Weak Poincaré Inequalities and L^2-Convergence Rates of Markov Semigroups *

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Abstract

In order to describe L^2-convergence rates slower than exponential, the weak Poincaré inequality is introduced. It is shown that the convergence rate of a Markov semigroup and the corresponding weak Poincaré inequality can be determined by each other. Conditions for the weak Poincaré inequality to hold are presented, which are easy to check and which hold in many applications. The weak Poincaré inequality is also studied by using isoperimetric inequalities for diffusion and jump processes. Some typical examples are given to illustrate the general results. In particular, our results are applied to the stochastic quantization of field theory in finite volume. Moreover, a sharp criterion of weak Poincaré inequalities is presented for Poisson measures on configuration spaces.

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1 Introduction

Let $(E, \mathcal{F}, \mu)$ be a probability space, and $(L, \mathcal{D}(L))$ a densely defined linear operator which generates a Markov C_0-semigroup P_t on $L^2(\mu)$ where μ is its invariant measure. The Poincaré inequality for L reads

$$\mu(f^2) \leq C\mathcal{E}(f, f), \quad \mu(f) = 0, f \in \mathcal{D}(L), \tag{1.1}$$

where $\mathcal{E}(f, f) := -\mu(fLf)$ for $f \in \mathcal{D}(L)$, and $C > 0$ is a constant. It is well known that (1.1) is equivalent to the exponential L^2-convergence of P_t

$$\mu((P_t f)^2) \leq \exp[-2t/C]\mu(f^2), \qquad f \in L^2(\mu), \mu(f) = 0, t \geq 0.$$

To describe also slower convergence rates of P_t, Liggett [19] introduced the following version of Nash type inequality

$$\mu(f^2) \leq C\mathcal{E}(f, f)^{1/p}\Phi(f)^{1/q}, \quad \mu(f) = 0, f \in \mathcal{D}(L), \tag{1.2}$$

where $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, C is a positive constant, and $\Phi : L^2(\mu) \to [0, \infty]$ satisfies $\Phi(cf) = c^2\Phi(f)$ for any $c \in \mathbb{R}$ and $f \in L^2(\mu)$. This inequality was used by Liggett [19] to prove an algebraic convergence of P_t, and was applied in [19] and [7] to some spin systems. The following result is taken from [19].

Theorem 1.1 (Liggett [19]). *If* (1.2) *holds with* Φ *satisfying* $\Phi(P_t f) \leq \Phi(f)$ *for* $f \in \mathcal{D}(L)$ *and* $t \geq 0$, *then there exists* $c > 0$ *such that*

$$\mu((P_t f)^2) \leq c\Phi(f)t^{1-q}, \qquad t > 0, f \in L^2(\mu), \mu(f) = 0. \tag{1.3}$$

Conversely, if P_t *is symmetric then* (1.3) *implies* (1.2) *for some* $C > 0$.

The main aim of this paper is to introduce a general version of (1.2)(which can be checked in many concrete cases) to describe general convergence rates of P_t. To do this, a very simple way is to replace $\mathcal{E}(f, f)^{1/p}$ in (1.2) by a general function of $\mathcal{E}(f, f)$, and to change $\Phi(f)^{1/q}$ there into another function of $\Phi(f)$ such that the whole inequality is homogeneous in f. Then, one may try to study the L^2-convergence of P_t using such a more general inequality. More precisely, one considers

$$\mu(f^2) \leq \Theta(\mathcal{E}(f, f)), \qquad \mu(f) = 0, f \in \mathcal{D}(L), \Phi(f) \leq 1, \tag{1.4}$$

where $\Theta \in C[0, \infty)$ is a nonnegative and increasing function with $\Theta(0) = 0$, $\Theta(r) > 0$ for $r > 0$ and $\Theta(r)/r \to 0$ as $r \to \infty$, and Φ is as in (1.2). For (1.2) we choose $\Theta(r) = r^{1/p}$.

To estimate the L^2-convergence rate of P_t, we let $\Theta^{-1}(r) := \inf\{s \geq 0 : \Theta(s) \geq r\}$. Then (1.4) implies

$$\Theta^{-1}(\mu(f^2)) \leq \mathcal{E}(f,f), \quad f \in \mathcal{D}(L), \mu(f) = 0, \Phi(f) \leq 1.$$

Assuming that $\Phi(P_t f) \leq \Phi(f)$, we obtain

$$\int_{\mu((P_t f)^2)}^{\mu(f^2)} \frac{\mathrm{d}r}{\Theta^{-1}(r)} \geq 2t, \quad t > 0, f \in \mathcal{D}(L), \Phi(f) \leq 1, \mu(f) = 0.$$

Therefore, to describe the convergence rate of P_t, one has to solve this possibly complicated inequality. To avoid this, we will use an alternative approach based on [16, 28, 29] in which semigroup properties and spectral estimates were studied by using the super-Poincaré inequality

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(L), \tag{1.5}$$

where β is a positive decreasing function on $(0, \infty)$. The advantage of such Poincaré type inequalities is that they imply semigroup estimates directly. Correspondingly to (1.5), we introduce the following weak Poincaré inequality as an extension of (1.2)

$$\mu(f^2) \leq \alpha(r)\mathcal{E}(f,f) + r\Phi(f), \quad \mu(f) = 0, f \in \mathcal{D}(L), r > 0, \tag{1.6}$$

where α is a nonnegative and decreasing function on $(0, \infty)$, and Φ is as in (1.2). It is easy to see that (1.2) is indeed equivalent to (1.6) for $\alpha(r) = cr^{1-p}$ and some $c > 0$. More generally, (1.4) implies (1.6) for $\alpha(r) := \sup_{s>0} s^{-1}[\Theta(s) - r]$, while, if $\lim_{r\to\infty} \alpha(r) = 0$, then (1.6) implies (1.4) with $\Theta(s) := \inf_{r>0}[\alpha(r)s + r]$.

The main general results of this paper are Theorems 2.1, 2.3 and Corollary 2.4. By the above considerations, Theorem 1.1 is hence an immediate consequence thereof, and this way we obtain a new proof for Theorem 1.1. Some criteria for (1.6) are presented in section 3 which are especially designed for diffusions on a Riemannian manifold. In sections 4 and 5 we study (1.6) by using isoperimetric inequalities for both diffusion and jump cases. Results obtained in these two sections extend known ones on the spectral gap via Cheeger's inequality (see e.g. [8, 15, 18, 31]). In section 6 the behaviour of the weak Poincaré inequality under perturbations of μ is studied. The main result in that section is applied to the stochastic quantization of field theory in a finite volume $\Lambda \subset \mathbb{R}^2$. In section 7 we obtained a sharp criterion of the weak Poincaré inequality for Poisson measures on configuration spaces. In particular, in the case where the intensity of the Poisson measure has positive smooth density, we prove that the spectral gap of the underling Dirichlet form on a configuration space coincides with the principal eigenvalue of the corresponding weighted Laplacian on the based manifold.

Moreover, we like to mention that, for a conservative Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ (cf. [20]), (1.6) for $\Phi(f) = \|f\|_\infty^2$ is equivalent to Kusuoka-Aida's "weak spectral gap property" (**WSGP** for short, see [1]): for any sequence $\{f_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\mu(f_n^2) \le 1, \mu(f_n) = 0$, and $\mathcal{E}(f_n, f_n) \to 0$ as $n \to \infty$, we have $f_n \to 0$ in probability.

Proposition 1.2. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a conservative Dirichlet form on $L^2(\mu)$. Then* **WSGP** *is equivalent to* (1.6) *for some α and $\Phi(f) := \|f\|_\infty^2$.*

Proof. First of all, by Theorem 2.13 in [20] (1.6) holds with $\Phi(f) = \|f\|_\infty^2$ for all $f \in \mathcal{D}(L)$ if and only if it holds for all $f \in \mathcal{D}(\mathcal{E})$. Assume that **WSGP** holds. If (1.6) does not hold for any α, then there exist $r > 0$ and a sequence $\{f_n\} \subset \mathcal{D}(\mathcal{E}) \cap L^\infty(\mu)$ such that $\mu(f_n) = 0, \mu(f_n^2) = 1$ and

$$n\mathcal{E}(f_n, f_n) + r\|f_n\|_\infty^2 < 1, \qquad n \ge 1.$$

Hence $\|f_n\|_\infty^2 < r^{-1}$ for all $n \to \infty$ and $\mathcal{E}(f_n, f_n) \to 0$ as $n \to \infty$. By **WSGP** it then follows that

$$1 = \lim_{n\to\infty} \mu(f_n^2) \le \varepsilon^2 + r^{-2} \lim_{n\to\infty} \mu(|f_n| > \varepsilon) = \varepsilon^2$$

for any $\varepsilon > 0$, which is impossible.

On the other hand, assume that (1.6) holds for $\Phi(f) = \|f\|_\infty^2$ and some α. Let $\{f_n\} \subset \mathcal{D}(\mathcal{E})$ with $\mu(f_n) = 0, \mu(f_n^2) \le 1$ and $\mathcal{E}(f_n, f_n) \to 0$ as $n \to \infty$. We have to prove that $\mu(|f_n| > \varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$. For $R > 0$, let $f_{n,R} := (f_n \wedge R) \vee (-R)$. By (1.6),

$$\mu(f_{n,R}^2) \le \mu(f_{n,R})^2 + rR^2 + \alpha(r)\mathcal{E}(f_n, f_n), \quad r > 0, n \ge 1, R > 0. \tag{1.7}$$

Since $\mu(f_n) = 0$ and $|f_{n,R} - f_n| \le 1_{\{|f_n|>R\}}(|f_n| - R)$, it follows that

$$\mu(f_{n,R})^2 \le \mu((|f_n| - R)1_{\{|f_n|>R\}})^2 \le \mu(f_n^2)\mu(|f_n| > R) \le R^{-2}. \tag{1.8}$$

Furthermore, $\mu(f_{n,R}^2) \ge \varepsilon^2\mu(|f_{n,R}| > \varepsilon) \ge \varepsilon^2\mu(|f_n| > \varepsilon)$ for $R > \varepsilon > 0$. Combining this with (1.7) and (1.8) we obtain

$$\mu(|f_n| > \varepsilon) \le \varepsilon^{-2}\big[\alpha(r)\mathcal{E}(f_n, f_n) + R^{-2} + rR^2\big], \quad r > 0, R > \varepsilon.$$

This implies $\mu(|f_n| > \varepsilon) \to 0$ as $n \to \infty$ since R and r are arbitrary and $\mathcal{E}(f_n, f_n) \to 0$. □

We would like to mention that, for conservative Dirichlet forms, Mathieu [21] proved the equivalence of **WSGP** and the L^1-convergence of P_t

$$\lim_{t\to\infty} \sup_{\mu(f^2)\le 1} \|P_t f - \mu(f)\|_1 = 0. \tag{1.9}$$

It is easy to see that (1.9) is equivalent to $\lim_{t\to\infty}\sup_{\|f\|_\infty\le 1}\|P_tf-\mu(f)\|_2=0$. Then one may prove Proposition 1.2 by Theorem 2.3 in the next section.

We also have the following result concerning the relation between (1.1) and (1.6).

Proposition 1.3. *Assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a conservative Dirichlet form. If (1.6) holds for $\Phi(f):=\|f\|_\infty$ and there exist $C_1,C_2>0$ such that*

$$\mu(f^2)\le C_1\mathcal{E}(f,f)+C_2\mu(|f|)^2,\quad f\in\mathcal{D}(L). \tag{1.10}$$

Then (1.1) holds for some $C>0$.

Proof. Assume that (1.6) holds for $\Phi(f):=\|f\|_\infty^2$. If (1.1) does not hold for any $C>0$, then there exists a sequence $\{f_n\}\subset\mathcal{D}(L)$ such that $\mu(f_n)=0,\mu(f_n^2)=1$ and $\mathcal{E}(f_n,f_n)\to 0$ as $n\to\infty$. By Proposition 1.2, for any $\varepsilon>0$ one has $\mu(|f_n|\ge\varepsilon)\to 0$ as $n\to\infty$. Therefore, (1.10) yields that

$$\begin{aligned}C_1\mathcal{E}(f_n,f_n)&\ge 1-C_2\mu(|f_n|)^2\ge 1-2C_2\mu(|f_n|1_{\{|f_n|\ge\varepsilon\}})^2-2C_2\varepsilon^2\\&\ge 1-2C_2\mu(|f_n|\ge\varepsilon)-2C_2\varepsilon^2,\quad \varepsilon>0.\end{aligned}$$

This implies $\liminf_{n\to\infty}\mathcal{E}(f_n,f_n)\ge 1/C_1$, which is a contradiction. $\square$

Finally, for motivation, we present below applications of our main results to diffusion processes on $\mathbb{R}^d$. See section 3 for the proofs.

Example 1.4. *Let $E=\mathbb{R}^d, L=\Delta+\nabla V,\mu(\mathrm{d}x)=\mathrm{e}^{V(x)}\mathrm{d}x/[\int\mathrm{e}^{V(x)}\mathrm{d}s]$, and $\mathcal{E}(f,f)=\mu(|\nabla f|^2)$ for the choices of V specified below. Let $\Phi(f)=\|f\|_\infty^2$.*
a) *For $p>0$, let $V(x)=-(d+p)\log(1+|x|)$ and $\tau=\min\{(d+p+2)/p,(4p+4+2d)/[p^2-4-2d-2p]^+\}$. Then (1.6) holds with $\alpha(r)=c(1+r^{-\tau})$ for some $c>0$, and there exists $c'>0$ such that*

$$\mu((P_tf)^2)\le c'\|f\|_\infty^2t^{-1/\tau},\quad t\ge 0,\mu(f)=0. \tag{1.11}$$

b) *Let $p>1$ and $V(x)=-d\log(1+|x|)-p\log\log(\mathrm{e}+|x|)$. Then (1.6) holds with $\alpha(r)=c_1\exp[c_2r^{-1/(p-1)}]$ for some $c_1,c_2>0$, and there exists $c>0$ such that*

$$\mu((P_tf)^2)\le c\|f\|_\infty^2[\log(1+t)]^{1-p},\quad t>0,\mu(f)=0. \tag{1.12}$$

c) *Let $V(x)=-\sigma|x|^\delta$ for some $\sigma,\delta>0$. We know from Corollary 1.4 in [27] that the Poincaré inequality holds if and only if $\delta\ge 1$, so we only consider the case $\delta\in(0,1)$. For*

$\delta \in (0,1)$, *there exist* $c, c_1, c_2 > 0$ *such that* (1.6) *holds with* $\alpha(r) = c[1+\log(1+r^{-1})]^{4(1-\delta)/\delta}$, *and*

$$\mu((P_t f)^2) \le c_1 \|f\|_\infty^2 \exp\big[-c_2 t^{\delta/(4-3\delta)}\big], \quad \mu(f) = 0, t \ge 0. \tag{1.13}$$

In particular, when $d = 1$ we obtain sharp choices of α for a) and b) at the end of section 4.

2 L^2-convergence of Markov semigroups

The aim in this section is to establish relationships between (1.6) and the L^2-convergence rate of P_t.

Theorem 2.1. *Assume that* (1.6) *holds. Then*

$$\mu((P_t f)^2) \le \inf_{r>0} \Big\{ r \sup_{s\in[0,t]} \Phi(P_s f) + \exp[-2t/\alpha(r)]\mu(f^2) \Big\}, \quad t > 0, \mu(f) = 0, f \in \mathcal{D}(L). \tag{2.1}$$

Consequently, if $\Phi(P_t f) \le \Phi(f)$ *for any* $t \ge 0$ *and* $f \in L^2(\mu)$, *then*

$$\mu((P_t f)^2) \le \xi(t)[\Phi(f) + \mu(f^2)], \quad t > 0, \mu(f) = 0, f \in \mathcal{D}(L), \tag{2.2}$$

where $\xi(t) := \inf\{r > 0 : -\frac{1}{2}\alpha(r)\log r \le t\}$ *for* $t > 0$. *In particular,* $\xi(t) \downarrow 0$ *as* $t \uparrow \infty$.

