_X$. Let $\ddot{\Gamma}_\infty := \bigcap_{n=1}^\infty \ddot{\Gamma}_n$. We extend the metric d_∞ to $\ddot{\Gamma}_\infty$ using the same formula (4.14). The Borel σ-algebra $\mathcal{B}(\ddot{\Gamma}_\infty)$ corresponding to the d_∞ metric coincides with the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma}_X)$ on $\ddot{\Gamma}_\infty$.

Let $\widehat{X}^{\mathbb{N}}$ denote the ($\mathcal{B}(X^{\mathbb{N}})$-measurable) subset of $X^{\mathbb{N}}$ consisting of those $(x_1, x_2, \dots) \in X^{\mathbb{N}}$ for which the number of the x_k's in any compactum in X is finite. Evidently, one can identify $\ddot{\Gamma}_X$ with the factor space $\widehat{X}^{\mathbb{N}}/S_\infty$. Analogously to (4.3), we define the corresponding quotient map $I\colon \widehat{X}^{\mathbb{N}} \to \ddot{\Gamma}_X$ by

$$\widehat{X}^{\mathbb{N}} \ni \mathbf{x} = (x_1, x_2, \dots) \to I\mathbf{x} := [x_1, x_, \dots] \in \ddot{\Gamma}_X, \tag{7.1}$$

which is measurable, as can be seen by similar arguments as those following (4.3).

For each $t > 0$ and $\gamma \in \ddot{\Gamma}_X$, we define the measure $\mathbf{P}_{t,\gamma}$ on $\ddot{\Gamma}_X$ as the image under the mapping (7.1) of the restriction to $\widehat{X}^{\mathbb{N}}$ of any measure $\mathbb{P}_{t,\mathbf{x}} := \bigotimes_{k=1}^\infty p_{t,x_k}$, $\mathbf{x} = (x_k)_{k=1}^\infty \in I^{-1}\{\gamma\}$ (the resulting measure is independent of the choice of $\mathbf{x} \in I^{-1}\{\gamma\}$). Thus, by Lemma 4.1, $\mathbf{P}_{t,\gamma}$ is either a probability measure or zero measure on $\ddot{\Gamma}_X$ depending on whether the series $\sum_{k=1}^\infty p_{t,x_k}(\Lambda)$ converges for each compact $\Lambda \subset X$, or not, and $\mathbf{P}_{t,\gamma}(\ddot{\Gamma}_X) = \mathbf{P}_{t,\gamma}(\Gamma_X)$. In the same way as we proved Proposition 4.2, we conclude that, for each $\gamma \in \ddot{\Gamma}_\infty$,

$$\mathbf{P}_{t,\gamma}(\ddot{\Gamma}_\infty) = \mathbf{P}_{t,\gamma}(\Gamma_\infty) = 1.$$

Following [34], we introduce the L^2-Wasserstein type distance ρ on $\ddot{\Gamma}_X$ setting, for any $\gamma_1 = [x_1, x_2, \dots]$ and $\gamma_2 = [y_1, y_2, \dots]$ from $\ddot{\Gamma}_X$,

$$\rho(\gamma_1, \gamma_2) := \inf\left\{ \left(\sum_{k=1}^\infty d(x_k, y_{\sigma(k)})^2 \right)^{1/2} \,\Big|\, \sigma \in S_\infty \right\}. \tag{7.2}$$

Notice that ρ is a pseudo-metric, i.e., it takes values in $[0, \infty]$. Obviously, convergence with respect to ρ implies vague convergence. We recall that ρ is the intrinsic metric of the Dirichlet form obtained as the closure of (2.7), see [34].

Analogously to (5.3), we set for a measurable function F on $\ddot{\Gamma}_X$:

$$(\mathbf{P}_t F)(\gamma) := \int_{\ddot{\Gamma}_X} F(\xi)\, \mathbf{P}_{t,\gamma}(d\xi), \qquad \gamma \in \ddot{\Gamma}_X,\ t > 0, \tag{7.3}$$

provided the integral on the right hand side of (7.3) exists.

In the rest of this section, we shall be concerned only with the case $X = \mathbb{R}^d$. Let us recall that the heat kernel has now the form

$$p(t, x, y) = (4\pi t)^{-d/2} \left[-\tfrac{1}{4t} \, |x - y|^2 \right]. \tag{7.4}$$

We shall show that the space $C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d})$ of all bounded functions on $\ddot{\Gamma}_{\mathbb{R}^d}$ which are continuous with respect to the ρ metric is invariant under $\mathbf{P}_t$ for all $t > 0$.

Theorem 7.1 *We have*:

$$\mathbf{P}_t(C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d})) \subset C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d}), \qquad t>0.$$

Proof. First, we note that the distance ρ can be extended from $\ddot{\Gamma}_{\mathbb{R}^d} = \widehat{\mathbb{R}^d{}^{\mathbb{N}}}/S_\infty$ to the bigger space $(\mathbb{R}^d)^{\mathbb{N}}/S_\infty$ by using the same formula (7.2) for calculating the distance between any $\gamma_1 = [x_1, x_2, \dots]$ and $\gamma_2 = [y_1, y_2, \dots]$ from $(\mathbb{R}^d)^{\mathbb{N}}/S_\infty$.

It follows directly from (7.2) that, if γ_1 and γ_2 are two elements of $(\mathbb{R}^d)^{\mathbb{N}}/S_\infty$ having finite ρ distance, then $\gamma_1 \in \ddot{\Gamma}_{\mathbb{R}^d}$ implies $\gamma_2 \in \ddot{\Gamma}_{\mathbb{R}^d}$. Therefore, any function $F \in C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d})$ can be extended to a continuous bounded function on $(\mathbb{R}^d)^{\mathbb{N}}/S_\infty$, again denoted by F, as follows:

$$F \restriction ((\mathbb{R}^d)^{\mathbb{N}}/S_\infty) \setminus \ddot{\Gamma}_{\mathbb{R}^d} := 0.$$

Then, (7.3) yields

$$(\mathbf{P}_t F)(\gamma) = \int_{(\mathbb{R}^d)^{\mathbb{N}}} F(y_1, y_2, \dots) \bigotimes_{k=1}^{\infty} p_{t,x_k}(dy_k), \qquad \gamma = [x_1, x_2, \dots] \in \ddot{\Gamma}_{\mathbb{R}^d}. \tag{7.5}$$

Notice that, in this formula, F is considered as an S_∞-invariant function on $(\mathbb{R}^d)^{\mathbb{N}}$.

Since F is bounded, so is the function $\mathbf{P}_t F$, and we only have to prove the continuity. To this end, let $\gamma^j \to \gamma$ in the ρ metric. By Lemma 4.1 in [34], there always exists a representative $(x_k^j)_{k=1}^\infty$ of γ^j such that

$$\rho(\gamma^j, \gamma) = \left(\sum_{k=1}^\infty |x_k^j - x_k|^2 \right)^{1/2}, \qquad j \in \mathbb{N}. \tag{7.6}$$

(7.4) and (7.5) imply

$$(\mathbf{P}_t F)(\gamma^j) = \int_{(\mathbb{R}^d)^{\mathbb{N}}} F(y_1 + x_1^j, y_2 + x_2^j, \dots) \bigotimes_{k=1}^{\infty} p_t(dy_k), \tag{7.7}$$
$$p_t(dy) = (4\pi t)^{-d/2} \exp\left[-\tfrac{1}{4t} |y|^2 \right] dy.$$

Since the integrand in (7.7) is a bounded function, it suffices to show that for any fixed $(y_k)_{k=1}^\infty \in (\mathbb{R}^d)^{\mathbb{N}}$

$$F(y_1 + x_1^j, y_2 + x_2^j, \dots) \to F(y_1 + x_1, y_2 + x_2, \dots) \quad \text{as } j \to \infty.$$

