TIME–DEPENDENT DIFFUSION OPERATORS ON L^1

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Abstract. We study the Cauchy problem for time–dependent diffusion operators with singular coefficients on L^1-spaces induced by infinitesimal invariant measures. We give sufficient conditions on the coefficients such that the Cauchy–Problem is well-posed. We construct associated diffusion processes with the help of the theory of generalized Dirichlet forms. We apply our results in particular to construct a large class of Nelson–diffusions that could not been constructed before.

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Running Head: Time–dependent diffusion operators on L^1

Contents

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0. Introduction

The purpose of this paper is to propose a new method for the construction of diffusion processes generated by time-dependent differential operators of type

$$(0.1)\qquad Lu(t,x)=\sum_{i=1}^{d}a_{ij}(t,x)\partial_{ij}u(t,x)+\sum_{i=1}^{d}b_i(t,x)\partial_i u(t,x)\ ,$$

$u\in C_0^\infty((0,T)\times U)$, on arbitrary open subsets $U\subset\mathbb{R}^d$ and $T>0$ with merely measurable coefficients.

We will be in particular interested in the following class of examples arising from stochastic mechanics: Let $d\geq 3$, $(a_{ij}(x))$ be measurable and uniformly strictly elliptic and $V\in L^1_{loc}(U)$ be a real potential with $V^-\in L^{\frac{d}{2}}(U)\cup L^\infty(U)$. Sobolev's inequality then implies that the quadratic form $\sum_{i,j=1}^d\int a_{ij}\partial_i u\partial_j u\,dx+\int u^2V\,dx$, $u\in H^1_0(U)\cap L^2(U,V^+\,dx)$, is well-defined and bounded from below. Its complexification uniquely determines a self-adjoint operator $L^V_{\mathbb{C}}$ and the wave-function $\psi(t):=e^{itL^V_{\mathbb{C}}}\psi_0$ solves the Schrödinger equation $i\partial_t\psi=-L_{\mathbb{C}}\psi$, $\psi(0)=\psi_0$. One of the basic problems in stochastic mechanics then consists of constructing a time-inhomogeneous diffusion process (sometimes called Nelson–diffusion (cf. [C], [Nag])) whose generator (now written in divergence form) extends

$$(0.2)\qquad Lu(t,x)=\sum_{i,j=1}^{d}\partial_i\left(a_{ij}(x)\partial_j u(t,x)\right)+\sum_{i=1}^{d}b_i(t,x)\partial_i u(t,x)\ ,u\in C_0^\infty((0,T)\times U)\ ,$$

where $b_i(t,x)=2|\psi(t,x)|^{-2}(g(t,x)\cdot\partial_i(h(t,x)+g(t,x))+h(t,x)\cdot\partial_i(h(t,x)-g(t,x))$, $1\leq i\leq d$, $g(t,x)=\ \mathrm{Re}\ \psi(t,x)$ and $h(t,x)=\ \mathrm{Im}\ \psi(t,x)$. It is well–known that the construction of the above mentioned diffusion process causes serious technical difficulties in that the operator has only unbounded measurable coefficients, so that it is not possible to consider the Cauchy-problem on spaces of continuous functions or to try to solve the corresponding stochastic differential equation directly. Even worse, since $b_i\notin L^2_{loc}([0,T]\times U,dx\,dt)$ for any $T>0$, it is not possible to use classical perturbation theory of the Laplacian or to construct weak solutions of the corresponding stochastic differential equation by means of the Girsanov transform. Note that however, using integration by parts, it is quite easy to see that under the assumption above it follows that for arbitrary $0\leq s<t$ and $u\in C_0^\infty(U)$

$$(0.3)\qquad \int_U u(x)\rho(t,x)\,dx-\int_U u(x)\rho(s,x)\,dx=\int_s^t\int_U L_r u(x)\rho(r,x)\,dx\,dr\ ,$$

where $\rho(t,x):=|\psi(t,x)|^2$ (cf. Example 1.e)(1)). We will therefore solve the Cauchy-problem in L^1–spaces induced by the measures $\rho(t,x)\,dx$. Note that this cannot been achieved with standard techniques from the theory of evolution equations since the L^1–spaces will depend on time.

In the particular stationary case, that is, the case where the density $\rho(t,x)=\rho_0(x)$ does not depend on time, hence $b_i(t,x)=b_i(x)$ is time-independent too, the problem reduces to construct a time-homogeneous Markov-process on U with invariant measure $\rho_0\,dx$ whose generator extends

$$Lu(x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j u(x)) + \sum_{i=1}^d b_i(x)\partial_i u(x)\ ,$$

$u \in C_0^\infty(U)$. Note that (0.3) reduces to

$$\int_U Lu(x)\rho_0(x)\,dx = 0\ , u \in C_0^\infty(U)\ ,$$

so that in this case $\rho_0(x)\,dx$ is an infinitesimally invariant measure for L. Hence, instead of trying to solve the Cauchy-problem associated to L in spaces of continuous functions the problem is studied in $L^1(\rho_0\,dx)$. Once, the existence of a sub-Markovian C_0-semigroup on $L^1(\rho_0\,dx)$ is settled, an associated diffusion process can then be constructed using the theory of generalized Dirichlet forms (cf. [St1]). This program has been carried out in a very general context in [St2]. In particular, the well–posedness of the Cauchy-problem in $L^1(\rho_0\,dx)$ has been studied thoroughly. Moreover, the same techniques have been applied to obtain similar results for a corresponding class of infinite dimensional diffusion operators. All this can be seen as part of a more general program on L^p–analysis of finite and infinite dimensional diffusion operators as it is outlined in [R].

The main results of this paper now will be generalizations of [St2] to a general time-dependent setting. Above all we will reduce the time-inhomogeneous Cauchy-problem associated to L to a time-homogeneous one by addding time to the state-space and passing to the differential operator

$$\bar{L}u(t,x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j u(t,x)) + \sum_{i=1}^d b_i(t,x)\partial_i u(t,x) + \partial_t u(t,x)\ , \tag{0.4}$$

$u \in C_0^\infty((0,T)\times U)$. Note that passing from (0.2) to (0.4) corresponds to the well–known procedure of constructing a semigroup of kernels $(\bar{p}_t)$ (called the *forward space-time homogenization*) to a given propagator of kernels $(p_{s,t})_{0\le s\le t}$ (that is, a family of kernels on a measurable space $(X,\mathcal{B})$ satisfying $p_{t,t}(x,\cdot) = \delta_x$, $x \in X$, $t \ge 0$, and the Chapman-Kolmogorov equation $p_{r,s}\circ p_{s,t} = p_{r,t}$, $0 \le r \le s \le t$) by defining $\bar{p}_t((r,x),\cdot) = \delta_{r+t}\otimes p_{r,r+t}(x,\cdot)$, $r,t \ge 0$, $x \in X$ (cf. [Sh, Section 16]). It is now easy to see that we have reduced the problem formally to a situation similar to the stationary case, since (0.3) immediately implies that

$$\int_0^T \int_U \bar{L}u(t,x)\rho(t,x)\,dx\,dt = 0\ ,\ \text{for all } u \in C_0^\infty((0,T)\times U)\ .$$

The disadvantage of this procedure consists in the fact that $\bar{L}$ is now a second-order differential operator with degenerate diffusion matrix. To choose the forward space–time homogenization, instead of the backward one, is arbitrary in our framework. Our choice is the usual one in stochastic analysis (cf. [StrV]). In particular, it corresponds to the standard formulation of the time–dependent Ito-formula.

Going back again to the general case we therefore assume the existence of a nonnegative measure $\overline{\mu} = \int_0^T \mu_t \, dt$ satisfying

$$\int_U u \, d\mu_t - \int_U u \, d\mu_s = \int_s^t \int_U L_r u \, d\mu_r \, dr \ , u \in C_0^\infty(U) \ . \tag{0.5}$$

A measure $\overline{\mu}$ satisfying (0.5) is called infinitesimal invariant for the operator L. We will specify exact conditions on a_{ij}, b_i and $\overline{\mu}$ later on that imply in particular that the integral in (0.5) is well-defined.

We are now mainly concerned with the following three problems:

(a) *Existence:* Give conditions on the coefficients a_{ij} and b_i of the operator L and on the measure $\bar{\mu}$ such that there exists a closed extension of $(L, C_0^\infty((0,T) \times U))$ in $L^1((0,T) \times U, \overline{\mu})$ generating a C_0–semigroup $(\overline{T}_t)_{t \geq 0}$ of (bounded) operators on $L^1((0,T) \times U, \overline{\mu})$ (henceforth such extensions will be called *maximal*) that are *sub–Markovian* (that is, $0 \leq \overline{T}_t u \leq 1$ if $0 \leq u \leq 1$) (cf. Subsection 1.a)).

(b) *Associated Processes:* Construct associated diffusion processes with the help of the theory of generalized Dirichlet forms and identify the associated processes as solutions of the martingale problem associated to $(L, C_0^\infty((0,T) \times U))$ (cf. Subsection 1.b)).

(c) *Uniqueness (in the case $U = \mathbb{R}^d$):* Give conditions on a_{ij}, b_i and $\overline{\mu}$ such that there exists only one maximal extension in $L^1((0,T) \times U, \overline{\mu})$ (cf. Subsection 1.c)).

In Subsection 1.d) we give several Examples. Besides the one mentioned above arising from stochastic mechanics (cf. Example 1.d)(2)) we will consider also time–dependent diffusion operators in divergence form (cf. Example 1.d)(1)). In particular, we will specify conditions on the coefficients that ensure the existence of a nontrivial infinitesimal invariant measure $\overline{\mu}$ satisfying (0.5) for a given L.

Sections 2–4 contain the proofs of the main theorems.

1. Framework and Main results

a) Existence.

Let us first introduce some notations. Fix $T > 0$. For an arbitrary subset U of $\mathbb{R}^d$ let $U_0 := (0,T] \times U$ and $U_T := [0,T) \times U$. Let $L^p(U)$, $p \in [1,\infty]$, be the usual L^p–spaces w.r.t. the Lebesgue measure, $H^1(U) = \{u \in L^2(U) | \partial_i u \in L^2(U), 1 \leq i \leq d\}$ be the Sobolev space in $L^2(U)$ of order 1 with Neumann boundary conditions and $H_0^1(U)$ be the Sobolev space of order 1 with Dirichlet boundary conditions, that is, the closure of $C_0^\infty(U)$ in $H^1(U)$. Furthermore, for an arbitrary Banach space E and $p \in [1,\infty]$ let $L^p([0,T]; E)$ be the space of all strongly measurable $u : [0,T] \to E$ with $\|u(\cdot)\|_E \in L^p([0,T])$. In the particular case $E = L^2(U)$ we will sometimes identify the two spaces $L^2([0,T]; L^2(U))$ and $L^2(U_T)$, so that any subspace of $L^2([0,T]; L^2(U))$ can be identified with a subspace of $L^2(U_T)$.

Throughout the whole Section fix an arbitrary nonempty open subset $U \subset \mathbb{R}^d$ and a σ–finite positive measure $\overline{\mu}$ on $\mathcal{B}(U_T)$. Denote by $\| \cdot \|_p$, $p \in [1,\infty]$, the usual norm on $L^p(U_T, \overline{\mu})$. If $W \subset L^p(U_T, \overline{\mu})$ is an arbitrary subset, denote by W_0 the subset of all elements $u \in W$ such that supp $(|u|\overline{\mu})$ is a compact subset contained in U_T, and by W_b the subset of all bounded elements in W. Finally, let $W_{0,b} := W_0 \cap W_b$.

Suppose that $\overline{\mu} \ll dx$ and that the density of $\overline{\mu}$ admits a representation φ^2, where φ is contained locally in the space $L^2([0,T];H_0^1(U))$, in the sense that $\varphi\chi \in L^2([0,T];H_0^1(U))$ for all $\chi \in C_0^\infty(U)$.

For any open subset V of U let $(\mathbb{D}, H_0^{0,1}(V_T,\overline{\mu}))$ be the closure of the bilinear form

$$\int_0^T \int \langle \nabla u(t,\cdot), \nabla v(t,\cdot)\rangle \, d\mu_t \, dt \, ; u, v \in C_0^\infty(V_T) \ ,$$

in $L^2(V_T,\overline{\mu})$, where ∇ denotes the gradient w.r.t. the space variables x_i, $1 \le i \le d$. Let $H_{loc}^{0,1}(V_T,\overline{\mu})$ be the space of all elements such that $u\chi \in H_0^{0,1}(V_T,\overline{\mu})$ for all $\chi \in C_0^\infty(V_T)$.

For $A = (a_{ij})_{1\le i,j\le d}$ and $B = (b_i)_{1\le i\le d}$, $a_{ij}, b_i : U_T \to \mathbb{R}$ measurable, let

$$L_{A,B}u := \sum_{i,j=1}^d a_{ij}(t,x)\partial_{ij}u(t,x) + \sum_{i=1}^d b_i(t,x)\partial_i u(t,x) \ .$$

Let us now specify conditions on A and B that will imply the existence of a maximal extension of $L_{A,B} + \partial_t$ generating a sub-Markovian C_0-semigroup in $L^1(\bar{\mu})$. For all V relatively compact in U we assume that

$$\partial_j a_{ij} \in L^2(V_T,\overline{\mu}) \, , 1 \le i,j \le d, \tag{1.1}$$

$$\nu_V^{-1}|h|^2 \le \langle A(t,x)h,h\rangle \le \nu_V|h|^2 \text{ for all } h \in \mathbb{R}^d, (t,x) \in V_T \ , \tag{1.2}$$

for some positive constant $\nu_V > 0$ and

$$B \in L^2(V_T;\mathbb{R}^d,\overline{\mu}) \ . \tag{1.3}$$

Finally we assume that

$$\int u \, d\mu_t - \int u \, d\mu_s = \int_s^t \int L_{A,B,r}u \, d\mu_r \, dr \text{ for all } u \in C_0^\infty(U), \tag{1.4}$$

where $L_{A,B,r}u(x) := \sum_{i,j=1}^d a_{ij}(r,x)\partial_{ij}u(x) + \sum_{i=1}^d b_i(r,x)\partial_i u(x)$, $u \in C_0^\infty(U)$. Note that (1.2) clearly implies that the bilinear form $\mathcal{A}^0(u,v) = \int \langle A\nabla u, \nabla v\rangle \, d\overline{\mu}$; $u, v \in H_0^{0,1}(V_T,\overline{\mu})$, is well-defined.

(1.4) has two immediate consequences. First,

$$\int (L_{A,B} + \partial_t)u \, d\overline{\mu} = 0 \text{ for all } u \in C_0^\infty((0,T)\times U) \ , \tag{1.5}$$

i.e., $\overline{\mu}$ is an infinitesimally invariant measure for $L_{A,B} + \partial_t$. Second, the operator $L_{A,2B^0-B} - \partial_t$ is the formal dual operator of $L_{A,B} + \partial_t$ w.r.t. the measure $\overline{\mu}$ in the sense that

$$\int (L_{A,B} + \partial_t)u \, v \, d\overline{\mu} = \int u \, (L_{A,2B^0-B} - \partial_t)v \, d\overline{\mu} \ ; u, v \in C_0^\infty((0,T)\times U) \ . \tag{1.6}$$

Here, $B^0 = (b_1^0, \dots, b_d^0) \in L^2_{loc}(U_T, \overline{\mu})$ is defined by

$$b_i^0 = \sum_{j=1}^{d} (\partial_j a_{ij} + a_{ij} 2\partial_j \varphi/\varphi)\,, 1 \le i \le d\,.$$

It is general theory of diffusion operators that (1.5) implies that $L_{A,B}u + \partial_t u$, $u \in C_0^\infty((0,T)\times U)$, is dissipative on all $L^p(\overline{\mu})$–spaces, $p \in [1,2]$, in particular negative definite on $L^2(\overline{\mu})$. Hence, there exist extensions of this operator in $L^2(\overline{\mu})$ that generate C_0–semigroups. Clearly, these extensions will never be unique, since we are considering a bounded domain. We will describe in the next step extensions of $L_{A,B} + \partial_t$ and of $L_{A,2B^0-B} - \partial_t$ that are still dual w.r.t. $\overline{\mu}$ and that generate sub-Markovian C_0–semigroups on all L^p–spaces.

To this end let

$$\beta := B - B^0$$

and note that (1.4) implies

$$\int u\, d\mu_t - \int u\, d\mu_s = \int_s^t \int \langle \beta, \nabla u\rangle\, d\mu_r\, dr \text{ for all } u \in C_0^\infty(U), \tag{1.7}$$

since $\int L_{A,B,r} u\, d\mu_r = \int \langle \beta(r,\cdot), \nabla u\rangle\, d\mu_r$ for all $u \in C_0^\infty(U)$.

If V is an arbitrary open subset of U, let $H_0^{1,1}(V_T, \overline{\mu})$ be the closure of $C_0^\infty(V_T)$ in $L^2(V_T, \overline{\mu})$ w.r.t. the norm $\left(\int u^2 + |\overline{\nabla} u|^2\, d\overline{\mu}\right)^{\frac{1}{2}}$. Here, $\overline{\nabla}$ denotes the gradient w.r.t. time– and space–variables. Similarly, let $H_0^{1,1}(V_0, \overline{\mu})$ be the closure of $C_0^\infty(V_0)$ in $L^2(V_0, \overline{\mu})$ w.r.t. the norm $\left(\int u^2 + |\overline{\nabla} u|^2\, d\overline{\mu}\right)^{\frac{1}{2}}$. Let $\overline{\beta}(t,x) := (1, \beta(t,x))$.

Lemma 1.1. *Let V be open and relatively compact in U. Then:*

(i) $\int \langle \overline{\beta}, \overline{\nabla} u\rangle\, d\overline{\mu} \le 0$ *for all* $u \in H_0^{1,1}(V_T, \overline{\mu})$, $u \ge 0$.

(ii) $-\int \langle \overline{\beta}, \overline{\nabla} u\rangle\, d\overline{\mu} \le 0$ *for all* $u \in H_0^{1,1}(V_0, \overline{\mu})$, $u \ge 0$.

(iii) $\int \langle \overline{\beta}, \overline{\nabla} u\rangle v\, d\overline{\mu} = -\int \langle \overline{\beta}, \overline{\nabla} v\rangle u\, d\overline{\mu}$ *for all* $u \in H_0^{1,1}(V_T, \overline{\mu})_b$, $v \in H_0^{1,1}(V_0, \overline{\mu})_b$.

Proof. For the proof of (i) let $u \in C_0^\infty(V_T)$ and extend u as well as β and μ to $[0,\infty) \times U$ by 0. For $\varepsilon \in (0,1)$ let $\psi_\varepsilon \in C^1(\mathbb{R})$, $\psi_\varepsilon(t) = 0$ if $t \le 0$, $\psi_\varepsilon(t) = t$ if $t \ge \varepsilon$ and $0 \le \psi_\varepsilon'(t) \le \frac{2}{\varepsilon}$. Then (1.7) implies that

$$\begin{aligned}
\int \partial_t \psi_\varepsilon(u)\, d\overline{\mu} &= \lim_{h\to 0} \frac{1}{h} \int_0^T \int \psi_\varepsilon(u)(t+h,\cdot) - \psi_\varepsilon(u)(t,\cdot)\, d\mu_t\, dt \\
&= \lim_{h\to 0} -\frac{1}{h} \int_0^h \int \psi_\varepsilon(u)(t,\cdot)\, d\mu_t\, dt \\
&\quad - \frac{1}{h} \int_0^T \left(\int \psi_\varepsilon(u)(t+h,\cdot)\, d\mu_{t+h} - \int \psi_\varepsilon(u)(t,\cdot)\, d\mu_t \right) dt \\
&\le -\lim_{h\to 0} \frac{1}{h} \int_0^T \int_0^h \int \langle \beta(t+r,\cdot), \nabla \psi_\varepsilon(u)(t+r,\cdot)\rangle\, d\mu_{t+r}\, dr\, dt \\
&= -\int \langle \beta, \nabla \psi_\varepsilon(u)\rangle\, d\overline{\mu}\,,
\end{aligned}$$

i.e., $\int\langle\overline{\beta},\overline{\nabla}\psi_\varepsilon(u)\rangle\,d\overline{\mu}\le 0$. If $u\in H_0^{1,1}(V_T,\overline{\mu})$, $u\ge 0$, let $(u_n)\subset C_0^\infty(V_T)$ be converging to u in $H_0^{1,1}(V_T,\overline{\mu})$. Then $\lim_{n\to\infty}\psi_\varepsilon(u_n)=\psi_\varepsilon(u)$ in $H_0^{1,1}(V_T,\overline{\mu})$ and thus $\int\langle\overline{\beta},\overline{\nabla}\psi_\varepsilon(u)\rangle\,d\overline{\mu}\le 0$. Since $\lim_{\varepsilon\to 0}\psi_\varepsilon(u)=u$ in $H_0^{1,1}(V_T,\overline{\mu})$ we finally conclude $\int\langle\overline{\beta},\overline{\nabla}u\rangle\,d\overline{\mu}\le 0$.

(ii) can be shown analoguously. For the proof of (iii) note that $uv\in H_0^{1,1}(V_T,\overline{\mu})\cap H_0^{1,1}(V_0,\overline{\mu})$ so that (i) and (ii) imply

$$\int\langle\overline{\beta},\overline{\nabla}u\rangle v\,d\overline{\mu}+\int\langle\overline{\beta},\overline{\nabla}v\rangle u\,d\overline{\mu}=\int\langle\overline{\beta},\overline{\nabla}(uv)\rangle\,d\overline{\mu}=0\ ,$$

if $u,v\ge 0$. The general case now follows from considering the cases u^+v^+, u^+v^-, u^-v^+ and u^-v^- separately. □

Let us state next some general results concerning diffusion operators on L^p-spaces. Let

$$\mathcal{M}:=\{\varphi\in C_b^1(\mathbb{R})|\varphi(0)=0\,,\varphi\text{ monotone increasing }\}\ .$$

Definition 1.2. Let (X,m) be a σ–finite measure space. A linear operator (A,D) on $L^1(X,m)$ is called an operator of *diffusion type* if $\int Au\varphi(u)\,dm\le 0$ for all $u\in D(A)$, $\varphi\in\mathcal{M}$.

Proposition 1.3. *Let (A,D) be a linear operator of diffusion type on $L^1(X,m)$. Then:*

(i) (A,D) is dissipative, in particular closable.

(ii) The closure $(\overline{A},D(\overline{A}))$ generates a C_0–semigroup $(T_t)_{t\ge 0}$ if and only if $(\lambda-A)(D)\subset L^1(X,m)$ dense for some $\lambda>0$. In this case, (T_t) is a sub–Markovian contraction semigroup. In particular, $T_{t|L^1\cap L^\infty}$ can be uniquely extended to a sub–Markovian contraction on T_t^p on $L^p(X,m)$ for all $t\ge 0$, $p\in[1,\infty]$. Moreover, (T_t^p) is a C_0–semigroup of contractions on $L^p(X,m)$ for all $p\in[1,\infty)$.

Proof. (i) Let $\psi_\varepsilon\in\mathcal{M}$, $\varepsilon>0$, be such that $\psi_\varepsilon(t)=\mathrm{sign}(t)$ if $|t|\ge\varepsilon$. Then

$$\int Au\left(1_{\{u>0\}}-1_{\{u<0\}}\right)\,dm=\lim_{\varepsilon\to 0}\int Au\psi_\varepsilon(u)\,dm\le 0\ .$$

Since $\|u\|_1(1_{\{u>0\}}-1_{\{u<0\}})\in L^\infty(X,m)=L^1(X,m)'$ is a normalized tangent functional to u we obtain the assertion.