Proof. For $f \in \mathcal{D}(L)$ with $\mu(f) = 0$, let $h(t) := \mu((P_t f)^2)$. By (1.6),

$$h'(t) = -2\mathcal{E}(P_t f, P_t f) \le -\frac{2}{\alpha(r)} h(t) + \frac{2r}{\alpha(r)} \Phi(P_t f), \quad t \ge 0.$$

This implies (2.1) immediately. □

To prove a converse of Theorem 2.1, i.e. to establish a functional inequality from the L^2-convergence rate of P_t, we need the following lemma due to [29] whose proof we include here for completeness.

Lemma 2.2. *If* L *is normal* (*i.e.* $LL^* = L^*L$), *then for any* $f \in L^2(\mu)$,

$$\mu((P_s f)^2) \le \mu((P_t f)^2)^{s/t} \mu(f^2)^{1-s/t}, \quad 0 \le s \le t. \tag{2.3}$$

Proof. Let $\sigma(L)$ be the spectrum of L and $\{E_\lambda : \lambda \in \sigma(L)\}$ the spectral family corresponding to L. We have

$$P_t = \int_{\sigma(L)} e^{\lambda t} dE_\lambda, \qquad t \geq 0. \tag{2.4}$$

For any f with $\mu(f^2) = 1, d\|E_\lambda(f)\|_2^2$ is a probability measure on $\sigma(L)$. Then (2.4) implies

$$\begin{aligned}\mu((P_s f)^2) &= \int_{\sigma(L)} \exp[2(\mathrm{Re}\lambda)s] d\|E_\lambda(f)\|_2^2 \\ &\leq \Big(\int_{\sigma(L)} \exp[2(\mathrm{Re}\lambda)t] d\|E_\lambda(f)\|_2^2\Big)^{s/t} = \mu((P_t f)^2)^{s/t}, \quad 0 \leq s \leq t.\end{aligned}$$

This proves (2.3). $\square$

Theorem 2.3. *Assume that L is normal. If there exist $\Psi : L^2(\mu) \to [0,\infty]$ and decreasing $\xi : [0,\infty) \to (0,\infty)$ such that $\Psi(cf) = c^2\Psi(f)$ for $c \in \mathbb{R}$ and $f \in L^2(\mu), \xi(t) \downarrow 0$ as $t \uparrow \infty$, and*

$$\mu((P_t f)^2) \leq \xi(t)\Psi(f), \qquad t > 0,\ \mu(f) = 0, f \in \mathcal{D}(L), \tag{2.5}$$

then (1.6) *holds with $\Phi = \Psi$ and*

$$\alpha(r) = 2r \inf_{s>0} \frac{1}{s}\xi^{-1}(s\exp[1 - s/r]), \quad \textit{where } \xi^{-1}(t) := \inf\{r > 0 : \xi(r) \leq t\}. \tag{2.6}$$

If in particular (2.5) *holds for $\xi(t) = \exp[-\delta t]$ for some $\delta > 0$, then the Poincaré inequality* (1.1) *holds for $C = 2/\delta$ and all $f \in \mathcal{D}(L)$ with $\Psi(f) < \infty$.*

Proof. Since (2.5) implies $\mu(f^2) \leq \xi(0)\Psi(f)$, we only need to prove the case where $r < \xi(0)$, where $\xi(0) := \lim_{t\downarrow 0} \xi(t)$. For any $t > 0$ and $f \in \mathcal{D}(L)$ with $\mu(f) = 0$ and $\mu(f^2) = 1$, let $h(s) := \mu((P_s f)^2), 0 \leq s \leq t$. By Lemma 2.2 and (2.5),

$$h(s) \leq \xi(t)^{s/t}\Psi(f)^{s/t}, \qquad s \in [0,t].$$

This implies

$$\begin{aligned}-2\mathcal{E}(f,f) = h'(0) &\leq \frac{1}{t}\log[\xi(t)\Psi(f)] \\ &\leq \frac{1}{t}\Big[\log\frac{\xi(t)}{u} - 1 + u\Psi(f)\Big], \quad u > 0.\end{aligned}$$

For $u > 0$, taking $t = \xi^{-1}(u \exp[1-u/r])$ which is positive since $u \exp[1-u/r] \le r < \xi(0)$, we obtain

$$\mu(f^2) = 1 \le \frac{2r}{u}\xi^{-1}(u \exp[1 - u/r])\mathcal{E}(f, f) + r\Psi(f), \quad u > 0.$$

This proves the first assertion.

If (2.5) holds for $\xi(t) = \exp[-\delta t]$, then $\alpha(0) := \lim_{r\to 0} \alpha(r) = 2/\delta$ for α determined by (2.6). □

The following is a consequence of Theorems 2.1 and 2.3, which recovers Theorem 1.1 since (1.2) is equivalent to (1.6) with $\alpha(r) = cr^{1-p}$ for some $c > 0$, see Appendix.

Corollary 2.4. 1) *Let $\varepsilon \in (0, 1)$. If* (1.6) *holds with Φ satisfying $\Phi(P_t f) \le \Phi(f)$ and $\alpha(r) = \delta_1 + \delta_2[\log(1 + r^{-1})]^{(1-\varepsilon)/\varepsilon}$ for some $\delta_1, \delta_2 > 0$. Then* (2.5) *holds for $\Psi(f) = \Phi(f) + \mu(f^2)$ and $\xi(t) = \exp[c_1 - c_2 t^\varepsilon]$ for some $c_1, c_2 > 0$. Conversely, if L is normal, then* (2.5) *with the above $\xi(t)$ implies* (1.6) *with $\Phi = \Psi$ and the above α for some $\delta_1, \delta_2 > 0$.*

2) *Let $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$. The assertions in* 1) *hold for $\alpha(r) = \delta r^{1-p}$ for some $\delta > 0$, and $\xi(t) = ct^{1-q}$ for some $c > 0$, where in the first assertion we may take $\Psi = \Phi$.*

3) *Let $p > 0$. The assertions in* 1) *hold for $\alpha(r) = \exp[\delta(1 + r^{-1/p})]$ for some $\delta > 0$, and $\xi(t) = c[\log(1 + t)]^{-p}$ for some $c > 0$.*

Proof. 1) Let $\varepsilon \in (0, 1)$. If (1.6) holds with Φ satisfying $\Phi(P_t f) \le \Phi(f)$ and $\alpha(r) = \delta_1 + \delta_2[\log(1+r^{-1})]^{(1-\varepsilon)/\varepsilon}$ for some $\delta_1, \delta_2 > 0$. Let $\xi(t) > 0$ be such that $\alpha(\xi(t)) \log \xi(t) = -2t$, we have $\xi(t) \le \exp[c_1 - c_2 t^\varepsilon]$ for some $c_1, c_2 > 0$. Then the first assertion follows from (2.2). Next, if L is normal and (2.5) holds with $\xi(t) = \exp[c_1 - c_2 t^\varepsilon]$ for some $c_1, c_2 > 0$, then $\xi^{-1}(t) = \{\frac{1}{c_2}[c_1 - \log t]^+\}^{1/\varepsilon}$. By Theorem 2.3, (1.6) holds with $\Phi = \Psi$ and

$$\alpha(r) = 2rc_2^{-1/\varepsilon} \inf_{s>0} \frac{1}{s}\{[c_1 - \log s - 1 + s/r]^+\}^{1/\varepsilon}.$$

Taking $s = r[\log(1 + r^{-1})]$, we prove the second assertion.

2) Let $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$. If (1.6) holds with $\alpha(r) = \delta r^{1-p}$ for some $\delta > 0$ and $\Phi(P_t f) \le \Phi(f)$, by (2.1) we have

$$\mu((P_t f)^2) \le r\Phi(f) + \exp[-2tr^{p-1}/\delta]\mu(f^2), \quad t > 0, \mu(f) = 0, f \in \mathcal{D}(L), r > 0.$$

Letting $c > 0$ be such that $\exp[-2c^{p-1}/\delta] = 2^{-q}$ and taking $r = ct^{1-q}$ in the above inequality, we obtain

$$\mu((P_t f)^2) \le ct^{1-q}\Phi(f) + 2^{-q}\mu(f^2), \quad t > 0, \mu(f) = 0, f \in \mathcal{D}(L).$$

Applying this inequality repeatly, we obtain

$$
\begin{aligned}
\mu((P_t f)^2) &\le c2^{q-1}t^{1-q}\Phi(f) + 2^{-q}\mu((P_{t/2}f)^2)\\
&\le c2^{q-1}t^{1-q}(1+2^{-1})\Phi(f) + 2^{-2q}\mu((P_{t/4}f)^2) \le \cdots\\
&\le c2^{q-1}t^{1-q}\Phi(f)\sum_{n=0}^{\infty} 2^{-n} = c2^q t^{1-q}\Phi(f), \quad t>0, \mu(f)=0.
\end{aligned}
$$

This proves the first assertion. The second assertion follows from Theorem 2.3 by taking $s=r$ in (2.6).

3) The first assertion follows immediately from (2.2), and the second one follows from (2.6) by taking $s=r$ in the expression of $\alpha(r)$. □

Finally, we present an analogue of Theorem 2.3 for a class of operators L, which are not necessarily normal, but are such that

$$
\mathcal{E}(P_t f, P_t f) \le h(t)\mathcal{E}(f,f), \quad t \ge 0, f \in \mathcal{D}(L) \tag{2.7}
$$

for some positive $h \in C[0,\infty)$. It is well-known that (2.7) holds for $h=1$ provided L is self-adjoint. Moreover, (2.7) holds for $h(t)=\exp[-2Kt]$ if the curvature of L is bounded below by $K \in \mathbb{R}$ (see e.g. [6] for details).

Theorem 2.5. *Assume that* (2.7) *holds. Then* (2.5) *implies* (1.6) *with* $\Phi = \Psi$ *and*

$$
\alpha(r) = 2\int_0^{\xi^{-1}(r)} h(s)\mathrm{d}s, \quad r>0.
$$

Proof. Noting that

$$
\mu(f^2) - \mu((P_t f)^2) = 2\int_0^t \mathcal{E}(P_{t-s}f, P_{t-s}f)\mathrm{d}s, \quad t>0, f \in \mathcal{D}(L),
$$

by (2.5) and (2.7) we obtain

$$
\mu(f^2) \le 2\mathcal{E}(f,f)\int_0^t h(s)\mathrm{d}s + \xi(t)\Psi(f), \quad f \in \mathcal{D}(L), \mu(f)=0, t>0.
$$

This completes the proof. □

3 Criteria of weak Poincaré inequalities

We first present a general criterion for the weak Poincaré inequality which applies in many cases. Then we go to estimate the function α in (1.6) for diffusions on a Riemannian manifold. To prove (1.6), we assume the following local Poincaré inequality

(A) *For any $\varepsilon \in (0,1)$, there exist $A \in \mathcal{F}$ and $c > 0$ such that $\mu(A) \geq 1 - \varepsilon$ and*

$$\mu(f^2 1_A) \leq c\mathcal{E}(f,f) + \mu(f1_A)^2/\mu(A), \quad f \in \mathcal{D}(L). \tag{3.1}$$

Theorem 3.1. *If* **(A)** *holds then*

$$\mu(f^2) \leq \alpha(r)\mathcal{E}(f,f) + r\|f\|_\infty^2, \quad r > 0, f \in \mathcal{D}(L), \mu(f) = 0, \tag{3.2}$$

where

$$\alpha(r) = \inf\{c > 0 : (3.1) \text{ holds for } \varepsilon = \frac{r}{1+r} \text{ with } c \text{ for some } A \in \mathcal{F} \text{ with } \mu(A) \geq \frac{1}{r+1}\}.$$

Proof. For any $\varepsilon \in (0,1)$, let $c > 0$ and $A \in \mathcal{F}$ be such that $\mu(A) \geq 1 - \varepsilon$ and (3.1) holds. For $f \in \mathcal{D}(L)$ with $\mu(f) = 0$, one has $\mu(f1_A)^2 = \mu(f1_{A^c})^2 \leq \varepsilon^2\|f\|_\infty^2$. Then

$$\begin{aligned}\mu(f^2) &\leq \mu(f^2 1_A) + \varepsilon\|f\|_\infty^2 \leq c\mathcal{E}(f,f) + \frac{\mu(f1_A)^2}{\mu(A)} + \varepsilon\|f\|_\infty^2 \\ &\leq c\mathcal{E}(f,f) + \frac{\varepsilon}{1-\varepsilon}\|f\|_\infty^2, \quad f \in \mathcal{D}(L), \mu(f) = 0.\end{aligned}$$

The proof is completed by taking $\varepsilon = r/(1+r)$ for $r > 0$. □

Remark. 1) **(A)** is not a strong assumption (in particular in the finite dimensional case). For example, **(A)** holds for $\mathcal{E}(f,f) := \mu(|\nabla f|^2)$ on a connected Riemannian manifold, where $\mathrm{d}\mu := \exp[V]\mathrm{d}x$ is a probability measure and V is locally bounded.

2) It is known that for a symmetric irreducible Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $P_t f$ converges to $\mu(f)$ in $L^2(\mu)$ as $t \to \infty$, see the Appendix in [3] for a simple proof. Theorems 2.1 , 2.3 and 3.1 imply estimates for the rate of L^2-convergence even for nonsymmetric semigroups. For instance, let $\mathrm{d}\mu = \mathrm{e}^V \mathrm{d}x$ be a probability measure on a connected Riemannian manifold with V locally bounded, then for any Markov semigroup P_t on $L^2(\mu)$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu((P_t f)^2) = -\mu(|\nabla P_t f|^2), \quad t \geq 0, f \in C_0^\infty, \tag{3.3}$$

it follows from Theorem 3.1 and the proof of Theorem 2.1 that $\sup_{\|f\|_\infty < \infty} \|P_t f - \mu(f)\|_2 \to 0$ as $t \to \infty$ provided μ is an invariant measure of P_t. For general diffusions on $\mathbb{R}^d$ with diffusion coefficient matrix a, $|\nabla f|^2$ is replaced by $\langle a\nabla f, \nabla f\rangle$.

In the remainder of this section, we consider Dirichlet forms with the local property for $E = M$, a connected noncompact Riemannian manifold M of dimension d. Let $\mathrm{d}\mu := \mathrm{e}^V \mathrm{d}x$ be a probability measure on M with V a locally bounded function, and

$\mathcal{E}(f,f) := \mu(|\nabla f|^2)$ for $f \in C_b^\infty(M)$. Below we use $C_b^\infty(M)$ to replace $\mathcal{D}(L)$ in (3.2). Let $o \in M$ be fixed, and denote by $\rho(x)$ the Riemannian distance between x and o. Let $B_r = \{\rho \le r\}$ for $r > 0$.

To obtain explicit estimates for α, one needs to estimate the local Poincaré constant c in (3.1). This is related to a well-known topic in geometry, namely, estimating the first Neumann eigenvalue on a regular domain, see e.g. [13, 26] and references therein.

Theorem 3.2. *Assume that B_r is convex for any $r > 0$. Let $K \in C(0,\infty)$ be a nonnegative increasing function such that the Ricci curvature on B_r is bounded below by $-K(r)$. Then* (3.2) *holds with*

$$\alpha(r) = \frac{4R_r^2 \cosh^{d-1}\left[R_r\sqrt{K(R_r)/(d-1)}\right]}{\pi^2\sqrt{1+8R_r^2K(R_r)/\pi^4}}\exp[\delta_{R_r}(V)], \tag{3.4}$$

where $R_r := \inf\{s > 0 : \mu(B_s^c) \le r/(1+r)\}, \delta_R(V) := \sup\{V(x) - V(y) : x, y \in B_R\}$. Where for $d = 1$ we put $K = K/(d-1) = 0$.