But this follows from the fact that F is continuous in the ρ metric and from the convergence

$$\rho\big([y_1 + x_1^j, y_2 + x_2^j, \dots], [y_1 + x_1, y_2 + x_2, \dots]\big) \le \left(\sum_{k=1}^\infty |y_k + x_k^j - y_k - x_k|^2 \right)^{1/2} = \rho(\gamma_j, \gamma) \to 0$$

as $j \to \infty$, which, in turn, is implied by (7.6). ■

Remark 7.1 Theorem 7.1, in particular, yields that if $\ddot{\Gamma}_{\mathbb{R}^d}$ is of full $\mathbf{P}_{t,\gamma}$ measure for some $\gamma \in \ddot{\Gamma}_{\mathbb{R}^d}$, then it is also of full $\mathbf{P}_{t,\gamma'}$ measure for each $\gamma' \in \ddot{\Gamma}_{\mathbb{R}^d}$ such that $\rho(\gamma, \gamma') < \infty$. For it suffices to note that $1 \in C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d})$ and $(\mathbf{P}_t 1)(\gamma) = \mathbf{P}_{t,\gamma}(\ddot{\Gamma}_X)$.

Finally, we shall present another version of the latter theorem. Let $C_{\rho,\mathrm{b}}(\ddot{\Gamma}_\infty)$ denote the space of all bounded functions on $\ddot{\Gamma}_\infty$ that are continuous with respect to the ρ metric. It is easy to see that each element of $(\mathbb{R}^d)^{\mathbb{N}}/S_\infty$ having a finite distance to $\ddot{\Gamma}_\infty$ itself belongs to $\ddot{\Gamma}_\infty$. Therefore, any function $F \in C_{\rho,\mathrm{b}}(\ddot{\Gamma}_\infty)$ can be extended to a function from $C_{\rho,\mathrm{b}}(\ddot{\Gamma}_{\mathbb{R}^d})$ by setting F to be equal to zero on $\ddot{\Gamma}_{\mathbb{R}^d} \setminus \ddot{\Gamma}_\infty$. We note that the convergence on $\ddot{\Gamma}_\infty$ with respect to the ρ metric implies the convergence with respect to the d_∞ metric.

Since for each $\gamma \in \ddot{\Gamma}_\infty$ the measure $\mathbf{P}_{t,\gamma}$ is concentrated on $\ddot{\Gamma}_\infty$, we get the following corollary of Theorem 7.1:

Corollary 7.1 *We have*:

$$\mathbf{P}_t(C_{\rho,\mathrm{b}}(\ddot{\Gamma}_\infty)) \subset C_{\rho,\mathrm{b}}(\ddot{\Gamma}_\infty), \qquad t > 0.$$

8 Brownian motion on the configuration space

We again consider the case of a general manifold X. Analogously to (5.4), (5.5), we define the family of kernels $(\mathbf{P}_t)_{t \in \mathbb{R}_+}$ on the space $(\ddot{\Gamma}_\infty, \mathcal{B}(\ddot{\Gamma}_\infty))$ setting

$$\mathbf{P}_t(\gamma, A) := \mathbf{P}_{t,\gamma}(A), \qquad \gamma \in \ddot{\Gamma}_\infty,\ A \in \mathcal{B}(\ddot{\Gamma}_\infty),\ t \in \mathbb{R}_+,$$

where $\mathbf{P}_{t,\gamma}$, $t > 0$, $\gamma \in \ddot{\Gamma}_X$, is defined as in the previous section and

$$\mathbf{P}_{0,\gamma} := \varepsilon_\gamma.$$

Analogously to Proposition 5.1, we conclude that $(\mathbf{P}_t)_{t \in \mathbb{R}_+}$ is a Markov semigroup of kernels on $(\ddot{\Gamma}_\infty, \mathcal{B}(\ddot{\Gamma}_\infty))$.

In this section, we shall give a direct construction of the independent infinite particle process. Under some additional conditions on the manifold X, we shall show that the resulting process is the unique continuous Markov process on $\ddot{\Gamma}_\infty$ with transition probabilities $\mathbf{P}_t(\gamma, \cdot)$. (We note that we are forced to deal with the space $\ddot{\Gamma}_\infty$, rather than Γ_∞, because in the general case we cannot exclude collision of the particles, see Corollary 8.1 below).

First, we strengthen a little bit condition (C1) by requiring the following stronger upper bound:

(C1$'$) For each $t > 0$, there exist $\vartheta_t \in (0, t)$, $C_t > 0$, and $\varepsilon_t > 0$ such that

$$p(s, x, y) \le C_t \exp\big[-d(x, y)^{1+\varepsilon_t}\big], \qquad s \in (t - \vartheta_t, t + \vartheta_t),\ x, y \in X.$$

Evidently, (4.9) and (4.11) imply (C1$'$).

Let us introduce the function

$$\tau(\delta, r) := \sup_{t\in(0,\delta]} \sup_{x\in X} \int_{B(x,r)^c} p(t, x, y)\, m(dy), \qquad \delta > 0,\ r > 0.$$

Because of (2.2), $\tau(\delta, r) \le 1$ for all $\delta, r > 0$, and for each fixed $r > 0$ $\tau(\cdot, r)$ is an increasing function on $(0, \infty)$.

Let $\Omega := C(\mathbb{R}_+; X)$ denote the space of all continuous functions (paths) from $\mathbb{R}_+$ to X, and let $\mathcal{F}$ be the product σ-algebra on Ω, i.e.,

$$\mathcal{F} := \sigma\{x_t,\ t \in \mathbb{R}_+\}, \tag{8.1}$$

where $\Omega \ni \omega \mapsto x_t(\omega) := \omega(t) \in X$. For each $x \in X$, let P_x denote the measure on $(\Omega, \mathcal{F})$ corresponding to Brownian motion on X starting at x.

We shall need the following lemma.

Lemma 8.1 *Let $0 \le a < b$ with $b - a \le \delta$. Then, for each $x \in X$ and $r > 0$,*

$$P_x\big(\exists s, t \in [a, b] : d(\omega(s), \omega(t)) > r\big) \le 2\tau(\delta, \tfrac{1}{4} r).$$

Proof. This lemma is a straightforward generalization of [30, Appendix A, Lemma 4], which deals with the usual Brownian motion on $\mathbb{R}^d$. However, for completeness, we present a proof of this lemma in the Appendix. ■

We suppose:

(C3) For each fixed $r > 0$,

$$\tau(\delta, r) \to 0 \qquad \text{as } \delta \to 0, \tag{8.2}$$

and there exist $\widetilde{\delta} > 0$ and $C > 0$ such that

$$\tau(\widetilde{\delta}, r) \le Ce^{-r}, \qquad r > 0. \tag{8.3}$$

The following simple lemma gives a sufficient condition for (C3) to hold.