(ii) By [ReSi, Th. X.48] the closure $(\overline{A},D(\overline{A}))$ generates a C_0–semigroup of contractions $(T_t)_{t\ge 0}$ if and only if $(\lambda-A)(D)\subset L^1(X,m)$ dense for some $\lambda>0$. In this case let $(G_\alpha)_{\alpha>0}$ be the resolvent associated with $(\overline{A},D(\overline{A}))$. We will show that $(G_\alpha)_{\alpha>0}$ is a sub–Markovian resolvent. Since $T_tu=\lim_{\alpha\to\infty}\exp(t\alpha(\alpha G_\alpha-1))u$ for all $u\in L^1(X,m)$ (cf. [Pa, 1.3.5]) it then follows that $(T_t)_{t\ge 0}$ is sub–Markovian too.

To this end let $\varphi_\varepsilon\in\mathcal{M}$, $\varepsilon>0$, be such that $\varphi_\varepsilon(t)=0$ if $t\le 1$, $\varphi_\varepsilon(t)=1$ if $t\ge 1+\varepsilon$. Then

$$\int Au1_{\{u>1\}}\,dm=\lim_{n\to\infty}\int Au\varphi_\varepsilon(u)\,dm\le 0$$

for all $u \in D(\overline{A})$.

Let $f \in L^1(X,m)$ and $u := \alpha G_\alpha f \in D(\overline{A})$. If $f \le 1$ then

$$\alpha \int u 1_{\{u>1\}}\, d\mu \le \int (\alpha u - \overline{A}u) 1_{\{u>1\}}\, d\mu = \alpha \int f 1_{\{u>1\}}\, d\mu \le \alpha \int 1_{\{u>1\}}\, d\mu \,.$$

Consequently, $\alpha \int (u-1)1_{\{u>1\}}\, d\mu \le 0$ which implies that $u \le 1$. If $f \ge 0$ then $-nf \le 1$, hence $-nu \le 1$ for all n, i.e., $u \ge 0$. Hence $(G_\alpha)_{\alpha>0}$ is sub–Markovian.

By the Riesz–Thorin Interpolation Theorem (cf. [ReSi, Th IX.17]) we conclude that $T_{t|L^1\cap L^\infty}$ can be uniquely extended to a contraction T_t^p on $L^p(X,m)$ for all $p \in [1,\infty]$. Clearly, $(T_t^p)_{t\ge 0}$ is again a C_0–semigroup of sub–Markovian contractions on $L^p(X,m)$ for $p \in [1,\infty)$. □

Definition 1.4. Let (A,D) be a linear operator of diffusion type on $L^1(X,m)$.

(i) (A,D) is called *maximal*, if $(\alpha - A)(D) = L^1(X,m)$ for one (hence all) $\alpha > 0$.

(ii) (A,D) is called L^1*–unique*, if there is only one extension of (A,D) on $L^1(X,m)$ that generates a C_0–semigroup.

Remark. (i) It follows from [Ar,Th. A–II, 1.33] that if (A,D) is L^1–unique, the unique extension $(\overline{A},\overline{D})$ of (A,D) generating a C_0–semigroup is obtained as the closure of (A,D). Equivalently, (A,D) is L^1–unique if and only if $(\alpha - A)(D) \subset L^1(X,m)$ dense for some $\alpha > 0$.

(ii) Let (A,D) be L^1–unique and $(\overline{A}, D(\overline{A}))$ be the closure in $L^1(X,m)$. By 1.3 (ii) the semigroup generated by $\overline{A}$ uniquely induces C_0–semigroups on $L^p(X,m)$, $p \in [1,\infty)$. The corresponding infinitesimal generators $(A_p, D(A_p))$ are called the L^p*–realization of* (A,D).

For the statement of the next Proposition recall that for a linear operator $T : D(T) \subset X \to Y$ with $X \subset Y$ its part on X is the linear operator $S : D(S) \subset X \to X$ defined by $Su := Tu$ for all $u \in D(S) := \{u \in D(T) | Tu \in X\}$.

For any open subset $V \subset U$ let us introduce the following norm $|\cdot|_V$ on $L^1(V_T,\overline{\mu})$:

$$|f|_V := \sup_{\substack{v \in H_0^{0,1}(V_T,\overline{\mu}) \\ \|v\|_\infty + \mathcal{A}^0(v,v)^{\frac{1}{2}} \le 1}} \int f v \, d\overline{\mu} \,. \tag{1.8}$$

Clearly, $|f|_V \le \|f\|_1$ and $L^1(V_T,\overline{\mu})$ is not complete w.r.t. $|\cdot|_V$. In the following denote by $\overline{L}^1(V_T,\overline{\mu})$ the completion of $L^1(V_T,\overline{\mu})$ w.r.t. $|\cdot|_V$.

Proposition 1.5. *Let $V \subset U$ be an open subset with $\partial V \in C^\infty$ relatively compact in U. Let $L := L_{A,B} + \partial_t$ and $L' := L_{A,2B^0-B} - \partial_t$. Then:*

(i) $L : C_0^\infty(V_T) \subset \overline{L}^1(V_T,\overline{\mu}) \to \overline{L}^1(V_T,\overline{\mu})$ (resp. $L' : C_0^\infty(V_0) \subset \overline{L}^1(V_T,\overline{\mu}) \to \overline{L}^1(V_T,\overline{\mu})$) is closable and the part $(\overline{L}^V, D(\overline{L}^V))$ (resp. $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$) in $L^1(V_T,\overline{\mu})$ of its closure is a maximal linear operator of diffusion type.

(ii) *For all* $u \in D(\overline{L}^V)_b$ *(resp.* $u \in D(\overline{L}^{V,\prime})_b$*) there exist* $(u_n) \subset C_0^\infty(V_T)$ *(resp.* $(u_n) \subset C_0^\infty(V_0)$*) such that* $\|u_n\|_\infty \le \|u\|_\infty + 2$ *and* $\lim_{n\to\infty} \|u_n - u\|_1 + \|Lu_n - \overline{L}^V u\|_{\overline{L^1}(V_T,\overline{\mu})} = 0$ *(resp.* $\lim_{n\to\infty} \|u_n - u\|_1 + \|L'u_n - \overline{L}^{V,\prime} u\|_{\overline{L^1}(V_T,\overline{\mu})} = 0$*).*

(iii) *(a)* $D(\overline{L}^V)_b \subset H_0^{0,1}(V_T,\overline{\mu})$ *(resp.* $D(\overline{L}^{V,\prime})_b \subset H_0^{0,1}(V_T,\overline{\mu})$*) and* $\mathcal{A}^0(u,u) \le -\int \overline{L}^V u\, u\, d\overline{\mu}$, $u \in D(\overline{L}^V)_b$ *(resp.* $\mathcal{A}^0(u,u) \le -\int \overline{L}^{V,\prime} u\, u\, d\overline{\mu}$, $u \in D(\overline{L}^{V,\prime})_b$*).*

(b) $D(\overline{L}^V)_b$ *(resp.* $D(\overline{L}^{V,\prime})_b$*) is an algebra and* $\overline{L}^V u^2 = 2u\overline{L}^V u + 2\langle A\nabla u, \nabla u\rangle$, $u \in D(\overline{L}^V)_b$ *(resp.* $\overline{L}^{V,\prime} u^2 = 2u\overline{L}^{V,\prime} u + 2\langle A\nabla u, \nabla u\rangle$, $u \in D(\overline{L}^{V,\prime})_b$*).*

(c) $\int \overline{L}^V u\, v\, d\overline{\mu} = \int u \overline{L}^{V,\prime} v\, d\overline{\mu}$ *for all* $u \in D(\overline{L}^V)_b$, $v \in D(\overline{L}^{V,\prime})_b$.

The proof of 1.5 is given in Section 2.

For all open subsets V relatively compact in U with $\partial V \in C^\infty$ let $\overline{G}_\alpha^V := (\alpha - \overline{L}^V)^{-1}$ (resp. $\overline{G}_\alpha^{V,\prime} := (\alpha - \overline{L}^{V,\prime})^{-1}$), $\alpha > 0$, be the resolvent of $(\overline{L}^V, D(\overline{L}^V))$ (resp. $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$). If we define

$$\overline{G}_\alpha^V f := \overline{G}_\alpha^V (f 1_V) (\text{ resp. } \overline{G}_\alpha^{V,\prime} f := \overline{G}_\alpha^{V,\prime}(f 1_V)) \ , f \in L^1(U_T,\overline{\mu}) \ , \alpha > 0,$$

then $\overline{G}_\alpha^V$ (resp. $\overline{G}_\alpha^{V,\prime}$) can be extended to a sub–Markovian contraction on $L^1(U_T,\overline{\mu})$.

Lemma 1.6. *Let* V, W *be open with* $\partial V, \partial W \in C^\infty$ *and relatively compact in* U *such that* $V \subset W$. *Let* $\alpha > 0$ *and* $f \in L^2(U_T,\overline{\mu})$, $f \ge 0$. *Then* $\overline{G}_\alpha^V f \le \overline{G}_\alpha^W f$ *(resp.* $\overline{G}_\alpha^{V,\prime} f \le \overline{G}_\alpha^{W,\prime} f$*).*

Proof. We will prove $\overline{G}_\alpha^V f \le \overline{G}_\alpha^W f$ only. The dual statement can be shown similarly. Let $v_n \in C_0^\infty(V_T)$, $w_n \in C_0^\infty(W_T)$, $n \ge 1$, be such that $\lim_{n\to\infty}(1 - \overline{L}^V)v_n = f1_V$ in $\overline{L^1}(V_T,\overline{\mu})$ and $\lim_{n\to\infty}(1 - \overline{L}^W)w_n = f1_W$ in $\overline{L^1}(W_T,\overline{\mu})$. Let $\varphi(t) = \frac{t^2}{1+t^2} 1_{[0,\infty)}(t) \in \mathcal{M}$. Then $\varphi(v_n - w_n) \in H_T^{1,1}(V_T,\overline{\mu})$ since $0 \le \varphi(v_n - w_n) \le v_n$ and $v_n \in H_T^{1,1}(V_T,\overline{\mu})$, $n \ge 1$. Now 1.1 implies that

$$\begin{aligned} \mathcal{A}_1^0(\varphi(v_n - w_n), \varphi(v_n - w_n)) &\le \mathcal{A}_1(v_n - w_n, \varphi(v_n - w_n)) \\ &\le \int (1 - L)(v_n - w_n)\varphi(v_n - w_n)\, d\overline{\mu}\ , \end{aligned} \tag{1.9}$$

and consequently, $(\varphi(v_n - w_n))_{n\ge1} \subset H_0^{0,1}(V_T,\overline{\mu})_b$ bounded. We can now find a subsequence $(\varphi(v_{n_k} - w_{n_k}))_{k\ge1}$ converging to $\varphi(\overline{G}_1^V f - \overline{G}_1^W f)$ weakly* both in $\overline{L^1}(V_T,\overline{\mu})'$ and $\overline{L^1}(W_T,\overline{\mu})'$. Taking the limit $n \to \infty$ in (1.9) one obtains that

$$\mathcal{A}_1^0(\varphi(\overline{G}_1^V f - \overline{G}_1^W f), \varphi(\overline{G}_1^V f - \overline{G}_1^W f)) \le \int (f1_V - f1_W)\varphi(\overline{G}_1^V f - \overline{G}_1^W f)\, d\overline{\mu} \le 0,$$

thus $\varphi(\overline{G}_1^V f - \overline{G}_1^W f) = 0$, i.e., $\overline{G}_1^V f \le \overline{G}_1^W f$, since $\varphi(t) > 0$ if $t > 0$. $\square$

For the statement of the main result let $(\mathcal{A}^0, D(\mathcal{A}^0))$ be the closure of the quadratic form $\int \langle A\nabla u, \nabla u\rangle\, d\overline{\mu}$, $u \in C_0^\infty(U_T)$, in $L^2(U_T,\overline{\mu})$.

Theorem 1.7. *Let (1.1)-(1.4) be satisfied. Then there exists a maximal extension $(\overline{L}, D(\overline{L}))$ of diffusion type of $L_{A,B}u + \partial_t u$, $u \in C_0^\infty(U_T)$, on $L^1(U_T, \overline{\mu})$ satisfying the following properties:*

(a) *If $(U^n)_{n\geq 1}$ is an increasing sequence of open subsets with $\partial U^n \in C^\infty$ such that $U = \bigcup_{n\geq 1} U^n$ then $\lim_{n\to\infty} \overline{G}_\alpha^{U^n} f = (\alpha - \overline{L})^{-1} f$ in $L^1(U_T, \overline{\mu})$ for all $f \in L^1(U_T, \overline{\mu})$ and $\alpha > 0$.*

(b) *$D(\overline{L})_b \subset D(\mathcal{A}^0)$ and for all $u \in D(\overline{L})_b$, $v \in C_0^\infty(U_0)$*

$$\mathcal{A}^0(u, v) + \int \langle \overline{\beta}, \nabla v \rangle u \, d\overline{\mu} = -\int \overline{L} u \, v \, d\overline{\mu} \,. \tag{1.10}$$

Moreover, $\mathcal{A}^0(u, u) \leq -\int \overline{L} u \, u \, d\overline{\mu}$ for all $u \in D(\overline{L})_b$.

(c) *$D(\overline{L})_b$ is an algebra and $\overline{L} u^2 = 2u\overline{L}u + 2\langle A\nabla u, \nabla u \rangle$, $u \in D(\overline{L})_b$.*

The proof of 1.7 is given in Section 2.

Definition 1.8. Let $p \in [1, \infty]$. A semigroup (T_t) on $L^p([0, T) \times X, m)$ is called a *semigroup of evolution type* if $T_t((f(\cdot - t)1_{[t,T)})g) = fT_t g$ for all $f \in \mathcal{B}_b([0, T))$, $g \in L^p([0, T) \times X, m)$ and $t \geq 0$.

Remark. Note that our definition of a semigroup of evolution type is different from the one given in [Ne]. This is due to the fact that in our setting we consider the *forward* space–time homogenization of a given time–dependent diffusion operator wheras in [Ne] one considers the *backward* space–time homogenization of a given propagator.

Proposition 1.9. *Let $(\overline{L}, D(\overline{L}))$ be the maximal extension of $(L_{A,B} + \partial_t, C_0^\infty(U_T))$ as specified in 1.7 and $(\overline{T}_t)_{t\geq 0}$ be the corresponding semigroup. Then:*

(i) *$fg \in D(\overline{L})$ and $\overline{L}(fg) = \dot{f}g + f\overline{L}g$ if $f \in C^1([0, T])$, $g \in D(\overline{L})$.*

(ii) *$(\overline{T}_t)_{t\geq 0}$ is of evolution type.*

Proof. (i) By taking appropriate limits we may assume $f \in C^\infty([0, T])$. Let $g = \overline{G}_1 h$, $h \in L^1(U_T, \overline{\mu})$, and $(U^n)_{n\geq 1}$ be an increasing sequence of open subsets as in 1.7 (a). Let $g_n := \overline{G}_1^{U^n} h$.

Claim: Let $n \in \mathbb{N}$. Then $fg_n \in D(\overline{L}^{U^n})$ and $\overline{L}^{U^n}(fg_n) = \dot{f}g + f\overline{L}^{U^n} g$.

Proof: Let $u_m \in C_0^\infty(U_T^n)$, $m \geq 1$, be such that $\lim_{m\to\infty} \|u_m - g_n\|_1 + |Lu_m - \overline{L}^{U^n} g_n|_{U^n} = 0$. Then $fu_m \in C_0^\infty(U_T^n)$ for all m and $\overline{L}^{U^n} fu_m = \dot{f}g + f\overline{L}^{U^n} g$. Note that $\lim_{m\to\infty} |\dot{f}u_m + f\overline{L}^{U^n} u_m - \dot{f}g_n - f\overline{L}^{U^n} g_n|_{U^n} \to 0$. Since $\lim_{m\to\infty} |fu_m - fg_n|_{U^n} = 0$ too, it follows that fg_n is an element of the closure $(A, D(A))$ of $(L, C_0^\infty(U_T^n))$ in $\overline{L^1}(U_T^n, \overline{\mu})$ and $A(fg_n) = \dot{f}g + f\overline{L}^{U^n} g$. $\dot{f}g + f\overline{L}^{U^n} g \in L^1(U_T, \overline{\mu})$ now implies that fg_n is an element of the part of $(A, D(A))$ in $L^1(U_T^n, \overline{\mu})$, hence the assertion follows.

The claim implies in particular that $fg_n = \overline{G}_1^{U^n}(-\dot{f}g_n + f(h1_{U_T^n}))$. Since $\lim_{n\to\infty} \|\dot{f}g_n + f(h1_{U_T^n})) + \dot{f}g - fh\|_{L^1(U_T,\overline{\mu})} = 0$ we conclude from 1.7 (a) that $fg = \lim_{n\to\infty} fg_n = \lim_{n\to\infty} \overline{G}_1^{U^n}(-\dot{f}g_n + f(h1_{U_T^n})) = \overline{G}_1(-\dot{f}g + fh)$ which implies the assertion.

(ii) We follow the proof of the corresponding statement in the theory of evolution semigroups (cf. [Ne]). By taking appropriate limits it is clearly enough to consider $f \in C_0^1((0,T])$ and $g \in D(\overline{L})$. Let $f_{-t}(s) := f(s-t)1_{[t,T)}(s)$, $t \ge 0$, and note that $f_{-t} \in C_0^1((0,T])$ again. For arbitrary $g \in L^1(U_T, \overline{\mu})$ define $\phi_f(g) : [0,T] \to L^1(U_T,\overline{\mu})$, $t \mapsto \overline{T}_t(f_{-t}g) - f\overline{T}_t g$. If $g \in D(\overline{L})$ (i) now implies that $\phi_f(g)$ is differentiable and $\dot{\phi}_f(g)(t) = \phi_f(\overline{L}g)$. Fix $t_0 > 0$ and define $\psi : [0,t_0] \to L^1(U_T,\overline{\mu})$, $t \mapsto \phi_f(\overline{T}_t g)(t_0 - t)$. Then ψ is differentiable and $\dot{\psi}(t) = -\dot{\phi}_f(\overline{T}_t g)(t_0 - t) + \phi_f(\overline{LT}_t g)(t_0 - t) = 0$. Consequently, $0 = \psi(t_0) = \psi(0) = \overline{T}_{t_0}(f_{-t_0}g) - f\overline{T}_{t_0}g$ which implies the assertion. $\square$

b) Construction of associated diffusion processes.

Once a maximal extension of diffusion type $(\overline{L}, D(\overline{L}))$ of $(L, C_0^\infty(U_T))$ is constructed, we now ask for the existence of an associated Markov process whose transition probabilities are given by the semigroup $(\overline{T}_t)$ generated by $\overline{L}$. A suitable technical tool for the construction of reasonable Markov processes to generators of sub–Markovian semigroups on L^2–spaces is provided by the *theory of generalized Dirichlet forms* (cf. [St1]). We intend to apply the main existence theorem ([St1, Th. IV.2.2]) to the generalized Dirichlet form induced by the L^2–realization $(L, D(L))$ of $(\overline{L}, D(\overline{L}))$ (cf. [St1, I.4.9 (ii)]). To state our main result properly we need to introduce some potential theoretic notions related to $(L, D(L))$. For an element $f \in L^2(U_T,\overline{\mu})$ let $\mathcal{L}_f := \{g \in L^2(U_T,\overline{\mu}) | g \ge f\}$. Let $G_\alpha := (\alpha - L)^{-1}$, $\alpha > 0$, be the resolvent of L. An element $f \in L^2(U_T,\overline{\mu})$ is called 1–excessive (w.r.t. (G_α)) if $\beta G_{\beta+1} f \le f$ for all $\beta \ge 0$. If $f \in L^2(U_T,\overline{\mu})$ such that $\mathcal{L}_f \cap D(L) \neq \emptyset$ there exists a 1–excessive element $e_f \in \mathcal{L}_f$ such that $e_f \le g$ for all $g \in \mathcal{L}_f$, g 1–excessive (cf. [St1, III.1.7]). e_f is called the 1*–reduced function of* f. An increasing sequence of closed subsets $F_k \subset U_T$, $k \ge 1$, is called an L*–nest* if $e_{f1_{F_k^c}} \to 0$ in $L^2(U_T,\overline{\mu})$ for all $f \in D(L)$, f 1–excessive. A subset $N \in \mathcal{B}(U_T)$ is called L*–exceptional* if there exists an L–nest (F_k) such that $N \subset \bigcap_{k\ge1} U_T \setminus F_k$. A property of points in U_T is said to hold L*–quasi everywhere (L–q.e.)* if it holds for all points in the complement of some L–exceptional set. Finally, a function $f : U_T \to \mathbb{R}$ is called L*–quasi continuous (L–q.c.)* if there exists an L–nest (F_k) such that $f_{|F_k}$ is continuous for all k. Our main result on the existence of Markov processes associated with $\overline{L}$ (resp. its L^2–realization L) is then stated as follows:

Theorem 1.10. *There exists a $\overline{\mu}$–tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (Y_t)_{t\ge0}, (P_{(s,x)})_{(s,x)\in U_T\cup\{\Delta\}})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E_\cdot\left[\int e^{-\alpha t} f(Y_t)\, dt\right]$ is an L–q.c. $\overline{\mu}$–version of $(\alpha - L)^{-1} f$ for all $f \in \mathcal{B}_b(U_T) \cap L^1(U_T,\overline{\mu})$, $\alpha > 0$.*

The proof of 1.10 is given in Section 3. For the precise definition of a $\overline{\mu}$–tight special standard process we refer to [St1, Ch. IV]. However, we will show in the next Proposition that any Markov process as in 1.10 is in fact a diffusion.

Proposition 1.11. *Let $\mathbb{M}$ be as in 1.10. Then*

$$P_{(s,x)}\,[t \mapsto Y_t \textit{ is continuous on } [0,\zeta)] = 1 \qquad L\text{–}q.e.\ (\textit{hence } \overline{\mu}\text{–}a.e.)$$

The proof of 1.11 is given in Section 3.

Remark 1.12. The fact that the semigroup $(\overline{T}_t)$ generated by $\overline{L}$ is of evolution type easily implies that the first component of any Markov process $\mathbb{M}$ associated with $\overline{L}$ as in 1.10 is a uniform motion to the right in the sense that

$$P_{(s,x)}[Y_t^1 \neq s+t, \zeta > t] = 0 \ \overline{\mu}\text{–a.e.} \tag{1.11}$$

Indeed, let $\mathbb{Q} \cap [0,T) = \{q_n | n \geq 1\}$, $f_n(t) = |t - q_n|$ for $n \geq 1$, $f_0 \equiv 1$ and $(\chi_n) \subset C_0(U)$ be an increasing sequence of nonnegative functions with $1_U = \sup_{n\geq 1} \chi_n$. Let (p_t) be the transition semigroup of $\mathbb{M}$. Since $p_t f$ is a $\overline{\mu}$–version of $\overline{T}_t f$ for all $f \in \mathcal{B}_b(U_T) \cap L^1(U_T, \overline{\mu})$ and $(\overline{T}_t)$ is of evolution type it follows that

$$p_t(f_m(\cdot - t)1_{[t,T)}\chi_n)(s,x) = f_m(s)p_t\chi_n(x) \ \overline{\mu}\text{–a.e.}$$

for all $m \geq 0$, $n \geq 1$. Taking the limit $n \to \infty$ we obtain

$$p_t(f_m(\cdot - t)1_{[t,T)}1_U)(s,x) = f_m(s)p_t 1_U(x) \tag{1.12}$$

$\overline{\mu}$–a.e. for all $m \geq 0$. Hence there exists a $\overline{\mu}$–null set N such that (1.12) holds for all $(s,x) \in U_T \setminus N$ and for all m. In particular, $p_t(1_{[t,T)}1_U)(s,x) = p_t 1_U(x)$, hence $E_{(s,x)}[Y_t^1 < t, Y_t^2 \in U] = 0$ for all $(s,x) \in U_T \setminus N$. For arbitrary $(s,x) \in U_T \setminus N$ we conclude

$$\begin{aligned}
p_t(|\cdot - t - s|1_U)(s,x) &= p_t(|\cdot - t - s|1_{[t,T)}1_U)(s,x) \\
&= \lim_{\substack{q\to s \\ q\in\mathbb{Q}\cap[0,T)}} p_t(|\cdot - t - q|1_{[t,T)}1_U)(s,x) \\
&= \lim_{\substack{q\to s \\ q\in\mathbb{Q}\cap[0,T)}} |s-q|p_t 1_U(x) = 0 \ ,
\end{aligned}$$

thus $P_{(s,x)}[Y_t^1 \neq s+t, \zeta > t] = 0$.