If $V \in C^2(M)$, let K_V be an increasing function such that $\mathrm{Ric} - \mathrm{Hess}_V$ *is bounded below by $-K_V(r)$ on B_r. Then* (3.2) *holds with*

$$\alpha(r) = \frac{\exp[\frac{1}{2}K_V(R_r)R_r^2] - 1}{K_V(R_r)}. \tag{3.5}$$

Proof. By Corollary 3 in [13], one has

$$\begin{aligned}\lambda(R) &:= \inf\left\{\mu(|\nabla f|^2 1_{B_R}) : f \in C^1(B_R), \mu(f1_{B_R}) = 0, \mu(f^2 1_{B_R}) = 1\right\}\\ &\ge \frac{\pi^2}{4R^2}\sqrt{1+8R^2K(R)/\pi^4}\cosh^{1-d}\left[R\sqrt{K(R)/(d-1)}\right],\end{aligned} \tag{3.6}$$

where for $d = 1$ we have $K = 0$ and set $K/(d-1) = 0$. Then, by a simple comparison argument, we see (3.1) holds for $A = B_R$ and

$$c = \frac{4R^2 \cosh^{d-1}\left[R\sqrt{K(R)/(d-1)}\right]}{\pi^2\sqrt{1+8R^2K(R)/\pi^4}}\exp[\delta_R(V)].$$

Therefore, the first assertion follows from Theorem 3.1.

For the case where $V \in C^2(M)$, by Corollary 1 in [13], (3.1) holds for $A = B_R$ and

$$c = \frac{\exp[\frac{1}{2}K_V(R)R^2] - 1}{K_V(R)}$$

which proves the second assertion. $\square$

It is clear that the assumption in Theorem 3.2 that B_r is convex was made to use known estimates for the first Neumann eigenvalue. This assumption is however not true in general. To treat the general case, we present a result below based on an idea from [27].

Theorem 3.3. *Assume that there exist $r_0 > 0$ and $\gamma \in C[r_0, \infty)$ such that*

$$L_0\rho := (\Delta + \nabla V)\rho \le \gamma(\rho) \quad \text{on } B_{r_0}^c \tag{3.7}$$

in the distribution sense. For any $\varepsilon > 0$ and $R \ge r_0$, let $\eta_\varepsilon(R) = \int_{r_0}^R (\varepsilon + \gamma(r))^+ \mathrm{d}r$. If there exists $\varepsilon > 0$ such that $\mu(B_R^c)\exp[\eta_\varepsilon(R+1) - \eta_\varepsilon(R)] \to 0$ as $R \to \infty$, then there exists $c(\varepsilon) > 0$ such that (3.2) *holds with $\alpha(r) = c(\varepsilon)\exp[\eta_\varepsilon(\tilde{R}_r + 1)]$, where*

$$\tilde{R}_r := \inf\big\{R \ge r_0 : \mu(B_R)^{-1} + c(\varepsilon)\exp[\eta_\varepsilon(R+1) - \eta_\varepsilon(R)] \le r/\mu(B_R^c)\big\}, \quad r > 0.$$

Especially, if $\eta_\varepsilon(\infty) < \infty$ for some $\varepsilon > 0$, then the Poincaré inequality (1.1) *holds for $C = c(\varepsilon)\exp[\eta_\varepsilon(\infty)]$.*

Proof. Let $L = L_0 - 1_{\{\rho \ge r_0\}}(\varepsilon + \gamma(\rho))^+\nabla\rho$, where $\nabla\rho(x) := 0$ if x is in the cut locus of o. Then $L\rho \le -\varepsilon$ on $B_{r_0}^c$ in the distribution sense. Therefore (see Corollary 1.4 in [27] and its remark), there exists $c_1(\varepsilon) > 0$ such that

$$\nu(f^2) - \nu(f)^2 \le c_1(\varepsilon)\nu(|\nabla f|^2) \tag{3.8}$$

for all Lipschitz continuous $f \in L^2(\mu)$, where $\mathrm{d}\nu = C\exp[-1_{\{\rho \ge r_0\}}\eta_\varepsilon(\rho)]\mathrm{d}\mu$ with $C > 0$ a normalizing constant. For $f \in C_b^\infty(M)$ with $\mu(f) = 0$ and $R > r_0$, let $h = (R+1-\rho)^+ \wedge 1$. We have

$$\begin{aligned}
\mu(f^2h^2) - \mu(fh)^2/\mu(B_{R+1}) &= \inf_{r\in\mathbb{R}} \mu((fh - r1_{B_{R+1}})^2) \\
&\le C^{-1}\exp[\eta_\varepsilon(R+1)] \inf_{r\in\mathbb{R}} \nu((fh - r1_{B_{R+1}})^2) \\
&= C^{-1}\exp[\eta_\varepsilon(R+1)]\big(\nu(f^2h^2) - \nu(fh)^2/\nu(B_{R+1})\big) \\
&\le C^{-1}\exp[\eta_\varepsilon(R+1)]\big(\nu(f^2h^2) - \nu(fh)^2\big).
\end{aligned}$$

Combining this with (3.8), we obtain (recalling that $\mu(f) = 0$)

$$\begin{aligned}
\mu(f^2h^2) &\le \mu(fh)^2/\mu(B_R) + C^{-1}c_1(\varepsilon)\exp[\eta_\varepsilon(R+1)]\nu(|\nabla(fh)|^2) \\
&\le \|f\|_\infty^2\mu(B_R^c)^2/\mu(B_R) + 2c_1(\varepsilon)\exp[\eta_\varepsilon(R+1)]\mu(|\nabla f|^2) \\
&\quad + 2c_1(\varepsilon)\exp[\eta_\varepsilon(R+1) - \eta_\varepsilon(R)]\mu(B_R^c)\|f\|_\infty^2
\end{aligned}$$

for some $c_1(\varepsilon) > 0$. This proves the theorem for $c(\varepsilon) = 2c_1(\varepsilon)$ since $\mu(f^2) \le \mu(f^2h^2) + \|f\|_\infty^2\mu(B_R^c)$. □

We note that the second assertion in Theorem 2.3 for the Poincaré inequality was already proved in [27]. In the case where γ is negative, we have the following result which provides better choices of α than Theorem 3.2 when $\mu(B_R^c)$ decays fast enough as $R \uparrow \infty$, see e.g. the proof of Proposition 1.4 c).

Theorem 3.4. *Assume that γ in (3.7) is negative and B_r is convex for any $r > 0$. For any $R \geq r_0$, let*

$$\begin{aligned}
&\psi(R) := \inf_{r \in [r_0, R]} [-\gamma(r)], \quad \phi(R) := \inf\{s \geq r_0 : \psi(R)^2 s^2 \geq 9\mu(B_s^c)\},\\
&\zeta(R) := \psi(R)^2 \Big[1 + \phi(R)\sqrt{K(\phi(R))}\Big] \cosh^{1-d} \Big[\phi(R)\sqrt{K(\phi(R))/(d-1)}\Big] \mathrm{e}^{-\delta_{\phi(R)}(V)}.
\end{aligned}$$

Then there exists $c > 0$ such that (3.2) holds for $\alpha(r) = c/\zeta(\bar{R}_r)$ provided

$$\bar{R}_r := \inf\big\{R \geq r_0 : \mu(B_{R-1}^c)[1 + c/\zeta(R)] \leq r\mu(B_{R-1})/(r+1)\big\} < \infty, \quad r > 0.$$

Proof. By Theorem 3.3, the Poincaré inequality (1.1) holds provided $\psi(\infty) > 0$. Hence we only consider the case where $\psi(\infty) = 0$. In this case one has $\phi(R) \uparrow \infty$ as $R \uparrow \infty$. Then there exist $\varepsilon > 0$ and $R(\varepsilon) \geq r_0 + 1$ such that

$$\psi(R)^2 \mu(B_{\phi(R)})(\phi(R) - r_0)^2 \geq 8(1+\varepsilon)\mu(B_{\phi(R)}^c), \qquad R \geq R(\varepsilon). \tag{3.9}$$

Next, since $L_0\rho \leq -\psi(R)$ in $B_R \setminus B_{r_0}$ in the distribution sense, by Cheeger's inequality we obtain (see e.g. page 398 in [27])

$$\lambda(B_R \setminus B_{r_0}) := \inf\big\{\mu(|\nabla f|^2 : f \in C_0^\infty(B_R \setminus B_{r_0}), \mu(f^2) = 1\big\} \geq \frac{\psi(R)^2}{4}. \tag{3.10}$$

Next, the proof of Theorem 1.1 in [27] yields that

$$\begin{aligned}
\bar{\lambda}(R) &:= \inf\big\{\mu(|\nabla f|^2) : f \in C_0^\infty(B_R), \mu(f^2) - \mu(f)^2 = 1\big\}\\
&\geq \frac{\lambda(B_R \setminus B_{r_0})\lambda(r)\mu(B_r)(r-r_0)^2 - 2\lambda(r)\mu(B_r^c)}{2\lambda(r)(r-r_0)^2 + \lambda(B_R \setminus B_{r_0})(r-r_0)^2\mu(B_r) + 2\mu(B_r)}, \quad r \in (r_0, R).
\end{aligned}$$

Combining this with (3.6), (3.9) and (3.10), we obtain

$$\bar{\lambda}(R) \geq \frac{c_1\lambda(\phi(R))\mu(B_{\phi(R)})\psi(R)^2(\phi(R)-r_0)^2}{\lambda(\phi(R))(\phi(R)-r_0)^2 + \psi(R)^2(\phi(R)-r_0)^2\mu(B_{\phi(R)}) + \mu(B_{\phi(R)})} \geq c_2\zeta(R),$$

for some $c_1, c_2 > 0$ and all $R \geq R(\varepsilon)$ such that $\phi(R) < R$. If $\phi(R) \geq R$, then $\bar{\lambda}(R)(\geq \lambda(R)) \geq c_2\zeta(R)$ still holds for some $c_2 > 0$ according to (3.6). Then for any $f \in C_b^\infty(M)$ with $\mu(f) = 0$ and any $R \geq R_\varepsilon$, let $h = (R-\rho)^+ \wedge 1$, we have

$$\mu(f^2) \leq \mu(f^2h^2) + \|f\|_\infty^2 \mu(B_{R-1}^c) \leq \frac{2\mu(|\nabla f|^2) + 2\|f\|_\infty^2 \mu(B_{R-1}^c)}{c_2\zeta(R)} + \frac{\|f\|_\infty^2 \mu(B_{R-1}^c)}{\mu(B_{R-1})}.$$

This proves the desired result by taking $c = 2/c_2$ for small $r > 0$ such that $R_r \geq R(\varepsilon)$. $\square$

Now we are ready to prove the results claimed in Example 1.4.

Proof of Example 1.4. By Corollary 2.4, it suffices to prove (1.6) with α as specified there. We note that $K = 0$ since $M = \mathbb{R}^d$. Let R_r and $\bar{R}_r$ be defined in Theorems 3.2 and 3.4 respectively.

a) In this case we have $\delta_R(V) = (d+p)\log(1+R)$ and $R_r \leq c(1 + r^{-1/p})$ for some $c > 0$ and any $r > 0$. By Theorem 3.2, (3.2) holds with

$$\alpha(r) = \frac{4R_r^2}{\pi^2}\exp[\delta_{R_r}(V)] \leq c_1 r^{-(d+p+2)/p} \tag{3.11}$$

for some $c_1 > 0$ and all $r \in (0,1]$ (hence for all $r > 0$ since $\mu(f^2) \leq \|f\|_\infty^2$). Next, assume that $p^2 - 4 - 2d - 2p > 0$. Let $r_0 \geq 1$ be such that $(d-1)/r_0 - (d+p)/(1+r_0) < 0$. It is easy to see that $\psi(R) \geq c_2 R^{-1}, \phi(R) \leq c_3 R^{2/(2+p)}, \zeta(R) \geq c_4 R^{-2(d+2+2p)/(2+p)}$ for some $c_2, c_3, c_4 > 0$ and all $R \geq r_0$. Moreover, we have $\bar{R}_r \leq c_5 r^{-(2+p)/(p^2-2d-4-2p)}$ for some $c_5 > 0$ and all $r \in (0,1]$. Therefore, by Theorem 3.4, (3.2) holds with

$$\alpha(r) = c/\zeta\big(\bar{R}_r\big) \leq c_6 r^{-2(d+2+2p)/(p^2-2d-4-2p)}$$

for some $c, c_6 > 0$. Combining this with (3.11) we prove (3.2) for $\alpha(r) = c_7 r^{-\tau}$ for some $c_7 > 0$.

b) Obviously, $\delta_R(V) = d\log(1+R) + p\log\log(\mathrm{e}+R)$, and $R_r \leq \exp[c(1 + r^{-1/(p-1)})]$ for some $c > 0$. By Theorem 3.2, (3.2) holds with

$$\alpha(r) = \frac{4R_r^2}{\pi^2}\exp[\delta_{R_r}(V)] \leq c_1 \exp[c_2 r^{-1/(p-1)}]$$

for some $c_1, c_2 > 0$.

c) In this case Theorem 3.4 provides better result than Theorem 3.2. Let $\delta \in (0,1)$. We have $\gamma(r) = \frac{d-1}{r} - \frac{\sigma\delta}{r^{1-\delta}}$ which is negative for big r. Taking $r_0 > 1$ such that $\gamma(r_0) < 0$. We see that $\psi(R) \geq c_1 R^{\delta-1}$ for some $c_1 > 0$ and all $R \geq r_0$. It is easy to see that there exists $c_2 > 0$ such that $\mu(B_s^c) \leq c_2\exp[-\sigma s^\delta]s^{1-\delta}$ for $s \geq r_0$. Let $s_R > 0$ solve $9c_2\exp[-\sigma s^\delta]s^{-(1+\delta)} = c_1^2 R^{2(\delta-1)}$, then $\phi(R) \leq s_R \vee r_0$. Hence

$$\exp[\delta_{\phi(R)}(V)] = \exp[\sigma\phi(R)^\delta] \le \exp[\sigma(s_R \vee r_0)^\delta] \le c_3 R^{2(1-\delta)}$$

for some $c_3 > 0$. Since $K = 0$,

$$\zeta(R) = \psi(R)^2 \exp[-\delta_{\phi(R)}(V)] \ge c_4 R^{4(\delta-1)}$$

for some $c_4 > 0$. Then $\mu(B_{R-1}^c)\zeta(R)^{-1} \le c_5 \exp[-\sigma R^\delta/2]$ for some $c_5 > 0$ and all $R > r_0$. We obtain $\bar{R}_r \le c_6(1+[\log(1+r^{-1})]^{1/\delta})$ for some $c_6 > 0$. Therefore, by Theorem 3.4, (3.2) holds for

$$\alpha(r) = c\psi(\bar{R}_r)^{-2} \exp[\delta_{\phi(\bar{R}_r)}(V)] \le c_7[\log(1+r^{-1})]^{4(1-\delta)/\delta}$$

for some $c_7 > 0$ and all $r \in (0, 1]$ (hence all $r > 0$).

On the other hand, by (3.4) one obtains (3.2) for $\alpha(r) = c(1 + r^{-\varepsilon})$ for some $c, \varepsilon > 0$. This choice of α is worse than the one above. $\square$

Before concluding this section, we consider general diffusions on $\mathbb{R}^d$. Consider $E = \mathbb{R}^d$ and let $a = (a_{ij})$ be uniformly positive definite on any compact domain. Assume that $\mathrm{d}\mu = \mathrm{e}^V \mathrm{d}x$ is a probability measure on $\mathbb{R}^d$ with V locally bounded. For any $R > 0$, let $B_R := \{x : |x| \le R\}$ and

$$\underline{a}(R) := \inf\{\langle a(x)y, y\rangle : |y| = 1, x \in B_R\}, \qquad \delta_R(V) := \sup\{V(x) - V(y) : x, y \in B_R\}.$$

Letting $\mu_R(\cdot) := \mu(\cdot \cap B_R)/\mu(B_R)$, we have

$$\begin{aligned}\mu_R(f^2) &\le \mu_R(f)^2 + \frac{4R^2}{\pi^2}\exp[\delta_R(V)]\mu_R(|\nabla f|^2)\\ &\le \mu_R(f)^2 + \frac{4R^2}{\pi^2\underline{a}(R)}\exp[\delta_R(V)]\mu_R(\langle a\nabla f, \nabla f\rangle),\end{aligned}$$

Then for any $f \in C_b^\infty(\mathbb{R}^d)$ with $\mu(f) = 0$,

$$\mu(f^2) \le \frac{\|f\|_\infty^2 \mu(B_R^c)}{\mu(B_R)} + \frac{4R^2}{\pi^2\underline{a}(R)}\exp[\delta_R(V)]\mu\langle a\nabla f, \nabla f\rangle.$$

Letting $R_r := \inf\{R \ge 0 : \mu(B_R^c) \le r/(1+r)\}$, we obtain

$$\mu(f^2) \le \alpha(r)\mu(\langle a\nabla f, \nabla f\rangle) + r\|f\|_\infty^2, \quad \mu(f) = 0, f \in C_b^\infty(\mathbb{R}^d), r > 0 \tag{3.12}$$

for $\alpha(r) = \frac{4R_r^2}{\pi^2\underline{a}(R_r)}\exp[\delta_{R_r}(V)]$.