Lemma 8.2 *Suppose that the manifold X has nonnegative Ricci curvature and the heat kernel $p(t, x, y)$ of X satisfies the Gaussian upper bound for small values of t:*

$$p(t, x, y) \le Ct^{-n/2} \exp\left[-\frac{d(x, y)^2}{Dt}\right], \qquad t \in (0, \widetilde{\delta}],\ x, y \in X, \tag{8.4}$$

where $n \in \mathbb{N}$ and $\widetilde{\delta}$, C, and D are positive constants. Then, (C3) *is satisfied.*

Remark 8.1 Concerning the Gaussian upper bound (8.4), see e.g. [10, 14, 15] and the references therein. In particular, (8.4) is implied by the estimate

$$p(x,x,t) \le Ct^{-n/2}, \qquad t>0,\ x\in X.$$

Proof of Lemma 8.2. Fix $r>0$, then for $t\in(0,\widetilde{\delta}]$ and $x\in X$ we get, by (4.10) and (8.4),

$$\begin{aligned}
\int_{B(x,r)^c} p(t,x,y)\,m(dy) &\le \int_{B(x,r)^c} Ct^{-n/2}\exp\left[-\frac{d(x,y)^2}{Dt}\right]m(dy)\\
&\le \mathrm{const}_1 \int_{B(x,r)^c}\exp\left[\frac{1}{t}\left(\frac{r^2}{2D}-\frac{d(x,y)^2}{D}\right)\right]m(dy)\\
&\le \mathrm{const}_1\sum_{n=1}^{\infty}\exp\left[\frac{1}{t}\left(\frac{r^2}{2D}-\frac{(r+n-1)^2}{D}\right)\right]m\big(B(x,r+n)\setminus B(x,r+n-1)\big)\\
&\le \mathrm{const}_2\sum_{n=1}^{\infty}\exp\left[\frac{1}{t}\left(\frac{r^2}{2D}-\frac{(r+n-1)^2}{D}\right)\right](r+n)^d. \qquad (8.5)
\end{aligned}$$

Since each term of the latter series monotonically converges to zero as $t\to 0$, we get (8.2).

Next, because $\tau(\delta,r)$ is bounded by 1, it is enough to verify that (8.3) holds for all $r\ge R$ with some $R>0$. Now, analogously to (8.5), we get for each $t\in(0,\widetilde{\delta}]$ and $r\ge 1$

$$\begin{aligned}
\int_{B(x,r)^c} p(t,x,y)\,m(dy) &\le \mathrm{const}_3\sum_{n=1}^{\infty}\exp\left[\frac{1}{\widetilde{\delta}}\left(\frac{1}{2D}-\frac{(r+n-1)^2}{D}\right)\right](r+n)^d\\
&\le \mathrm{const}_4\sum_{n=1}^{\infty}\exp\big[-2(r+n-1)+(r+n)\big] = \mathrm{const}_4\, e^{-r}\sum_{n=1}^{\infty}e^{-n},
\end{aligned}$$

which yields the statement. ■

Theorem 8.1 *Let* (C1′), (C2), *and* (C3) *hold. Then, the independent infinite particle process can be realized as the unique continuous, time homogeneous Markov process*

$$\mathbf{M} = (\boldsymbol{\Omega},\mathbf{F},(\mathbf{F}_t)_{t\in\mathbb{R}_+},(\boldsymbol{\theta}_t)_{t\in\mathbb{R}_+},(\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_\infty},(\mathbf{X}_t)_{t\in\mathbb{R}_+})$$

on the state-space $(\ddot{\Gamma}_\infty,\mathcal{B}(\ddot{\Gamma}_\infty))$ *with transition probability function* $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$ (*cf. e.g.* [9]).

Proof of Theorem 8.1. Let us consider the set $\Omega^{\mathbb{N}}$ and the product σ-algebra $\mathcal{C}_\sigma(\Omega^{\mathbb{N}})$ on it that is constructed from the σ-algebra $\mathcal{F}$ on Ω.

We fix any $\mathbf{x} = (x_k)_{k=1}^{\infty} \in \widehat{X}^{\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, x_k)\big] < \infty \qquad \forall n \in \mathbb{N} \tag{8.6}$$

and define the product measure

$$\mathbb{P}_{\mathbf{x}} := \bigotimes_{k=1}^{\infty} P_{x_k} \tag{8.7}$$

on $(\Omega^{\mathbb{N}}, \mathcal{C}_\sigma(\Omega^{\mathbb{N}}))$.

By using (4.16), we conclude from (8.6) that

$$\begin{aligned} &\int_{X^{\mathbb{N}}} \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, y_k)\big] \bigotimes_{k=1}^{\infty} p_{t,x_k}(dy_k) \\ &\qquad = \sum_{k=1}^{\infty} \int_X \exp\big[-\tfrac{1}{n}\, d(x_0, y)\big]\, p_{t,x_k}(dy) < \infty \end{aligned}$$

for all $t > 0$ and $n \in \mathbb{N}$, which yields that, for each fixed $t \in \mathbb{R}_+$,

$$\begin{aligned} &\sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, \omega_k(t))\big] < \infty \qquad \text{for all } n \in \mathbb{N} \text{ and } \mathbb{P}_{\mathbf{x}}\text{-a.e. } \omega = (\omega_k)_{k=1}^{\infty} \in \Omega^{\mathbb{N}}, \\ &\text{in particular, } (\omega_k(t))_{k=1}^{\infty} \in \widehat{X}^{\mathbb{N}} \text{ for such } \omega \in \Omega^{\mathbb{N}}. \end{aligned} \tag{8.8}$$

Lemma 8.3 *For each fixed $t \in \mathbb{R}_+$ and $\mathbf{x} = (x_k)_{k=1}^{\infty} \in \widehat{X}^{\mathbb{N}}$ satisfying (8.6), we have*

$$\mathbb{P}_{\mathbf{x}}\bigg(\bigcup_{i=1}^{\infty}\bigcap_{k=1}^{\infty}\big\{\, d(\omega_k(s), \omega_k(t)) \le \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t))\}\ \forall s \in (t, t+\tfrac{1}{i}]\,\big\}\bigg) = 1. \tag{8.9}$$

Proof. First, we will prove (8.9) for $t = 0$. Thus, we have to show that

$$\mathbb{P}_{\mathbf{x}}\bigg(\bigcup_{i=1}^{\infty}\bigcap_{k=1}^{\infty}\big\{\, d(\omega_k(s), x_k) \le \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\}\ \forall s \in (0, \tfrac{1}{i}]\,\big\}\bigg) = 1, \tag{8.10}$$

or equivalently

$$\mathbb{P}_{\mathbf{x}}\bigg(\bigcap_{i=1}^{\infty}\bigcup_{k=1}^{\infty}\big\{\, \exists s \in (0, \tfrac{1}{i}] : d(\omega_k(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\}\,\big\}\bigg) = 0. \tag{8.11}$$

Since

$$\mathbb{P}_{\mathbf{x}}\bigg(\bigcap_{i=1}^{\infty}\bigcup_{k=1}^{\infty}\big\{\, \exists s \in (0, \tfrac{1}{i}] : d(\omega_k(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\}\,\big\}\bigg)$$

$$= \lim_{i\to\infty} \mathbb{P}_{\mathbf{x}}\Bigg(\bigcup_{k=1}^{\infty} \big\{ \exists s \in (0, \tfrac{1}{i}] : d(\omega_k(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\} \big\} \Bigg), \tag{8.12}$$

we have, by Lemma 8.1,

$$\begin{aligned} &\mathbb{P}_{\mathbf{x}}\Bigg(\bigcup_{k=1}^{\infty} \big\{ \exists s \in (0, \tfrac{1}{i}] : d(\omega_k(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\} \big\} \Bigg) \\ &\qquad \le \sum_{k=1}^{\infty} \mathbb{P}_{\mathbf{x}}\big(\exists s \in (0, \tfrac{1}{i}] : d(\omega_k(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\}\big) \\ &\qquad = \sum_{k=1}^{\infty} P_{x_k}\big(\exists s \in (0, \tfrac{1}{i}] : d(\omega(s), x_k) > \max\{1, \tfrac{1}{2}\, d(x_0, x_k\}\big) \\ &\qquad \le 2\sum_{k=1}^{\infty} \tau\big(\tfrac{1}{i}, \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, x_k)\}\big). \end{aligned} \tag{8.13}$$

By (8.2),

$$\tau\big(\tfrac{1}{i}, \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, x_k)\}\big) \to 0 \qquad \text{as } i \to \infty \tag{8.14}$$

for each x_k. On the other hand, it follows from (8.3) that, for any $i \in \mathbb{N}$ satisfying $\frac{1}{i} \le \widetilde{\delta}$, the latter series in (8.13) is majorized by the series

$$2C \sum_{k=1}^{\infty} \exp\big[-\max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, x_k)\}\big],$$

which converges due to (8.6). Hence, (8.11) follows from (8.12)–(8.14), and the monotone convergence theorem.