Remark 1.13. (i) Note that any process $\mathbb{M}$ as in 1.10 solves the martingale problem associated to $(L, C_0^\infty(U_T))$ in the following sense: Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $\mathbb{M}$ and denote by ζ its lifetime. Then $u(Y_{t\wedge\zeta}) - u(Y_0) - \int_0^{t\wedge\zeta} Lu(Y_s)\,ds$, $t \geq 0$, is an $(\mathcal{F}_t)$–martingale under $P_{v\overline{\mu}} = \int P_x v(x)\,\overline{\mu}(dx)$ for all $v \in \mathcal{B}_b^+(U_T)$ such that $\int v\,d\overline{\mu} = 1$ and all $u \in C_0^\infty(U_T)$.

(ii) For results on solutions of the martingale problem in the particular case $U = \mathbb{R}^d$, $\mu_t(\mathbb{R}^d) = 1$ for all $t \in [0,T]$ and (a_{ij}) locally Hölder–continuous we refer to [CaL]. Note that in contrast to our framework, no absolute continuity of $(\mu_t)_{t\in[0,T]}$ is needed. On the other hand, in contrast to [CaL], we obtain the full Markov process associated to $(L, C_0^\infty(U_T))$ rather than just a solution to the martingale problem.

c) Uniqueness in the case $U = \mathbb{R}^d$.

Let $U = \mathbb{R}^d$. Throughout this subsection we assume in addition to assumptions (1.1)–(1.4) on $\overline{\mu}$, A and B that for all compact $V \subset \mathbb{R}^d$ there exist $L_V \geq 0$ and $\alpha_V \in (0,1)$ such that

$$|a_{ij}(s,x) - a_{ij}(t,y)| \leq L_V \left(|x-y|^{\alpha_V} + |s-t|^{\frac{\alpha_V}{2}} \right) \tag{1.13}$$

for all $s,t \in [0,T]$, $x,y \in V$. The following is the main regularity result on which our L^1–uniqueness result is based on.

Theorem 1.14. *Let $(a_{ij})_{1\leq i,j\leq d}$ satisfy (1.13) and $\alpha_0 > 0$. Let $h \in \mathcal{B}(\mathbb{R}^d_T)$, h locally bounded, $h^+ \in \mathcal{B}_b(\mathbb{R}^d_T)$, be such that $\int (\alpha_0 - L_{A,B} - \partial_t) u\, h\, d\overline{\mu} \leq 0$ for all $u \in C_0^\infty(\mathbb{R}^d_T)$, $u \geq 0$. Then $h^+ \in D(\mathcal{A}^0)_{loc}$ and*

(1.14)

$$\int \langle A\nabla\chi h^+, \nabla\chi h^+\rangle + \alpha_0 \chi^2 (h^+)^2\, d\overline{\mu} \leq \int \langle A\nabla\chi, \nabla\chi\rangle (h^+)^2\, d\overline{\mu}$$
$$+ \int \langle B - B^0, \nabla\chi\rangle \chi (h^+)^2\, d\overline{\mu} + \int \chi(\partial_t\chi)(h^+)^2\, d\overline{\mu}$$

for all $\chi \in C_0^\infty(\mathbb{R}^d_T)$.

The proof of 1.14 is given in Section 4.

Corollary 1.15. *Let (a_{ij}) satisfy (1.13). Then each of the following assumptions (a) and (b) imply that $(L_{A,B} + \partial_t, C_0^\infty(\mathbb{R}^d_T))$ is L^1–unique:*

(a) $a_{ij}, b_i - b_i^0 \in L^1(\mathbb{R}^d_T, \overline{\mu})$, $1 \leq i,j \leq d$.

(b) There exists $V \in C^{1,2}([0,T] \times \mathbb{R}^d)$, such that $\lim_{|x|\to\infty} V(t,x) = +\infty$ uniformly in t and $L_{A,2B^0-B} V - \partial_t V \leq \alpha_0 V$ for some $\alpha_0 > 0$.

Proof. Fix $\alpha_0 > 0$ as in (b) and let $h \in L^\infty(\mathbb{R}^d_T, \overline{\mu})$ be such that $\int (\alpha_0 - L_{A,B} - \partial_t) uh\, d\overline{\mu} = 0$ for all $u \in C_0^\infty(\mathbb{R}^d_T)$.

For the proof of (a) fix $\chi_n \in D(\mathcal{A}^0)_{loc}$, $0 \leq \chi_n \leq 1$, with $\chi_n \uparrow 1$, $\|\nabla\chi_n\|_\infty \leq \frac{c}{n}$ and $\partial_t \chi_n \leq 0$. 1.14 now implies that $h^+ \in D(\mathcal{A}^0)_{loc}$ and by (1.14)

$$\int \langle A\nabla\chi_n h^+, \nabla\chi_n h^+\rangle + \alpha_0 \chi_n^2 (h^+)^2\, d\overline{\mu} \leq \int \langle A\nabla\chi_n, \nabla\chi_n\rangle (h^+)^2\, d\overline{\mu}$$
$$+ \int \langle B - B^0, \nabla\chi_n\rangle \chi_n (h^+)^2\, d\overline{\mu}$$
$$\leq \frac{c}{n} \|h\|_\infty^2 \left(\frac{c}{n} \sum_{ij} \|a_{ij}\|_1 + \frac{c}{n} \||B - B^0|\|_1\right),$$

which implies $h^+ = 0$ taking the limit $n \to \infty$. Similarly, $h^- = 0$ hence the assertion.

For the proof of (b) first note that adding a positive constant if necessary we may suppose that $V \geq 0$. Let $\chi_n := h - \frac{V}{n}$. Then χ_n is locally bounded, χ_n^+ globally bounded and $\int(\alpha_0 - L_{A,B} - \partial_t)u\,\chi_n\,d\overline{\mu} \le 0$ for all $u \in C_0^\infty(\mathbb{R}_T^d)$, $u \ge 0$. Hence $\chi_n^+ \in D(\mathcal{A}^0)_{loc}$ by 1.14. Since $\mathrm{supp}\,(\chi_n^+\bar{\mu})$ is a compact subset of $\mathbb{R}_T^d$ there exists $\chi \in C_0^\infty(\mathbb{R}_T^d)$, $0 \le \chi \le 1$, such that $\chi \equiv 1$ on $\mathrm{supp}\,(\chi_n^+\overline{\mu})$ and thus by (1.14)

$$\int \langle A\nabla\chi_n^+, \nabla\chi_n^+\rangle + \alpha_0(\chi_n^+)^2\,d\overline{\mu} \le 0\ ,$$

i.e., $h \le \frac{V}{n}$. Since n was arbitrary we conclude that $h \le 0$. Similarly, $-h \le 0$ and consequently, $h = 0$. $\square$

d) Examples.

1) Time–dependent singular diffusion operators in divergence form

Let $U \subset \mathbb{R}^d$ be an arbitrary open subset and on U_T consider the following time-dependent diffusion operator in divergence form

$$Lu(t,x) = \sum_{i,j=1}^d \partial_i(a_{ij}(t,x)\partial_j u(t,x)) + \sum_{i=1}^d b_i(t,x)\partial_i u(t,x)\ .$$

(i) Time-dependent linear drifts

Let $B = (b_1, \dots, b_d)$ be a time–dependent linear drift, i.e., $B(t,x) = B(t)x$. This is a simple example, of course. However, we include it here to demonstrate the precise scope of our techniques. In the following we assume on the coefficients of L that for all V relatively compact in U

$$\partial_j a_{ij} \in L^2(V_T)\ , 1 \le i,j \le d\ , \tag{1.15}$$

$$\nu_V^{-1}|h|^2 \le \langle A(t,x)h,h\rangle \le \nu_V|h|^2\,; h \in \mathbb{R}^d, (t,x) \in V_T\ , \tag{1.16}$$

for some positive constant ν_V and

$$B_{ij} \in L^1([0,T])\ , 1 \le i \le d\ . \tag{1.17}$$

Let $\mu_t(dx) := \alpha(t)\,dt$, $t \in [0,T]$, where $\alpha(t) := \exp(-\int_0^t\ \mathrm{tr}(B(r))\,dr)$. If $u \in C_0^\infty(U)$ it is easy to see that

$$\begin{aligned}
\int_s^t\int L_r u\,d\mu_r\,dr &= \int_s^t\int \sum_{i,j=1}^d \partial_i(a_{ij}(r,x)\partial_j u(x)) + \langle B(r)x, \nabla u(x)\rangle\,\alpha(r)\,dx\,dr \\
&= -\int_s^t\int u\,dx\,\alpha(r)\ \mathrm{tr}(B(r))\,dr \\
&= \int_s^t\int u\,dx\frac{d\alpha}{dr}(r)\,dr = \int u\,dx(\alpha(t)-\alpha(s)) = \int u\,d\mu_t - \int u\,d\mu_s\ .
\end{aligned}$$

The results of Subsections 1.a)-c) now imply the following:

Theorem 1.16. *There exists a maximal extension of diffusion type $(\overline{L}, D(\overline{L}))$ of $(L + \partial_t, C_0^\infty(U_T))$ in $L^1(U_T, \alpha(t)\,dt \otimes dx)$ satisfying 1.7 (a)-(c). Moreover, there exists a strong Markov process $\mathbb{M}$ with life–time ζ on U_T whose transition probabilities are given by the semigroup generated by $(\overline{L}, D(\overline{L}))$. $\mathbb{M}$ is a diffusion in the sense that $P_{(s,x)}[t \mapsto Y_t$ is continuous on $[0,\zeta)] = 1$ $dt \otimes dx$–a.e. and $\mathbb{M}$ is a solution of the martingale problem to $(L + \partial_t, C_0^\infty(U_T))$ in the sense of 1.13. If $U = \mathbb{R}^d$, the (a_{ij}) are locally Hölder-continuous and satisfy (1.13) and for some positive constant M*

$$(1.18)\quad -2\frac{\langle A(t,x)x, x\rangle}{|x|^2+1} + tr(A(t,x)) + \langle B^0(t,x) - B(t)x, x\rangle \le M(|x|^2 \ln(|x|^2+1)+1)$$

(here $B_i^0(t,x) = \sum_{j=1}^d \partial_j a_{ij}(t,x)$, $1 \le i \le d$) it follows that $(L, C_0^\infty(\mathbb{R}^d_T))$ is L^1–unique, hence $(\overline{L}, D(\overline{L}))$ is given by the closure of $(L, C_0^\infty(\mathbb{R}^d_T))$ in $L^1(\mathbb{R}^d_T, \alpha(t)\,dt \otimes dx)$ (or equivalently in $L^1(\mathbb{R}^d_T)$).

Proof. Clearly, (1.15)–(1.17) imply (1.1)–(1.4) on $\overline{\mu}(dt,dx) = \alpha(t)\,dt\,dx$, A and B. Hence 1.7 implies the existence of $(\overline{L}, D(\overline{L}))$ and 1.10, 1.11 and 1.13 the existence of $\mathbb{M}$ with the stated properties. If (1.18) holds, let $V(t,x) := \ln(|x|^2+1)+1$. Then $L_{A,B^0-B}V - \partial_t V \le 2MV$, hence $Lu + \partial_t u = L_{A,B^0+B}u + \partial_t u$, $u \in C_0^\infty(\mathbb{R}^d_T)$, is L^1–unique by 1.15. $\square$

(ii) Time-dependent non-linear drifts

In the case of a time–dependent non–linear drift the first problem consists of constructing a nontrivial measure $\overline{\mu}$. We will do this in the following with the help of Fredholm perturbation theory. To this end let $U \subset \mathbb{R}^d$, $d \ge 3$, be a relatively compact subset with $\partial U \in C^\infty$.

Lemma 1.17. *$H^1([0,T]; H^{-1}(U)) \cap L^2([0,T]; H^1(U)) \hookrightarrow L^2(U_T)$ is compact.*

Proof. Let $(u_n) \subset H^{1,2}([0,T]; H^{-1}(U)) \cap L^2([0,T]; H^1(U))$ be a weakly convergent sequence, denote by u its limit and fix an orthonormal basis (e_n) of $H^1(U)$. Note that the trace theorem (cf. [LMa, I.3.1]) implies that each u_k admits a continuous version $\tilde{u}_{n_k} : [0,T] \to L^2(U)$ and for some positive constant c_0

$$(1.19)\qquad \sup_{t\in[0,T]} \|\tilde{u}_k(t)\|_2 \le c_0 \|\tilde{u}_k\|_{L^2([0,T];H^1(U))} + \|\frac{du_k}{dt}\|_{L^2([0,T];H^{-1}(U))}\ , k \ge 1\ .$$

Since $g_{n,m}(t) = \int u_n(t,\cdot)e_m\,dm \in H^1([0,T])$, $n,m \ge 1$, is bounded we can find by compactness of $H^1([0,T]) \hookrightarrow L^2([0,T])$ a subsequence (n_k) for which $\lim_{k\to\infty} g_{n_k,m} = g_m$ in $L^2([0,T])$ for all m. Here, $g_m(t) := \int u(t,\cdot)e_m\,dt$. Passing to a further subsequence if necessary we may assume in addition that $\lim_{k\to\infty} g_{n_k,m} = g_m$ a.e. for all m. Hence $u_{n_k}(t,\cdot) \to u(t,\cdot)$ weakly in $H^1(U)$ a.e. Since $H^1(U) \hookrightarrow L^2(U)$ is compact by the Rellich-Kondrachov theorem (cf. [Ad, Th. 6.2]), we obtain that $u_{n_k}(t,\cdot) \to u(t,\cdot)$ strongly in $L^2(U)$ a.e. Consequently,

$$\lim_{k\to\infty} \|u_{n_k} - u\|^2_{L^2(U_T)} = \int_0^T \|u_{n_k}(t,\cdot) - u(t,\cdot)\|^2_{L^2(U)}\,dt = 0$$

by Lebesgue's theorem and (1.19). $\square$

We will now assume that for all V relatively compact in U

$$\partial_j a_{ij} \in L^{\frac{d+2}{d}}([0,T]; L^{4\frac{d}{d+2}}(V)) \ , 1 \le i,j \le d \ , \tag{1.20}$$

$$\nu^{-1}|h|^2 \le \langle A(t,x)h,h\rangle \le \nu|h|^2 \, ; h \in \mathbb{R}^d, (t,x) \in U_T \ , \tag{1.21}$$

for some positive constant ν,

$$b_i = \tilde{b}_i + \hat{b}_i \text{ with } \tilde{b}_i \in L^\infty([0,T]; L^d(U)) \ , \hat{b}_i \in L^\infty(U_T) \ , 1 \le i \le d \ , \tag{1.22}$$

and for some positive constant c

$$-\frac{1}{2}\sum_{i=1}^d \int_U b_i(t,\cdot)\partial_i u \, dx + c\int_U u \, dx \ge 0; u \in H^1(U) \, , u \ge 0 \, , t \in [0,T]. \tag{1.23}$$

Proposition 1.18. *There exists a nontrivial measure $\overline{\mu}$ on $\mathcal{B}(U_T)$ such that its density φ^2 admits a representation $\varphi \in L^2_{loc}([0,T]; H^1_0(U))$ for which*

$$\int u \, d\mu_t - \int u \, d\mu_s = \int_s^t \int L_r u \, d\mu_r \, dr$$

for all $u \in C_0^\infty(U)$.

Proof. Modifying b_i on a set of Lebesgue-measure zero we may assume that $\sup_{t\in[0,T]} \|\tilde{b}_i(t,\cdot)\|_d + \|\hat{b}_i(t,\cdot)\|_\infty < +\infty$. By (1.21), (1.22), (1.23) and Sobolev's imbedding theorem (cf. [Ad, Th. 5.4]), the quadratic form

$$\mathcal{E}^{(t)}(u,v) = \sum_{i,j=1}^d \int a_{ij}\partial_i u \partial_j v \, dx - \sum_{i=1}^d \int b_i \partial_i uv \, dx + c\int uv \, dx \ ; u,v \in H^1(U) \ ,$$

is a positive definite closed sectorial form on $L^2(U)$. Let $(\frac{d}{dt}, H^1_{per}([0,T]; H^{-1}(U)))$ be the time–derivative on $L^2([0,T]; H^{-1}(U))$ with periodic boundary conditions, that is, the generator of the C_0–semigroup $U_t u(s,\cdot) = u((s+t) \bmod T, \cdot)$, $t \ge 0$. Let $\mathcal{E}$ be the time–dependent Dirichlet form corresponding to $(\mathcal{E}^{(t)})_{t\in[0,T]}$ and $\frac{d}{dt}$, given by

$$\mathcal{E}(u,v) := \begin{cases} \int_0^T \mathcal{E}^{(t)}(u(t,\cdot), v(t,\cdot)) \, dt + \int_0^T \langle \frac{du}{dt}(t,\cdot), v(t,\cdot)\rangle \, dt & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \int_0^T \mathcal{E}^{(t)}(u(t,\cdot), v(t,\cdot)) \, dt - \int_0^T \langle \frac{dv}{dt}(t,\cdot), u(t,\cdot)\rangle \, dt & \text{if } u \in \mathcal{V}, v \in \mathcal{F} \ . \end{cases}$$

Here $\mathcal{V} = L^2([0,T]; H^1(U))$, $\mathcal{F} := H^1([0,T]; H^{-1}(U)) \cap \mathcal{V}$ and $\langle\cdot,\cdot\rangle$ denotes the dualization between $H^{-1}(U)$ and $H^1(U)$ (cf. [St1, I.4.9. (iii)]). $\mathcal{E}$ uniquely determines a generator $(A, D(A))$ of a sub–Markovian C_0–semigroup of contractions (S_t) on $L^2([0,T]; L^2(U)) \equiv L^2(U_T)$. In particular, ind $(A - \alpha) = 0$ for $\alpha > 0$. Since

$D(A) \subset \mathcal{F}$ and the latter one is compactly embedded into $L^2(U_T)$ by 1.17 we conclude that ind $(A + c) =$ ind $(A - 1) = 0$ by [Ka, IV.5.26]. Since $1 \in D(A)$ (this is the reason why we have chosen $H^1(U)$ instead of $H^1_0(U)$ and the time–derivative with periodic boundary conditions) and $(A + c)1 = 0$, thus dim $\ker(A + c) > 0$, it follows that dim koker$(A + c) =$ dim $\ker(A' + c) > 0$, where A' is the adjoint of A. Now fix $\rho \in \ker\,(A' + c) \setminus \{0\}$. We will show in the following that we may assume $\rho \geq 0$. To this end let (S'_t) be the adjoint semigroup of (S_t) and note that (S'_t) is positivity preserving, that is $S'_t u \geq 0$ if $u \geq 0$ (cf. [St1, I.4.2]). Then $(A' + c)\rho = 0$ implies $S'_t\rho = e^{-ct}\rho$, hence $S'_t(\rho^+) \geq (S'_t\rho)^+ = e^{-ct}\rho^+$. On the other hand $\int_0^T \int_U S'_t(\rho^+) - e^{-ct}\rho^+ \, dx\, dt = \int_0^T \int_U \rho^+(S_t 1 - e^{-ct}1)\, dx\, dt = 0$, since $(A + c)1 = 0$ implies $S_t 1 = e^{-ct}1$. Hence $e^{-ct}S'_t(\rho^+) = \rho^+$ and similarly $e^{-ct}S'_t(\rho^-) = \rho^-$. Consequently, $\rho^+, \rho^- \in \ker\,(A' + c)$, too. Since at least one, ρ^+ or ρ^-, $\neq 0$, we may assume $\rho \geq 0$.

Fix $g \in C_0^\infty(U)$ and $f \in C^\infty((0, T))$. Then $(A' + c)\rho = 0$ implies for $f \in C^\infty((0, T))$ and $g \in C_0^\infty(U)$

$$\int_0^T f \sum_{i,j=1}^d \int a_{ij}\partial_i g \partial_j \rho \, dx\, dt - \int_0^T f \sum_{i=1}^d \int b_i \partial_i g \rho \, dx\, dt + \int_0^T \dot{f} \int g\rho \, dx\, dt = 0 \ ,$$

hence $\frac{d}{dt} \int g\rho(t, \cdot)\, dx = \int L_t g\rho(t, \cdot)\, dx$. Since $\sqrt{\rho} \in L^2([0, T]; H^1(U))$, hence in particular in $L^2_{loc}([0, T]; H^1_0(U))$, by the following Lemma, the assertion now follows. □

Lemma 1.19. *Let ρ be as in 1.18. Then $\sqrt{\rho} \in L^2([0, T]; H^1(U))$ and*

$$\int_0^T \int_U \langle A\nabla\sqrt{\rho}, \nabla\sqrt{\rho}\rangle \, dx\, dt \leq \frac{c}{4} \int_0^T \int_U \rho + e^{-1} \, dx\, dt \ . \tag{1.24}$$

Proof. Let us first prove the following

Claim: Let $\varphi \in \mathcal{B}(\mathbb{R})$, $|\varphi(t)| \leq M(|t| + 1)$. Then $\int_0^T \langle \frac{du}{dt}(t, \cdot), \varphi(u)(t, \cdot)\rangle \, dt = 0$ for all $u \in \mathcal{F}$.

Proof: The assertion is obvious for $u \in H^1_{per}([0, T]; L^2(U))$, since then, for $\psi \in H^1_{loc}(\mathbb{R})$ with $\dot{\psi} = \varphi$, $\int_0^T \langle \frac{du}{dt}(t, \cdot), \varphi(u)(t, \cdot)\rangle \, dt = \int_0^T \int_U \frac{d\psi(u)}{dt}(t, \cdot)\, dx\, dt = 0$. But $H^1_{per}([0, T]; H^1(U)) \subset H^1_{per}([0, T]; L^2(U)) \subset \mathcal{F}$ dense by [LMa, Sect. 3.1, Lemma 1.2], hence the assertion follows by taking limits.