4 Isoperimetric inequalities: the diffusion case

The aim of this section is to prove (1.6) using isoperimetric inequalities for diffusions on a manifold. The study goes back to Cheeger's inequality for estimating the principal eigenvalue and the spectral gap of the Laplacian, see e.g. [8, 31]. Isoperimetric inequalities have also been developed by Ledoux in [18] for the log-Sobolev inequality, by Röckner and Wang in [23] for the F-Sobolev inequality (i.e., using an increasing function F to replace log in the log-Sobolev inequality), and by Wang in [28] for the super-Poincaré inequality (1.5).

Let M be a connected noncompact Riemannian manifold, and μ a probability measure on M. Define

$$k(r) := \inf_{\mu(A)\in[r,1/2]} \frac{\mu_\partial(\partial A)}{\mu(A)}, \quad r \in (0, 1/2], \tag{4.1}$$

where A runs over all open smooth domains (according to Yau [31], we may also assume that A is connected), and $\mu_\partial(\partial A)$ denotes the area of ∂A induced by μ.

Theorem 4.1. *If $k(r) > 0$ for any $r \in (0, 1/2]$, then*

$$\mu(f^2) \le \alpha(r)\mu(|\nabla f|^2) + r\delta_\mu(f)^2, \quad r > 0, f \in C_b^\infty(M), \mu(f) = 0, \tag{4.2}$$

where $\alpha(r) = 4k(r/2)^{-2}$ and $\delta_\mu(f) = \inf_{\mu(A)=1} \sup\{f(x) - f(y) : x, y \in A\}$. In particular, if $k(0) := \lim_{r\to 0} k(r) > 0$, then (1.1) holds for $C = 4/k(0)^2$.

Proof. Assume that $k(r) > 0$ for $r \in (0, 1/2]$. Let $f \in C_b^\infty(M)$ be such that $\mu(f) = 0$. Take $r_0 \in [\inf f, \sup f]$ such that $\mu(f > r_0) \vee \mu(f < r_0) \le 1/2$. For $s > 0$, let $t_s := \inf\big\{t \ge 0 : \mu\big((f - r_0)^{+2}\big) \ge s\big\}$. By (4.1) and the coarea formula, we obtain

$$\begin{aligned}
\mu((f - r_0)^{+2}) &= \int_0^{\|(f-r_0)^+\|_\infty^2} \mu\big((f - r_0)^{+2} > t\big)\mathrm{d}t \\
&\le \int_0^{t_s} \frac{\mu_\partial((f - r_0)^{+2} = t)}{k(s)}\mathrm{d}t + s\|(f - r_0)^+\|_\infty^2 \\
&\le \frac{1}{k(s)}\mu(|\nabla(f - r_0)^{+2}|) + s\|(f - r_0)^+\|_\infty^2, \quad s > 0.
\end{aligned}$$

The same estimate holds for $(f - r_0)^-$ in place of $(f - r_0)^+$. Then

$$\begin{aligned}
&s\delta_\mu(f)^2 + \frac{1}{k(s)}\mu\big(\big|\nabla(f - r_0)^{+2}\big| + \big|\nabla(f - r_0)^{-2}\big|\big) \\
&\ge \mu\big((f - r_0)^{+2} + (f - r_0)^{-2}\big) = \mu\big((f - r_0)^2\big).
\end{aligned} \tag{4.3}$$

Noting that

$$\mu\big(\big|\nabla(f-r_0)^{+2}\big|+\big|\nabla(f-r_0)^{-2}\big|\big)=\mu(|\nabla(f-r_0)^2|)\leq 2\sqrt{\mu(|\nabla f|^2)\mu((f-r_0)^2)},$$

by (4.3) we obtain

$$\mu(f^2)\leq\mu((f-r_0)^2)\leq\frac{4}{k(s)^2}\mu(|\nabla f|^2)+2s\delta_\mu(f)^2,\quad s>0.$$

This implies (4.2) by taking $s=r/2$. $\square$

It is well-known that (1.1) holds for $C=4/k(0)^2$ provided $k(0)>0$ (see e.g. [8, 18, 31]). Theorem 4.1 extends this result to weak Poincaré inequalities.

Theorem 4.2. *Assume that $V\in C^2(M)$ such that $\mathrm{d}\mu:=\mathrm{e}^V\mathrm{d}x$ is a probability measure and that $|\nabla P_tf|^2\leq h(t)P_t|\nabla f|^2$ holds for some positive $h\in C[0,\infty)$ and all $t>0, f\in C_b^\infty(M)$, where P_t is generated by $\Delta+\nabla V$ on $L^2(\mu)$. If (4.2) holds, then for any $\varepsilon\in(0,1/2)$ there exists $c(\varepsilon)>0$ such that $k(r)\geq c(\varepsilon)/\alpha(\varepsilon r)$.*

Proof. (The idea for this proof originates from [18]) We first note that the assumed gradient estimate implies $P_t1=1$. Moreover, this assumption implies (see e.g. Lemma 4.2 in [6])

$$P_tf^2-(P_tf)^2\geq 2\int_0^t\frac{\mathrm{d}s}{h(s)}|\nabla P_tf|^2:=c(t)|\nabla P_tf|^2,\quad f\in C_b^\infty(M).$$

Then

$$\|\nabla P_tf\|_\infty\leq\|f\|_\infty\sqrt{1/c(t)},\quad t>0.$$

Hence for any smooth g with $\|g\|_\infty\leq 1$,

$$\begin{aligned}\mu(g(f-P_tf))&=-\int_0^t\mu(g(\Delta+\nabla V)P_sf)\mathrm{d}s=\int_0^t\mu(\langle\nabla P_sg,\nabla f\rangle)\mathrm{d}s\\&\leq\mu(|\nabla f|)\int_0^t\|\nabla P_sg\|_\infty\mathrm{d}s\leq ct\mu(|\nabla f|)\end{aligned}$$

for some $c>0$ and all $t\geq 1$. Therefore,

$$\mu(|f-P_tf|)\leq ct\mu(|\nabla f|),\quad t\geq 1. \tag{4.4}$$

For any $r\in(0,1/2]$ and any smooth domain A with $\mu(A)\in[r,1/2]$. Take $\{f_n\}\subset C_0^\infty(M)$ such that $f_n|_A=1, f_n(x)=0$ if $\mathrm{dist}(x,A)\geq 1/n$, and $|\nabla f_n|\leq n+1/n$. Applying (4.4) to f_n and letting $n\uparrow\infty$, we arrive at

$$\begin{aligned} ct\mu_\partial(\partial A) &\geq \mu(1_A(1-P_t1_A)) + \mu(1_{A^c}P_t1_A) = 2[\mu(A) - \mu(1_AP_t1_A)] \\ &= 2\mu(A) - 2\mu((P_{t/2}1_A)^2). \end{aligned} \tag{4.5}$$

If (4.2) holds, by Theorem 2.1 we have

$$\mu((P_{t/2}1_A)^2) \leq s + \exp[-t/\alpha(s)]\mu(A) + \mu(A)^2, \quad s > 0.$$

Therefore, (4.5) implies

$$\begin{aligned} \frac{\mu_\partial(\partial A)}{\mu(A)} &\geq \sup_{s,t>0} \frac{2}{ct}\big\{1 - s/\mu(A) - \exp[-t/\alpha(s)] - \mu(A)\big\} \\ &\geq \sup_{s,t>0} \frac{1 - 2s/r - 2\exp[-t/\alpha(s)]}{ct}. \end{aligned}$$

For any $\varepsilon \in (0, 1/2)$, taking $s = \varepsilon r$ and $t = \alpha(\varepsilon r) \log \frac{4}{1-2\varepsilon}$, we obtain $\frac{\mu_\partial(\partial A)}{\mu(A)} \geq \frac{c(\varepsilon)}{\alpha(\varepsilon r)}$ for some $c(\varepsilon) > 0$. $\square$

Corollary 4.3. *consider the situation of Theorem* 4.2. *If* (4.2) *holds then for any* $\varepsilon \in (0, 1/2)$ *there exists* $c(\varepsilon) > 0$ *such that*

$$\int_{\mu(\rho \geq R)}^1 \frac{\alpha(\varepsilon r)}{r} \mathrm{d}r \geq c(\varepsilon) R. \tag{4.6}$$

Proof. Let $h(s) = \mu(\rho \geq s)$. By Theorem 4.3, (4.2) implies that $-\frac{h'(s)\alpha(\varepsilon s)}{h(s)} \geq c(\varepsilon)$. This proves (4.6). $\square$

Obviously, for a given function α, (4.6) provides an estimate of the decay of $\mu(\rho \geq R)$ as $R \uparrow \infty$. In particular, if (1.1) holds then by (4.6) there exists $c > 0$ such that $\mu(\exp[c\rho]) < \infty$. This is a well-known result according to Herbst's argument, see e.g. [2].

Next, let us consider Dirichlet forms on $\mathbb{R}^d$ with the local property. Let μ be a probability measure on $\mathbb{R}^d$ and $\mathcal{E}(f,f) := \mu(\langle a\nabla f, \nabla f\rangle), f \in C_b^\infty(M)$, where $a(x) = (a_{ij}(x))_{d\times d}$ is positive definite for any $x \in \mathbb{R}^d$. Let ϕ_1, ϕ_2 be two positive continuous function such that

$$\phi_1(x)|y|^2 \leq \langle a(x)y, y\rangle \leq \phi_2(x)|y|^2, \qquad x, y \in \mathbb{R}^d.$$

Finally, let $\mathrm{d}\bar{\mu}_\partial = \phi_2 \mathrm{d}\mu_\partial$ and $\mathrm{d}\underline{\mu}_\partial = \phi_1 \mathrm{d}\mu_\partial$ be defined on the boundary of any smooth domain, where $\mathrm{d}\mu_\partial$ is induced by μ and the standard Euclidean metric. The proofs of Theorems 4.1 and 4.2 imply the following result.

Theorem 4.4. *Let $\bar{k}(r)$(resp. $\underline{k}(r)$) be defined in* (4.1) *with μ_∂ replaced by $\bar{\mu}_\partial$(resp. $\underline{\mu}_\partial$).*

1) *If $\underline{k}(r) > 0$ for $r \in (0, 1/2]$, then*

$$\mu(f^2) \leq \alpha(r)\mu(\langle a\nabla f, \nabla f\rangle) + r\delta_\mu(f)^2, \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f) = 0, r > 0 \tag{4.7}$$

for $\alpha(r) = 4\underline{k}(r/2)^{-2}$.

2) *Assume that $a_{ij} \in C^2(\mathbb{R}^d), \mathrm{d}\mu = \mathrm{e}^V \mathrm{d}x$ for some $V \in C^2(\mathbb{R}^d)$, and that there exist $K \geq 1$ and a matrix-valued function σ such that $K^{-1}I \leq a = \sigma\sigma^* \leq KI$ and*

$$\sup_{x \neq y} |x-y|^{-2}[\|\sigma(x) - \sigma(y)\|^2 + \langle b(x) - b(y), x-y\rangle] \leq K, \tag{4.8}$$

where $b_i = \sum_{j=1}^d \left[a_{ij}\frac{\partial}{\partial x_j}V + \frac{\partial}{\partial x_j}a_{ij}\right]$. If (4.7) *holds then for any $\varepsilon \in (0, 1/2)$ there exists $c(\varepsilon) > 0$ such that $\bar{k}(r) \geq c(\varepsilon)/\alpha(\varepsilon r)$.*

Proof. Let P_t be the Markov semigroup generated by the closure of $\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$. One has $\langle a\nabla P_t f, \nabla P_t f\rangle \leq K^2 \mathrm{e}^{2Kt} P_t \langle a\nabla f, \nabla f\rangle$ for $f \in C_b^\infty(\mathbb{R}^d)$, see (9.1) in [14]. Hence the proof of Theorem 4.2 applies to the manifold $\mathbb{R}^d$ with the metric induced by a^{-1}. $\square$

The following is a simple consequence of Theorem 4.4 in the one dimensional case.

Corollary 4.5. *Consider the situation of Theorem 4.4. Let $d = 1$ and $\mathrm{d}\mu = \mathrm{e}^V \mathrm{d}x$ for some $V \in C(\mathbb{R})$. For any $r \in (0, 1)$, let $c_r > 0$ be such that $\mu([-c_r, c_r]) = 1 - r$. Then*

$$\bar{k}(r) = \underline{k}(r) \geq \inf_{s \in [r, 1/2]} \frac{1}{s} \inf_{t \in [-c_s, c_s]} \sqrt{a(x)} \exp[V(x)] := \kappa(r). \tag{4.9}$$

Consequently, (4.7) *holds for $\alpha(r) = 4\kappa(r/2)^{-2}$ provided it is finite. On the other hand, if $a, V \in C^2(\mathbb{R})$ such that $aV'' + \frac{1}{2}a'V' + \frac{1}{2}a'' - \frac{a'^2}{4a}$ is bounded from above, then* (4.7) *implies $\bar{k}(r) = \underline{k}(r) \geq c(\varepsilon)/\alpha(\varepsilon r)$ for any $\varepsilon \in (0, 1/2)$ and some $c(\varepsilon) > 0$.*

Proof. In the present case we have $\bar{\mu}_\partial(x) = \underline{\mu}_\partial(x) = (\sqrt{a}\mathrm{e}^V)(x)$. Then for any $r \in (0, 1/2]$ and connected $I \subset \mathbb{R}$ with $\mu(I) = r$, we have $\partial I \cap [-c_r, c_r] \neq \emptyset$. This proves (4.9). To prove the second assertion, we consider the metric induced by a^{-1}: $|\frac{\partial}{\partial x}|_a^2 = a^{-1}$. Therefore, under this metric $\mathrm{Ric} = 0$ (since $d = 1$) and the unit vector field is $X = \sqrt{a}\frac{\partial}{\partial x}$. Next, let $\mathrm{d}\bar{x} = a^{-1/2}\mathrm{d}x$ which is the Riemannian volume element. Then we see that $\mathrm{d}\mu = \exp[V + \frac{1}{2}\log a]\mathrm{d}\bar{x}$. Therefore,

$$\mathrm{Hess}_V(X, X) := X^2 V = aV'' + \frac{1}{2}a'V' + \frac{1}{2}a'' - \frac{a'^2}{4a}.$$

The proof is completed by Theorem 4.4. $\square$

Finally, we apply Corollary 4.5 to the first two cases in Example 1.4 to obtain better choices of α for $d = 1$.

a) For $p > 0$ and $V(x) = -(1+p)\log(1+|x|)$, we have $c_r \le cr^{-1/p}$ for some $c > 0$. Then $\kappa(r) \ge cr^{1/p}$ and hence (4.7) holds for $\alpha(r) = c_1 r^{-2/p}$ and some $c_1 > 0$. Moreover, it is easy to see that $\bar{k}(r) \le c' r^{1/p}$ for some $c' > 0$ and V'' is bounded above. Hence, by Corollary 4.5, (4.7) does not hold for any α with $\alpha(r) r^{1/p} \to 0$ as $r \to 0$.

b) Let $p > 1$ and $V(x) = -\log(1+|x|) - p\log\log(\mathrm{e}+|x|)$. Similarly to a), we have $\kappa(r) \ge cr^{1/(p-1)}\exp[-r^{-1/(p-1)}]$ for some $c > 0$ and $r \in (0, 1/2]$. By Corollary 4.5, (4.7) holds for $\alpha(r) = c_1 r^{-2/(p-1)}\exp[2^{p/(p-1)} r^{-1/(p-1)}]$ for some $c_1 > 0$. Moreover, there exists $c' > 0$ such that $\bar{k}(r) \le c' r^{1/(p-1)}\exp[-r^{-1/(p-1)}]$. Hence, by Corollary 4.5, (4.7) does not hold for any α with $a(r)\exp[-sr^{-1/(p-1)}] \to 0$ as $r \to 0$ for some $s \in (0, 2^{1/(p-1)})$.