Next, using the Markov property of Brownian motion on X, we easily conclude that

$$\begin{aligned} &\mathbb{P}_{\mathbf{x}}\Bigg(\bigcup_{i=1}^{\infty}\bigcap_{k=1}^{\infty} \big\{ d(\omega_k(s), \omega_k(t)) \le \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t))\}\ \forall s \in (t, t+\tfrac{1}{i}] \big\} \Bigg) \\ &\quad = \int_{\widehat{X}^{\mathbb{N}}} \mathbb{P}_{t,\mathbf{x}}(d\mathbf{y})\, \mathbb{P}_{\mathbf{y}}\Bigg(\bigcup_{i=1}^{\infty}\bigcap_{k=1}^{\infty} \big\{ d(\omega_k(s), y_k) \le \max\{1, \tfrac{1}{2}\, d(x_0, y_k)\}\ \forall s \in (0, \tfrac{1}{i}] \big\} \Bigg), \end{aligned} \tag{8.15}$$

where $\mathbb{P}_{t,\mathbf{x}}$ is the distribution of $\omega(t) = (\omega_k(t))_{k=1}^{\infty}$ under $\mathbb{P}_{\mathbf{x}}$. Now, (8.9) follows from (8.8), (8.10), and (8.15). ■

Lemma 8.4 *For each fixed $t > 0$ and $\mathbf{x} = (x_k)_{k=1}^{\infty} \in \widehat{X}^{\mathbb{N}}$ satisfying* (8.6), *we have*

$$\mathbb{P}_{\mathbf{x}}\Big(\bigcup_{i=\mathcal{I}_t}^{\infty} \bigcap_{k=1}^{\infty} \big\{ d(\omega_k(t-\tfrac{1}{i}), \omega_k(s)) \le \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t-\tfrac{1}{i}))\} \ \forall s \in (t-\tfrac{1}{i}, t] \big\} \Big) = 1,$$

where $\mathcal{I}_t := [t^{-1}] + 1$ ($[a]$ denoting the integer part of $a > 0$).

Proof. It is enough to show that

$$\lim_{i\to\infty} \mathbb{P}_{\mathbf{x}}\Big(\bigcup_{k=1}^{\infty} \big\{ \exists s \in (t-\tfrac{1}{i}, t] : d(\omega_k(t-\tfrac{1}{i}), \omega_k(s)) > \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t-\tfrac{1}{i}))\} \big\} \Big) = 0. \tag{8.16}$$

Using Lemma 8.1, we get

$$\begin{aligned}
&\mathbb{P}_{\mathbf{x}}\Big(\bigcup_{k=1}^{\infty} \big\{ \exists s \in (t-\tfrac{1}{i}, t] : d(\omega_k(t-\tfrac{1}{i}), \omega_k(s)) > \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t-\tfrac{1}{i}))\} \big\} \Big) \\
&\quad \le \sum_{k=1}^{\infty} P_{x_k}\big(\exists s \in (t-\tfrac{1}{i}, t] : d(\omega(t-\tfrac{1}{i}), \omega(s)) > \max\{1, \tfrac{1}{2}\, d(x_0, \omega(t-\tfrac{1}{i}))\} \big) \\
&\quad = \sum_{k=1}^{\infty} \int_X m(dy)\, p(t-\tfrac{1}{i}, x_k, y) P_y\big(\exists s \in (0, \tfrac{1}{i}] : d(y, \omega(s)) > \max\{1, \tfrac{1}{2}\, d(x_0, y)\} \big) \\
&\quad \le \sum_{k=1}^{\infty} \int_X m(dy)\, p(t-\tfrac{1}{i}, x_k, y)\, 2\tau(\tfrac{1}{i}, \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\}).
\end{aligned} \tag{8.17}$$

Now, it follows from (8.2) that

$$p(t-\tfrac{1}{i}, x_k, y)\, 2\tau(\tfrac{1}{i}, \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\}) \to 0 \qquad \text{as } i \to \infty \tag{8.18}$$

for each x_k and $y \in X$. Next, by (C1′) and (8.3), we have, for $i > \max\{\vartheta_t^{-1}, \widetilde{\delta}^{-1}\}$,

$$\begin{aligned}
&p(t-\tfrac{1}{i}, x_k, y)\, 2\tau(\tfrac{1}{i}, \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\}) \\
&\qquad \le C_t \exp\big[- d(x_k, y)^{1+\varepsilon_t} \big] 2C \exp\big[- \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\} \big] \\
&\qquad \le \text{const} \exp\big[- \tfrac{1}{16}\, d(x_k, y) - \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\} \big] \\
&\qquad \le \text{const} \exp\big[- \tfrac{1}{16}\, d(x_0, x_k) \big] \exp\big[\tfrac{1}{16}\, d(x_0, y) - \max\{\tfrac{1}{4}, \tfrac{1}{8}\, d(x_0, y)\} \big].
\end{aligned} \tag{8.19}$$

Hence, by (8.6), (8.17)–(8.19), (4.15), and the dominated convergence theorem, we get (8.16). ■

¿From (8.6), (8.8), and Lemmas 8.3 and 8.4, we get the central lemma of the proof:

Lemma 8.5 *Let $x \in \widehat{X}^{\mathbb{N}}$ satisfy* (8.6)*. For $t > 0$ and $i \geq \mathcal{I}_t$, we set*

$$\mathbb{A}_{t,i} := \Bigg[\bigcup_{l=\mathcal{I}_t}^{i} \Big\{ \omega \in \Omega^{\mathbb{N}} : \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, \omega_k(t - \tfrac{1}{l})) \big] < \infty \ \forall n \in \mathbb{N},$$

$$d(\omega_k(t-\tfrac{1}{l}), \omega_k(s)) \leq \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t-\tfrac{1}{l}))\} \ \forall s \in (t - \tfrac{1}{l}, t] \ \forall k \in \mathbb{N} \Big\} \Bigg]$$

$$\bigcap \Big\{ \omega \in \Omega^{\mathbb{N}} : \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, \omega_k(t)) \big] < \infty \ \forall n \in \mathbb{N},$$

$$d(\omega_k(t), \omega_k(s)) \leq \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t)) \ \forall s \in (t, t + \tfrac{1}{i}], \ \forall k \in \mathbb{N} \Big\},$$

and for $t = 0$ and $i \in \mathbb{N}$, we set

$$\mathbb{A}_{t,i} = \mathbb{A}_{0,i} := \big\{ \omega \in \Omega^{\mathbb{N}} : d(x_k, \omega_k(s)) \leq \max\{1, \tfrac{1}{2}\, d(x_0, x_k)\} \ \forall s \in (0, \tfrac{1}{i}] \ \forall k \in \mathbb{N} \big\}.$$

Then,

$$\mathbb{P}_{\mathbf{x}}\Bigg(\bigcup_{i=\mathcal{I}_t}^{\infty} \mathbb{A}_{t,i} \Bigg) = \lim_{i \to \infty} \mathbb{P}_{\mathbf{x}}(\mathbb{A}_{t,i}) = 1$$

for each $t \in \mathbb{R}_+$.