For $\varepsilon > 0$ let $\psi_\varepsilon(t) = t - \varepsilon \log(t + \varepsilon) + e^{-1}$, $t \geq 0$, and $\psi_\varepsilon(t) = -\varepsilon \log \varepsilon + e^{-1}$ if $t < 0$. Then $\psi_\varepsilon(\rho) \geq 0$, $\nabla\psi_\varepsilon(\rho) = \frac{\rho}{\rho+\varepsilon}\nabla\rho$, and (1.23) now implies

$$-\frac{1}{2} \int\int \langle B, \nabla \log(\rho + \varepsilon)\rangle \rho \, dx\, dt + c \int\int \rho - \varepsilon \log(\rho + \varepsilon) + e^{-1} \, dx\, dt \geq 0 \ .$$

Moreover, since $\int_0^T \langle \frac{d\rho}{dt}(t,\cdot), \log(\rho+\varepsilon)(t,\cdot)\rangle\, dt = 0$ by the Claim, it follows that

$$
\begin{aligned}
\int\int \langle A\nabla\rho, \nabla\log(\rho+\varepsilon)\rangle\, dx\, dt &\le \mathcal{E}(\log(\rho+\varepsilon),\rho) \\
&\quad + c\int\int \rho - (\rho+\varepsilon)\log(\rho+\varepsilon) + e^{-1}\, dx\, dt \\
&= -\int\int (A'\rho)\log(\rho+\varepsilon)\, dx\, dt + c\int\int \rho - (\rho+\varepsilon)\log(\rho+\varepsilon) + e^{-1}\, dx\, dt \\
&= c\int\int \rho - \varepsilon\log(\rho+\varepsilon) + e^{-1}\, dx\, dt\ .
\end{aligned}
\tag{1.25}
$$

Clearly, $\sqrt{\rho+\varepsilon} \in L^2([0,T];H^1(U))$ for $\varepsilon > 0$. Since

$$
\begin{aligned}
\int\int \langle A\nabla\sqrt{\rho+\varepsilon}, \nabla\sqrt{\rho+\varepsilon}\rangle\, dx\, dt &= \frac{1}{4}\int\int \langle A\nabla\rho,\nabla\rho\rangle/(\rho+\varepsilon)\, dx\, dt \\
&= \frac{1}{4}\int\int \langle A\nabla\rho, \nabla\log(\rho+\varepsilon)\rangle\, dx\, dt\ ,
\end{aligned}
$$

we conclude from (1.25) $\sup_{\varepsilon>0}\int\int\langle A\nabla\sqrt{\rho+\varepsilon}, \nabla\sqrt{\rho+\varepsilon}\rangle\, dx\, dt < +\infty$, hence $\sqrt{\rho} \in L^2([0,T];H^1(U))$, $\sqrt{\rho+\varepsilon} \to \sqrt{\rho}$ weakly in $L^2([0,T];H^1(U))$ and (1.24) now follows from (1.25) by taking the limit $\varepsilon \to 0$. $\square$

The results of Subsections 1.a)-c) now imply the following:

Theorem 1.20. *There exists a maximal extension of diffusion type $(\overline{L}, D(\overline{L}))$ of $(L+\partial_t, C_0^\infty(U_T))$ in $L^1(U_T,\overline{\mu})$ satisfying 1.7 (a)-(c). Moreover, there exists a strong Markov process $\mathbb{M}$ with life–time ζ on U_T whose transition probabilities are given by the semigroup generated by $(\overline{L}, D(\overline{L}))$. $\mathbb{M}$ is a diffusion in the sense that $P_{(s,x)}[t \mapsto Y_t$ is continuous on $[0,\zeta)] = 1$ $\overline{\mu}$–a.e. and $\mathbb{M}$ is a solution of the martingale problem to $(L+\partial_t, C_0^\infty(U_T))$ in the sense of 1.13.*

Proof. It remains to verify (1.1)–(1.3). Clearly, (1.2) is implied by (1.21). Since $\rho \in L^2([0,T];H^1(U))$ and $\|\rho(t,\cdot)\|_{2\frac{d}{d-2}} \le c_0\|\rho(t,\cdot)\|_{H^1(U)}$ for some positive constant c_0 by Sobolev's imbedding theorem ([Ad, Th. 5.4]), we obtain for arbitrary $V \subset U$ and $f \in \mathcal{B}(V_T)$,

$$
\begin{aligned}
\int_0^T\int_V f^2\rho\, dx\, dt &\le \int_0^T \|f(t,\cdot)\|^2_{4\frac{d}{d+2}}\, \|\rho(t,\cdot)\|_{2\frac{d}{d-2}}\, dt \\
&\le \left(\int_0^T \|f(t,\cdot)\|^4_{4\frac{d}{d+2}}\, dt\right)^{\frac{1}{2}} \left(\int_0^T \|\rho(t,\cdot)\|^2_{2\frac{d}{d-2}}\, dt\right)^{\frac{1}{2}} \\
&\le c_0\|\rho\|_{L^2([0,T];H^1(U))}\left(\int_0^T \|f(t,\cdot)\|^4_{4\frac{d}{d+2}}\, dt\right)^{\frac{1}{2}}\ ,
\end{aligned}
\tag{1.26}
$$

hence $L^{\frac{d+2}{d}}([0,T];L^{4\frac{d}{d+2}}(V)) \subset L^2(V_T,\overline{\mu})$. Clearly, (1.1) and (1.3) now follow from (1.20) and (1.22). $\square$

2) Conservative Diffusions

Let $d \geq 3$ and $U \subset \mathbb{R}^d$ be an arbitrary nonempty open subset. Let $A = (a_{ij})$ be uniformly strictly elliptic. Then the bilinear form $\sum_{i,j=1}^d \int a_{ij}\partial_i u \partial_j u\, dx$, $u \in H_0^1(U)$, is well–defined and uniquely determines a self–adjoint operator L^0 in $L^2(U)$. In the following suppose that

$$\partial_j a_{ij} \in L^d_{loc}(U)\ , 1 \leq i,j \leq d\ . \tag{1.27}$$

Let $V \in L^1_{loc}(U)$ be a real potential with $V^- \in L^{\frac{d}{2}}(U)$. Sobolev's imbedding theorem then implies that the bilinear form

$$\mathcal{E}^V(u,u) = \sum_{i,j=1}^d \int a_{ij}\partial_i u\partial_j u\, dx + \int u^2 V\, dx\ , u \in D(\mathcal{E}^V) := H_0^1(U) \cap L^2(U, V^+\, dx),$$

is a well–defined closed bilinear form on $L^2(U)$. Moreover, there exists a positive constant b such that $\mathcal{E}^V(u,u) + b\int u^2\, dx \geq 0$, for all $u \in D(\mathcal{E}^V)$. Let L^V be its uniquely determined self–adjoint generator. Let $L^2(U;\mathbb{C})$ (resp. $D(\mathcal{E}^V;\mathbb{C})$) be the complexification of $L^2(U)$ (resp. $D(\mathcal{E}^V)$), and $L^V_{\mathbb{C}}$ be the complexification of L^V. $L^V_{\mathbb{C}}$ is self–adjoint and Stone's theorem implies that for arbitrary $\psi_0 \in D(L^V_{\mathbb{C}})$, $\psi_t := e^{itL^V_{\mathbb{C}}}\psi_0$, $t \in \mathbb{R}$, solves the Schrödinger equation $i\partial_t\psi = -L^V_{\mathbb{C}}\psi$, $\psi(t,\cdot) = \psi_0$, which is equivalent to $\partial_t g = -L^V h$, $\partial_t h = L^V g$, where $g := Re(\psi)$ and $h := Im(\psi)$. From now on suppose that $\psi_0 \in D(\mathcal{E}^V;\mathbb{C})$. Then $\mathbb{R} \to D(\mathcal{E}^V;\mathbb{C})$, $t \mapsto e^{itL^V_{\mathbb{C}}}\psi_0$, is continuous, in particular $g, h \in L^2([0,T]; D(\mathcal{E}^V)) \subset L^2([0,T]; H_0^1(U))$ for finite $T > 0$. We can now define the time–dependent diffusion operator

$$Lu(t,x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j u(t,x)) + \sum_{i=1}^d b_i(t,x)\partial_i u(t,x)\ , u \in C_0^\infty(U_T)\ ,$$

where b_i, $1 \leq i \leq d$, are the components of the vector–field $B := 2|\psi|^{-2}(gA\nabla(h + g) + hA\nabla(h-g))$. In stochastic mechanics one is now interested in the existence of non–homogeneous diffusion processes whose generator extend L (cf. [C], [Nag], [O]). The example fits exactly in our framework if we let $d\overline{\mu} := |\psi|^2\, dx\, dt = g^2 + h^2\, dx\, dt$. Indeed, first note that the density of $\overline{\mu}$ clearly admits a representation φ^2 for $\varphi \in L^2([0,T]; H_0^1(U))$ and that $B \in L^2(U_T; \mathbb{R}^d, \overline{\mu})$. (1.2) is obvious and (1.1) follows from Sobolev's imbedding theorem, since for all V relatively compact in U an analoguous computation to that of (1.26) shows that $L^\infty([0,T]; L^d(V)) \subset L^2(V_T, \overline{\mu})$, hence $\partial_j a_{ij} \in L^2(V_T, \overline{\mu})$ by (1.27). Moreover, (1.3) is satisfied, since both b_i and $\sum_{j=1}^d \partial_j a_{ij} \in L^2(V_T, \overline{\mu})$. To verify (1.4) note that for $u \in C_0^\infty(U) \subset D(L^0)$

$$\begin{aligned}
\int_s^t \int L_r u\, d\mu_r\, dr &= -2\int_s^t \int \langle A\nabla u, \nabla g\rangle g + \langle A\nabla u, \nabla h\rangle h\, dx\, dr \\
&\quad + 2\int_s^t \int \langle gA\nabla(h+g), \nabla u\rangle + \langle hA\nabla(h-g), \nabla u\rangle\, dx\, dr \\
&= 2\int_s^t \int \langle A\nabla u, \nabla h\rangle g - \langle A\nabla u, \nabla g\rangle h\, dx\, dr \qquad (1.28) \\
&= 2\int_s^t \int \langle A\nabla(ug), \nabla h\rangle - \langle A\nabla(uh), \nabla g\rangle\, dx\, dr\ .
\end{aligned}$$

Suppose now for the moment that $\psi_0 \in D(L^V_{\mathbb{C}})$, hence $g(t,\cdot), h(t,\cdot) \in D(L^V)$ for all t. Then

(1.29)
$$
\begin{aligned}
2\int_s^t \int \langle A\nabla(ug), \nabla h\rangle - \langle A\nabla(uh), \nabla g\rangle \, dx\, dr &= -2\int_s^t \int L^V h\, gu - L^V g\, hu\, dx\, dr \\
&= 2\int_s^t \int (\partial_r g)gu + (\partial_r h)hu\, dx\, dr = \int u\, d\mu_t - \int u\, d\mu_s \ .
\end{aligned}
$$

Combining (1.28) and (1.29) implies

$$
(1.30) \qquad \int_s^t \int L_r u\, d\mu_r\, dr = \int u\, d\mu_t - \int u\, d\mu_s
$$

in the particular case $\psi_0 \in D(L^V_{\mathbb{C}})$. The general case $\psi_0 \in D(\mathcal{E}^V; \mathbb{C})$ can now be obtained from (1.30) by taking limits in $D(\mathcal{E}^V; \mathbb{C})$.

Remark. The procedure above very much reminds one of the *h–transform of Doob* of a given Markov process (cf. [Sh, Section 62]), used in probability theory to construct bridge–processes. Indeed, let $\mathbb{M}$ be a Markov process associated with L^0, fix $\rho \in L^2(U)$, $\rho \geq 0$, and let $\rho(t,\cdot) := e^{(T-t)L^0}\rho$, $t \in [0,T)$. The bridge–process $\mathbb{M}^\rho$ corresponding to ρ is then obtained as the time–inhomogeneous Markov process with generator

$$
L^\rho u(t,x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j u(t,x)) + \rho(t,x)^{-1}\sum_{i,j=1}^d a_{ij}(x)\partial_i\rho(t,x)\partial_j u(t,x) \ .
$$

Let $d\mu_t = \rho(t,\cdot)\, dx$, $t \in [0,T]$. Then $\partial_t\rho(t,\cdot) = -L^0\rho(t,\cdot)$, $t \in [0,T)$, hence $(L^\rho + \partial_t)u = \rho^{-1}(L^0+\partial_t)(u\rho)$ for all $u \in C_0^\infty(U)$, implies

$$
\begin{aligned}
\int_s^t \int L^\rho_r u\, d\mu_r\, dr &= \int_s^t \int (L^0 + \partial_r)(u\rho)\, dx\, dr - \int_s^t \int \partial_r u\, d\mu_r\, dr \\
&= \int_s^t \int u\partial_r\rho\, dx\, dr = \int u\, d\mu_t - \int u\, d\mu_s \ , 0 \leq s \leq t < T \ ,
\end{aligned}
$$

for all $u \in C_0^\infty(U)$. However, instead of using now the results of Subsections 1.a) and 1.b) to construct the (forward space–time homogenization of the) bridge–process $\mathbb{M}^\rho$, this particular case can be treated more naturally in an L^2–setting as the h–transformation of the time–dependent Dirichlet form corresponding to $(\mathcal{E}^0, D(\mathcal{E}^0))$ (in the sense of [St1, I.4.9 (iii)]) with the excessive function ρ (cf. [St1, II.5 (b)]).

The results of Subsection 1a)-c) now imply the following:

Theorem 1.21. *There exists a maximal extension of diffusion type $(\overline{L}, D(\overline{L}))$ of $(L+\partial_t, C_0^\infty(U_T))$ in $L^1(U_T, \overline{\mu})$ satisfying 1.7 (a)-(c). Moreover, there exists a strong Markov process $\mathbb{M}$ with life–time ζ on U_T whose transition probabilities are given by the semigroup generated by $(\overline{L}, D(\overline{L}))$. $\mathbb{M}$ is a diffusion in the sense that $P_{(s,x)}[t \mapsto Y_t$ is continuous on $[0,\zeta)] = 1$ $\overline{\mu}$–a.e. and $\mathbb{M}$ is a solution of the martingale problem* *to $(L+\partial_t, C_0^\infty(U_T))$ in the sense of 1.13. If $U=\mathbb{R}^d$, the (a_{ij}) are locally Hölder-continuous and satisfy (1.13) and*

$$\partial_j a_{ij} \in L^{\frac{d}{2}}(\mathbb{R}^d) \cup L^\infty(\mathbb{R}^d)\,, 1 \le i,j \le d, \tag{1.31}$$

it follows that $(L, C_0^\infty(\mathbb{R}^d_T))$ is L^1–unique, hence $(\overline{L}, D(\overline{L}))$ is given by the closure of $(L, C_0^\infty(\mathbb{R}^d_T))$ in $L^1(\mathbb{R}^d_T, \overline{\mu})$.

Proof. It remains to check that in the case $U=\mathbb{R}^d$ and (1.31) $(L, C_0^\infty(\mathbb{R}^d_T))$ is L^1–unique. But this follows from Corollary 1.15 (a) since $b_i \in L^2(\mathbb{R}^d_T, \overline{\mu}) \subset L^1(\mathbb{R}^d_T, \overline{\mu})$, $1 \le i,j \le d$, $a_{ij} 2\partial_j\varphi/\varphi \in L^2(\mathbb{R}^d_T, \overline{\mu}) \subset L^1(\mathbb{R}^d_T, \overline{\mu})$ and finally (1.31) implies $\partial_j a_{ij} \in L^1(\mathbb{R}^d_T, \overline{\mu})$ by Sobolev's imbedding theorem. $\square$

Remark. (i) We emphasize that all the results of 1.21 remain true if $V \in L^1_{loc}(U)$ is such that for all $u \in H^1_0(U)$

$$|\int u^2 V\,dx| \le a \sum_{i,j=1}^d \int a_{ij}\partial_i u \partial_j u\,dx + b\int u^2\,dx$$

for $a \in (0,1)$ and $b \ge 0$. Moreover, the dimension d can be arbitrary under this assumption.

(ii) For a construction of the Markov process $\mathbb{M}$ associated to $(L+\partial_t, C_0^\infty(U_T))$ in the particular case where $U=\mathbb{R}^d$, $a_{ij}=\delta_{ij}$ and V is a Rellich class potential, we refer to [C].

2. Proof of Prop. 1.5 and Theorem 1.7

Let the assumptions be as in 1.5. To simplify notations let $\mathcal{V} := H_0^{0,1}(V_T, \overline{\mu})$ and $\|v\|_{\mathcal{V}_b} := \|v\|_\infty + \mathcal{A}^0(v,v)^{\frac{1}{2}}$, $v \in \mathcal{V}_b$, so that $|f|_V = \sup_{\|v\|_{\mathcal{V}_b}\le 1}\int fv\,d\overline{\mu}$. Moreover, let $\overline{L}^1 := \overline{L}^1(V_T, \overline{\mu})$.

Lemma 2.1. $\overline{L}^1(V_T, \overline{\mu})' = H_0^{0,1}(V_T, \overline{\mu})_b$.

Proof. For $v \in \mathcal{V}_b$ it follows that

$$\ell_v(f) := \langle f, v\rangle \le |f|_V(\|v\|_\infty + \mathcal{A}^0(v,v)^{\frac{1}{2}})\ ,\ f \in L^1(V_T, \overline{\mu})\ ,$$

which implies that ℓ_v can be extended uniquely to a continuous linear functional on $\overline{L^1}$ which implies that $\mathcal{V}_b \subset \overline{L^1}'$. To see the converse implication fix $\ell \in \overline{L^1}'$. Then $\ell_{|L^1(V_T,\overline{\mu})} \in L^1(V_T, \overline{\mu})' = L^\infty(V_T, \overline{\mu})$. Hence there exists $h \in L^\infty(V_T, \overline{\mu})$ such that $\ell(f) = \int fh\,d\overline{\mu}$, $f \in L^1(V_T, \overline{\mu})$. We will show next that $h \in \mathcal{V}$. To this end denote by $(R_\alpha)_{\alpha>0}$ the resolvent associated to the quadratic form $\mathcal{A}^0(u,v)$, $u,v \in \mathcal{V}$. Then $\alpha R_\alpha h \in \mathcal{V}_b$, $\alpha > 0$ and

$$\begin{aligned}\mathcal{A}^0(\alpha R_\alpha h, \alpha R_\alpha h) &= \alpha\int (h-\alpha R_\alpha h)\alpha R_\alpha h\ , d\overline{\mu} \\ &\le \alpha \int (h-\alpha R_\alpha h) h\,d\overline{\mu} = \ell(\alpha(h-\alpha R_\alpha h)) \\ &\le \|\ell\|_{\overline{L^1}'}|\alpha(h-\alpha R_\alpha h)|_V\ .\end{aligned} \tag{2.1}$$

Note that

(2.2)

$$|\alpha(h-\alpha R_\alpha h)|_V = \sup_{\|v\|_{\mathcal{V}_b}\le 1} \alpha \int (h-\alpha R_\alpha h)v\, d\overline{\mu} = \sup_{\|v\|_{\mathcal{V}_b}\le 1} \mathcal{A}^0(\alpha R_\alpha h, v)$$
$$\le cA^0(\alpha R_\alpha h, \alpha R_\alpha h)^{\frac{1}{2}} \ .$$

Hence, combining (2.1) and (2.2) we conclude that

$$\sup_{\alpha>0} \mathcal{A}^0(\alpha R_\alpha h, \alpha R_\alpha h) < +\infty \ ,$$

hence $h \in \mathcal{V}$. $\square$

Lemma 2.2. *$(L, C_0^\infty(V_T))$ (resp. $(L', C_0^\infty(V_0))$) is closable in $\overline{L^1}(U_T, \overline{\mu})$.*

Proof. We will prove closability of $(L, C_0^\infty(V_T))$ only. Closability of $(L', C_0^\infty(V_0))$ can be shown similarly. To this end let $(u_n) \subset C_0^\infty(V_T)$ be such that $\lim_{n\to\infty} u_n = 0$ in $\overline{L^1}$ and $\lim_{n\to\infty} Lu_n = f$ in $\overline{L^1}$ for some $f \in \overline{L^1}$. Let $\varphi \in \mathcal{M}$, $v \in C_0^\infty(V_T)$ and $\lambda \in \mathbb{R}$. Then $(\varphi(u_n+\lambda v)) \subset \mathcal{V}_b = \overline{L^1}'$ bounded, since $\sup_{n\ge 1} \|\varphi(u_n+\lambda v)\|_\infty \le \|\varphi\|_\infty$ and

$$\mathcal{A}^0(\varphi(u_n+\lambda v), \varphi(u_n+\lambda v)) \le -\int L(u_n+\lambda v)(u_n+\lambda v)\, d\overline{\mu}$$
$$\le |L(u_n+\lambda v)|_V \|\varphi(u_n+\lambda v)\|_{\mathcal{V}_b} \ .$$

Since $\overline{L^1}$ is separable we can find a subsequence such that $\lim_{k\to\infty} \varphi(u_{n_k}+\lambda v) = \varphi(\lambda v)$ weakly* in $\overline{L^1}'$. Since $L(u_n+\lambda v) \to f + \lambda Lv$ in $\overline{L^1}$ it now follows that

$$\langle f+\lambda Lv, \varphi(\lambda v)\rangle = \lim_{k\to\infty} \langle L(u_{n_k}+\lambda v), \varphi(u_{n_k}+\lambda v)\rangle \le 0 \ .$$

In particular, if we choose $\varphi \in \mathcal{M}$ such that $\varphi(t) = t$ for $|t| \le |\lambda| \|v\|_\infty$ we conclude that $\langle f+\lambda Lv, \lambda v\rangle \le 0$ and taking the limit $\lambda \to 0$ we obtain $\langle f, v\rangle \le 0$. Passing from v to $-v$ implies $\langle f, v\rangle = 0$ for all $v \in C_0^\infty(V_T)$. For arbitrary $v \in \mathcal{V}_b$ we can now find a sequence $(v_n) \subset C_0^\infty(V_T)$ bounded in $\mathcal{V}_b$ and converging weakly* to v in $\overline{L^1}'$. Hence $\langle f, v\rangle = \lim_{n\to\infty} \langle f, v_n\rangle = 0$ which implies $f = 0$ and hence the assertion. $\square$

By 2.2 we can define the closure $(A, D(A))$ (resp. $(A', D(A'))$) of $(L, C_0^\infty(V_T))$ (resp. $(L', C_0^\infty(V_0))$) in $\overline{L^1}$. In the following let Δ be the Laplacian on V with Dirichlet boundary conditions.

Lemma 2.3. *Let $f \in C_b^\infty(V)$, $\alpha > 0$ and $u := (\alpha - \Delta)^{-1} f \in H_0^1(V)$. Then $u \in C^\infty(\overline{V})$, and there exist $v_n \in C_0^\infty(V)$, $\|v_n\|_\infty \le \|u\|_\infty$, $n \ge 1$, converging to u almost everywhere such that $\||\nabla v_n|\|_\infty \le L$ for some constant L.*

Proof. By [GT, 8.14] $u \in C^\infty(\overline{V})$. Note that u can be extended by 0 to a function in $H_0^{1,2}(\mathbb{R}^d)$ such that $\int_{B_r(x)} |\nabla u|\, dx \le Mr^d$ for all $x \in \mathbb{R}^d$, $r > 0$, and some constant M. By the theorem of Morrey it follows that $|u(x)-u(y)| \le L|x-y|$ for all $x, y \in \mathbb{R}^d$ and some constant L.