5 Isoperimetric inequalities: the jump case

In this section we study the weak Poincaré inequality for general symmetric Dirichlet forms following the line of [15, 30] in which the Poincaré and Sobolev type inequalities are considered (see also [11, 12] for estimates of the constants in the log-Sobolev and Nash inequalities).

Let J be a symmetric measure on $(E \times E, \mathcal{F} \times \mathcal{F})$. Define

$$\mathcal{E}(f,f) := \frac{1}{2}\int_{E\times E} [f(x) - f(y)]^2 J(\mathrm{d}x, \mathrm{d}y), \quad \mathcal{D}(\mathcal{E}) = \{f \in L^2(\mu) : \mathcal{E}(f,f) < \infty\}.$$

We consider the inequality

$$\mu(f^2) \le \alpha(r)\mathcal{E}(f,f) + r\delta_\mu(f), \quad r > 0, \mu(f) = 0. \tag{5.1}$$

If (5.1) holds, then for $r \in (0, 1/2]$ and A with $\mu(A) = r \in (0, 1/2]$, taking $f = 1_A$ in (5.1) we obtain

$$k(r) := \inf_{\mu(A)\in[r,1/2]} \frac{J(A \times A^c)}{\mu(A)} \ge \sup_{s>0} \frac{1 - s/r}{\alpha(s)} \ge \frac{1-\varepsilon}{\alpha(\varepsilon r)}, \quad \varepsilon \in (0,1). \tag{5.2}$$

Therefore, the main task is to prove (5.1) using isoperimetric inequalities. To do this, we take a nonnegative symmetric measurable function γ on $E \times E$ such that

$$\int_{A\times E} \frac{1_{\{\gamma(x,y)>0\}} J(\mathrm{d}x, \mathrm{d}y)}{\gamma(x,y)} \le \mu(A), \quad A \in \mathcal{F}.$$

Define $\tilde{J}(\mathrm{d}x, \mathrm{d}y) = 1_{\{\gamma(x,y)>0\}} J(\mathrm{d}x, \mathrm{d}y)/\sqrt{\gamma(x,y)}$, and let $\tilde{k}$ be defined in (5.2) with J replaced by $\tilde{J}$. We have the following result.

Theorem 5.1. *If $\tilde{k}(r) > 0$ for $r \in (0, 1/2]$, then (5.1) holds for $\alpha(r) = 2\tilde{k}(r/2)^{-2}$. Consequently, if $\tilde{k}(0) > 0$ then (1.1) holds for $C = 2\tilde{k}(0)^{-2}$.*

Proof. Let $r \in (0, 1/2]$. For bounded f with $\mu(f) = 0$ and $\mathcal{E}(f, f) < \infty$, let r_0 be such that $\mu(f > r_0) \vee \mu(f < r_0) \leq 1/2$. For any $t \geq 0$, let $A_t := \{(f - r_0)^{+2} > t\}$ and $p(t) := \mu(A_t)$. Then we have $p(t) \leq 1/2$. Let $t_r := \inf\{t \geq 0 : p(t) \leq r\}$, then

$$\begin{aligned}
&\sqrt{2\mathcal{E}((f - r_0)^+, (f - r_0)^+)\mu((f - r_0)^{+2})} \\
&\geq \frac{1}{2}\int \left|(f(x) - r_0)^{+2} - (f(y) - r_0)^{+2}\right| \tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
&= \int_{\{(f(x)-r_0)^+ > (f(y)-r_0)^+\}} \left[(f(x) - r_0)^{+2} - (f(y) - r_0)^{+2}\right] \tilde{J}(\mathrm{d}x, \mathrm{d}y) \\
&\geq \int_0^{t_r} \tilde{J}(A_t \times A_t^c)\mathrm{d}t \geq \tilde{k}(r)\int_0^{t_r} p(t)\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu((f - r_0)^{+2}) &= \int_0^{\|(f-r_0)^+\|_\infty^2} p(t)\mathrm{d}t \\
&\leq \frac{1}{\tilde{k}(r)}\sqrt{2\mathcal{E}((f - r_0)^+, (f - r_0)^+)\mu((f - r_0)^{+2})} + r\|(f - r_0)^+\|_\infty^2
\end{aligned}$$

for all $r > 0$. This implies

$$\mu((f - r_0)^{+2}) \leq \frac{2}{\tilde{k}(r)^2}\mathcal{E}((f - r_0)^+, (f - r_0)^+) + 2r\|(f - r_0)^+\|_\infty^2, \quad r > 0. \tag{5.3}$$

Similarly, (5.3) holds for $(f - r_0)^-$ in place of $(f - r_0)^+$. Then the proof is completed by noting that $\mu(f^2) \leq \mu((f - r_0)^2)$ and

$$\begin{aligned}
&|(f(x) - r_0)^+ - (f(y) - r_0)^+|^2 + |(f(x) - r_0)^- - (f(y) - r_0)^-|^2 \\
&\leq |f(x) - f(y)|^2.
\end{aligned} \tag{5.4}$$

Indeed, (5.4) is obvious when $(f(x) - r_0)(f(y) - r_0) > 0$. In the case where $(f(x) - r_0)(f(y) - r_0)) \leq 0$, we have $|f(x) - f(y)| = |f(x) - r_0| + |f(y) - r_0|$ and hence (5.4) holds. □

Corollary 5.2. *Assume that there exists $R > 0$ such that $J(A \times E) \leq R\mu(A), A \in \mathcal{F}$. Taking $\gamma \equiv R$ we obtain $\tilde{k}(r) = k(r)/\sqrt{R}$. Therefore, (5.1) holds for some α if and only if $k(r) > 0$ for $r \in (0, 1/2]$, and in this case (5.1) holds for $\alpha(r) = 2Rk(r/2)^{-2}$.*

Corollary 5.3. *Consider the birth-death process:* $E = \mathbb{Z}_+, \mu(i) > 0$ *for* $i \geq 0$,

$$J(i,j) := \begin{cases} \mu(i)a_i, & \text{if } j = i-1, \\ \mu(i)b_i, & \text{if } j = i+1, \\ 0, & \text{otherwise,} \end{cases}$$

where $a_0 = 0, b_0 = 1, a_i, b_i > 0$ *for* $i \geq 1$ *such that* J *is symmetric. We call* a_i *and* b_i *respectively the death rate and the birth rate. Since* J *is symmetric, we have* $\mu(i) = \frac{b_0 \cdots b_{i-1}}{a_1 \cdots a_i}\mu(0), i \geq 1$. *Obviously, we may take* $\gamma(i,j) = (a_i+b_i) \vee (a_j+b_j)$. *For any* $r \in (0, 1/2]$, *let* $i_r := \inf\{i > 0 : \sum_{j \geq i} \mu(j) \leq r\}$ *and*

$$p(r) := \inf\left\{\frac{(\mu(i+1)a_{i+1}) \wedge (\mu(i)b_i)}{\sqrt{(a_i+b_i) \vee (a_{i+1}+b_{i+1})}} : 0 \leq i \leq i_r\right\}.$$

Then $\tilde{k}(r) \geq \inf_{s \in [r,1/2]} p(s)/s$ *and* (5.1) *holds for* $\alpha(r) = 2\{\inf_{s \in [r/2,1/2]} p(s)/s\}^{-2}$ *provided it is finite for any* $r \in [0,1]$.

Proof. Let $r \in (0,1/2]$. For any $s \in [r,1/2]$ and $I \subset \mathbb{Z}_+$ with $\mu(I) = s$, we have $[0, i_s] \cap I \neq \emptyset, [0, i_s+1] \cap I^c \neq \emptyset$. Then, for $\gamma(i,j) = (a_i+b_i) \vee (a_j+b_j)$ we have

$$\frac{\tilde{J}(I \times I^c)}{\mu(I)} = \frac{1}{s} \sum_{i \in I, j \notin I} \tilde{J}(i,j) \geq \frac{p(s)}{s}.$$

□

Finally, we present some examples for birth-death processes which have the same convergence rates as the ones given in Corollary 2.4.

Example 5.4. Let $a_i = 1$ for $i \geq 1$. We consider the following three choices of b_i.

1) Let $b_i = (\frac{i}{i+1})^\delta$ for some $\delta > 1$ and all $i \geq 1$. We have $\mu(i) = \mu(0)i^{-\delta}$ for $i \geq 1$. Obviously, by Corollary 5.3 $\tilde{k}(r) \geq cr^{-1/(\delta-1)}$ for some $c > 0$, hence (5.1) holds for $\alpha(r) = c' r^{2/(1-\delta)}$ for some $c' > 0$.

2) Let $b_i = \frac{i}{i+1}(\frac{\log(1+i)}{\log(2+i)})^\delta$ for some $\delta > 1$. We have $\mu(i) = i^{-1}(\log(1+i))^{-\delta}\mu(0), i \geq 1$. By Corollary 5.3 there exists $c_1, c_2 > 0$ such that $\tilde{k}(r) \geq c_1 \exp[-c_2 r^{1/(1-\delta)}]$, and hence (5.1) holds for $\alpha(r) = \exp[c(1 + r^{1/(1-\delta)})]$ for some $c > 0$.

3) Let $b_i = \exp[\sigma(i^\delta - (i+1)^\delta)]$ for some $\sigma > 0, \delta \in (0,1)$ and all $i \geq 1$. We have $\mu(i) = \exp[-\sigma i^\delta]\mu(0), i \geq 1$. Since $\sum_{j \geq i} \exp[-\sigma j^\delta] \leq c_1 i^{1-\delta} \exp[-\sigma i^\delta]$ for some $c_1 > 0$ and all $i \geq 1$, we have $i_r \leq i'_r \leq c_2[\log(1+r^{-1})]^{1/\delta}$ for some $c_2 > 0$, where $i'_r > 0$ satisfies $c_1(i'_r)^{1-\delta}\exp[-\sigma(i'_r)^\delta] = r$. Then by Corollary 5.3,

$$\tilde{k}(r) \geq \frac{c_3}{r}\exp[-\sigma(i'_r)^\delta] = \frac{c_3}{c_1}(i'_r)^{\delta-1} \geq c_4[\log(1+r^{-1})]^{(\delta-1)/\delta}$$

for some $c_3, c_4 > 0$. Therefore (5.1) holds for $\alpha(r) = c[\log(1+r^{-1})]^{2(1-\delta)/\delta}$ for some $c > 0$. Finally, it is easy to see that (1.1) holds if $\delta \geq 1$.

In the next example we consider some birth-death processes with unbounded rates.

Example 5.5. Letting $a_i = b_i$ for $i \geq 1$, we have $\mu(i) = a_i^{-1}\mu(0), i \geq 1$.

1) Let $a_i = i^\delta$ for some $\delta > 1$ and all $i \geq 1$. Then $i_r \leq c_1 r^{1/(1-\delta)}$ for some $c_1 > 0$. By Corollary 5.3, $\tilde{k}(r) \geq c_2 r^{(2-\delta)^+/(2(\delta-1))}$ for some $c_2 > 0$. Hence (5.1) holds for $\alpha(r) = c_3 r^{(2-\delta)^+/(1-\delta)}$ for some $c_3 > 0$. Especially, if $\delta \geq 2$ then (1.1) holds.

2) Let $a_i = i[\log(1+i)]^\delta$ for some $\delta > 1$ and all $i \geq 1$. Then $i_r \leq \exp[c_1 r^{1/(1-\delta)}]$ for some $c_1 > 0$. By Corollary 5.3, there exists $c_2 > 0$ such that $\tilde{k}(r) \geq \exp[-c_2 r^{1/(1-\delta)}]$ and (5.1) holds with $\alpha(r) = \exp[c(1 + r^{1/(1-\delta)})]$ for some $c > 0$.

3) Let $a_i = i^2[\log(1+i)]^{-\delta}$ for some $\delta > 0$ and all $i \geq 1$. Then $i_r \leq c_1 r^{-1}[\log(1+r^{-1})]^\delta$ for some $c_1 > 0$. By Corollary 5.3 $\tilde{k}(r) \geq c_2[\log(1+r^{-1})]^{-\delta/2}$ for some $c_2 > 0$ and (5.1) holds with $\alpha(r) = c[\log(1+r^{-1})]^\delta$ for some $c > 0$.

6 Perturbations of μ with application to the stochastic quantization of field theory

We first study the behaviour of the weak Poincaré inequality under perturbations of the probability measure μ. Then we apply the corresponding results to the stochastic quantization of field theory.

Let $\mathcal{M}$ denote the class of measurable functions on $(E, \mathcal{F})$, and let $\Gamma : \mathcal{D}(\Gamma) \times \mathcal{D}(\Gamma) \to \mathcal{M}$ be a symmetric bilinear mapping satisfying

1) $\mathcal{D}(\Gamma)$ is a sub-algebra of $\mathcal{M}, \Gamma(f,f) \geq 0$ for $f \in \mathcal{D}(\Gamma), 1 \in \mathcal{D}(\Gamma)$.

2) If $f, g \in \mathcal{D}(\Gamma)$ then $\Gamma(fg, h) = f\Gamma(g,h) + g\Gamma(f,h),\ h \in \mathcal{D}(\Gamma)$.

3) If $f, g \in \mathcal{D}(\Gamma)$, then $f \wedge g \in \mathcal{D}(\Gamma)$ and $\Gamma(f\wedge g, f \wedge g) \leq 1_{\{f \leq g\}}\Gamma(f,f) + 1_{\{f \geq g\}}\Gamma(g,g)$.

Assume that there exists a decreasing function $\alpha : (0,\infty) \to [0,\infty)$ such that

$$\mu(f^2) \leq \alpha(r)\mu(\Gamma(f,f)) + r\|f\|_\infty^2, \quad f \in \mathcal{D}(\Gamma) \cap L^2(\mu), \mu(f) = 0, r > 0. \tag{6.1}$$

Let V be a measurable function such that $\mathrm{d}\bar{\mu} := \mathrm{e}^V \mathrm{d}\mu$ is a probability measure on E. Our first aim is to establish the weak Poincaré inequality for $\bar{\mu}$.