Let $D := \mathbb{R}_+ \cap \mathbb{Q}$ ($\mathbb{Q}$ denoting the set of rational numbers), and $D = \{t_l\}_{l=1}^{\infty}$. We consider arbitrary numbers $\varepsilon_{lp} > 0$, $l, p \in \mathbb{N}$, such that $\sum_{l=1}^{\infty} \varepsilon_{lp} < \infty$ for each $p \in \mathbb{N}$, and

$$\lim_{p \to \infty} \sum_{l=1}^{\infty} \varepsilon_{lp} = 0. \tag{8.20}$$

For each $l, p \in \mathbb{N}$, we choose an $i_{lp} \in \mathbb{N}$ such that

$$\mathbb{P}_{\mathbf{x}}\big(\mathbb{A}^{\mathrm{c}}_{t_l, i_{lp}}\big) \leq \varepsilon_{lp}\,, \tag{8.21}$$

which exists due to Lemma 8.5. Then, we set

$$\mathbb{A}_p := \bigcap_{l=1}^{\infty} \mathbb{A}_{t_l, i_{lp}}. \tag{8.22}$$

By (8.20), (8.21), and (8.22),

$$\lim_{p \to \infty} \mathbb{P}_{\mathbf{x}}(\mathbb{A}_p) = 1. \tag{8.23}$$

For each $p \in \mathbb{N}$, let us consider the set

$$T_p := \mathbb{R}_+ \cap \Big[\bigcup_{l=1}^{\infty} (t_l - i_{lp}^{-1}, t_l + i_{lp}^{-1}) \Big].$$

Since the set $\bigcup_{l=1}^{\infty} (t_l - i_{lp}^{-1}, t_l + i_{lp}^{-1})$ is open in $\mathbb{R}$, and since T_p is dense in $\mathbb{R}_+$, we have $T_p = \mathbb{R}_+ \setminus T_p^{\mathrm{c}}$, where T_p^{c} is countable. We set

$$\mathbf{A}_p := \mathbb{A}_p \cap \Big[\bigcap_{t \in T_p^{\mathrm{c}}} \bigcup_{i=\mathcal{I}_t}^{\infty} \mathbb{A}_{t,i} \Big].$$

By Lemma 8.5, we get

$$\mathbb{P}_{\mathbf{x}}(\mathbf{A}_p) = \mathbb{P}_{\mathbf{x}}(\mathbb{A}_p). \tag{8.24}$$

Finally, we set

$$\mathbf{A} := \bigcup_{p=1}^{\infty} \mathbf{A}_p. \tag{8.25}$$

Therefore, by (8.23) and (8.24), we get

$$\mathbb{P}_{\mathbf{x}}(\mathbf{A}) = 1. \tag{8.26}$$

Lemma 8.6 *For any $\omega \in \mathbf{A}$ and $n \in \mathbb{N}$, we have*

$$\forall t \in \mathbb{R}_+ : \quad B_n(\omega(t)) := \sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, \omega_k(t)) \big] < \infty,$$

and moreover, the mapping

$$\mathbb{R} \ni t \mapsto B_n(\omega(t)) \in \mathbb{R}$$

is continuous.

Proof. We note that, if $B_{2n}(\omega(t)) < \infty$ and if there exists an interval $(a, b) \subset \mathbb{R}_+$ such that $t \in [a, b]$ and

$$d(\omega_k(s), \omega_k(t)) \le \max\{1, \tfrac{1}{2}\, d(x_0, \omega_k(t))\}, \qquad s \in (a, b),$$

then the series

$$\sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{n}\, d(x_0, \omega_k(s)) \big], \qquad s \in (a, b),$$

are majorized by the convergent series

$$\sum_{k=1}^{\infty} \exp\big[-\tfrac{1}{2n}\, d(x_0, \omega_k(t))\big].$$

Hence, the statement follows from the construction of the set $\mathbf{A}$. ■

Now, we define the action of the group S_∞ on $\Omega^{\mathbb{N}}$ by

$$\sigma((\omega_k)_{k=1}^{\infty}) := (\omega_{\sigma(k)})_{k=1}^{\infty}, \qquad \sigma \in S_\infty.$$

Evidently, the set $\mathbf{A}$ is invariant under the action of S_∞. We introduce the factor space $\Omega^{\mathbb{N}}/S_\infty$ consisting of factor classes $[\omega_1, \omega_2, \dots]$. Analogously to (4.3) and (7.1), we introduce then the mapping

$$\Omega^{\mathbb{N}} \ni \omega = (\omega_k)_{k=1}^{\infty} \mapsto \mathbf{I}\omega = [\omega_1, \omega_2, \dots] \in \Omega^{\mathbb{N}}/S_\infty.$$

Lemma 8.7 *We have*

$$\mathbf{IA} \subset C(\mathbb{R}_+; \ddot{\Gamma}_\infty),$$

where $C(\mathbb{R}_+; \ddot{\Gamma}_\infty)$ denotes the set of continuous mappings from $\mathbb{R}_+$ into $\ddot{\Gamma}_\infty$.

Proof. Taking notice of Lemma 8.6 and of the definition of the metric space $\ddot{\Gamma}_\infty$, it remains only to show that, for each fixed $(\omega_k)_{k=1}^{\infty} \in \mathbf{A}$, the mapping

$$\mathbb{R}_+ \ni t \mapsto \{\omega_k(t)\}_{k=1}^{\infty} \in \ddot{\Gamma}_\infty \tag{8.27}$$

is vaguely continuous.

To this end, let us fix any $t \in \mathbb{R}_+$ and any ball $B(x_0, r)$ of radius $r > 0$. Then, there exist $\varepsilon > 0$ and $K \in \mathbb{N}$ such that

$$\sum_{k=K}^{\infty} \exp\big[-d(x_0, \omega_k(s))\big] < e^{-r}, \qquad s \in \mathbb{R}_+ \cap (t-\varepsilon, t+\varepsilon)$$

(see the proof of Lemma 8.6). Hence,

$$\omega_k(s) \notin B(x_0, r), \qquad k \geq K,\ s \in \mathbb{R}_+ \cap (t-\varepsilon, t+\varepsilon),$$

which, together with the continuity of each ω_k as a mapping from $\mathbb{R}_+$ into X, implies the vague continuity of (8.27). ■

Thus, by Lemma 8.7, we have that

$$\mathbf{I}\colon \mathbf{A} \to \mathbf{\Omega} := C(\mathbb{R}_+; \ddot{\Gamma}_\infty).$$

Denote the trace σ-algebra of $\mathcal{C}_\sigma(\Omega^{\mathbb{N}})$ on $\mathbf{A}$ by $\mathcal{C}_\sigma(\mathbf{A})$. Let $\mathbf{F}$ be the product σ-algebra on $\mathbf{\Omega} = C(\mathbb{R}_+; \ddot{\Gamma}_\infty)$ generated by the σ-algebra $\mathcal{B}(\ddot{\Gamma}_\infty)$:

$$\mathbf{F} := \sigma\{\mathbf{X}_t,\, t \in \mathbb{R}_+\},$$

where

$$\mathbf{X}_t(\omega) := \omega(t).$$

Since the mapping $I\colon \widehat{X}^{\mathbb{N}} \to \ddot{\Gamma}_X$ defined by (7.1) is measurable and since $\mathcal{B}(\ddot{\Gamma}_\infty)$ is the trace σ-algebra of $\mathcal{B}(\ddot{\Gamma}_X)$ on $\ddot{\Gamma}_\infty$, we easily conclude that the mapping $\mathbf{I}$ is $\mathcal{C}_\sigma(\mathbf{A})$-$\mathbf{F}$-measurable. Because of (8.26), we can consider $\mathbb{P}_{\mathbf{x}}$ as a probability measure on $(\mathbf{A}, \mathcal{C}_\sigma(\mathbf{A}))$ and let $\mathbf{P}_{\mathbf{x}}$ denote the image of this measure under the mapping $\mathbf{I}$. Thus, $\mathbf{P}_{\mathbf{x}}$ is a probability measure on $(\mathbf{\Omega}, \mathbf{F})$.