Let $V_\varepsilon := \{x \in V | d(x, \partial V) > \varepsilon\}$. Let $\psi \in C_0^\infty(B_1(0))$, $\psi \ge 0$, with $\int \psi\, dx = 1$, $\psi_\varepsilon(x) := \varepsilon^{-d}\psi(\varepsilon^{-1}x)$, $\chi_\varepsilon(x) := \int 1_{V_\varepsilon}(x-y)\psi_{\frac{\varepsilon}{2}}(y)\, dy$ and $u_\varepsilon := \chi_\varepsilon u$. Then $\chi_\varepsilon \in C_0^\infty(V)$, $\nabla\chi_\varepsilon(x) = 0$ if $x \in V_{\frac{3}{2}\varepsilon}$ and $|\nabla\chi_\varepsilon(x)| \le c\varepsilon^{-1}$ if $x \in V \setminus V_{\frac{3}{2}\varepsilon}$, with $c := \sqrt{d}\int |\psi'|\, dy$. Since $u(x) = 0$ if $x \in \mathbb{R}^d \setminus \overline{V}$ we conclude that $|u(x)| \le L\frac{3}{2}\varepsilon$ if $x \in V \setminus V_{\frac{3}{2}\varepsilon}$ and consequently,

$$|\nabla u_\varepsilon(x)| \le |\nabla\chi_\varepsilon(x)||u(x)| + |\nabla u(x)| \le L(\frac{3}{2}c + 1)\ .$$

Since $u_\varepsilon \in C_0^\infty(V)$ we can now take $v_n := u_{\frac{1}{n}}$, $n \ge 1$. $\square$

Lemma 2.4. *Let Δ be the Laplacian on V with Dirichlet boundary conditions, $f \in C_0^\infty((0,T))$, $g \in C_0^\infty(V)$ and $\alpha > 0$. Then $(\alpha - \Delta - \partial_t)^{-1}(fg) \in H_0^{1,1}(V_T, \overline{\mu}) \cap D(A)$ (resp. $(\alpha - \Delta + \partial_t)^{-1}(fg) \in H_0^{1,1}(V_0, \overline{\mu}) \cap D(A')$.*

Proof. We will prove the statement for A only. The dual statement can be shown similarly. First note that $H_0^{1,1}(V_T, \overline{\mu}) \subset D(A)$. Indeed, for arbitrary $u \in H_0^{1,1}(V_T, \overline{\mu})$ we can find $(u_n) \subset C_0^\infty(V_T)$ converging to u in $H_0^{1,1}(V_T, \overline{\mu})$. Then

$$\begin{aligned}|L(u_n - u_m)|_V &= \sup_{\|v\|_{\mathcal{V}_b} \le 1} \int L(u_n - u_m) v\, d\overline{\mu} \\ &= \sup_{\|v\|_{\mathcal{V}_b} \le 1} \mathcal{A}^0(u_n - u_m, v) - \int \langle \overline{\beta}, \overline{\nabla}(u_n - u_m)\rangle\, v\, d\overline{\mu} \\ &\le \|u_n - u_m\|_{H_0^{1,1}(V_T, \overline{\mu})}\, (1 + \||\beta|\|_2) \to 0\ ; n, m \to \infty\ ,\end{aligned}$$

which implies that $u \in D(A)$. It is therefore enough to prove that $(\alpha - \Delta - \partial_t)^{-1}(fg) \in H_0^{1,1}(V_T, \overline{\mu})$.

Step 1: Let $f \in C^\infty([0,T])$, $f(T) = 0$, and $g \in C_0^\infty(V)$. Then $f(\alpha - \Delta)^{-1}g \in H_0^{1,1}(V_T, \overline{\mu})$.

Proof: Let $u := (\alpha - \Delta)^{-1}g$. By 2.3 there exist $(v_n) \subset C_0^\infty(V)$ converging to u a.e., $\|v_n\|_\infty \le \|u\|_\infty$ and $\sup_{n \ge 1}\|\nabla v_n|\|_\infty < +\infty$. Then $fv_n \in C_0^\infty(V_T) \subset H_0^{1,1}(V_T, \overline{\mu})$ and $\sup_{n \ge 1}\int |\overline{\nabla}(fv_n|^2\, d\overline{\mu} < +\infty$, which implies that $fu \in H_0^{1,1}(V_T, \overline{\mu})$.

Step 2: Let $f \in C^\infty([0,T])$, $f(T) = 0$, and $u \in D(\Delta)$ with $u, \Delta u \in L^\infty(V)$. Then $fu \in H_0^{1,1}(V_T, \overline{\mu})$.

Proof: Let $g := (\alpha - \Delta)u \in L^\infty(dx)$ and $(g_n) \subset C_0^\infty(V)$ be such that $\lim_{n\to\infty} g_n = g$ a.e. and $\sup_{n \ge 1}\|g_n\|_\infty < \infty$. For $u_n := (\alpha - \Delta)^{-1}g_n$ it then follows from Step 1 that $fu_n \in H_0^{1,1}(V_T, \overline{\mu})$. Since $fu_n \to fu$ in $L^2(V_T, \overline{\mu})$,

$$\begin{aligned}\int |\nabla(fu_n)|^2\, d\overline{\mu} &\le -\int f^2\Delta u_n\, u_n\, d\overline{\mu} - 2\int f^2\langle \frac{\nabla\varphi}{\varphi}, \nabla u_n\rangle\, u_n\, d\overline{\mu} \\ &\le -\int f^2\Delta u_n\, u_n\, d\overline{\mu} + 2\int f^2 u_n^2 \left|\frac{\nabla\varphi}{\varphi}\right|^2\, d\overline{\mu} \\ &\quad + \frac{1}{2}\int |\nabla(fu_n)|^2\, d\overline{\mu}\ ,\end{aligned}$$

so that

$$\int |\nabla(fu_n)|^2 \, d\overline{\mu} \leq -2 \int f^2 \Delta u_n \, u_n \, d\overline{\mu} + 4 \int f^2 u_n^2 \left| \frac{\nabla \varphi}{\varphi} \right|^2 \, d\overline{\mu} \ ,$$

it follows that $(fu_n)_{n \geq 1} \subset H_0^{1,1}(V_T, \overline{\mu})$ bounded, hence $fu \in H_0^{1,1}(V_T, \overline{\mu})$ and

$$\begin{aligned} \int |\overline{\nabla}(fu_n)|^2 \, d\overline{\mu} &\leq \liminf_{n\to\infty} \int |\overline{\nabla}(fu_n)|^2 \, d\overline{\mu} \\ &\leq \liminf_{n\to\infty} \int (\partial_t f u_n)^2 \, d\overline{\mu} - 2 \int f^2 \Delta u_n \, u_n \, d\overline{\mu} + 4 \int f^2 u_n^2 \left| \frac{\nabla \varphi}{\varphi} \right|^2 \, d\overline{\mu} \qquad (2.3) \\ &= \int (\partial_t f u)^2 \, d\overline{\mu} - 2 \int f^2 \Delta u \, u \, d\overline{\mu} + 4 \int f^2 u^2 \left| \frac{\nabla \varphi}{\varphi} \right|^2 \, d\overline{\mu} \ . \end{aligned}$$

Step 3: $(\alpha - \Delta - \partial_t)^{-1}(fg) \in H_0^{1,1}(V_T, \overline{\mu})$.

Proof: Let $p_t = e^{t\Delta}$, $t \geq 0$, be the semigroup generated by Δ. Fix $f \in C_0^\infty((0,T))$, and extend f by 0 to $\mathbb{R}^+$. Let $g \in C_0^\infty(V)$. By Step 2

$$u_n := \sum_{k=1}^{n2^n} \frac{1}{n} e^{-\alpha \frac{k}{n}} f(\cdot + \frac{k}{n}) p_{\frac{k}{n}} g \in H_0^{1,1}(V_T, \overline{\mu}) \ .$$

Since $\|u_n\|_\infty \leq \frac{1}{\alpha}\|f\|_\infty \|g\|_\infty$, $\|\partial_t u_n\|_\infty \leq \frac{1}{\alpha}\|\dot{f}\|_\infty \|g\|_\infty$, $\|\Delta u_n\|_\infty \leq \frac{1}{\alpha}\|f\|_\infty \|\Delta g\|_\infty$ (2.3) implies that

$$\int |\overline{\nabla} u_n|^2 \, d\overline{\mu} \leq \int (\partial_t u_n)^2 \, d\overline{\mu} - 2 \int \Delta u_n \, u_n \, d\overline{\mu} + 4 \int u_n^2 \left| \frac{\nabla \varphi}{\varphi} \right|^2 \, d\overline{\mu}$$

is bounded in n, hence $(u_n) \subset H_0^{1,1}(V_T, \overline{\mu})$ bounded. Consequently, $(\alpha - \Delta - \partial_t)^{-1}(fg) = \int_0^\infty e^{-\alpha t} f(\cdot + t) p_t g \, dt = \lim_{n\to\infty} u_n \in H_0^{1,1}(V_T, \overline{\mu})$ which implies the assertion. $\square$

Proof of 1.5 (i). Step 1: $(1-A)(D(A)) \subset \overline{L^1}$ (resp. $(1-A')(D(A')) \subset \overline{L^1}$) dense.

Proof: Again, we will prove the statement for A only. The proof of the dual statement is similar. By the Hahn–Banach theorem we only need to show that $\ell \in \overline{L^1}'$ with $\ell((1-A)u) = 0$ for all $u \in D(A)$ implies $\ell = 0$. By 2.1 we may identify ℓ with the linear functional $f \mapsto \langle f, h \rangle$ for some $h \in \mathcal{V}_b$ and we have to show that $h = 0$. Let $h_\alpha \in \operatorname{span}\{fg | f \in C_0^\infty((0,T)), g \in C_0^\infty(V)\}$, be such that $\|h_\alpha\|_\infty \leq \|h\|_\infty + 1$ and $\|h_\alpha - h\|_2 \leq \frac{1}{\alpha}$. By 2.4 $u_\alpha := (\alpha - \Delta - \partial_t)^{-1} h_\alpha \in H_0^{1,1}(V_T, \overline{\mu}) \cap D(A)$ and

$\|\alpha u_\alpha\|_\infty \le \|h\|_\infty + 1$ by the maximum principle. Observe that

$$
\begin{aligned}
\mathbb{D}(\alpha u_\alpha,\alpha u_\alpha) &\le -\int (\Delta+\partial_t)(\alpha u_\alpha)\,(\alpha u_\alpha)\,d\overline{\mu} - \int \langle \beta + 2\frac{\nabla\varphi}{\varphi}, \nabla \alpha u_\alpha\rangle \alpha u_\alpha\,d\overline{\mu} \\
&= -\int (\Delta+\partial_t)(\alpha u_\alpha)((\Delta+\partial_t)u_\alpha + h_\alpha)\,d\overline{\mu} \\
&\quad - \int \langle \beta + 2\frac{\nabla\varphi}{\varphi}, \nabla \alpha u_\alpha\rangle \alpha u_\alpha\,d\overline{\mu} \\
&= -\alpha\int ((\Delta+\partial_t)u_\alpha)^2\,d\overline{\mu} - \alpha\int (\Delta+\partial_t)u_\alpha(h_\alpha - h)\,d\overline{\mu} \\
&\quad - \int (\Delta+\partial_t)(\alpha u_\alpha)h\,d\overline{\mu} - \int \langle \beta + 2\frac{\nabla\varphi}{\varphi}, \nabla \alpha u_\alpha\rangle \alpha u_\alpha\,d\overline{\mu} \\
&\le -\frac{\alpha}{2}\int ((\Delta+\partial_t)u_\alpha)^2\,d\overline{\mu} + \frac{\alpha}{2}\|h_\alpha - h\|_2^2 - \int L(\alpha u_\alpha)\,h\,d\overline{\mu} - \mathcal{A}^0(\alpha u_\alpha, h) \\
&\quad + \mathbb{D}(\alpha u_\alpha, h) + \int \langle \beta + 2\frac{\nabla\varphi}{\varphi}, \nabla \alpha u_\alpha\rangle (h - \alpha u_\alpha)\,d\overline{\mu} \\
&\le -\frac{\alpha}{2}\int ((\Delta+\partial_t)u_\alpha)^2\,d\overline{\mu} + \frac{1}{2\alpha} - \int \alpha u_\alpha\,h\,d\overline{\mu} - \mathcal{A}^0(\alpha u_\alpha, h) \\
&\quad + \mathbb{D}(\alpha u_\alpha, h) + \int \langle \beta + 2\frac{\nabla\varphi}{\varphi}, \nabla \alpha u_\alpha\rangle (h - \alpha u_\alpha)\,d\overline{\mu}\ .
\end{aligned}
\tag{2.4}
$$

Hence

$$
\sup_{\alpha>0} \mathbb{D}(\alpha u_\alpha, \alpha u_\alpha) + \frac{\alpha}{2}\int ((\Delta+\partial_t)u_\alpha)^2\,d\overline{\mu} < +\infty\ ,
$$

in particular, $\lim_{\alpha\to\infty}\|\alpha u_\alpha - h\|_2 \le \lim_{\alpha\to\infty}\|(\Delta+\partial_t)u_\alpha\|_2 + \|h_\alpha - h\|_2 = 0$, and then $\lim_{\alpha\to\infty}\alpha u_\alpha = h$ weakly in $\mathcal{V}$. Now (2.4) implies that

$$
\begin{aligned}
\mathbb{D}(h,h) &\le \liminf_{\alpha\to\infty} \mathbb{D}(\alpha u_\alpha, \alpha u_\alpha) \\
&\le \liminf_{\alpha\to\infty} -\int \alpha u_\alpha h\,d\overline{\mu} - \mathcal{A}^0(\alpha u_\alpha, h) + \mathbb{D}(\alpha u_\alpha, h) \\
&\qquad + \int \langle (\beta + 2\frac{\nabla\varphi}{\varphi}), \nabla(\alpha u_\alpha)\rangle (h - \alpha u_\alpha)\,d\overline{\mu} \\
&= -\int h^2\,d\overline{\mu} - \mathcal{A}^0(h,h) + \mathbb{D}(h,h)\ ,
\end{aligned}
$$

hence $h = 0$.

Let $(\overline{L}^V, D(\overline{L}^V))$ (resp. $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$ be the part of $(A, D(A))$ (resp. $(A', D(A'))$) in $L^1(V_T, \overline{\mu})$.

Step 2: $(1-\overline{L}^V)(D(\overline{L}^V)) = L^1(V_T,\overline{\mu})$ (resp. $(1-\overline{L}^{V,\prime})(D(\overline{L}^{V,\prime})) = L^1(V_T,\overline{\mu})$), i.e., $(\overline{L}^V, D(\overline{L}^V))$ (resp. $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$) is maximal.

Proof: Again, we will prove the statement for L only. The proof of the dual statement is similar. By Step 1 and the definition of $(A, D(A))$ there exist $(u_n) \subset C_0^\infty(V_T)$ such that $f_n := (1-L)u_n \to f$ in $\overline{L^1}$. Let $\varphi \in \mathcal{M}$ be such that $0 \le \dot{\varphi} \le 1$,

$|\varphi(t)| > 0$ if $t \neq 0$ and $\varphi(t) = \mathrm{sgn}\ (t)$ if $|t| \geq 1$. Then

$$
(2.5)\qquad
\begin{aligned}
\mathcal{A}^0(\varphi(u_m - u_n), \varphi(u_m - u_n)) + \int \varphi(u_m - u_n)\,(u_m - u_n)\,d\overline{\mu} \\
\leq \int (1-L)(u_m - u_n)\,\varphi(u_m - u_n)\,d\overline{\mu} \\
\leq |(1-L)(u_m - u_n)|_V \|\varphi(u_m - u_n)\|_{\mathcal{V}_b}\ .
\end{aligned}
$$

Consequently, $(\varphi(u_m - u_n))_{m,n} \subset \mathcal{V}_b$ bounded, and now (2.5) implies that

$$
\lim_{n,m\to\infty} \int \varphi(u_n - u_m)\,(u_n - u_m)\,d\overline{\mu} = 0\ .
$$

Hence there exists $u \in L^1(V_T, \overline{\mu})$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty}(1-L)u_n = f$ in $\overline{L^1}$ which implies $u \in D(A)$ and $(1-A)u = f$. Since $(\overline{L}^V, D(\overline{L}^V))$ is the part of $(A, D(A))$ in $L^1(V_T, \overline{\mu})$ we conclude that $f \in D(\overline{L}^V)$ and $(1-\overline{L}^V)u = (1-A)u = f$.

Step 3: $(\overline{L}^V, D(\overline{L}^V))$ (resp. $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$) is of diffusion type.

Again, we will prove the statement for L only. The proof of the dual statement is similar. Fix $u \in D(\overline{L}^V)$, $\varphi \in \mathcal{M}$ and let $(u_n) \subset C_0^\infty(V_T)$ be such that $\lim_{n\to\infty}(1-L)u_n = (1-\overline{L}^V)u$ in $\overline{L^1}$. Similar to the proof of the corresponding statement in Step 2 it follows that $\lim_{n\to\infty} u_n = u$ in $L^1(V_T, \overline{\mu})$. In particular, $\lim_{n\to\infty} \int (\varphi(u_n) - \varphi(u))\,d\overline{\mu} = 0$ and $(\varphi(u_n)) \subset \mathcal{V}_b$ bounded. Consequently,

$$
\begin{aligned}
|\int \overline{L}^V u\,\varphi(u)\,d\overline{\mu} - \int \overline{L}^V u_n\,\varphi(u_n)\,d\overline{\mu}| &\leq |\int \overline{L}^V u\,(\varphi(u) - \varphi(u_n))\,d\overline{\mu}| \\
&+ |\int \overline{L}^V (u - u_n)\,\varphi(u_n)\,d\overline{\mu}| \to 0\ , n \to \infty\ ,
\end{aligned}
$$

which implies that

$$
\int \overline{L}^V u\,\varphi(u)\,d\overline{\mu} = \lim_{n\to\infty} \int \overline{L}^V u_n\,\varphi(u_n)\,d\overline{\mu} \leq 0\ .
$$

Clearly, 2.2 and Step 1-3 now imply Prop. 1.5 (i).

Proof of 1.5 (ii): We will prove the statement again for L only. The proof of the dual statement is similar. Let $u \in D(\overline{L}^V)_b$ and $(u_n) \subset C_0^\infty(V_T)$ be such that $\lim_{n\to\infty}(1-L)u_n = (1-\overline{L}^V)u$ in $\overline{L^1}$.

Claim: $\lim_{n\to\infty} \int_{\{M_1 \leq |u_n| \leq M_2\}} \langle A\nabla u_n, \nabla u_n\rangle = 0$ for all $\|u\|_\infty < M_1 < M_2$.

Proof: Let $\psi \in \mathcal{M}$ be such that $\psi(t) = 0$ if $t \leq \|u\|_\infty$ and $\dot{\psi}(t) = 1$ if $t \in [M_1, M_2]$. Then $(\psi(u_n)) \subset \mathcal{V}_b$ bounded, $\lim_{n\to\infty} \psi(u_n) = 0$ in $L^1(V_T, \overline{\mu})$ and consequently,

$$
\begin{aligned}
\lim_{n\to\infty} \int_{\{u \in [M_1, M_2]\}} \langle A\nabla u_n, \nabla u_n\rangle\,d\overline{\mu} &\leq \lim_{n\to\infty} \int \dot{\psi}(u_n)\langle A\nabla u_n, \nabla u_n\rangle\,d\overline{\mu} \\
&= \lim_{n\to\infty} \mathcal{A}^0(u_n, \psi(u_n)) \leq \lim_{n\to\infty} -\int L u_n\,\psi(u_n)\,d\overline{\mu} = 0
\end{aligned}
$$

(cf. the proof of Step 3). Similarly, $\lim_{n\to\infty}\int_{\{-u\in[M_1,M_2]\}}\langle A\nabla u_n,\nabla u_n\rangle\,d\overline{\mu}=0$ and the claim is proved.

Let $\varphi\in\mathcal{M}$ be smooth, $0\le\dot\varphi\le 1$, $\|\varphi\|_\infty\le\|u\|_\infty+2$ and $\varphi(t)=t$ if $|t|\le\|u\|_\infty+1$. Similarly to the proof of the corresponding statement in Step 1 it follows that $(\varphi(u_n))\subset\mathcal{V}_b$ bounded and $\lim_{n\to\infty}u_n=u$ in $L^1(V_T,\overline{\mu})$.

Clearly, $\lim_{n\to\infty}\ddot\varphi(u_n)\langle A\nabla u_n,\nabla u_n\rangle=0$ in $L^1(V_T,\overline{\mu})$ hence in particular in $\overline{L^1}$ by the last statement. Furthermore, $\lim_{n\to\infty}\dot\varphi(u_n)Lu_n=Lu$ in $\overline{L^1}$ since

$$|\dot\varphi(u_n)Lu_n-\overline{L}u|_V\le|\dot\varphi(u_n)\overline{L}(u_n-u)|_V+|(\dot\varphi(u_n)-1)\overline{L}u|_V\to 0\ ,$$

$$\begin{aligned}|\dot\varphi(u_n)\overline{L}^V(u_n-u)|_V&=\sup_{v\in\mathcal{V}_b:\|v\|_{\mathcal{V}_b}\le 1}\int\overline{L}^V(u_n-u)\,\dot\varphi(u_n)v\,d\overline{\mu}\\&\le|\overline{L}^V(u_n-u)|_V\|\varphi(u_n)\|_{\mathcal{V}_b}\to 0,\end{aligned}$$

and $\lim_{n\to\infty}(\dot\varphi(u_n)-1)\overline{L}^V u=0$ in $L^1(V_T,\overline{\mu})$, hence in particular in $\overline{L^1}$. Consequently,

$$\overline{L}^V\varphi(u_n)=\dot\varphi(u_n)\overline{L}^V u_n+\ddot\varphi(u_n)\langle A\nabla u_n,\nabla u_n\rangle\to\overline{L}^V u$$

in $\overline{L^1}$ and $(\varphi(u_n))$ is the desired sequence.

Proof of 1.5 (iii): We will prove the statements for $\overline{L}^V$ only. The dual statements can be shown similarly. Fix $u\in D(\overline{L}^V)_b$, $v\in D(\overline{L}^{V,\prime})_b$ and let $(u_n)\subset C_0^\infty(V_T)$ and $(v_n)\subset C_0^\infty(V_0)$ as in 1.5 (ii). Similarly to the proof of the corresponding statements in Step 1 it follows that $\lim_{n\to\infty}u_n=u$ in $L^1(V_T,\overline{\mu})$ and $(\varphi(u_n))\subset\mathcal{V}_b$ bounded for all $\varphi\in\mathcal{M}$. In particular, $(u_n)\subset\mathcal{V}_b$ bounded itself, since $\sup_{n\ge1}\|u_n\|_\infty<+\infty$. Hence $u\in\mathcal{V}_b$ and $\lim_{n\to\infty}u_n=u$ weakly in $\mathcal{V}$. Passing to a subsequence we may assume in addition that $\lim_{n\to\infty}u_n=u$ weakly* in $\overline{L^1}'$. Hence

$$\mathcal{A}^0(u,u)\le\liminf_{n\to\infty}\mathcal{A}(u_n,u_n)\le\liminf_{n\to\infty}-\int\overline{L}^V u_n\,u_n\,d\overline{\mu}=-\int\overline{L}^V u\,u\,d\overline{\mu}\ .$$

This proves (a). Moreover,

$$\begin{aligned}\mathcal{A}^0(u_n-u,u_n-u)&\le\liminf_{m\to\infty}\mathcal{A}^0(u_n-u_m,u_n-u_m)\\&\le\liminf_{m\to\infty}-\int L(u_n-u_m)(u_n-u_m)\,d\overline{\mu}\\&=-\int L(u_n-u)(u_n-u)\,d\overline{\mu}\ ,\end{aligned}$$

and consequently, $\lim_{n\to\infty}u_n=u$ in $\mathcal{V}$.