Theorem 6.1. *Assume* (6.1). *Let*

$$\theta(r) := \mu(V > r),\ \ \theta(-r) := \mu(V < -r),\ \ \bar\theta(r) := \bar\mu(V > r),\ \ \bar\theta(-r) := \bar\mu(V < -r),\ \ r > 0.$$

If $V \in \mathcal{D}(\Gamma)$ with $\Gamma(V,V) \in L^p(\mu)$ for some $p > 1$, and $\mathrm{e}^{r_1}[\theta(r_1) - \theta(r_1+1)]^{(p-1)/p} \to 0$ as $r_1 \to \infty$, then (6.1) holds for $\bar\mu$ and $\bar\alpha$ in place of μ and α for

$$\bar\alpha(r) := \inf\{2\alpha(\varepsilon)\exp[r_1 + r_2 + 1] : r_1, r_2, \varepsilon > 0, \vartheta(r_1, r_2, \varepsilon) \le r\},$$

where

$$\begin{aligned}\vartheta(r_1, r_2, \varepsilon) :=& 4[\bar\theta(r_1) + \bar\theta(-r_2) + \varepsilon \mathrm{e}^{r_1}] + 2\mathrm{e}^{r_1}\alpha(\varepsilon)\|\Gamma(V,V)\|_{L^p(\mu)}\\ &\cdot \big[\theta(r_1) + \theta(-r_2) - \theta(r_1+1) - \theta(-r_2-1)\big]^{(p-1)/p}.\end{aligned}$$

Proof. For $r_1, r_2 > 0$ let $\varphi(r) := (r+r_2+1)^+ \wedge 1 \wedge (r_1+1-r)^+,\ r \in \mathbb{R}$. For $f \in L^\infty(\mu) \cap \mathcal{D}(\Gamma)$ with $\bar\mu(f) = 0$ and $\Gamma(f,f) \in L^1(\bar\mu)$, we have

$$\begin{aligned}\bar\mu(f^2) &\le \inf_{|c| \le \|f\|_\infty} \bar\mu((f-c)^2 1_{\{-r_2 \le V \le r_1\}}) + 4\|f\|_\infty^2(\bar\theta(r_1) + \bar\theta(-r_2))\\ &\le \mathrm{e}^{r_1} \inf_{|c| \le \|f\|_\infty} \mu((f-c)^2 1_{\{-r_2 \le V \le r_1\}}) + 4\|f\|_\infty^2(\bar\theta(r_1) + \bar\theta(-r_2))\\ &\le \mathrm{e}^{r_1} \inf_{|c| \le \|f\|_\infty} \mu((f\varphi(V) - c)^2) + 4\|f\|_\infty^2(\bar\theta(r_1) + \bar\theta(-r_2))\\ &\le \mathrm{e}^{r_1}\alpha(\varepsilon)\mu(\Gamma(f\varphi(V), f\varphi(V)) + 4\|f\|_\infty^2(\bar\theta(r_1) + \bar\theta(-r_2) + \varepsilon\mathrm{e}^{r_1}),\end{aligned} \tag{6.2}$$

for all $\varepsilon > 0$. On the other hand,

$$\begin{aligned}\mu(\Gamma(f\varphi(V), f\varphi(V))) \le& 2\big[\mu(f^2\varphi'(V)^2\Gamma(V,V)) + \mu(\varphi(V)^2\Gamma(f,f))\big]\\ \le& 2\mathrm{e}^{r_2+1}\bar\mu(\Gamma(f,f)) + 2\|f\|_\infty^2 \cdot \|\Gamma(V,V)\|_{L^p(\mu)} \cdot\\ &[\theta(r_1) + \theta(-r_2) - \theta(r_1+1) - \theta(-r_2-1)]^{(p-1)/p}.\end{aligned}$$

Combining this with (6.2) we obtain

$$\bar\mu(f^2) \le 2\mathrm{e}^{r_1+r_2+1}\alpha(\varepsilon)\bar\mu(\Gamma(f,f)) + \|f\|_\infty^2\vartheta(r_1, r_2, \varepsilon),$$

for all $r_1, r_2, \varepsilon > 0$. This completes the proof. □

Next, we apply Theorem 6.1 to the stochastic quantization of ($P(\Phi)_2$-) field theory in finite volume studied by Jona-Lasinio and Mitter [17]. We use the notation in Röckner and Zhang [24].

Let Λ be an open rectangle in $\mathbb{R}^2$, and $(-\Delta+1)_N$ the generator of the quadratic form

$$(u,v) \to \int_\Lambda \langle\nabla u, \nabla v\rangle \mathrm{d}x + \int_\Lambda uv\mathrm{d}x, u, v \in \{g \in L^2(\Lambda; \mathrm{d}x) : |\nabla g| \in L^2(\Lambda; \mathrm{d}x)\}.$$

Let $\{\lambda_n : n \geq 1\}$ be all (Neumann) eigenvalues of $(-\Delta+1)_N$ on Λ and e_n the corresponding normalized eigenfunction of λ_n. For $\delta \in \mathbb{R}$, define

$$H_\delta = \Big\{ u \in L^2(\Lambda, \mathrm{d}x) : \sum_{n=1}^\infty \lambda_n^\delta \langle u, e_n \rangle_{L^2(\Lambda;\mathrm{d}x)}^2 < \infty \Big\},$$

$$\langle u, v \rangle_{H_\delta} = \sum_{n=1}^\infty \lambda_n^\delta \langle u, e_n \rangle_{L^2(\Lambda;\mathrm{d}x)} \langle v, e_n \rangle_{L^2(\Lambda;\mathrm{d}x)}, \quad u, v \in H_\delta.$$

We now fix $\delta, \tilde{\delta} > 0$, let μ be the mean zero Gaussian probability measure on $E := H_{-\tilde{\delta}}$ such that

$$\int_E {}_{E'}\langle l, z \rangle_E^2 \mu(\mathrm{d}z) = \|l\|_{-1}^2, \qquad l \in E' := H_{\tilde{\delta}}.$$

For $n \geq 1$, let $H_n(t) := \sum_{m=0}^{[n/2]} (-\frac{1}{2})^m \frac{n! t^{n-2m}}{(n-2m)! m!}$, let $\varrho \in C_0^\infty(\mathbb{R}^2)$ with $\varrho \geq 0, \int \varrho(x)\mathrm{d}x = 1$ and $\varrho(x) = \varrho(-x)$. For $\kappa \geq 1$ let $\varrho_{\kappa,x}(y) := 2^{2\kappa} \varrho(2^\kappa (x-y)), z_\kappa(x) := {}_{E'}\langle \varrho_{\kappa,x}, z \rangle_E$ and $c_\kappa(x) := \int z_\kappa(x)^2 \mu(\mathrm{d}z), x \in \Lambda, z \in E$. Then for any $h \in L^2(\Lambda; \mathrm{d}x)$,

$$\int_\Lambda H_n(c_\kappa(x)^{-1/2} z_\kappa(x)) c_\kappa(x)^{n/2} h(x) \mathrm{d}x$$

converges in $L^p(\mu)$ for any $p \geq 1$, and the limit is independent of the choice of $\varrho(x)$ (see e.g. [24]). Denote the limit by $: z^n : (h)$ which is known as the Wick power of a random variable (see e.g. [25]).

Now, fix $N \geq 1, a_n \in \mathbb{R}, 0 \leq n \leq 2N$ with $a_{2N} > 0$. Define

$$V(z) = C - \sum_{n=0}^{2N} a_n : z^n : (1_\Lambda), \qquad z \in E,$$

where C is a constant such that $\mathrm{d}\bar{\mu} = \mathrm{e}^V \mathrm{d}\mu$ is a probability measure. V has the following properties.

a) (see e.g. Theorems V. 2 and V.7 in [25]). There exists $c > 0$ such that $\|V\|_{L^p(\mu)} \leq c(p-1)^N, \quad p \geq 2$, and $\exp[V] \in L^p(\mu)$ for all $p \geq 1$.

b) (see e.g. Theorem V.5 in [25]). There exist $a, b > 0$ such that $\mu(\{V > b(\log K)^N\}) \leq \exp[-K^a]$ for big K.

c) (see Proposition 7.2 and (7.19) in [24]). For any $p \geq 1$, $|\nabla^{(\delta)} V|^2 := \sum_{j=1}^\infty (\frac{\partial V}{\partial k_j})^2 \in L^p(\mu)$, where $k_j = \lambda_j^{-\delta/2} e_j$.

Theorem 6.2. *Let $\delta \in [0, \frac{1}{2})$. For any $c > (\mathrm{e}/N)^N$, there exists $c' > 0$ such that*

$$\bar\mu(f^2) \le \bar\alpha(r)\bar\mu(|\nabla^{(\delta)} f|^2) + r\|f\|_\infty^2,\ \ f \in \mathcal{F}C_b^\infty, \bar\mu(f) = 0, r > 0, \tag{6.3}$$

for $\bar\alpha(r) = c' \exp\{c[\log(1 + r^{-1})]^N\}, r > 0$.

Proof. By b) there exist $a, r_0 > 0$ such that

$$\theta(r) \le \exp\big\{ -\exp[ar^{1/N}]\big\}, \quad r \ge r_0. \tag{6.4}$$

By a) there exists $c_1 > 0$ such that

$$\theta(-r) \le \mu(|V| > r) \le \inf_{p\ge 2} c_1(p-1)^{Np} r^{-p}, \quad r > 0.$$

Taking $p = \mathrm{e}^{-1} r^{1/N}$ for $r \ge (2\mathrm{e})^N$, we have $p^{Np} r^{-p} = e^{-Np}$ and hence

$$\theta(-r) \le c_1 \exp[-N\mathrm{e}^{-1} r^{1/N}], \quad r \ge (2\mathrm{e})^N. \tag{6.5}$$

Now, for fixed $s \in (1/2, 1)$, let $p > 1$ be such that $(p-1)/p = s$. By c) we have $|\nabla^{(\delta)} V|^2 \in L^p(\mu)$. Since $\delta < 1/2$ the Poincaré inequality holds for μ and $\mathcal{E}(f, f) := \mu(|\nabla^{(\delta)} f|^2)$, i.e. (6.1) holds for some α with $\alpha(0) := \lim_{r\downarrow 0} \alpha(r) < \infty, \Gamma(f, f) = |\nabla^{(\delta)} f|^2$ and $\mathcal{D}(\Gamma) = \mathcal{F}C_b^\infty$. Hence using that $\exp[V] \in \cap_{p\ge1} L^p(\mu)$ (cf. a) above) we see that there exists $c_2 > 0$ such that $\vartheta(r_1, r_2, 0) \le c_2 \mathrm{e}^{r_1}[\theta(r_1) + \theta(-r_2)]^s, r_1, r_2 > 0$. Combining this with (6.4) and (6.5), we obtain

$$\vartheta(r_1, r_2, 0) \le c_2 \mathrm{e}^{r_1}\big\{ \exp\big[-\exp[ar_1^{1/N}]\big] + c_1 \exp[-Nr_2^{1/N}\mathrm{e}^{-1}]\big\}^s \tag{6.6}$$

for $r_1, r_2 \ge r_0 \vee (2\mathrm{e})^N$. Obviously, there exists $r_3 > 0$ such that for any $r \in (0, r_3)$, there exist $r_1, r_2 \ge r_0 \vee (2\mathrm{e})^N$ such that

$$\exp\{r_1 - s\exp[ar_1^{1/N}]\} = c_1^s \exp[r_1 - sNr_2^{1/N}\mathrm{e}^{-1}] = \frac{r}{2^s c_2}. \tag{6.7}$$

By (6.6) we have $\vartheta(r_1, r_2, 0) \le r$ for $r_1, r_2 > 0$ solving (6.7). It is easy to see that for any $\varepsilon \in (0, 1)$, there exists $c_3 > 0$ (independent of r_1, r_2, r_3, s) such that

$$r_1 \le \varepsilon \log r^{-1} + c_3, \quad r_2 \le \Big[\frac{(1+\varepsilon)\mathrm{e}}{sN} \log r^{-1}\Big]^N + c_3,$$

for all $r \in (0, r_3)$ and the corresponding $r_1, r_2 > 0$ solving (6.7). Hence we obtain (6.3) from Theorem 6.1 for

$$\bar{\alpha}(r) = c(\varepsilon) r^{-\varepsilon} \exp\Big[\Big(\frac{(1+\varepsilon)\mathrm{e}}{sN}\log r^{-1}\Big)^N\Big]$$

for some $c(\varepsilon) > 0$. Then the proof is completed since $\varepsilon > 0$ is arbitrary and $s \uparrow 1$ as $p \uparrow \infty$. □

The following is a direct consequence of Theorems 2.1 and 6.2.

Corollary 6.3. *Let $\bar{P}_t$ be the semigroup on $L^2(\bar{\mu})$ associated to $\bar{\mathcal{E}}(f,f) := \bar{\mu}(|\nabla^{(\delta)} f|^2)$. For any $c \in (0, N/\mathrm{e})$ there exists $c' > 0$ such that*

$$\bar{\mu}((\bar{P}_t f)^2) \le c' \exp\big[-c(\log(1+t))^{1/N}\big]\|f\|_\infty^2, \quad \bar{\mu}(f) = 0,\ f \in L^\infty(\bar{\mu}),\ t > 0.$$

7 Weak Poincaré inequalities on configuration spaces

In this section we study weak Poincaré inequalities for Dirichlet forms determined by the gradient operator and Poisson measures on a configuration space. We refer to [4, 5, 22] for previous results concerning analysis and geometry on configuration spaces. We first recall some basic notions in the literature.

Let M be a connected noncompact Riemannian manifold, and σ an infinite Radon measure on M with $\sigma(K) < \infty$ for any compact $K \subset M$. The configuration space over M is defined by

$$\Gamma := \{\gamma \subset M : |\gamma \cap K| < \infty \text{ for any compact } K \subset M\},$$

where $|A|$ denotes the cardinality of A. As usual, we identify γ with the measure $\sum_{x\in\gamma}\delta_x$. For any $f \in C_0^\infty(M)$ and any $\gamma \in \Gamma$, denote $\langle f, \gamma\rangle := \gamma(f) := \sum_{x\in\gamma} f(x)$. Let

$$\mathcal{F}C_b^\infty = \{g(\langle f_1,\cdot\rangle, \cdots, \langle f_N,\cdot\rangle) : N \ge 1, g \in C_b^\infty(\mathbb{R}^N), f_1, \cdots, f_N \in C_0^\infty(M)\}.$$

For $\Lambda \subset M$, let $\mathcal{F}C_b^\infty(\Lambda)$ be defined as $\mathcal{F}C_b^\infty$ with Λ in place of M. We consider the vague topology on Γ. Denote by $\mathcal{B}(\Gamma)$ the corresponding Borel σ-field. Let π_σ denote the (pure) Poisson measure on Γ with intensity σ, i.e. π_σ is the unique measure on $(\Gamma, \mathcal{B}(\Gamma))$ with Laplace transform

$$\int_\Gamma \exp[\gamma(f)]\pi_\sigma(\mathrm{d}\gamma) = \exp\big[\sigma(\mathrm{e}^f - 1)\big], \quad f \in C_0^\infty(M).$$

For $F \in \mathcal{F}C_b^\infty$ and $v \in V_0(M)$, the set of smooth vector fields on M with compact supports, define

$$(\nabla_v^\Gamma F)(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t} F\big(\exp_\gamma(tv)\big)\big|_{t=0},$$

where $\exp_\gamma(tv) = \{\exp_x(tv_x) : x \in \gamma\}$. If $F = g(\langle f_1, \cdot\rangle, \cdots, \langle f_N, \cdot\rangle)$, then

$$\nabla_v^\Gamma F(\gamma) = \sum_{i=1}^N \partial_i g(\langle f_1, \gamma\rangle, \cdots, \langle f_N, \gamma\rangle)\gamma(\langle v, \nabla f_i\rangle) = \gamma\big(\langle v, \nabla^\Gamma F(\gamma)\rangle\big),$$

where $\nabla^\Gamma F(\gamma) = \sum_{i=1}^N \partial_i g(\langle f_1, \gamma\rangle, \cdots, \langle f_N, \gamma\rangle)\nabla f_i \in V_0(M)$.

For $\gamma \in \Gamma$, let $T_\gamma \Gamma := L^2(M \to TM; \gamma)$ be the tangent space at γ, equipped with the product $\langle \cdot, \cdot\rangle_{T_\gamma\Gamma} = \langle \cdot, \cdot\rangle_{L^2(M\to TM;\gamma)}$. We have

$$\begin{aligned}\big|\nabla^\Gamma F(\gamma)\big|^2 &:= \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma F(\gamma)\rangle_{T_\gamma\Gamma} \\ &= \sum_{i,j=1}^N \partial_i g(\langle f_1, \gamma\rangle, \cdots \langle f_N, \gamma\rangle)\partial_j g(\langle f_1, \gamma\rangle, \cdots, \langle f_N, \gamma\rangle)\gamma(\langle \nabla f_i, \nabla f_j\rangle).\end{aligned}$$

For $F, G \in \mathcal{F}C_b^\infty$, define $\mathcal{E}(F, G) = \pi_\sigma(\langle \nabla^\Gamma F, \nabla^\Gamma G\rangle_{T.\Gamma})$ which is a pre-Dirichlet form on $L^2(\pi_\sigma)$. It is known that (see e.g. Proposition 4.3 in [22]) $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable provided $\sigma(\mathrm{d}x) = p(x)\mathrm{d}x$ with $p, |\nabla p|/p \in L^1_{loc}(\mathrm{d}x)$ and $\sqrt{p} \in H^{1,2}_{loc}(\mathrm{d}x)$, where $\mathrm{d}x$ denotes the Riemannian volume element. We assume that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable and let $(\mathcal{E}, D(\mathcal{E}))$ denote the closure.