For each $\sigma \in S_\infty$, the measures $\mathbf{P}_{\mathbf{x}}$ and $\mathbf{P}_{\sigma(\mathbf{x})}$ evidently coincide, and so for each $\gamma \in \ddot{\Gamma}_\infty$, we can introduce the probability measure $\mathbf{P}_\gamma := \mathbf{P}_{\mathbf{x}}$, where $\mathbf{x}$ is an arbitrary element of the set $I^{-1}\gamma$.

Finally, we introduce the sub-σ-algebras $\mathbf{F}_t := \sigma\{\mathbf{X}_s,\, s \le t\}$ and the translations $(\boldsymbol{\theta}_t\omega)(s) := \omega(s+t)$, $t \in \mathbb{R}_+$. Thus, we get

$$(\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\in\mathbb{R}_+}, (\boldsymbol{\theta}_t)_{t\in\mathbb{R}_+}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_\infty}, (\mathbf{X}_t)_{t\in\mathbb{R}_+}). \tag{8.28}$$

It follows directly from our construction that (8.28) is a realization of the independent infinite particle process,

For a fixed $\gamma \in \ddot{\Gamma}_\infty$, the finite-dimensional distributions of the process $\mathbf{X}_t$ under $\mathbf{P}_\gamma$ are given by

$$\begin{aligned}
\mathbf{P}_\gamma(\mathbf{X}_{t_1} \in A_1, \mathbf{X}_{t_2} \in A_2, \dots, \mathbf{X}_{t_n} \in A_n) &= \mathbf{P}_{\mathbf{x}}(\mathbf{X}_{t_1} \in A_1, \mathbf{X}_{t_2} \in A_2, \dots, \mathbf{X}_{t_n} \in A_n)\\
&= \mathbb{P}_{\mathbf{x}}(\omega(t_1) \in I^{-1}A_1, \omega(t_2) \in I^{-1}A_2, \dots, \omega(t_n) \in I^{-1}A_n)\\
= \int_{I^{-1}A_1} \mathbb{P}_{t_1}(\mathbf{x}, d\mathbf{x}_1) \int_{I^{-1}A_2} &\mathbb{P}_{t_2-t_1}(\mathbf{x}_1, d\mathbf{x}_2)\cdots \int_{I^{-1}A_n} \mathbb{P}_{t_n-t_{n-1}}(\mathbf{x}_{n-1}, d\mathbf{x}_n)\\
= \int_{A_1} \mathbf{P}_{t_1}(\gamma, d\gamma_1) \int_{A_2} &\mathbf{P}_{t_2-t_1}(\gamma_1, d\gamma_2)\cdots \int_{A_n} \mathbf{P}_{t_n-t_{n-1}}(\gamma_{n-1}, d\gamma_n),\\
0 < t_1 < t_2 < \cdots < t_n, &\quad A_1, A_2, \dots, A_n \in \mathcal{B}(\ddot{\Gamma}_\infty)
\end{aligned}$$

where $\mathbb{P}_t(\mathbf{x}_i, d\mathbf{x}_j) := \mathbb{P}_{t,\mathbf{x}_i}(d\mathbf{x}_j)$ and $\mathbf{x}$ is an arbitrary element of $I^{-1}\{\gamma\}$. Thus, the finite-dimensional distributions of $\mathbf{X}_t$ are determined by the Markov semigroup of kernels $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$. Hence, it follows that (8.28) is a time homogeneous Markov process on $(\ddot{\Gamma}_\infty, \mathcal{B}(\ddot{\Gamma}_\infty))$ with transition probability function $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$ (see e.g. [8, Ch. 1, Sect. 3]).

Finally, we note that any measure on the space $(\mathbf{\Omega}, \mathbf{F})$ is uniquely determined by its finite-dimensional distributions, and therefore the constructed continuous Markov process is unique. ■

Remark 8.2 It is easy to see that the process $(\mathbf{X}_t)_{t\in\mathbb{R}_+}$constructed in the course of the proof of Theorem 8.1 is even Markov with respect to the filtration $(\mathbf{F}_{t+})_{t\in\mathbb{R}_+}$, where $\mathbf{F}_{t+} := \bigcap_{s>t} \mathbf{F}_t$.

The following corollary states that, if the dimension d of the manifold X is ≥ 2, then the process $\mathbf{X}_t$ starting at $\gamma \in \Gamma_\infty$ lives with $\mathbf{P}_\gamma$-probability one in Γ_∞, i.e., the particles never collide (compare with [35]).

Corollary 8.1 *Let* (C1′), (C2), *and* (C3) *hold, and let* $d \geq 2$. *Then, the independent infinite particle process can be realized as the unique continuous, time homogeneous Markov process*

$$\mathbf{M} = (\boldsymbol{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\in\mathbb{R}_+}, (\boldsymbol{\theta}_t)_{t\in\mathbb{R}_+}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma_\infty}, (\mathbf{X}_t)_{t\in\mathbb{R}_+})$$

on the state-space $(\Gamma_\infty, \mathcal{B}(\Gamma_\infty))$ *with transition probability function* $(\mathbf{P}_t)_{t\in\mathbb{R}_+}$.

Proof. First, we claim that, if $d \geq 2$, then

$$P_{x_1} \otimes P_{x_2}\big(\exists t > 0 : \omega_1(t) = \omega_2(t)\big) = 0, \qquad x_1, x_2 \in X, \tag{8.29}$$

i.e., two independent Brownian motions on X never collide.

In the Euclidean case $X = \mathbb{R}^d$, this is a direct consequence of a classical result from potential theory. Indeed, $\omega_1(\frac{t}{2}) - \omega_2(\frac{t}{2})$ is standard Brownian motion on $\mathbb{R}^d$ starting at $x_1 - x_2$, and therefore (8.29) is equivalent to the equality

$$P_{x_1-x_2}\big(\exists t > 0 : \omega(t) = 0\big) = 0,$$

which is true since points are polar for Brownian motion on $\mathbb{R}^d$ if $d \geq 2$ (see e.g. [31, Proposition 2.5]).

In the general case, to prove (8.29) one can follow the idea of [35]. First, we note that (8.29) is equivalent to

$$P_{x_1} \otimes P_{x_2}\big(\exists t > \tfrac{1}{n} : \omega_1(t) = \omega_2(t)\big) = 0 \qquad \forall n \in \mathbb{N}. \tag{8.30}$$

Using the Markov property, we have

$$P_{x_1} \otimes P_{x_2}\big(\exists t > \tfrac{1}{n} : \omega_1(t) = \omega_2(t)\big) =$$
$$= \int_{X^2} m(dy_1)\, m(dy_2)\, p(\tfrac{1}{n}, x_1, y_1) p(\tfrac{1}{n}, x_2, y_2)\, P_{y_1} \otimes P_{y_2}\big(\exists t > 0 : \omega_1(t) = \omega_2(t)\big). \tag{8.31}$$

By virtue of (8.30) and (8.31), it suffices to verify that the equality (8.29) holds only for $m^{\otimes 2}$-a.a. $(x_1, x_2) \in X^2$.

There exists a countable, locally finite covering $\{U^{(i)}\}_{i=1}^\infty$ of the manifold X such that each $U^{(i)}$ is an open set in X diffeomorphic to the open cube $(-1, 1)^d$ in $\mathbb{R}^d$.