To prove (b) note that $u_n^2\in C_0^\infty(V_T)$ and $L(u_n^2)=2u_nLu_n+2\langle A\nabla u_n,\nabla u_n\rangle$. Hence it suffices to prove that $\lim_{n\to\infty}2u_nLu_n+2\langle A\nabla u_n,\nabla u_n\rangle=2uLu+2\langle A\nabla u,\nabla u\rangle$ in $\overline{L^1}$. Clearly, $\lim_{n\to\infty}\langle A\nabla u_n,\nabla u_n\rangle=\langle A\nabla u,\nabla u\rangle$ in $L^1(V_T,\overline{\mu})$, hence in particular in $\overline{L^1}$. Moreover,

$$\begin{aligned}|\int u\overline{L}^V u-u_nLu_n)v\,d\overline{\mu}&\le|\int(\overline{L}^V u)(u_n-u)v\,d\overline{\mu}|+|\int\overline{L}^V(u-u_n)u_nv\,d\overline{\mu}|\\&\le\|\overline{L}^V u(u-u_n)\|_1\|v\|_\infty+|\overline{L}^V(u-u_n)|_V\|u_nv\|_{\mathcal{V}_b}\\&\le\|\overline{L}^V u(u-u_n)\|_1\|v\|_\infty+|\overline{L}^V(u-u_n)|_V\|u_n\|_{\mathcal{V}_b}\|v\|_{\mathcal{V}_b}\ ,\end{aligned}$$

hence

$$|u\overline{L}^V u - u_n L u_n|_V = \sup_{\|v\|_{\mathcal{V}_b}\le 1} \int (u\overline{L}^V u - u_n L u_n)\, d\overline{\mu}$$
$$\le \|\overline{L}^V u(u-u_n)\|_1 + |\overline{L}^V (u-u_n)|_V \left(\sup_{n\ge 1} \|u_n\|_{\mathcal{V}_b}\right) \to 0, n\to\infty \ .$$

This proves (b). For the proof of (c) note that $\int \overline{L}^V u_n\, v_n\, d\overline{\mu} = \int u_n \overline{L}^{V,\prime} v_n\, d\overline{\mu}$ by 1.1 (iii), hence

$$\int \overline{L}^V u\, v\, d\overline{\mu} = \lim_{n\to\infty} \int \overline{L}^V u_n\, v_n\, d\overline{\mu} = \lim_{n\to\infty} \int u_n \overline{L}^{V,\prime} v_n\, d\overline{\mu} = \int u\overline{L}^{V,\prime} v\, d\overline{\mu} \ . \quad \square$$

Proof of 1.7. Let $(V^n)_{n\ge 1}$ be an increasing sequence of open subsets with $\partial V^n \in C^\infty$ relatively compact in U such that $\overline{V^n} \subset V^{n+1}$, $n\ge 1$, and $U = \bigcup_{n\ge 1} V^n$. Let $f\in L^1(U_T,\overline{\mu})$, $f\ge 0$. Then $\overline{G}_\alpha f := \lim_{n\to\infty} \overline{G}_\alpha^{V^n} f$ exists $\overline{\mu}$–a.e. by 1.6. Moreover, $\sup_{n\ge 1} \int \alpha \overline{G}_\alpha^{V^n} f\, d\overline{\mu} \le \int f\, d\overline{\mu}$ implies $\overline{G}_\alpha f \in L^1(U_T,\overline{\mu})$ and $\lim_{n\to\infty} \overline{G}_\alpha^{V^n} f = \overline{G}_\alpha f$ in $L^1(U_T,\overline{\mu})$. Consequently, $\lim_{\alpha\to\infty} \overline{G}_\alpha^{V^n} f = \lim_{\alpha\to\infty} \overline{G}_\alpha^{V^n} (f^+ - f^-) = \overline{G}_\alpha f$ in $L^1(U_T,\overline{\mu})$ for all $f\in L^1(U_T,\overline{\mu})$ too. It is easy to see that $(\overline{G}_\alpha)_{\alpha>0}$ is a sub–Markovian C_0–resolvent of contractions on $L^1(U_T,\overline{\mu})$ (cf. the proof of the similar statement in [St2, Th. 1.5]). To see that the corresponding generator $(\overline{L}, D(\overline{L}))$ extends $(L, C_0^\infty(U_T))$ fix $u\in C_0^\infty(U_T)$ and let n be such that $u\in H_0^{1,1}(V_T^n,\overline{\mu})$. Then $u\in D(\overline{L}^{V^m})$ and $\overline{L}^{V^m} u = Lu$, $m\ge n$, implies that $u = \overline{G}_1^{V^m}(1-L)u \to \overline{G}_1(1-L)u$. Hence $u\in D(\overline{L})$ and $\overline{L}u = Lu$. Clearly, $(\overline{L}, D(\overline{L}))$ is of diffusion type, since for $\varphi\in\mathcal{M}$ and $u_n = \overline{G}_1^{V^n}(1-\overline{L})u$ 1.5 (i) implies that $\int \overline{L}^{V^n} u_n \varphi(u_n)\, d\overline{\mu} \le 0$ and consequently, $\int \overline{L}u\varphi(u)\, d\overline{\mu} = \lim_{n\to\infty} \int \overline{L}^{V^n} u_n\varphi(u_n)\, d\overline{\mu} \le 0$.

We will show in the following that $(\overline{L}, D(\overline{L}))$ satisfies (a)–(c) as stated in the theorem. For the proof of (a) let $(U^n)_{n\ge 1}$ be as in (a) and $f\in L^1(U_T,\overline{\mu})$, $f\ge 0$. By compactness of $\overline{V^n}$ there exist m such that $V^n\subset U^m$ and therefore $\overline{G}_\alpha^{V^n} f \le \overline{G}_\alpha^{U^m} f$ by 1.6. Hence $\overline{G}_\alpha f \le \lim_{n\to\infty} \overline{G}_\alpha^{U^n} f$. Similarly, $\lim_{n\to\infty} \overline{G}_\alpha^{U^n} f \le \overline{G}_\alpha f$ hence (a) is satisfied.

For the proof of (b) fix $u\in D(\overline{L})_b$. Then $\alpha\overline{G}_\alpha^{V^n} u \in H_0^{0,1}(V_T^n,\overline{\mu}) \subset D(\mathcal{A}^0)$ for all n and by 1.5 (iii)

$$\limsup_{n\to\infty} \mathcal{A}^0(\alpha\overline{G}_\alpha^{V^n} u, \alpha\overline{G}_\alpha^{V^n} u) \le \liminf_{\alpha\to\infty} \alpha \int (u - \alpha\overline{G}_\alpha^{V^n} u)\, \alpha\overline{G}_\alpha^{V_n} u\, d\overline{\mu}$$
$$= \alpha \int (u-\alpha\overline{G}_\alpha u)\, \alpha\overline{G}_\alpha u\, d\overline{\mu} = \int (\alpha\overline{G}_\alpha \overline{L}u)\alpha\overline{G}_\alpha u\, d\overline{\mu} \ .$$

In particular, $\sup_{n\ge 1} \mathcal{A}^0(\alpha\overline{G}_\alpha^{V^n} u, \alpha\overline{G}_\alpha^{V^n} u) < +\infty$, hence $\alpha\overline{G}_\alpha u\in D(\mathcal{A}^0)$ and

$$\mathcal{A}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) \le -\int \alpha\overline{G}_\alpha(\overline{L}u)\alpha\overline{G}_\alpha u\, d\overline{\mu} \le \|\overline{L}u\|_1 \|u\|_\infty \ ,$$

which now implies $\sup_{\alpha>0}\mathcal{A}^0(\alpha\overline{G}_\alpha u,\alpha\overline{G}_\alpha u)<+\infty$, hence $u\in D(\mathcal{A}^0)$ and $\mathcal{A}^0(u,u)\le-\int\overline{L}u\,u\,d\overline{\mu}$.

For the proof of (1.10) fix $v\in C_0^\infty(U_0)$ and assume first that $u:=\overline{G}_1 f$, $f\in L^1(U_T,\overline{\mu})_b$. Since $v\in D(\overline{L}^{V^n,\prime})$ for big n, 1.5 (iii) implies for $u_n:=\overline{G}_1^{V^n}f$ that

$$\mathcal{A}^0(u_n,v)+\int\langle\overline{\beta},\overline{\nabla}v\rangle u_n\,d\overline{\mu}=\int\overline{L}^{V^n,\prime}v\,u_n\,d\overline{\mu}=\int\overline{L}^{V^n}u_n\,v\,d\overline{\mu}\,. \tag{2.6}$$

Since $\lim_{n\to\infty}u_n=u$ in $L^1(U_T,\overline{\mu})$ and weakly in $D(\mathcal{A}^0)$ we can take the limit on both sides of (2.6) to obtain (1.10) in this case. For general $u\in D(\overline{L})_b$ consider $\alpha\overline{G}_\alpha u=\overline{G}_1(u+(1-\alpha)\overline{G}_\alpha u)$, $\alpha>0$. Then

$$\mathcal{A}^0(\alpha\overline{G}_\alpha u,v)+\int\langle\overline{\beta},\overline{\nabla}v\rangle\alpha\overline{G}_\alpha u\,d\overline{\mu}=\int\alpha\overline{G}_\alpha\overline{L}u\,v\,d\overline{\mu} \tag{2.7}$$

for all $\alpha>0$. Taking the limit $\alpha\to\infty$ in (2.7) implies (1.10).

For the proof of (c) it is enough to show that $u\in D(\overline{L})_b$ implies $u^2\in D(\overline{L})_b$ and $\overline{L}u^2=g_u$, where $g_u:=2u\overline{L}u+2\langle A\nabla u,\nabla u\rangle$. To this end it suffices to prove that

$$\int\overline{L}'v\,u^2\,d\mu=\int gv\,d\mu \text{ for all } v=\overline{G}_1'h\,,h\in L^1(U_T,\overline{\mu})_b\,, \tag{2.8}$$

since then $\int\overline{G}_1(u^2-g)h\,d\overline{\mu}=\int(u^2-g)\overline{G}_1'h\,d\overline{\mu}=\int u^2(\overline{G}_1'h-\overline{L}'\overline{G}_1'h)\,d\overline{\mu}=\int u^2\,h\,d\overline{\mu}$ for all $h\in L^1(U_T,\overline{\mu})_b$. Consequently, $u^2=\overline{G}_1(u^2-g)\in D(\overline{L})_b$.

For the proof of (2.8) fix $v=\overline{G}_1'h$, $h\in L^1(U,\mu)_b$. Suppose first that $u=\overline{G}_1f$ for some $f\in L^1(U_T,\overline{\mu})_b$. Let $\overline{u}_n:=\overline{G}_1^{V^n}f$ and $\overline{v}_n=\overline{G}_1^{V^n,\prime}h$. By 1.5 (ii) we can find $u_n\in C_0^\infty$ with $\|u_n\|_\infty\le\|u\|_\infty+2$ and $\|\overline{u}_n-u_n\|_1+|\overline{L}^{V^n}(\overline{u}_n-u_n)|_{V^n}\le\frac{1}{n}$ and $v_n\in C_0^\infty(V_0)$ such that $\|\overline{v}_n-v_n\|_1+|\overline{L}^{V^n,\prime}(\overline{v}_n-v_n)|_{V^n}\le\frac{1}{n}$. Note that $(u_n),(v_n)\subset D(\mathcal{A}^0)$ bounded, so that $\lim_{n\to\infty}u_n=u$ weakly in $D(\mathcal{A}^0)$ and thus $\lim_{n\to\infty}\int\langle A\nabla u_n,\nabla u\rangle v_n\,d\overline{\mu}=\int\langle A\nabla u,\nabla u\rangle v\,d\overline{\mu}$. Clearly,

$$\begin{aligned}\int\overline{L}^{V^n,\prime}v_n\,uu_n\,d\overline{\mu}&=-\mathcal{A}^0(v_n,uu_n)-\int\langle\overline{\beta},\overline{\nabla}v_n\rangle uu_n\,d\overline{\mu}\\&=-\mathcal{A}^0(v_nu_n,u)-\int\langle A\nabla v_n,\nabla u_n\rangle u\,d\overline{\mu}+\int\langle A\nabla u_n,\nabla u\rangle v_n\,d\overline{\mu}\\&\qquad-\int\langle\overline{\beta},\overline{\nabla}(v_nu_n)\rangle u\,d\overline{\mu}+\int\langle\overline{\beta},\overline{\nabla}u_n\rangle v_nu\,d\overline{\mu}\\&=\int\overline{L}u\,v_nu_n\,d\overline{\mu}+\int\overline{L}^{V^n}u_n\,v_nu\,d\overline{\mu}+\int\langle A\nabla u_n,\nabla(v_nu)\rangle\,d\overline{\mu}\\&\qquad-\int\langle A\nabla v_n,\nabla u_n\rangle u\,d\overline{\mu}+\int\langle A\nabla u_n,\nabla u\rangle v_n\,d\overline{\mu}\\&=\int\overline{L}u\,v_nu_n\,d\overline{\mu}+\int\overline{L}^{V^n}u_n\,v_nu\,d\overline{\mu}+2\int\langle A\nabla u_n,\nabla u\rangle v_n\,d\overline{\mu}\,.\end{aligned} \tag{2.9}$$

Since $|\int\overline{L}^{V^n,\prime}v_nuu_n\,d\overline{\mu}-\int\overline{L}^{V^n,\prime}\overline{v}_nuu_n\,d\overline{\mu}|\le|\overline{L}^{V^n}(v_n-\overline{v}_n)|_{V^n}\|uu_n\|_{H_0^{1,1}(V_T^n,\overline{\mu})_b}\to 0$, $n\to\infty$, and $\lim_{n\to\infty}\int\overline{L}^{V^n,\prime}\overline{v}_nuu_n\,d\overline{\mu}=\int\overline{L}'vu^2\,d\overline{\mu}$ it follows that

$\lim_{n\to\infty} \int \overline{L}^{V^n,\prime} v_n\, u u_n\, d\overline{\mu} = \int \overline{L} u\, v u\, d\overline{\mu}$. Similarly, $\lim_{n\to\infty} \int \overline{L}^{V^n} u_n\, v_n u\, d\overline{\mu}$ $= \int \overline{L} u\, v u\, d\overline{\mu}$. Hence we may pass to the limit in (2.9) to obtain (2.8) in this particular case.

For arbitrary $u \in D(\overline{L})_b$ let $g_\alpha := 2(\alpha \overline{G}_\alpha u)\overline{L}(\alpha \overline{G}_\alpha u) + 2\langle A\nabla \alpha \overline{G}_\alpha u, \nabla \alpha \overline{G}_\alpha u\rangle$, $\alpha > 0$. Then (1.10) implies

$$\begin{aligned} \mathcal{A}^0(\alpha\overline{G}_\alpha u - u, \alpha\overline{G}_\alpha u - u) &\leq -\int \overline{L}(\alpha\overline{G}_\alpha u - u)(\alpha\overline{G}_\alpha u - u)\, d\overline{\mu} \\ &\leq 2\|u\|_\infty \|\alpha\overline{G}_\alpha \overline{L}u - \overline{L}u\|_1 \to 0 \end{aligned}$$

if $\alpha \to \infty$, which implies that $\lim_{\alpha\to\infty} \alpha\overline{G}_\alpha u = u$ in $D(\mathcal{A}^0)$ and thus $\lim_{\alpha\to\infty} g_\alpha = g$ in $L^1(U_T, \overline{\mu})$. Since $\alpha u + (1-\alpha)\overline{G}_\alpha u \in L^1(U,\mu)_b$ and $\overline{G}_1(\alpha u + (1-\alpha)\overline{G}_\alpha u) = \alpha\overline{G}_\alpha u$ by the resolvent equation it follows from what we have just proved that

$$\int \overline{L}' v(\alpha\overline{G}_\alpha u)^2\, d\mu = \int g_\alpha v\, d\overline{\mu}$$

for all $\alpha > 0$ and thus, taking the limit $\alpha \to \infty$,

$$\int \overline{L}' v\, u^2\, d\overline{\mu} = \int g v\, d\overline{\mu}$$

and (2.8) is shown. $\square$

3. Proof of Theorem 1.10 and Prop. 1.11

Throughout the whole section fix an increasing sequence of open sets V^n as in Theorem 1.5 (a). Let $(L^n, D(L^n))$ be the L^2–realization of $(\overline{L}^{V^n}, D(\overline{L}^{V^n}))$ (cf. Remark following 1.4) and let $(G^n_\alpha)_{\alpha>0}$ be the resolvent of L^n.

Lemma 3.1. *(i) $D(L^n) \subset H^{0,1}_0(V^n_T, \overline{\mu})$ and $\mathcal{A}^0(f,f) \leq -\int L^n f\, f\, d\overline{\mu}$, $f \in D(L^n)$.*

(ii) $D(L) \subset D(\mathcal{A}^0)$ and $\mathcal{A}^0(f,f) \leq -\int Lf\, f\, d\overline{\mu}$, $f \in D(L)$.

Proof. (i) Let $g_k \in L^1(V^n_T, \overline{\mu})_b$ be such that $\lim_{k\to\infty} \|g_k - (1-L^n)f\|_2 = 0$. Then $G^n_1 g_k \in D(\overline{L}^{V^n})_b \subset H^{0,1}(V^n_T, \overline{\mu})$ and $\mathcal{A}^0_1(G^n_1 g_k - G^n_1 g_l, G^n_1 g_k - G^n_1 g_l) \leq \int (g_k - g_l)(G^n_1 g_k - G^n_1 g_l)\, d\overline{\mu}$. Now $\lim_{k\to\infty} \|G^n_1 g_k - f\|_2 = 0$ implies that $(G^n_1 g_k)_{k\geq 1}$ is an $H^{0,1}_0(V^n_T, \overline{\mu})$–Cauchy sequence, hence $f \in H^{0,1}_0(V^n_T, \overline{\mu})$ and $\mathcal{A}^0(f,f) \leq -\int L^n f f\, d\overline{\mu}$.

(ii) By 1.7 (b) the proof of (ii) is similar to (i). $\square$

To unify notations in the following let $V^\infty := U$, $(L^\infty, D(L^\infty)) = (L, D(L))$ and $(G^\infty_\alpha) = (G_\alpha)$. Furthermore, denote by the superscript ^{n} all potential theoretic concepts that are meant w.r.t. L^n, $n \in \mathbb{N} \cup \{+\infty\}$. In particular, let e^n_F for $f \in L^2(V^n_T, \overline{\mu})$ with $\mathcal{L}_f \cap D(L^n) \neq \emptyset$ be the 1–reduced function of f w.r.t. L^n, that is, the smallest 1–excessive element (1–excessive w.r.t. (G^n_α)) dominating f. Furthermore, an increasing sequence of closed subsets $F_k \subset V^n_T$, $k \geq 1$, is called an L^n–nest if $e^n_{f 1_{F^c_k}} \to 0$ in $L^2(V^n_T, \overline{\mu})$ for all $f \in D(L^n)$, f 1–excessive w.r.t. (G^n_α).

Lemma 3.2. *Let $f \in L^2(V_T^\infty, \overline{\mu})$, $f \geq 0$; $m, n \in \mathbb{N} \cup \{+\infty\}$, $m \geq n$. Then* $e^m_{(G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}} = G_1^m f - G_1^n f$.

Proof. We first show that $h := G_1^m f - G_1^n f$ is 1–excessive (w.r.t. (G_α^m)). Since $h \geq G_1^m f$ on $V_T^m \setminus \overline{V}_T^n$ it then follows that $e^m_{(G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}} \leq G_1^m f - G_1^n f$. To see that h is 1–excessive w.r.t. (G_α^m) note that

$$\beta G^m_{\beta+1} G_1^n f \geq \beta G^n_{\beta+1} G_1^n f = G_1^n f - G^n_{\beta+1} f \ ,$$

and thus

$$\beta G^m_{\beta+1}(G_1^m f - G_1^n f) \leq G_1^m f - G^m_{\beta+1} f + G^n_{\beta+1} f - G_1^n f \leq G_1^m f - G_1^n f \ .$$

To prove the converse inequality, let $g := e^m_{(G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}}$ and denote by g_α the unique element in $D(L^m)$ such that $(1 - L^m) g_\alpha = \alpha (g_\alpha - g)^-$. Then $0 \leq g_\alpha \leq g (\leq h)$ and $\lim_{\alpha \to \infty} g_\alpha = g$ in $L^2(V_T^m, \overline{\mu})$ (cf. [St1,III.1.7]). Now $h = G_1^m f$ on $V_T^m \setminus \overline{V}_T^n$ implies that

$$\begin{aligned}
\int (g_\alpha - \alpha G^m_{\alpha+1} h)^2 \, d\overline{\mu} &\leq \int (1 - L^m)(g_\alpha - \alpha G^m_{\alpha+1} h)(g_\alpha - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \\
&= \alpha \int ((g_\alpha - (G_1^m f)1_{V_T^m \setminus \overline{V}_T^n})^- - (h - \alpha G^m_{\alpha+1} h))(g_\alpha - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \\
&= -\alpha \int_{\{g_\alpha \geq (G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}\}} (h - \alpha G^m_{\alpha+1} h)(g_\alpha - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \\
&\quad + \alpha \int_{\{g_\alpha < (G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}\}} ((G_1^m f)1_{V_T^m \setminus \overline{V}_T^n} - g_\alpha - (h - \alpha G^m_{\alpha+1} h)) \\
&\qquad\qquad (g_\alpha - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \\
&\leq -\alpha \int_{\{g_\alpha \geq (G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}\}} (h - \alpha G^m_{\alpha+1} h)(h - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \\
&\quad + \alpha \int_{\{g_\alpha < (G_1^m f)1_{V_T^m \setminus \overline{V}_T^n}\}} (\alpha G^m_{\alpha+1} h - g_\alpha)(g_\alpha - \alpha G^m_{\alpha+1} h) \, d\overline{\mu} \leq 0 \ .
\end{aligned}$$

Consequently, $g_\alpha = \alpha G^m_{\alpha+1} h$ for all $\alpha > 0$, which implies the assertion taking the limit $\alpha \uparrow \infty$. $\square$

In the following, fix an element $\varphi \in L^1(U_T, \overline{\mu})$ such that $0 < \varphi \leq 1$.

Definition 3.3. Let $n \in \mathbb{N} \cup \{+\infty\}$. For $U \subset V_T^n$, U open, let $\mathrm{Cap}^n_\varphi(U) := \int e^n_{(G_1^n \varphi)1_U} \varphi \, d\overline{\mu}$ and for arbitrary $A \subset V_T^n$ let $\mathrm{Cap}^n_\varphi(A) := \inf_{\substack{A \subset U \\ U \text{open}}} \mathrm{Cap}^n_\varphi(U)$.

The main feature of the capacities just defined is that an increasing sequence of closed subsets $F_k \subset V_T^n$, $k \geq 1$, is an L^n–nest if and only if $\lim_{k \to \infty} \mathrm{Cap}^n_\varphi(V_T^n \setminus F_k) = 0$ (cf. [St1, III.2.10]).

Lemma 3.4. *Let $V \subset V_T^\infty$ be open; $m, n \in \mathbb{N} \cup \{+\infty\}$, $m \geq n$. Then:*

(i) $Cap_\varphi^m(V \cap V_T^m) \leq Cap_\varphi^n(V \cap V_T^n) + Cap_\varphi^m(V_T^m \setminus \overline{V}_T^n)$.