Let $\delta(F) := \sup F - \inf F$ for a bounded function F, and let

$$\begin{aligned}\lambda_\sigma(r) &:= \inf\{\sigma(|\nabla f|^2) : f \in C_0^\infty(M), \sigma(f^2) = 1, \|f\|_\infty^2 \le r\}, \\ \mathrm{gap}(\mathcal{E}) &:= \inf\{\mathcal{E}(F, F) : F \in \mathcal{F}C_b^\infty, \pi_\sigma(F^2) - \pi_\sigma(F)^2 = 1\}.\end{aligned}$$

The main result in this section is the following.

Theorem 7.1. *We have $\lambda_\sigma(r) > 0$ for any $r > 0$ if and only if there exists $\alpha : (0, \infty) \to (0, \infty)$ such that*

$$\pi_\sigma(F^2) - \pi_\sigma(F)^2 \le \alpha(r)\mathcal{E}(F, F) + r\delta(F)^2, \quad r > 0, F \in \mathcal{F}C_b^\infty, \tag{7.1}$$

and (7.1) *holds for $\alpha(r) = \lambda_\sigma(1/r)^{-1}$ provided $\lambda_\sigma(r) > 0$ for all $r > 0$. In particular,* $\mathrm{gap}(\mathcal{E}) = \lambda_\sigma(\infty)$.

To prove Theorem 7.1 we need some preparations. We will frequently use the following local representation of π_σ. For $\Lambda \in \mathcal{O}_c(M)$ (i.e. Λ is a relatively compact open subset of M), and for $F \in \mathcal{F}C_b^\infty(\Lambda)$, let $F_0 = F(\emptyset), F_n \in C_b^\infty(M^n)$ with $F_n(x_1, \cdots, x_n) := F(\{x_1, \cdots, x_n\})$ if $x_i \ne x_j$ for $i \ne j$. We have

$$\pi_\sigma(F)=\sum_{n=0}^\infty \frac{\sigma_\Lambda^n(F_n)}{\exp[\sigma(\Lambda)]n!}, \tag{7.2}$$

where σ_Λ^n on M^n is the product measure of $\sigma_\Lambda := \sigma(\cdot\cap\Lambda)$, and $\sigma_\Lambda^0(F_0) := F_0 = F(\emptyset)$. The following lemma is well-known. We include a proof for completeness.

Lemma 7.2. *For $f\in C_0^\infty(M)$, let $F=\langle f,\cdot\rangle-\sigma(f)$. We have $\pi_\sigma(F^4)=\sigma(f^4)+3\sigma(f^2)^2$.*

Proof. Let $G=\langle f,\cdot\rangle$. We have

$$F^4 = G^4-4G^3\sigma(f)+6G^2\sigma(f)^2-4G\sigma(f)^3+\sigma(f)^4.$$

Let $h(r):=\pi_\sigma(\mathrm{e}^{rG})=\exp[\sigma(\mathrm{e}^{rf}-1)]$, we have $\frac{\mathrm{d}^n}{\mathrm{d}r^n}h|_{r=0}=\pi_\sigma(G^n), n\geq 1$. Then $\pi_\sigma(G)=\sigma(f), \pi_\sigma(G^2)=\sigma(f)^2+\sigma(f^2)$, and

$$\begin{aligned}&\pi_\sigma(G^3)=\sigma(f)^3+3\sigma(f)\sigma(f^2)+\sigma(f^3),\\&\pi_\sigma(G^4)=\sigma(f)^4+6\sigma(f^2)\sigma(f)^2+3\sigma(f^2)^2+4\sigma(f)\sigma(f^3)+\sigma(f^4).\end{aligned}$$

Therefore, the desired result follows immediately. □

Lemma 7.3. *If $\lambda_\sigma(c)>0$ for some $c>0$, then*

$$\sigma(f^2)\leq \frac{1}{\lambda_\sigma(c)}\sigma(|\nabla f|^2)+\frac{1}{c}\|f\|_\infty^2,\quad f\in C_0^\infty(M). \tag{7.3}$$

Conversely, if (7.3) *holds with some $c'>0$ replacing $\lambda_\sigma(c)$, then for any $r\in(0,c)$, we have $\lambda_\sigma(r)\geq (c-r)c'/c$.*

Proof. For any $f\in C_0^\infty(M)$, let $\bar f = f/\sqrt{\sigma(f^2)}$. We have $\sigma(\bar f^2)=1, \|\bar f\|_\infty^2=\|f\|_\infty^2/\sigma(f^2)$. If $\lambda_\sigma(c)>0$ and $\sigma(f^2)\geq \|f\|_\infty^2/c$, then $\|\bar f\|_\infty^2\leq c$ and $\sigma(|\nabla\bar f|^2)\geq\lambda_\sigma(c)$, hence $\sigma(|\nabla f|^2)\geq \lambda_\sigma(c)\sigma(f^2)$. Therefore (7.3) holds.

On the other hand, if (7.3) holds with c' replacing $\lambda_\sigma(c)$, then for any $f\in C_0^\infty(M)$ with $\sigma(f^2)=1$ and $\|f\|_\infty^2\leq r\in(0,c)$, we have $\sigma(|\nabla f|^2)\geq c'[1-r/c]$. This proves the second assertion. □

Lemma 7.4. *Let $\mu_\Lambda:=\sigma_\Lambda/\sigma(\Lambda)$. Let ∇^n denote the gradient operator on M^n. If* (7.3) *holds for all $c>0$ then*

$$\mu_\Lambda^n(F_n^2)-\mu_\Lambda^n(F_n)^2\leq \frac{\mu_\Lambda^n(|\nabla^nF_n|^2)}{\lambda_\sigma(1/r)}+\frac{rn\delta(F)^2}{\sigma(\Lambda)},\quad r>0, F\in\mathcal{F}C_b^\infty(\Lambda), n\geq 1. \tag{7.4}$$

Proof. Let $F = g(\langle f_1, \cdot\rangle, \cdots, \langle f_N, \cdot\rangle) \in \mathcal{F}C_b^\infty(\Lambda)$. Noting that $F_1 - F_0 \in C_0^\infty(\Lambda)$, by (7.3) for $c = 1/r$ we obtain

$$\mu_\Lambda(F_1^2) - \mu_\Lambda(F_1)^2 \le \mu_\Lambda\big((F_1 - F_0)^2\big) \le \frac{1}{\lambda_\sigma(1/r)}\mu_\Lambda(|\nabla F_1|^2) + \frac{r\delta(F)^2}{\sigma(\Lambda)}.$$

Hence (7.4) holds for $n = 1$. Assume that (7.4) holds for $n = k$, it suffices to prove it for $n = k + 1$. Since

$$h := \int_{\Lambda^k} F_{k+1}(x_1, \cdots, x_k, \cdot)\mu_\Lambda^k(\mathrm{d}x_1, \cdots, \mathrm{d}x_k) - \int_{\Lambda^k} F_k \mathrm{d}\mu_\Lambda^k \in C_0^\infty(\Lambda),$$

we have

$$\begin{aligned}&\int_\Lambda \bigg(\int_{\Lambda^k} F_{k+1}(x_1, \cdots, x_k, \cdot)\mu_\Lambda^k(\mathrm{d}x_1, \cdots, \mathrm{d}x_k)\bigg)^2 \mathrm{d}\mu_\Lambda - \mu_\Lambda^{k+1}(F_{k+1})^2\\&\le \mu_\Lambda(h^2) \le \frac{\mu_\Lambda(|\nabla h|^2)}{\lambda_\sigma(1/r)} + \frac{r\delta(F)^2}{\sigma(\Lambda)}.\end{aligned}$$

By this and applying (7.4) with $n = k$ to

$$F^x(\gamma) := g(\langle f_1, \gamma\rangle + f_1(x), \cdots, \langle f_N, \gamma\rangle + f_N(x))$$

for each $x \in M$, we obtain

$$\begin{aligned}&\mu_\Lambda^{k+1}(F_{k+1}^2) - \mu_\Lambda^{k+1}(F_{k+1})^2\\
=&\int_\Lambda \big[\mu_\Lambda^k\big((F_k^x)^2\big) - \mu_\Lambda^k(F_k^x)^2\big]\mu_\Lambda(\mathrm{d}x)\\
&+ \int_\Lambda \mu_\Lambda(\mathrm{d}x_{k+1})\bigg(\int_{\Lambda^k} F_{k+1}(x_1, \cdots, x_{k+1})\mu_\Lambda^k(\mathrm{d}x_1, \cdots, \mathrm{d}x_k)\bigg)^2 - \mu_\Lambda^{k+1}(F_{k+1})^2\\
\le&\frac{1}{\lambda_\sigma(1/r)}\int_\Lambda \mu_\Lambda^k(|\nabla^k F_k^x|^2)\mu_\Lambda(\mathrm{d}x) + \frac{rk\delta(F)^2}{\sigma(\Lambda)}\\
&+ \frac{1}{\lambda_\sigma(1/r)}\mu_\Lambda\bigg(\bigg|\nabla\int_{\Lambda^k} F_{k+1}(x_1, \cdots, x_k, \cdot)\mu_\Lambda^k(\mathrm{d}x_1, \cdots, \mathrm{d}x_k)\bigg|^2\bigg) + \frac{r\delta(F)^2}{\sigma(\Lambda)}\\
\le&\frac{1}{\lambda_\sigma(1/r)}\mu_\Lambda^{k+1}(|\nabla^{k+1}F_{k+1}|^2) + \frac{r(k+1)\delta(F)^2}{\sigma(\Lambda)}.\end{aligned}$$

□

Lemma 7.5. *Let $p_\lambda(i) := \exp[-\lambda]\lambda^i/i!, \lambda \ge 1, i \ge 0$. For any $\varepsilon > 0$ and $r \in (0, 1/2]$, there exists $\lambda_{\varepsilon,r} \ge 1$ such that for all $\lambda \in [\lambda_{\varepsilon,r}, \infty)$ and all bounded $\alpha : \mathbb{Z}_+ \to \mathbb{R}$,*

$$
\begin{aligned}
&\sum_{i=0}^{\infty} p_\lambda(i)\alpha_i^2 - \Big(\sum_{i=0}^{\infty} p_\lambda(i)\alpha_i\Big)^2 \\
&\le 8r^2 e^{2/r}(\pi+\varepsilon)\sum_{i=1}^{\infty} i p_\lambda(i)(\alpha_i-\alpha_{i-1})^2 + 2r[\sup\alpha_i - \inf\alpha_i]^2.
\end{aligned}
$$

Proof. Let ξ_λ be the Poisson random variable with intensity λ, then $E\xi_\lambda = E|\xi_\lambda-\lambda|^2 = m$. For any $I \subset \mathbb{Z}_+$ with $s := P(\xi_\lambda \in I) \in [r, 1/2]$, we have $P\big(|\xi_\lambda - \lambda| \ge \sqrt{\lambda/s + 1/2}\big) < s$. Then

$$
\begin{aligned}
&(\mathbb{Z}_+ \setminus I) \cap \Big(\lambda - \sqrt{\lambda/s} - 1, \lambda + \sqrt{\lambda/s} + 1\Big) \ne \emptyset, \\
&I \cap \Big(\lambda - \sqrt{\lambda/s} - 1, \lambda + \sqrt{\lambda/s} + 1\Big) \ne \emptyset.
\end{aligned}
\tag{7.5}
$$

It is easy to check that $p_\lambda(i)$ is increasing in i for $i \le \lambda$ and decreasing in i for $i \ge \lambda$, and for any $\varepsilon' > 0$,

$$
\begin{aligned}
&\log p_\lambda\big([\lambda + \sqrt{\lambda/s}]_z + 1\big) - \log p_\lambda([\lambda]_z) \\
&= \big([\lambda + \sqrt{\lambda/s}]_z + 1 - [\lambda]_z\big)\log\lambda - \log\frac{\big([\lambda + \sqrt{\lambda/s}]_z + 1\big)!}{[\lambda]_z!} \\
&\ge -\big(\sqrt{\lambda/s} + 2\big)\log\left(1 + \frac{1 + \sqrt{\lambda/s}}{\lambda}\right) \\
&\ge -\frac{1}{\lambda}\big(\sqrt{\lambda/s} + 1\big)\big(2 + \sqrt{\lambda/s}\big) \ge -\frac{1}{s} - \varepsilon'
\end{aligned}
$$

for big enough λ, where $[r]_z = \max\{i \in \mathbb{Z} : i \le r\}$ for $r > 0$. We have the same estimate for $[\lambda - \sqrt{\lambda/s}]_z - 1$ in place of $[\lambda + \sqrt{\lambda/s}]_z + 1$. Then, for sufficiently big λ, we have

$$
p_\lambda(i) \ge \exp[-s^{-1} - \varepsilon']p_\lambda([\lambda]_z),\ \ i \in \mathbb{Z}_+ \cap \big[\lambda - \sqrt{\lambda/s} - 1, \lambda + \sqrt{\lambda/s} + 1\big]. \tag{7.6}
$$

To apply Theorem 5.1, let $a_i = i, b_i = \lambda, i \ge 0$. Define

$$
J(i,j) = \begin{cases} a_i p_\lambda(i), & \text{if } j = i-1, \\ b_i p_\lambda(i), & \text{if } j = i+1 \\ 0, & \text{otherwise,} \end{cases} \qquad i, j \in \mathbb{Z}_+.
$$

Then J is symmetric and

$$\sum_{i=1}^{\infty} ip_\lambda(i)(\alpha_i-\alpha_{i-1})^2 = \frac{1}{2}\sum_{i,j=0}^{\infty}(\alpha_i-\alpha_j)^2 J(i,j). \tag{7.7}$$

Let $\tilde J = 1_{\{\gamma>0\}}J/\sqrt{\gamma}$ for $\gamma(i,j) := (a_i+b_i)\vee(a_j+b_j)$. By (7.5) and (7.6), for big λ and any $I\subset\mathbb{Z}_+$ with $P(\xi_\lambda\in I) =: s\in[r,1/2]$, we have

$$\begin{aligned}\frac{J(I\times I^c)}{\mu(I)} &\geq \min\left\{\frac{ip_\lambda(i)}{s\sqrt{2\lambda+\sqrt{\lambda/s}+1}} : i\in\mathbb{Z}_+\cap\big[\lambda-\sqrt{\lambda/s}-1,\lambda+\sqrt{\lambda/s}+1\big]\right\}\\ &\geq \frac{\big(\lambda-\sqrt{\lambda/s}-1\big)\exp[-s^{-1}-\varepsilon']p_\lambda([\lambda]_z)}{s\sqrt{2\lambda+\sqrt{\lambda/s}+1}}.\end{aligned} \tag{7.8}$$

Noting that $\sqrt{\lambda}p_\lambda([\lambda]_z)\to 1/\sqrt{2\pi}$ as $m\to\infty$, and $s^{-1}\exp[-s^{-1}]\geq r^{-1}\exp[-r^{-1}]$ for $s\in[r,1/2]$, we obtain from (7.8) that, for any $\varepsilon>0$, there exists $m_{\varepsilon,r}\geq 1$ such that $\tilde k(r)\geq \big\{2r\exp[r^{-1}]\sqrt{\pi+\varepsilon}\big\}^{-1}$ provided $\lambda\geq\lambda_{\varepsilon,r}$. Then the proof is completed by Theorem 5.1 and (7.7). $\square$

Proof of Theorem 7.1 Assume that (7.1) holds for some α, we are going to prove $\lambda_\sigma(c)>0$ for any $c>0$. If $\lambda_\sigma(c)=0$ for some $c>0$, then there exists $\{f_n\}\subset C_0^\infty(M)$ such that $\sigma(f_n^2)=1, \|f_n\|_\infty^2\leq c$ and $\sigma(|\nabla f_n|^2)\to 0$ as $n\to\infty$. By an approximation argument (cf. the last paragraph of this proof), we may apply (7.1) to functions $F^{(n)} := \langle f_n,\cdot\rangle-\sigma(f_n)$. Then $\pi_\sigma(F^{(n)})=0, \pi_\sigma(F^{(n)^2})=1$ and $\mathcal{E}(F^{(n)},F^{(n)})=\sigma(|\nabla f_n|^2)\to 0$ as $n\to\infty$. By Proposition 1.2, for any $\varepsilon\in(0,1)$ one has

$$\lim_{n\to\infty}\pi_\sigma(|F^{(n)}|>\varepsilon)=0. \tag{7.9}$$

On the other hand, we have

$$1=\pi_\sigma(F^{(n)^2})\leq\varepsilon+\pi_\sigma(F^{(n)^2}1_{\{|F^{(n)}|>\varepsilon\}})\leq\varepsilon+\sqrt{\pi_\sigma(F^{(n)^4})\pi_\sigma(|F^{(n)}|>\varepsilon)}.$$

Therefore, Lemma 7.2 implies that

$$\pi_\sigma(|F^{(n)}|>\varepsilon)\geq\frac{(1-\varepsilon)^2}{\pi_\sigma(F^{(n)^4})}=\frac{(1-\varepsilon)^2}{\sigma(f_n^4)+3\sigma(f_n^2)^2}\geq\frac{(1-\varepsilon)^2}{c+3}$$

since $\|f_n\|_\infty^2\leq c$ and $\sigma(f_n^2)=1$. This is a contradiction to (7.9).