Furthermore, two independent Brownian motions on X which start respectively at x_1 and x_2 form a Brownian motion on the manifold X^2 starting at the point (x_1, x_2). Hence, our problem can be reduced to the following one: Show that

$$P_{(x_1,x_2)}\big(\exists t>0 : \omega_1(t)=\omega_2(t)\in U_j^{(i)}\big)=0, \qquad \text{for } m^{\otimes 2}\text{-a.a. } (x_1,x_2)\in X^2,\ i,j\in\mathbb{N}, \tag{8.32}$$

where $U_j^{(i)}$ is the subset of $U^{(i)}$ that is diffeomorphic to the open cube

$$\mathcal{C}_j := (-1+(1+j)^{-1}, 1-(1+j)^{-1})^d.$$

Let us consider the Dirichlet form that corresponds to Brownian motion on X^2:

$$\begin{aligned}\mathcal{E}(f,g) := \int_{X^2} \big[&\langle \nabla_{x_1}^X f(x_1,x_2), \nabla_{x_1}^X g(x_1,x_2)\rangle_{T_{x_1}(X)} \\ &+ \langle \nabla_{x_2}^X f(x_1,x_2), \nabla_{x_2}^X g(x_1,x_2)\rangle_{T_{x_2}(X)}\big]\, m(dx_1)\, m(dx_2).\end{aligned} \tag{8.33}$$

The bilinear form $\mathcal{E}$ is defined first for $f, g\in \mathcal{D}^{\otimes 2} = C_0^\infty(X^2)$, and then it is closed.

Since $m^{\otimes 2}\big(\{(x_1,x_2)\in X^2 : x_1=x_2\}\big)=0$, to prove (8.32) it is enough to construct a sequence $\{u_n\}_{n=1}^\infty \subset \mathrm{Dom}(\mathcal{E})$ such that u_n's converge pointwisely to the indicator function of the set $\{(x_1,x_2)\in X^2 : x_1=x_2\in U_j^{(i)}\}$ and $\sup_n \mathcal{E}(u_n,u_n)<\infty$ (see [35]).

By using the representation of the Dirichlet form $\mathcal{E}$ in local coordinates on $U^{(i)}$, we get, for any function $f\in\mathrm{Dom}(\mathcal{E})$ having support in $(U^{(i)})^2$:

$$\begin{aligned}\mathcal{E}(f,f) = \frac{1}{2}\int_{(-1,1)^{2d}} \sum_{k,l=1}^d &\bigg[g^{kl}(x_1)\, \frac{\partial f}{\partial x_1^k}(x_1,x_2)\, \frac{\partial f}{\partial x_1^l}(x_1,x_2) \\ &+ g^{kl}(x_2)\, \frac{\partial f}{\partial x_2^k}(x_1,x_2)\, \frac{\partial f}{\partial x_2^l}(x_1,x_2)\bigg] \sqrt{\mathbf{g}(x_1)}\,\sqrt{\mathbf{g}(x_2)}\, dx_1^1\cdots dx_1^d\, dx_2^1\cdots dx_2^d,\end{aligned} \tag{8.34}$$

where $\mathbf{g}$ denotes the determinant of the matrix $(g_{kl})_{k,l=1}^d := (\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\rangle)_{k,l=1}^d$, and $(g^{kl})_{k,l=1}^d$ is its inverse. We conclude from (8.34) that there exists a constant $C>0$ such that, for each function $f\in\mathrm{Dom}(\mathcal{E})$ having support in a fixed $(U_j^{(i)})^2$,

$$\mathcal{E}(f,f)\le C\mathcal{E}_{\mathrm{E}}(f,f),$$

where $\mathcal{E}_{\mathrm{E}}$ is the (Euclidean) Dirichlet form on $\mathbb{R}^{2d}$:

$$\mathcal{E}_{\mathrm{E}}(f,f) = \int_{\mathbb{R}^{2d}} \langle \nabla f(x_1,x_2), \nabla f(x_1,x_2)\rangle\, dx_1^1\cdots dx_1^d\, dx_2^1\cdots dx_2^d,$$

∇ denoting the usual gradient on $\mathbb{R}^{2d}$. Hence, it suffices to construct for any fixed $j\in\mathbb{N}$ a sequence $\{u_n\}_{n=1}^\infty\subset\mathrm{Dom}(\mathcal{E}_{\mathrm{E}})$ such that each u_n has support in $\mathcal{C}_{j+1}^2$, the

u_n's converge pointwisely to the indicator of the set $\{(x_1,x_2)\in\mathbb{R}^{2d}: x_1=x_2\in\mathcal{C}_j\}$, and $\sup_n \mathcal{E}_{\mathrm{E}}(u_n,u_n)<\infty$. But the existence of such sequence can be seen by a trivial modification of the proof of Proposition 1 in [35]. Thus, (8.32) and hence also the claim (8.29) are proven.

The rest of the proof follows from that of Theorem 8.1. Instead of the set $\mathbf{A}$ given by (8.25), one should use its subset

$$\mathbf{A}' := \mathbf{A}\cap\Bigg[\bigcap_{\{i,j\}\subset\mathbb{N}}\big\{\,\omega_i(t)\ne\omega_j(t)\ \forall t\in\mathbb{R}_+\,\big\}\Bigg]. \tag{8.35}$$

By (8.7), (8.26), (8.29), and (8.35), we get

$$\mathbb{P}_{\mathbf{x}}(\mathbf{A}')=1$$

for each $\mathbf{x}\in\widetilde{X}^{\mathbb{N}}$ satisfying (8.6), so that the measure $\mathbb{P}_{\mathbf{x}}$ can be considered as a probability measure on $(\mathbf{A}',\mathcal{B}(\mathbf{A}'))$. Finally, noting that

$$\mathbf{I}\mathbf{A}'\subset C(\mathbb{R}_+;\Gamma_\infty)$$

(compare with Lemma 8.7), we get the corollary by a corresponding modification of the last part of the proof of Theorem 8.1. ■

Remark 8.3 (Path-wise construction of the independent particle process on Γ_∞) Let $d\ge 2$ and let us consider the probability space $(\mathbf{\Omega},\mathcal{C}_\sigma(\mathbf{\Omega}),\mathbf{P})$, where $\mathbf{\Omega}:=\Omega^X$, $\Omega:=C(\mathbb{R}_+;X)$, $\mathcal{C}_\sigma(\mathbf{\Omega})$ is the product σ-algebra on $\mathbf{\Omega}$ constructed from the σ-algebra $\mathcal{F}$ on Ω defined by (8.1), and $\mathbf{P}:=\bigotimes_{x\in X}P_x$. For any fixed $\gamma\in\Gamma_\infty$ and $t\in\mathbb{R}_+$, we define

$$\mathbf{\Omega}\ni\omega=(\omega_x)_{x\in X}\mapsto\mathbf{X}_t^\gamma(\omega):=\sum_{x\in\gamma}\varepsilon_{X_t^x(\omega)},$$

where $X_t^x(\omega):=\omega_x(t)$. Thus, for any $\gamma\in\Gamma_\infty$ we have constructed a process $\mathbf{X}^\gamma:=(\mathbf{X}_t^\gamma)_{t\in\mathbb{R}_+}$ which takes values in the space of all measures on X. Let us fix any $\mathbf{x}=(x_k)_{k=1}^\infty\in I^{-1}\{\gamma\}$. Then, $\mathbb{P}_{\mathbf{x}}=\mathbf{P}\circ\mathbf{I}_{\mathbf{x}}^{-1}$, where $\mathbb{P}_{\mathbf{x}}$ is defined by (8.7) and

$$\mathbf{\Omega}\ni\omega=(\omega_x)_{x\in X}\mapsto\mathbf{I}_{\mathbf{x}}\omega:=(\omega_{x_k})_{k=1}^\infty\in\Omega^{\mathbb{N}}.$$

Hence, it follows from the proof of Theorem 8.1 (respectively Corollary 8.1) that with $\mathbf{P}$ probability one the independent particle process $\mathbf{X}^\gamma$ starts at γ, never leaves Γ_∞, i.e., $\mathbf{P}\big(\forall t>0:\mathbf{X}_t^\gamma\in\Gamma_\infty\big)=1$, and has sample paths which are continuous in the d_∞ metric.