(ii) $Cap_\varphi^n(V \cap V_T^n) \leq Cap_\varphi^m(V \cap V_T^m)$.

Proof. (i) Let us show first that $e^n_{(G_1^n\varphi)1_{V\cap V_T^n}} + (G_1^m\varphi - G_1^n\varphi)$ is 1–excessive w.r.t. $(G_\alpha^m)_{\alpha>0}$. To simplify notations let $e^n := e^n_{(G_1^n\varphi)1_{V\cap V_T^n}}$. The resolvent equation for $(G_\alpha^n)_{\alpha>0}$ (resp. $(G_\alpha^m)_{\alpha>0}$) then implies

$$\begin{aligned}\beta G^m_{\beta+1}(e^n + G_1^m\varphi - G_1^n\varphi) &= \beta G^n_{\beta+1}e^n + \beta G^m_{\beta+1}(e^n - G_1^n\varphi) - \beta G^n_{\beta+1}(e^n - G_1^n\varphi) \\ &\quad + (G_1^m\varphi - G_1^n\varphi) - (G^m_{\beta+1}\varphi - G^n_{\beta+1}\varphi) \\ &\leq \beta G^n_{\beta+1}e^n + (G_1^m\varphi - G_1^n\varphi) \leq e^n + (G_1^m\varphi - G_1^n\varphi)\ ,\end{aligned}$$

since $G_1^n\varphi - e^n \geq 0$, hence $\beta G^n_{\beta+1}(G_1^n\varphi - e^n) \leq \beta G^m_{\beta+1}(G_1^n\varphi - e^n)$, and e^n is 1–excessive w.r.t. $(G_\alpha^n)_{\alpha>0}$. Since $e^n_{(G_1^n\varphi)1_{V\cap V_T^n}} + (G_1^m\varphi - G_1^n\varphi) \geq (G_1^m\varphi)1_{V\cap V_T^m}$ it follows that $e^m_{(G_1^m\varphi)1_{V_T^m\setminus V}} \leq e^n_{(G_1^n\varphi)1_{V\cap V_T^n}} + (G_1^m\varphi - G_1^n\varphi)$ and 3.2 now implies

$$\begin{aligned}\mathrm{Cap}_\varphi^m(V \cap V_T^m) &= \int e^m_{(G_1^m\varphi)1_{V\cap V_T^m}}\varphi\, d\overline{\mu} \\ &\leq \int e^n_{(G_1^n\varphi)1_{V\cap V_T^n}}\varphi\, d\overline{\mu} + \int e^m_{(G_1^m\varphi)1_{V_T^m\setminus V_T^n}}\varphi\, d\overline{\mu} \\ &= \mathrm{Cap}_\varphi^n(V \cap V_T^n) + \mathrm{Cap}_\varphi^m(V_T^m \setminus \overline{V}_T^n)\ .\end{aligned}$$

(ii) For the proof of (ii) first note that if u is 1–excessive w.r.t. $(G_\alpha^m)_{\alpha>0}$ then $u1_{V_T^n}$ is 1–excessive w.r.t. $(G_\alpha^n)_{\alpha>0}$. Indeed, since $u \geq 0$ it follows that $\beta G^n_{\beta+1}(u1_{V_T^n}) \leq (\beta G^m_{\beta+1}u)1_{V_T^n} \leq u1_{V_T^n}$. But then $e^m_{(G_1^m\varphi)1_{V\cap V_T^m}}1_{V_T^n}$ is 1–excessive w.r.t. $(G_\alpha^n)_{\alpha>0}$ which implies that $e^n_{(G_1^n\varphi)1_{V\cap V_T^n}} \leq e^m_{(G_1^m\varphi)1_{V\cap V_T^m}}1_{V_T^n} \leq e^m_{(G_1^m\varphi)1_{V\cap V_T^m}}$, and thus

$$\mathrm{Cap}_\varphi^n(V \cap V_T^n) = \int e^n_{(G_1^n\varphi)1_{V\cap V_T^n}}\varphi\, d\overline{\mu} \leq \int e^m_{(G_1^m\varphi)1_{V\cap V_T^m}}\varphi\, d\overline{\mu} = \mathrm{Cap}_\varphi^m(V \cap V_T^m)\ .$$

□

Lemma 3.5. *(i) Let $f \in D(L^n)_b, n \in \mathbb{N}$. Then $e_f^n \in H_0^{0,1}(V_T, \overline{\mu})$ and $\mathcal{A}_1^0(e_f^n, e_f^n) \leq 16|(1-L^n)f|_{V^n}(|(1-L^n)f|_{V^n} + \|f\|_\infty)$.*

(ii) Let $f \in D(\overline{L})_b$ with $\mathcal{L}_{|f|} \cap D(L^\infty) \neq \emptyset$. Then $e_f^\infty \in D(\mathcal{A}^0)$ and $\mathcal{A}_1^0(e_f^\infty, e_f^\infty) \leq 20\|(1-\overline{L}^\infty)f\|_1\|f\|_\infty$.

Proof. (i) By assumption e_f^n exists. Let $f_\alpha \in D(L^n)$, $\alpha > 0$, be the uniquely determined element in $D(L^n)$ with $(1-L^n)f_\alpha = \alpha(f_\alpha - f)^-$ (cf. [St1, III.1.6]). Then $0 \leq f_\alpha \leq e_f^n$, $\alpha > 0$, and $\lim_{\alpha\to\infty} f_\alpha = e_f^n$ in $L^2(V_T^n, \overline{\mu})$ (cf. [St1, III.1.7]). By 3.1 (i)

$$\begin{aligned}\mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) &\leq \int (1-L^n)(f_\alpha - f)(f_\alpha - f)\, d\overline{\mu} \\ &= \alpha\int (f_\alpha - f)^-(f_\alpha - f)\, d\overline{\mu} - \int (1-L^n)f(f_\alpha - f)\, d\overline{\mu} \\ &\leq \|(1-L^n)f\|_2\|f_\alpha - f\|_2\ ,\end{aligned}$$

which implies that $\sup_{\alpha>0} \mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) < +\infty$, therefore $e_f^n \in H_0^{0,1}(V_T^n, \overline{\mu})$, $\lim_{\alpha\to\infty} f_\alpha = e_f^n$ weakly in $H_0^{0,1}(V_T^n, \overline{\mu})$ and

$$\begin{aligned}
\mathcal{A}_1^0(e_f^n, e_f^n) &\leq \liminf_{\alpha\to\infty} \mathcal{A}_1^0(f_\alpha, f_\alpha) \\
&\leq \liminf_{\alpha\to\infty} 2\mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) + 2\mathcal{A}_1^0(f, f) \\
&\leq \liminf_{\alpha\to\infty} -2\int (1-L^n)f\,(f_\alpha - f)\, d\overline{\mu} + 2\int (1-L^n) f\, f\, d\overline{\mu} \\
&= -2\int (1-L^n) f(e_f^n - f)\, d\overline{\mu} + 2\int (1-L^n) f\, f\, d\overline{\mu}\ .
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathcal{A}_1^0(e_f^n, e_f^n) \leq |(1-L^n)f|_{V^n}(2\mathcal{A}^0(e_f^n, e_f^n)^{\frac{1}{2}} + 4\mathcal{A}^0(f,f)^{\frac{1}{2}} + 6\|f\|_\infty) \\
&\quad \leq 2|(1-L^n)f|_{V^n}^2 + \frac{1}{2}\mathcal{A}^0(e_f^n, e_f^n) + |(1-L^n)f|_{V^n}(4\mathcal{A}^0(f,f)^{\frac{1}{2}} + 6\|f\|_\infty)\ ,
\end{aligned}$$

hence

$$\mathcal{A}_1^0(e_f^n, e_f^n) \leq 20|(1-L^n)f|_{V^n}(|(1-L^n)f|_{V^n} + \|f\|_\infty)\ .$$

Here we used the fact that

$$\mathcal{A}^0(f,f) \leq \int (1-L^n) f\, f\, d\overline{\mu} \leq |(1-L^n)f|_{V^n}(\mathcal{A}^0(f,f)^{\frac{1}{2}} + \|f\|_\infty)\ ,$$

hence $\mathcal{A}^0(f,f) \leq 2|(1-L^n)f|_{V^n}^2 + \|f\|_\infty^2$.

(ii) Similarly to (i) let $f_\alpha \in D(L^\infty)$, $\alpha > 0$, be the uniquely determined element in $D(L^\infty)$ with $(1-L^\infty)f_\alpha = \alpha(f_\alpha - f)^-$. Since $0 \leq f_\alpha \leq e_f^\infty \leq \|f\|_\infty$, $\alpha > 0$, 3.1 (ii) implies

$$\begin{aligned}
\mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) &\leq \int (1-L^\infty)(f_\alpha - f)(f_\alpha - f)\, d\overline{\mu} \\
&= \alpha \int (f_\alpha - f)^-(f_\alpha - f)\, d\overline{\mu} - \int (1-L^\infty) f(f_\alpha - f)\, d\overline{\mu} \\
&\leq 2\|(1-L^\infty)f\|_1 \|f\|_\infty\ ,
\end{aligned}$$

hence $\sup_{\alpha>0} \mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) < +\infty$, therefore $e_f^\infty \in D(\mathcal{A}^0)$ and $\lim_{\alpha\to\infty} f_\alpha = e_f^\infty$ weakly in $D(\mathcal{A}^0)$. Finally,

$$\begin{aligned}
\mathcal{A}_1^0(e_f^\infty, e_f^\infty) &\leq \liminf_{\alpha\to\infty} \mathcal{A}_1^0(f_\alpha, f_\alpha) \\
&\leq \liminf_{\alpha\to\infty} 2\mathcal{A}_1^0(f_\alpha - f, f_\alpha - f) + 2\mathcal{A}_1^0(f, f) \\
&\leq \liminf_{\alpha\to\infty} 6\|(1-L^\infty)f\|_1 \|f\|_\infty\ . \quad \square
\end{aligned}$$

Lemma 3.6. *Let $f \in D(L^n)_b$. Then f admits an L^n–q.c. $\overline{\mu}$–version $\tilde{f}$. Moreover, if we extend $\tilde{f}$ by 0 to $V_T^\infty \setminus V_T^n$ then there exists for all $\varepsilon > 0$ a subset $F_\varepsilon \subset V_T^\infty$, closed in V_T^∞, such that $\tilde{f}_{|F_\varepsilon}$ is continuous and $Cap_\varphi^n(V_T^n \setminus F_\varepsilon) \le \varepsilon$.*

Proof. Since $D(L^n)_b \subset D(\overline{L}^{V^n})_b$ there exist $f_k \in C_0^\infty(V_T^n)$ with $\sup_{k\ge 1} \|f_k\|_\infty < +\infty$ and $\lim_{k\to\infty} |(1-L^n)(f_k - f)|_{V^n} = 0$ by 1.5 (ii). Consequently, $\lim_{k\to\infty} e^n_{f_k - f} + e^n_{f-f_k} = 0$ in $H_0^{0,1}(V_T^n, \overline{\mu})$, in particular in $L^2(V_T^n, \overline{\mu})$ by 3.5. Since each $f_k \in C_0^\infty(V_T^n)$ is clearly continuous, [St1, III.3.7] now implies the existence of an L^n–q.c. $\overline{\mu}$–version $\tilde{f}$ of f such that $f_{k_l} \to \tilde{f}$ L^n–quasi uniformly along some subsequence, i.e., uniformly on F_n, $n \ge 1$, for an L^n–nest $(F_k)_{k\ge 1}$. Note that f_k can be extended by 0 to a continuous function on $V^\infty \setminus V_T^n$. If we extend $\tilde{f}$ similarly by 0 on $V^\infty \setminus V_T^n$ we obtain that $f_{k_l} \to \tilde{f}$ uniformly on $F_k' := F_k \cup V_T^\infty \setminus F_k$ for all k. Clearly, $F_k' \subset V_T^\infty$ is closed and $\mathrm{Cap}_\varphi^n(V_T^n \setminus F_k') = \mathrm{Cap}_\varphi^n(V_T^n \setminus F_k) \to 0$, $k \to \infty$, which proves the second assertion. $\square$

Lemma 3.7. *Let $f \in L^1(V_T^\infty, \overline{\mu})_b$, $f \ge 0$. Then $G_1^\infty f$ admits an L^∞–q.c. $\overline{\mu}$–version.*

Proof. For all $n \in \mathbb{N}$ fix an L^n–q.c. $\overline{\mu}$–version $\tilde{h}_n$ of $G_1^n f$. We extend $\tilde{h}_n$ to V_T^∞ by letting $\tilde{h}_n(x) = 0$ for $x \in V_T^\infty \setminus V_T^n$.

Step 1: For all $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbb{N}$ and a closed subset $F_\varepsilon \subset V_T^\infty$ such that $\tilde{h}_{n|F_\varepsilon}$ is continuous for all $n \ge n_\varepsilon$ and $\mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus F_\varepsilon) \le \varepsilon$.

Proof: Fix n_ε such that $\mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus \overline{V}_T^{n_\varepsilon}) \le \frac{\varepsilon}{2}$. By 3.6 we can find a subset F_n, closed in V_T^∞, such that $\tilde{h}_{n|F_n}$ is continuous and $\mathrm{Cap}_\varphi^n(V_T^n \setminus F_n) \le \varepsilon 2^{-(n+1)}$. Let $F_\varepsilon := \bigcap_{n\ge n_\varepsilon} F_n$. Then $F_\varepsilon \subset V_T^\infty$ closed and 3.4 now implies that

$$\begin{aligned}
\mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus F_\varepsilon) &\le \mathrm{Cap}_\varphi^{n_\varepsilon}((V_T^\infty \setminus F_\varepsilon) \cap V_T^{n_\varepsilon}) + \mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus \overline{V}_T^{n_\varepsilon}) \\
&\le \mathrm{Cap}_\varphi^{n_\varepsilon}(\bigcup_{n\ge n_\varepsilon} V_T^{n_\varepsilon} \setminus F_n) + \frac{\varepsilon}{2} \le \sum_{n\ge n_\varepsilon} \mathrm{Cap}_\varphi^{n_\varepsilon}(V_T^{n_\varepsilon} \setminus F_n) + \frac{\varepsilon}{2} \\
&\le \sum_{n \ge n_\varepsilon} \mathrm{Cap}_\varphi^n(V_T^n \setminus F_n) + \frac{\varepsilon}{2} \le \sum_{n\ge n_\varepsilon} \varepsilon 2^{-(n+1)} + \frac{\varepsilon}{2} \le \varepsilon \ .
\end{aligned}$$

Step 2: There exists an L^∞–q.c. $\overline{\mu}$–version $\tilde{h}$ of $G_1^\infty f$ such that $\tilde{h}_n \to \tilde{h}$ L^∞–quasi uniformly along some subsequence.

Proof: By Step 1 there exists a sequence of closed subsets (F_k) such that $\tilde{h}_{n|F_k}$ is continuous for all $n \ge \alpha_k$ (where α_k is sufficiently big) and $\mathrm{Cap}_\varphi^\infty(F_k^c) \le 2^{-k}$. Without loss of generality we may assume (F_k) and (α_k) to be increasing. Since $\{|\tilde{h}_n - \tilde{h}_m| > \lambda\} \cup F_k^c$ is open for all $\lambda > 0$; $n, m \ge \alpha_k$, $v_n := G_1^\infty f - G_1^n f$ is 1–excessive w.r.t. $(G_\alpha^\infty)_{\alpha>0}$ for all n and

$$G_1^\infty \varphi \le (1/\lambda)(v_n + v_m) + (G_1^\infty \varphi)_{F_k^c} \text{ on } \{|\tilde{h}_n - \tilde{h}_m| > \lambda\} \cup F_k^c$$

for all k, n, m, it follows that

$$\begin{aligned}
\mathrm{Cap}_\varphi^\infty(\{|\tilde{h}_n - \tilde{h}_m| > \lambda\}) &\le (1/\lambda) \int (v_n + v_m)\, \varphi \, d\overline{\mu} + \int e^\infty_{(G_1^\infty \varphi) 1_{F_k^c}} \varphi \, d\overline{\mu} \\
&\le (1/\lambda)(\|v_n\|_2 + \|v_m\|_2) \|\varphi\|_2 + 2^{-k} \ .
\end{aligned}$$

Since $\lim_{n\to\infty} v_n = 0$ in $L^2(V_T^\infty, \overline{\mu})$ we can find an increasing subsequence $(n_k)_{k\geq 1}$ with $n_k \geq \alpha_k$ and $\sum_{k=1}^\infty 2^k \|v_{n_k}\|_2 < \infty$. Let

$$A_k := \bigcup_{i\geq k} \{|\tilde{h}_{n_{i+1}} - \tilde{h}_{n_i}| > 2^{-i}\}\,, k \geq 1\,,$$

then $F_k' := F_k \cap A_k^c$ is an increasing sequence of closed subsets with

$$\begin{aligned}
\mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus F_k') &\leq \mathrm{Cap}_\varphi^\infty(V_T^\infty \setminus F_k) + \mathrm{Cap}_\varphi^\infty(A_k) \\
&\leq 2^{-k} + \sum_{i\geq k} \mathrm{Cap}_\varphi^\infty(\{|\tilde{h}_{n_{i+1}} - \tilde{h}_{n_i}| > 2^{-i}\}) \\
&\leq 2^{-k} + \sum_{i\geq k} 2^i(\|v_{n_{i+1}}\|_2 + \|v_{n_i}\|_2)\|\varphi\|_2 + 2^{-i} \\
&\leq 3/2^k + (3/2)\|\varphi\|_2 \sum_{i\geq k} 2^i \|v_{n_i}\|_2 \to 0\,, k \to \infty\,,
\end{aligned}$$

hence $(F_k')_{k\geq 1}$ is an L^∞–nest. Since $(\tilde{h}_{n_k})_{k\geq 1}$ converges uniformly on F_k' we obtain that

$$\tilde{h}(x) := \begin{cases} \lim_{k\to\infty} \tilde{h}_{n_k}(x) & \text{if } x \in \bigcup_{k\geq 1} F_k' \\ 0 & \text{else} \end{cases}$$

is L^∞–q.c. Clearly, $\tilde{h}$ is a $\overline{\mu}$–version of $G_1^\infty f$. $\square$

Proof (of Theorem 1.10). To apply the main existence result in the theory of *generalized Dirichlet forms* we need to show that $(L^\infty, D(L^\infty))$ is quasi–regular and that there exists a linear subspace $\mathcal{Y} \subset L^\infty(V_T^\infty, \overline{\mu})$ such that $\mathcal{Y} \cap D(L^\infty) \subset D(L^\infty)$ dense, $\lim_{\alpha\to\infty} e^\infty_{\alpha G_\alpha^\infty f - f} = 0$ in $L^2(V_T^\infty, \overline{\mu})$ for all $f \in \mathcal{Y}$ and $\alpha \wedge f \in \overline{\mathcal{Y}}$ (= the closure of $\mathcal{Y}$ in $L^\infty(V_T^\infty, \overline{\mu})$) for all $f \in \overline{\mathcal{Y}}$ and $\alpha \geq 0$.

We will show first the existence of an L^∞–nest consisting of compact sets. To this end let $F_n := [0, \frac{n-1}{n}T] \times \overline{V}^n$, $n \geq 1$. We will show that $e^\infty_{(G_1^\infty \varphi)1_{V^\infty \setminus F_n}} \to 0$, $n \to \infty$. To this end fix n and note that $V_T^\infty \setminus F_n = U_1 \cup U_2$ for the open sets $U_1 = (V^\infty \setminus \overline{V}^n)_T$ and $U_2 = (\frac{n-1}{n}T, T) \times V^\infty$. Note that $G_1^\infty \varphi - G_1^n \varphi \geq G_1^\infty \varphi$ on U_1 and $G_1^\infty \varphi \wedge (T/n) \geq G_1^\infty \varphi$ on U_2. Indeed, since the semigroup $(\overline{T}_t^\infty)$ generated by $(\overline{L}^\infty, D(\overline{L}^\infty))$ is of evolution type by 1.9 it follows that $1_{[T-t,T)} \overline{T}_t^\infty \varphi = \overline{T}_t^\infty (1_{[T-t,T)}(\cdot - t) 1_{[t,T)} \varphi) = \overline{T}_t^\infty 0 = 0$. Hence

$$\begin{aligned}
1_{[\frac{n-1}{n}T,T)} G_1^\infty \varphi &= \int_0^\infty e^{-t} 1_{[\frac{n-1}{n}T,T)} \overline{T}_t^\infty \varphi \, dt = \int_0^\infty e^{-t} 1_{[\frac{n-1}{n}T,T-t)} \overline{T}_t^\infty(\varphi)\, dt \\
&\leq \int_0^\infty e^{-t} 1_{[\frac{n-1}{n}T,T-t)}\, dt \leq T/n\ ,
\end{aligned}$$

which implies that $G_1^\infty \varphi \leq T/n$ on U_2. Since both, $G_1^\infty \varphi - G_1^n \varphi$ and $G_1^\infty \varphi \wedge (T/n)$ are 1–excessive w.r.t. (G_α^∞) it follows that $e^\infty_{(G_1^\infty \varphi)1_{V^\infty \setminus F_n}} \leq G_1^\infty \varphi - G_1^n \varphi + G_1^\infty \varphi \wedge (T/n)$. Since the right hand side of the last inequality converges to 0 for $n \to \infty$ we obtain that $\mathrm{Cap}_\varphi^\infty(V^\infty \setminus F_n) = \int e^\infty_{(G_1^\infty \varphi)1_{V^\infty \setminus F_n}} \varphi \, d\overline{\mu} \to 0$, hence (F_n) is an L^∞–nest.

Since $G_1(L^1(V_T^\infty,\overline{\mu})_b) \subset D(L^\infty)$ dense there exists by 3.7 a dense subset of $D(L^\infty)$ whose elements have L^∞–q.c. $\overline{\mu}$–versions. Finally, since $C_0^\infty(V_T^\infty) \subset D(L)$ there exists a countable subset of $\overline{\mu}$–versions of elements in $D(L)$ separating the points of V_T^∞.

Finally, let $\mathcal{Y} := \{f \in D(\overline{L}^\infty)_b | \mathcal{L}_{|f|} \cap D(L^\infty) \neq \emptyset\}$. $\mathcal{Y}$ is an algebra, since $D(\overline{L}^\infty)_b$ is an algebra by 1.7 (c). Hence $f \wedge \alpha \in \overline{\mathcal{Y}}$ if $f \in \mathcal{Y}$ and $\alpha \geq 0$. Clearly, $\mathcal{Y} \cap D(L^\infty) = D(L^\infty)_b \subset D(L^\infty)$ dense. Let $f \in \mathcal{Y}$. Then $e^\infty_{\alpha G_\alpha f - f} \in D(\mathcal{A}^0)$ by 3.5 (ii) and the strong continuity of $(\overline{G}_\alpha)_{\alpha>0}$ in $L^1(V_T^\infty,\overline{\mu})$ now implies that

$$\begin{aligned}\mathcal{A}_1^0(e^\infty_{\alpha G_\alpha f - f}, e^\infty_{\alpha G_\alpha f - f}) &\leq 6\|(1-\overline{L}^\infty)(\alpha G_\alpha f - f)\|_1 \|\alpha G_\alpha f - f\|_\infty \\ &\leq 12\|\alpha \overline{G}_\alpha (1-\overline{L}^\infty)f - (1-\overline{L}^\infty)f\|_1 \|f\|_\infty \to 0 \ , \alpha \to \infty \ . \quad \square\end{aligned}$$

Let us now turn to the proof of 1.11. We will need one lemma. Let $\mathbb{M}$ be as in 1.10. Denote by $(p_t)_{t\geq 0}$ (resp. $(R_\alpha)_{\alpha>0}$) the corresponding transition semigroup (resp. resolvent). For $f \in \mathcal{B}_b(U_T)$ and $V \subset U_T$, V open, let

$$H^V f(s,x) = E_{(s,x)}\left[e^{-\sigma_V} f(Y_{\sigma_V})\right] \ .$$

Here, $\sigma_V = \inf\{t > 0 | Y_t \in V\}$ is the first hitting time of V. To simplify notations in the following let $f_V := e^\infty_{f1_V}$ for $f \in L^2(U_T,\overline{\mu})$, $V \subset U_T$ open.