Conversely, if $\lambda_\sigma(c) > 0$ for any $c > 0$, Lemmas 7.3 and 7.4 imply (7.4). It follows from (7.2) and (7.4) that

$$\begin{aligned}&\pi_\sigma(F^2) - \sum_{n=0}^\infty \frac{\mu_\Lambda^n(F_n)^2\sigma(\Lambda)^n}{\exp[\sigma(\Lambda)]n!}\\ &\le \frac{1}{\lambda_\sigma(r^{-1})}\mathcal{E}(F,F) + r\delta(F)^2, \quad r>0, F\in \mathcal{F}C_b^\infty(\Lambda).\end{aligned} \tag{7.10}$$

Now, for $\Lambda_0 \in \mathcal{O}_c(M)$ and $F \in \mathcal{F}C_b^\infty(\Lambda_0)$, take $\Lambda_k \supset \Lambda_0$ such that $\Lambda_k \uparrow M$ as $k \uparrow \infty$. Noting that $F_n(x_1,\cdots,x_n) = F_{n-1}(x_1,\cdots,x_{n-1})$ if $x_n \notin \Lambda_0$, we obtain

$$\begin{aligned}&|\mu_{\Lambda_k}^n(F_n) - \mu_{\Lambda_k}^{n-1}(F_{n-1})|\\ &\le \int_{\Lambda_k^n} |F_n(x_1,\cdots,x_n) - F_{n-1}(x_1,\cdots,x_{n-1})|\mu_{\Lambda_k}^n(\mathrm{d}x_1,\cdots,\mathrm{d}x_n)\\ &\le \delta(F)\mu_{\Lambda_k}(\Lambda_0) = \frac{\delta(F)\sigma(\Lambda_0)}{\sigma(\Lambda_k)}.\end{aligned} \tag{7.11}$$

By (7.2) one has $\pi_\sigma(F) = \sum_{n=0}^\infty \frac{\mu_{\Lambda_k}^n(F_n)\sigma(\Lambda_k)^n}{\exp[\sigma(\Lambda_k)]n!}$. Then, by applying Lemma 7.5 with $\varepsilon = 1, \lambda = \sigma(\Lambda_k), \alpha_i = \mu_{\Lambda_k}^i(F_i)$, and using (7.10) and (7.11), we obtain

$$\begin{aligned}&\pi_\sigma(F^2) - \pi_\sigma(F)^2\\ &\le \frac{1}{\lambda_\sigma(r^{-1})}\mathcal{E}(F,F) + \Big(\frac{8r_1^2\mathrm{e}^{2/r_1}(\pi+1)\sigma(\Lambda_0)^2}{\sigma(\Lambda_k)} + r + 2r_1\Big)\delta(F)^2\end{aligned} \tag{7.12}$$

for all $r_1, r \in (0,1/2]$ and sufficiently big k. By letting first $k\uparrow\infty$ then $r_1 \downarrow 0$ in (7.12), we obtain (7.1) for $\alpha(r) = \lambda_\sigma(1/r)^{-1}$.

Finally, we obtain $\mathrm{gap}(\mathcal{E}) \ge \lambda_\sigma(\infty)$ by letting $r\to 0$ in (7.1) with $\alpha(r) = \lambda_\sigma(1/r)^{-1}$. It remains to prove $\mathrm{gap}(\mathcal{E}) \le \lambda_\sigma(\infty)$. For any $f \in C_0^\infty(M)$ and $n \ge 1$, let $F^{(n)} = g_n(\langle f,\cdot\rangle) - \pi_\sigma(g_n(\langle f,\cdot\rangle)$, where $g_n \in C_b^\infty(\mathbb{R})$ satisfying $g_n(r) = r$ for $|r| \le n$, $g_n(r) = \mathrm{sign}(r)(n+1)$ for $|r| \ge n+2$, and $|g_n'| \le 1$. We have $\pi_\sigma(F^{(n)}) = 0, \pi_\sigma({F^{(n)}}^2) \to \sigma(f^2)$ as $n\to\infty$, and

$$\pi_\sigma(|\nabla^\Gamma F^{(n)}|^2) \le \sum_{i=1}^\infty \frac{i\sigma(\Lambda)^{i-1}}{\exp[\sigma(\Lambda)]i!}\sigma(|\nabla f|^2) = \sigma(|\nabla f|^2),$$

where $\Lambda \in \mathcal{O}_c(M)$ be such that $\mathrm{supp} f \subset \Lambda$. □

As an extension of Cheeger's inequality for $\lambda_\sigma(\infty)$ which is well-known in geometry (cf. [9]), we present the following result for $\lambda_\sigma(r)$.

Proposition 7.6. *For $r > 0$, let $k(r) := \inf_{\sigma(A)\in(r,\infty)} \sigma_\partial(\partial A)/\sigma(A)$, where A runs over all bounded smooth domains, and $\sigma_\partial(\partial A)$ denotes the area of ∂A induced by σ. If $k(r) > 0$,* *then $\lambda_\sigma(s) \geq (1-rs)^2 k(r)^2/4$ for $s \in (0, 1/r)$. On the other hand, assume that $p := \frac{\mathrm{d}\sigma}{\mathrm{d}x}$ is positive and C^2. Let P_t denote the semigroup generated by $L := \Delta + \nabla \log p$ on $L^2(\sigma)$. If $|\nabla P_t f|^2 \leq h(t) P_t |\nabla f|^2$ for some positive $h \in C[0,\infty)$ and all $f \in C_b^\infty(M)$, then $\lambda_\sigma(s) > 0$ implies $k(r) > 0$ for $r > 1/s$.*

Proof. Assume that $k(r) > 0$. For any $f \in C_0^\infty(M)$ with $\sigma(f^2) = 1$, by the coarea formula,

$$\sigma(|\nabla f^2|) = \int_0^\infty \sigma_\partial(\{f^2 = t\})\mathrm{d}t \geq k(r) \int_0^{t_r} \sigma(f^2 > t)\mathrm{d}t,$$

where $t_r = \sup\{t > 0 : \sigma(f^2 > t) \geq r\}$. Then

$$\begin{aligned}\sigma(f^2) &= \int_0^\infty \sigma(f^2 > t)\mathrm{d}t \leq \int_{t_r}^{\|f\|_\infty^2} \sigma(f^2 > t)\mathrm{d}t + \frac{1}{k(r)}\sigma(|\nabla f^2|) \\ &\leq \frac{2}{k(r)}\sqrt{\sigma(|\nabla f|^2)} + r\|f\|_\infty^2.\end{aligned}$$

Then, for $s \in (0, 1/r)$ and $\|f\|_\infty^2 \leq s$, we have

$$1 = \sigma(f^2) \leq \frac{4\sigma(|\nabla f|^2)}{k(r)^2(1-rs)^2}.$$

Therefore, $\lambda_\sigma(s) \geq k(r)^2(1-rs)^2/4$.

On the other hand, assume $\lambda_\sigma(s) > 0$. By (7.3) we obtain

$$\sigma((P_t f)^2) \leq \exp[-2\lambda_\sigma(s)t]\sigma(f^2) + \|f\|_\infty^2/s, \quad f \in C_0^\infty(M). \tag{7.13}$$

If $|\nabla P_t f|^2 \leq h(t) P_t |\nabla f|^2$ for some positive $h \in C[0,\infty)$ and all $f \in C_b^\infty(M)$, then P_t is conservative and (4.4) holds. For any bounded smooth domain A with $\sigma(A) := r > 1/s$, by taking $f = 1_A$ in (4.4) and (7.13) we obtain

$$\begin{aligned}c_1 t\sigma_\partial(\partial A) &\geq \sigma\big(1_A(1 - P_t 1_A)\big) + \sigma(1_{A^c} P_t 1_A) \\ &= 2\sigma(A) - 2\sigma\big((P_{t/2}1_A)^2\big) \geq 2\sigma(A) - 2\exp[-\lambda_\sigma(s)t]\sigma(A) - 2/s \\ &\geq 2\big(1 - \exp[-\lambda_\sigma(s)t] - 1/(sr)\big)\sigma(A).\end{aligned}$$

Therefore,

$$\inf_{\sigma(A)\geq r} \frac{\sigma_\partial(\partial A)}{\sigma(A)} \geq \sup_{t\geq 1} \frac{2}{c_1 t}\big(1 - \exp[-\lambda_\sigma(s)t] - 1/(sr)\big) > 0.$$

□

Finally, we present two typical examples, where in the first example $\mathcal{E}$ is irreducible but (7.1) does not hold, and in the second (7.1) holds but $\mathrm{gap}(\mathcal{E}) = 0$.

Example 7.7. Let $M = \mathbb{R}^d$ and let σ be the Lebesgue measure. Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible (see e.g. [22]) but (7.1) does not hold. Indeed, for any $R > 0$ with $\sigma(B_R) \geq 1$, taking $f = (R + 1 - \rho)^+ \wedge 1$, where $\rho(x) := |x|$, we have $f^2/\sigma(f^2) \leq 1$ and hence

$$\lambda_\sigma(1) \leq \frac{\sigma(|\nabla f|^2)}{\sigma(f^2)} \leq \frac{\sigma(B_{R+1}) - \sigma(B_R)}{\sigma(B_R)}$$

which goes to zero as $R \to \infty$. Therefore (7.1) does not hold according to Theorem 7.1.

Example 7.8. Let M be the d-dimensional hyperbolic space with $d > 4$. Let $o \in M$ be fixed and ρ the distance function from a fixed point o. Take a sequence $\{x_n\} \subset M$ such that $\rho(x_n) = n$. Let $\sigma_0(\mathrm{d}x) = \mathrm{d}x$ be the Riemannian volume element. Let $\sigma_n(\mathrm{d}x) = p_n(x)\mathrm{d}x$ for some smooth $p_n \geq 0$ satisfying $p_n|_{B_{n^{-2}}(x_n)^c} = 0, p_n|_{B_{n^{-2}/2}(x_n)} = n^{4+\delta} = \max p_n$, where $\delta \in (0, 2d - 9)$ is a constant and $B_r(x)$ denotes the geodesic ball with center x and radius r. Let $\sigma = \sum_{n=0}^{\infty} \sigma_n$, then (7.1) holds for some α but $\mathrm{gap}(\mathcal{E}) = 0$.

Proof. For any $n \geq 1$, let $f_n \in C_0^\infty(M)$ be such that $f_n|_{B_{n^{-2}}(x_n)} = 1, f_n|_{B_{2n^{-2}}(x_n)^c} = 0, |\nabla f_n| \leq 2n^2$. We have

$$\frac{\sigma(|\nabla f_n|^2)}{\sigma(f_n^2)} \leq \frac{4\sigma_0(B_{2n^{-2}}(x_n) \setminus B_{n^{-2}}(x_n))}{n^\delta \sigma_0(B_{n^{-2}/2}(x_n))} \leq cn^{-\delta}$$

for some $c > 0$ and all $n \geq 1$. Therefore, $\lambda_\sigma(\infty) = 0$. By Theorem 7.1 we have $\mathrm{gap}(\mathcal{E}) = 0$.

It remains to prove (7.1). For $n \geq 1$, let $M_n = M \setminus \cup_{i>n} B_{i^{-2}}(x_i)$. Obviously, we have $1 \leq p := 1 + \sum_{n=1}^{\infty} p_n \leq 1 + n^{4+\delta}$ on M_n. Since $\Delta\rho \geq d - 1$, by the integration by parts formula, we obtain

$$\frac{\sigma_\partial(\partial A)}{\sigma(A)} \geq \frac{(\sigma_0)_\partial(\partial A)}{(1 + n^{4+\delta})\sigma_0(A)} \geq \frac{d-1}{1 + n^{4+\delta}}$$

for all bounded smooth $A \subset M_n$. By the coarea formula (cf. the proof of Proposition 7.6), we arrive at

$$\sigma(f^2) \leq \frac{1 + n^{4+\delta}}{d-1} \sigma(|\nabla f^2|), \qquad f \in C_0^\infty(M_n). \tag{7.14}$$

For $i > n$, let $h_i \in C_0^\infty(M)$ such that $h_i|_{B_{i^{-2}}(x_i)} = 0, h_{B_{2i^{-2}}(x_i)^c} = 1$ and $|\nabla h_i| \leq 2i^2$. For any $f \in C_0^\infty(M)$, let $f_n = f \prod_{i>n} h_i$. By (7.14),

$$\begin{aligned} \sigma(f_n^2) &\leq \frac{1 + n^{4+\delta}}{d-1} \sigma(|\nabla f^2|) + \frac{4(1 + n^{4+\delta})}{d-1} \|f\|_\infty^2 \sum_{i>n} \sigma_0(B_{2i^{-2}}(x_i) \setminus B_{i^{-2}}(x_i)) i^4 \\ &\leq \frac{1 + n^{4+\delta}}{d-1} \sigma(|\nabla f^2|) + cn^{-2d+9+\delta} \|f\|_\infty^2, \end{aligned} \tag{7.15}$$

for some $c > 0$ independent of n and f. Therefore,

$$\begin{aligned}\sigma(f^2) &\le \sigma(f_n^2) + \|f\|_\infty^2 \sum_{i>n} \sigma(B_{2i^{-2}}(x_i)) \\ &\le \frac{1+n^{4+\delta}}{d-1}\sigma(|\nabla f^2|) + cn^{-2d+9+\delta}\|f\|_\infty^2 + c_1\|f\|_\infty^2 \sum_{i>n} i^{4+\delta-2d} \\ &\le \frac{2(1+n^{4+\delta})}{d-1}\sqrt{\sigma(|\nabla f|^2)\sigma(f^2)} + c_2 n^{9+\delta-2d}\|f\|_\infty^2, \quad n \ge 1, f \in C_0^\infty(M),\end{aligned}$$

for some $c_1, c_2 > 0$. Then for any $r > 0$ and any $f \in C_0^\infty(M)$ with $\sigma(f^2) = 1$ and $\|f\|_\infty^2 \le r$, we have

$$\sigma(|\nabla f|^2) \ge \sup_{n\ge 1}\left[\frac{(d-1)(1-rc_2n^{9+\delta-2d})^+}{2(1+n^{4+\delta})}\right]^2 =: c(r) > 0.$$

Therefore, $\lambda_\sigma(r) \ge c(r) > 0$ for any $r > 0$, and the assertion follows by Theorem 7.1. $\square$

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