In the case $d=1$, in order to give a corresponding path-wise construction of the independent particle process on $\ddot{\Gamma}_\infty$, we proceed as follows. We consider the probability space $(\mathbf{\Omega},\mathcal{C}_\sigma(\mathbf{\Omega}),\mathbf{P})$, where

$$\mathbf{\Omega}:=\mathop{\times}_{n(\cdot)\in\mathbb{N}^X}\ \mathop{\times}_{x\in X}\Omega^{n(x)},$$

$\mathcal{C}_\sigma(\mathbf{\Omega})$ is the corresponding product σ-algebra on $\mathbf{\Omega}$, and

$$\mathbf{P} := \bigotimes_{n(\cdot)\in\mathbb{N}^X} \bigotimes_{x\in X} P_x^{\otimes n(x)}.$$

For any $\gamma \in \ddot{\Gamma}_X$, we define $\hat{\gamma} \in \Gamma_X$ and a mapping $n_\gamma : \hat{\gamma} \to \mathbb{N}$ by

$$\hat{\gamma} := \operatorname{supp}\gamma, \qquad \hat{\gamma} \ni x \mapsto n_\gamma(x) := \gamma(\{x\}) \in \mathbb{N}.$$

We extend the mapping $n_\gamma(\cdot)$ to the whole of X by $n_\gamma(x) := 1$ for all $x \in X \setminus \hat{\gamma}$. Now, for any fixed $\gamma \in \ddot{\Gamma}_\infty$, we define a process $\mathbf{X}^\gamma := (\mathbf{X}_t^\gamma)_{t\in\mathbb{R}_+}$ setting for each $t \in \mathbb{R}_+$

$$\mathbf{\Omega} \ni \omega = (\omega^1_{n(\cdot),\,x}, \dots, \omega^{n(x)}_{n(\cdot),\,x})_{n(\cdot)\in\mathbb{N}^X,\,x\in X} \mapsto \mathbf{X}_t^\gamma(\omega) := \sum_{x\in\hat{\gamma}} \sum_{i=1}^{n_\gamma(x)} \varepsilon_{\omega^i_{n_\gamma(\cdot),\,x}(t)}.$$

Analogously to the above, we conclude that with $\mathbf{P}$ probability one the independent particle process $\mathbf{X}^\gamma$ starts at γ, never leaves $\ddot{\Gamma}_\infty$, and has sample paths continuous in the d_∞ metric.

Remark 8.4 The Markov process on the state space Γ_∞ that was constructed in Corollary 8.1 is a strong Markov process. This can be shown by a modification of the proof of [12, Theorem 5.10] using Corollary 6.2. Furthermore, by proving a corresponding Feller property of $\mathbf{P}_t$ on $\ddot{\Gamma}_\infty$ with respect to the metric d_1, one can show that the Markov process on the state space $\ddot{\Gamma}_\infty$ that was constructed in Theorem 8.1 also possesses the strong Markov property.

9 Appendix: Proof of Lemma 8.1

Let us fix $r > 0$, $\delta > 0$, $x \in X$, $0 \le t_1 < \cdots < t_n$, $n \ge 2$, with $t_n - t_1 \le \delta$. Let

$$A := \big\{\, \omega \in \Omega : d(\omega(t_1), \omega(t_j)) > r \text{ for some } j = 2, \dots, n \,\big\}, \tag{9.1}$$

and let us show that

$$P_x(A) \le 2\tau(\delta, \tfrac{1}{2}\, r). \tag{9.2}$$

Let

$$\begin{aligned} B :&= \big\{\, \omega \in \Omega : d(\omega(t_1), \omega(t_n)) > \tfrac{1}{2}\, r \,\big\},\\ C_j :&= \big\{\, \omega \in \Omega : d(\omega(t_j), \omega(t_n)) > \tfrac{1}{2}\, r \,\big\},\\ D_j :&= \big\{\, \omega \in \Omega : d(\omega(t_1), \omega(t_j)) > r, \text{ and } d(\omega(t_1), \omega(t_k)) \le r \text{ for } k = 1, \dots, j-1 \,\big\}. \end{aligned}$$

Then,

$$A \subset B \cup \Big[\bigcup_{j=1}^{n} (C_j \cap D_j) \Big].$$

Therefore,

$$P_x(A) \le P_x(B) + \sum_{j=1}^{n} P_x(C_j \cap D_j).$$

Define $\widetilde{D}_j \subset X^j$, $\widetilde{C}_j \subset X^2$ by

$$\begin{aligned}
\widetilde{D}_j &:= \big\{ (x_1, \dots, x_j) \in X^j : d(x_1, x_j) > r \text{ and } d(x_1, x_k) \le r \text{ for } k = 1, \dots, j-1 \big\}, \\
\widetilde{C}_j &:= \big\{ (x_1, x_2) \in X^2 : d(x_1, x_2) > \tfrac{1}{2}\, r \big\}.
\end{aligned}$$

Then,

$$\begin{aligned}
P_x(C_j \cap D_j) &= \int_X \cdots \int_X p(t_1, x, dx_1) \cdots p(t_j - t_{j-1}, x_{j-1}, dx_j) \\
&\quad \times p(t_n - t_j, x_j, dx_n) 1_{\widetilde{D}_j}(x_1, \dots, x_j) 1_{\widetilde{C}_j}(x_j, x_n) \\
&\le \tau(\delta, \tfrac{1}{2}\, r) \int_X \cdots \int_X p(t_1, x, dx_1) \cdots p(t_j - t_{j-1}, x_{j-1}, dx_j) 1_{\widetilde{D}_j}(x_1, \dots, x_j) \\
&= \tau(\delta, \tfrac{1}{2}\, r) P_x(D_j).
\end{aligned}$$

Since the sets D_j are disjoint, we have

$$\sum_{j=1}^{n} P_x(C_j \cap D_j) \le \sum_{j=1}^{n} \tau(\delta, \tfrac{1}{2}\, r) P_x(D_j) \le \tau(\delta, \tfrac{1}{2}\, r),$$

and since $P_x(B) \le \tau(\delta, \frac{1}{2}\, r)$, we get (9.2).

It follows from (9.1), (9.2) that

$$P_x(d(\omega(t_j), \omega(t_k)) > 2r \text{ for some } j, k,\ 1 \le j, k \le n) \le 2\tau(\delta, \tfrac{1}{2}\, r). \tag{9.3}$$

Indeed, if $d(\omega(t_j), \omega(t_k)) > 2r$, then $d(\omega(t_1), \omega(t_j)) > r$ or $d(\omega(t_1), \omega(t_k)) > r$.

Since the estimate (9.3) is independent of n and $t_1, \dots, t_n$, we get

$$P_x(d(\omega(t_1), \omega(t_2)) > 2r \text{ for some } t_1, t_2 \in \mathbb{R}_+ \cap \mathbb{Q},\ |t_1 - t_2| \le \delta) \le 2\tau(\delta, \tfrac{1}{2}\, r). \tag{9.4}$$

Due to the continuity of the trajectories $\omega \in \Omega$, (9.4) implies the statement of the lemma. ■

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