Lemma 3.8. *Let $K \subset U_T$ be compact, and $V := U_T \setminus K$. Let $f \in C_0^\infty(V)$, $f \geq 0$. Then $H^V f = 0$ L–q.e. on K.*

Proof. Let $g := (1-L)f$, $h^+ := R_1(g^+)$ and $h^- := R_1(g^-)$. For $n \in \mathbb{N}$ let $h_n^+ := R_1(g^+ \wedge n)$ (resp. $h_n^- := R_1(g^- \wedge n)$). Then $H^V h_n^+$ (resp. $H^V h_n^-$) is an L–quasi lower semicontinuous (= L–q.l.s.c.) $\overline{\mu}$–version of $(h_n^+)_V$ (resp. $(h_n^-)_V$) (that is, $H^V h^+_{n|F_k}$ (resp. $H^V h^-_{n|F_k}$) is a $\overline{\mu}$–version of $(h_n^+)_V$ (resp. $(h_n^+)_V$) and there exists an L–nest (F_k) such that $H^V h^+_{n|F_k}$ (resp. $H^V h^-_{n|F_k}$) is lower semicontinuous for all k)(cf. [St1, IV.3.9]). Since $h^+ = \sup_{n\geq 1} h_n^+$ (resp. $h^- = \sup_{n\geq 1} h_n^-$) we conclude that $H^V h^+$ (resp. $H^V h^-$) too is an L–q.l.s.c. $\overline{\mu}$–version of h_V^+ (resp. $\bar{}_V$).

We will show next that $H^V h^+ = h^+$ $\overline{\mu}$–a.e. To this end let h_α^+, $\alpha > 0$, be the unique element in $D(L)$ such that $(1-L)h_\alpha^+ = \alpha(h_\alpha^+ - h^+ 1_V)^-$. Since $0 \leq h_\alpha^+ \leq h_V^+$, $\lim_{\alpha\to\infty} h_\alpha^+ = h_V^+$ in $L^2(U_T,\overline{\mu})$ (cf. [St1, III.1.7]) we obtain that

$$\begin{aligned}\int (H^V h^+ - h^+)^2 \, d\overline{\mu} &= \lim_{\alpha\to\infty} \int (h_\alpha^+ - h^+)^2 \, d\overline{\mu} \\ &\leq \lim_{\alpha\to\infty} \int (1-L)(h_\alpha^+ - h^+)(h_\alpha^+ - h^+) \, d\overline{\mu} \\ &= \lim_{\alpha\to\infty} \int (\alpha(h_\alpha^+ - h^+ 1_V)^- - (1-L)h^+ (h_\alpha^+ - h^+) \, d\overline{\mu} \\ &\leq \lim_{\alpha\to\infty} - \int g^+ (h^+ - h_\alpha^+) \, d\overline{\mu} \\ &= - \int ((1-L)f)^+ (h^+ - h_V^+) \, d\overline{\mu} = 0 \ ,\end{aligned}$$

since $h_V^+ = h^+$ on V and $(1-L)f = 0$ on K. This is where we use the fact that the operator $(L, C_0^\infty(U_T))$ is local. Similarly, $H^V h^- = h^-$ in $L^2(U_T, \overline{\mu})$ and thus $H^V h = h$ $\overline{\mu}$–a.e. Since $H^V h^+$ (resp. $H^V h^-$) is L–q.l.s.c. it follows that $H^V h^+ \le 0$ L–q.e. (resp. $H^V h^- \le 0$ L–q.e.) on K. Thus $0 \le H^V f = H^V(h^+ - h^-) \le H^V(h^+ + h^-) \le 0$ L–q.e. on K which implies the assertion. $\square$

Proof (of 1.11). Fix $K \subset U_T$ compact and let $u_n \in C_0^\infty(U_T \setminus K)$, $n \ge 1$, $u_n \ge 0$, such that $\sup_{n\ge 1} u_n > 0$ on $U_T \setminus K$. Then $H^V u_n = 0$ L–q.e. on K for all n by 3.9 implies that $P_{(s,x)}\left[Y_{\sigma_V} \in V\right] = 0$ L–q.e. on K.

Now, let (K_n) be a countable family of compact subsets of U_T such that $(U_T \setminus K_n)_{n\ge 1}$ separates the points of U_T and let $\mathcal{U} = \{U_T \setminus K_n | n \in \mathbb{N}\}$. Then there exists some L–exceptional set N such that $P_{(s,x)}\left[Y_{\sigma_V} \in V\right] = 0$ for all $(s,x) \in U_T \setminus N$, $V \in \mathcal{U}$. Let $(F_k)_{k\ge 1}$ be an L–nest such that $N \subset \bigcap_{k\ge 1} U_T \setminus F_k$ and $\Omega_0 := \{\lim_{k\to\infty} \sigma_{U_T\setminus F_k} \ge \zeta\}$. Let

$$\Omega_d := \{\omega | Y_{t-}(\omega) \ne Y_t(\omega) \text{ for some } t \in (0, \zeta(\omega))\} .$$

Then

$$\Omega_d \cap \Omega_0 \subset \bigcup_{V\in\mathcal{U}} \bigcup_{\substack{t\in \\ \mathbb{Q}\cap(0,\infty)}} \{\omega | Y_t(\omega) \in V^c \setminus N, Y_{\sigma_V}(\theta_t\omega) \in V\} .$$

Since

$$P_{(s,x)}\left[Y_t \in V^c \setminus N, Y_{\sigma_V} \circ \theta_t \in V\right] = P_{(s,x)}\left[P_{Y_t}\left[Y_{\sigma_V} \in V\right], Y_t \in V^c \setminus N\right] = 0$$

for all $(s,x) \in U_T$, $V \in \mathcal{U}$, it follows that $P_{(s,x)}\left[\Omega_d \cap \Omega_0\right] = 0$ for all $(s,x) \in U_T$, and thus $P_{(s,x)}\left[\Omega_d\right] = 0$ L–q.e., since $P_{(s,x)}\left[\Omega_0\right] = 1$ L–q.e. by [St1, IV.3.10]. $\square$

4. Proof of Theorem 1.14

Proof of 1.14. First note that $\int (\alpha_0 - L_{A,B}) uh\, d\overline{\mu} \le 0$ for all $u \in C_0^\infty(\mathbb{R}_T^d)$, $u \ge 0$, implies the same inequality for all $u \in C_0^{1,2}(\mathbb{R}_T^d)$, $u \ge 0$. To simplify notations let $\mathcal{V} = D(\mathcal{A}^0)$. Fix $\chi \in C_0^\infty(B_r(0)_T)$. We have to show that $\chi h^+ \in \mathcal{V}$. Fix $L \ge 0$ and $\alpha \in (0,1)$ with $|a_{ij}(s,x) - a_{ij}(t,y)| \le L\left(|x-y|^\alpha + |s-t|^{\frac{\alpha}{2}}\right)$ for all $(s,x), (t,y) \in B_r(0)_T$ and define

$$\overline{a}_{ij}(s,x) := a_{ij}(s, (\frac{r}{|x|} \wedge 1)x) \ , (s,x) \in \mathbb{R}_T^d .$$

Then $\overline{a}_{ij}(s,x) := a_{ij}(s,x)$ for all $s \in [0,T]$, $x \in B_r(0)$, and $|\overline{a}_{ij}(s,x) - \overline{a}_{ij}(t,y)| \le 2L\left(|x-y|^\alpha + |s-t|^{\frac{\alpha}{2}}\right)$ for all $(s,x), (t,y) \in \mathbb{R}_T^d$. Let $L^{\overline{A}} = \sum_{i,j=1}^d \overline{a}_{ij}\partial_{ij} + \partial_t$. By [Fr, Th. 1.12, p.25] (and the time reversal $t \mapsto T-t$, $t \in [0,T]$) there exists for all $f : \mathbb{R}_T^d \to \mathbb{R}$, f bounded and Lipschitz–continuous, a function $R_\alpha f \in C^{1,2}(\mathbb{R}_T^d)$, $R_\alpha f(T,\cdot) = 0$, for which $(\alpha - L^{\bar{A}})R_\alpha f = f$ and $|R_\alpha f(t,x)| \le c\exp(c|x|^2)$ for some constant $c > 0$.

If $f \ge 0$ then $R_\alpha f \ge 0$ by [Fr, Th. 2.9, p.43]. Moreover, since $(\alpha - L^{\overline{A}})(\frac{1}{\alpha}\|f\|_\infty \pm R_\alpha f = \|f\|_\infty \pm f \ge 0$, also $|R_\alpha f| \le \frac{1}{\alpha}\|f\|_\infty$, i.e., $\|\alpha R_\alpha f\|_\infty \le \|f\|_\infty$ by [Fr, Th. 2.9, p.43] again.

Since $C_0^\infty(\mathbb{R}^d_T) \subset C_\infty(\mathbb{R}^d_T)$ dense, where $C_\infty(\mathbb{R}^d_T)$ is the space of all bounded continuous functions on $\mathbb{R}^d_T$ vanishing at infinity, we obtain that $f \mapsto \alpha R_\alpha f$, $f \in C_0^\infty(\mathbb{R}^d_T)$, can be uniquely extended to a positive linear contraction $\alpha\overline{R}_\alpha : C_\infty(\mathbb{R}^d_T) \to C_b(\mathbb{R}^d_T)$, hence a sub–Markovian kernel on $(\mathbb{R}^d_T, \mathcal{B}(\mathbb{R}^d_T))$, which will be denoted again by $\alpha\overline{R}_\alpha$.

Let $f_n \in C_0^\infty(\mathbb{R}^d_T)$, $n \geq 1$, such that $0 \leq f_n \leq \|h\|_\infty$ and $g := \lim_{n\to\infty} f_n$ is a $\overline{\mu}$–version of h^+. Then $\lim_{n\to\infty} \alpha\overline{R}_\alpha f_n(s,x) = \alpha\overline{R}_\alpha g(s,x)$ for all $(s,x) \in \mathbb{R}^d_T$ by Lebesgue's theorem and $\|\alpha\overline{R}_\alpha g\|_\infty \leq \|h\|_\infty$.

Estimates for $\mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n)$, uniform in n for fixed α:

Note that

$$
\begin{aligned}
\mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) &\leq -\int L^{\overline{A}}(\chi\alpha\overline{R}_\alpha f_n)\chi\alpha\overline{R}_\alpha f_n \, d\overline{\mu} \\
&\qquad - \int \langle B, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle \chi(\alpha\overline{R}_\alpha f_n)\, d\overline{\mu} \\
&= -\int \chi L^{\overline{A}}\chi\, (\alpha\overline{R}_\alpha f_n)^2 \, d\overline{\mu} - 2\int \langle A\nabla\chi, \nabla\alpha\overline{R}_\alpha f_n\rangle \chi\alpha\overline{R}_\alpha f_n \, d\overline{\mu} \\
&\qquad - \int L^{\overline{A}}(\alpha\overline{R}_\alpha f_n)\, \chi^2\, \alpha\overline{R}_\alpha f_n \, d\overline{\mu} - \int \langle B, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle \chi\alpha\overline{R}_\alpha f_n \, d\overline{\mu} \\
&= -\int \chi L^{\overline{A}}\chi\, (\alpha\overline{R}_\alpha f_n)^2 \, d\overline{\mu} - 2\int \langle A\nabla\chi, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle \alpha\overline{R}_\alpha f_n \, d\overline{\mu} \\
&\qquad + 2\int \langle A\nabla\chi, \nabla\chi\rangle (\alpha\overline{R}_\alpha f_n)^2 \, d\overline{\mu} - \alpha \int (\alpha\overline{R}_\alpha f_n - f_n)\chi^2\, \alpha\overline{R}_\alpha f_n \, d\overline{\mu} \\
&\qquad - \int \langle B, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle \chi\, \alpha\overline{R}_\alpha f_n \, d\overline{\mu}\ .
\end{aligned}
\tag{4.1}
$$

Hence $\mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) \leq c\,\mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n)^{1/2} + M$ for positive constants c and M independent of n. Thus, $\sup_{n\geq 1} \mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) < +\infty$, hence $\chi\alpha\overline{R}_\alpha g \in \mathcal{V}$ and $\lim_{n\to\infty} \chi\alpha\overline{R}_\alpha f_n = \chi\alpha\overline{R}_\alpha g$ weakly in $\mathcal{V}$. Taking the limit $n \to \infty$ in (4.1) we conclude that

$$
\begin{aligned}
\mathcal{A}^0(\chi\alpha\overline{R}_\alpha g, \chi\alpha\overline{R}_\alpha g) &\leq \liminf_{n\to\infty} \mathcal{A}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) \\
&\leq -\int \chi L^{\overline{A}}\chi\, (\alpha\overline{R}_\alpha g)^2 \, d\overline{\mu} - 2\int \langle A\nabla\chi, \nabla(\chi\alpha\overline{R}_\alpha g)\rangle \alpha\overline{R}_\alpha g \, d\overline{\mu} \\
&\qquad + 2\int \langle A\nabla\chi, \nabla\chi\rangle (\alpha\overline{R}_\alpha g)^2 \, d\overline{\mu} - \alpha\int (\alpha\overline{R}_\alpha g - g)\chi^2\, \alpha\overline{R}_\alpha g \, d\overline{\mu} \\
&\qquad - \int \langle B, \nabla(\chi\alpha\overline{R}_\alpha g)\rangle \chi\, \alpha\overline{R}_\alpha g \, d\overline{\mu}\ .
\end{aligned}
\tag{4.2}
$$

Estimates for $\mathcal{A}^0(\chi\alpha\overline{R}_\alpha g, \chi\alpha\overline{R}_\alpha g)$, uniform in α:

Since $\alpha(g-\alpha\overline{R}_\alpha g)(h-g) = -\alpha(h^+-\alpha\overline{R}_\alpha g)h^- \geq 0$ $\overline{\mu}$–a.e. we obtain that

$$
\begin{aligned}
-\alpha\int(\alpha\overline{R}_\alpha g-g)(\alpha\overline{R}_\alpha g)\chi^2\,d\overline{\mu} &= -\alpha\int(\alpha\overline{R}_\alpha g-g)g\chi^2\,d\overline{\mu} \\
&\quad -\alpha\int(\alpha\overline{R}_\alpha g-g)^2\chi^2\,d\overline{\mu} \\
&\leq -\alpha\int(\alpha\overline{R}_\alpha g-g)h\chi^2\,d\overline{\mu} - \alpha\int(\alpha\overline{R}_\alpha g-g)^2\chi^2\,d\overline{\mu} \qquad (4.3)\\
&= \lim_{n\to\infty} -\alpha\int(\alpha\overline{R}_\alpha f_n-f_n)h\chi^2\,d\overline{\mu} - \alpha\int(\alpha\overline{R}_\alpha g-g)^2\chi^2\,d\overline{\mu} \\
&= \lim_{n\to\infty} -\int L^{\overline{A}}(\alpha\overline{R}_\alpha f_n)h\chi^2\,d\overline{\mu} - \alpha\int(\alpha\overline{R}_\alpha g-g)^2\chi^2\,d\overline{\mu} \\
&= \lim_{n\to\infty} -\int L^{\overline{A}}(\chi^2\alpha\overline{R}_\alpha f_n)h\,d\overline{\mu} + 4\int\langle A\nabla\chi,\nabla\alpha\overline{R}_\alpha f_n\rangle\chi h\,d\overline{\mu} \\
&\quad + \int L^{\overline{A}}(\chi^2)\alpha\overline{R}_\alpha f_n\,h\,d\overline{\mu} - \alpha\int(\alpha\overline{R}_\alpha g-g)^2\chi^2\,d\overline{\mu}\ .
\end{aligned}
$$

Observe that $L^{\overline{A}}(\chi^2\alpha\overline{R}_\alpha f_n) = (L_{A,B}+\partial_t)(\chi^2\alpha\overline{R}_\alpha f_n) - \langle B,\nabla(\chi^2\alpha\overline{R}_\alpha f_n)\rangle$, so that

$$
\begin{aligned}
-\int L^{\overline{A}}(\chi^2\alpha\overline{R}_\alpha f_n)h\,d\overline{\mu} &= -\int(L_{A,B}+\partial_t)(\chi^2\alpha\overline{R}_\alpha f_n)h\,d\overline{\mu} \\
&\quad + \int\langle B,\nabla(\chi^2\alpha\overline{R}_\alpha f_n)\rangle h\,d\overline{\mu} \qquad (4.4)\\
&\leq -\alpha_0\int\chi^2(\alpha\overline{R}_\alpha f_n)h\,d\overline{\mu} + \int\langle B,\nabla(\chi^2\alpha\overline{R}_\alpha f_n)\rangle h\,d\overline{\mu}\ .
\end{aligned}
$$

Inserting (4.4) into (4.3) and using $g=h^+$ $\overline{\mu}$–a.e. we obtain that

$$
\begin{aligned}
&-\alpha\int(\alpha\overline{R}_\alpha g-g)(\alpha\overline{R}_\alpha g)\chi^2\,d\overline{\mu} \\
&\quad\leq \lim_{n\to\infty} -\alpha_0\int\chi^2(\alpha\overline{R}_\alpha f_n)h\,d\overline{\mu} + \int\langle B,\nabla(\chi^2\alpha\overline{R}_\alpha f_n)\rangle h\,d\overline{\mu} \\
&\qquad + 4\int\langle A\nabla\chi,\nabla\alpha\overline{R}_\alpha f_n\rangle\chi h\,d\overline{\mu} + \int L^{\overline{A}}(\chi^2)\alpha\overline{R}_\alpha f_n\,h\,d\overline{\mu} \\
&\qquad - \alpha\int(\alpha\overline{R}_\alpha g-h^+)^2\chi^2\,d\overline{\mu} \qquad (4.5)\\
&\quad= -\alpha_0\int\chi^2(\alpha\overline{R}_\alpha g)h\,d\overline{\mu} + \int\langle B,\nabla(\chi^2\alpha\overline{R}_\alpha g)\rangle h\,d\overline{\mu} \\
&\qquad + 4\int\langle A\nabla\chi,\nabla(\chi\alpha\overline{R}_\alpha g)\rangle h\,d\overline{\mu} - 4\int\langle A\nabla\chi,\nabla\chi\rangle(\alpha\overline{R}_\alpha g)h\,d\overline{\mu} \\
&\qquad + \int L^{\overline{A}}(\chi^2)\alpha\overline{R}_\alpha g\,h\,d\overline{\mu} - \alpha\int(\alpha\overline{R}_\alpha g-h^+)^2\chi^2\,d\overline{\mu}\ .
\end{aligned}
$$

Combining (4.2) and (4.5) we obtain that

$$
\mathcal{A}^0(\chi\alpha\overline{R}_\alpha g,\chi\alpha\overline{R}_\alpha g) + \alpha\int(\alpha\overline{R}_\alpha g-h^+)^2\chi^2\,d\overline{\mu} \leq \tilde{c}\,\mathcal{A}^0(\chi\alpha\overline{R}_\alpha g,\chi\alpha\overline{R}_\alpha g)^{1/2} + \tilde{M}
$$

for positive constants $\tilde{c}$ and $\tilde{M}$ independent of α. Hence $(\chi\alpha\overline{R}_\alpha g)_{\alpha>0}$ is bounded in $\mathcal{V}$ and $\lim_{\alpha\to\infty}\chi\alpha\overline{R}_\alpha g = \chi h^+$ in $L^2(\mathbb{R}^d_T,\overline{\mu})$. Consequently, $\chi h^+\in\mathcal{V}$ and $\lim_{\alpha\to\infty}\chi\alpha\overline{R}_\alpha g = \chi h^+$ weakly in $\mathcal{V}$.

Verification of (1.14):

Combining (4.2) and (4.5) and taking the limit $\alpha\to\infty$ we obtain

$$
\begin{aligned}
\mathcal{A}^0(\chi h^+,\chi h^+) &\le \liminf_{\alpha\to\infty}\mathcal{A}^0(\chi\alpha\overline{R}_\alpha g,\chi\alpha\overline{R}_\alpha g)\\
&\le \liminf_{\alpha\to\infty} -\int \chi L^{\overline{A}}\chi(\alpha\overline{R}_\alpha g)^2\,d\overline{\mu} - 2\int\langle A\nabla\chi,\nabla(\chi\alpha\overline{R}_\alpha g)\rangle\alpha\overline{R}_\alpha g\,d\overline{\mu}\\
&\qquad + 2\int\langle A\nabla\chi,\nabla\chi\rangle(\alpha\overline{R}_\alpha g)^2\,d\overline{\mu} - \alpha_0\int\chi^2(\alpha\overline{R}_\alpha g)h\,d\overline{\mu}\\
&\qquad + \int\langle B,\nabla(\chi^2\alpha\overline{R}_\alpha g)\rangle h\,d\overline{\mu} + 4\int\langle A\nabla\chi,\nabla(\chi\alpha\overline{R}_\alpha g)\rangle h\,d\overline{\mu}\\
&\qquad - 4\int\langle A\nabla\chi,\nabla\chi\rangle(\alpha\overline{R}_\alpha g)h\,d\overline{\mu} + \int L^{\overline{A}}(\chi^2)(\alpha\overline{R}_\alpha g)h\,d\overline{\mu}\\
&\qquad - \int\langle B,\nabla(\chi\alpha\overline{R}_\alpha g)\rangle\chi\alpha\overline{R}_\alpha g\,d\overline{\mu}\ ,
\end{aligned}
\tag{4.6}
$$

and the right hand side of (4.6) is equal to

$$
\begin{aligned}
&-\int\chi L^{\overline{A}}\chi(h^+)^2\,d\overline{\mu} - 2\int\langle A\nabla\chi,\nabla(\chi h^+)\rangle h^+\,d\overline{\mu}\\
&\qquad + 2\int\langle A\nabla\chi,\nabla\chi\rangle(h^+)^2\,d\overline{\mu} - \alpha_0\int\chi^2(h^+)^2\,d\overline{\mu}\\
&\qquad + \int\langle B,\nabla(\chi^2 h^+)\rangle h\,d\overline{\mu} + 4\int\langle A\nabla\chi,\nabla(\chi h^+)\rangle h\,d\overline{\mu}\\
&\qquad - 4\int\langle A\nabla\chi,\nabla\chi\rangle(h^+)^2\,d\overline{\mu} + \int L^{\overline{A}}(\chi^2)(h^+)^2\,d\overline{\mu}\\
&\qquad - \int\langle B,\nabla(\chi h^+)\rangle\chi h^+\,d\overline{\mu}\\
&\quad = -\alpha_0\int\chi^2(h^+)^2\,d\overline{\mu} + 2\int\langle A\nabla\chi,\nabla(\chi h^+)\rangle h^+\,d\overline{\mu}\\
&\qquad + \int\langle B,\nabla\chi\rangle\chi(h^+)^2\,d\overline{\mu} + \int\chi L^{\overline{A}}\chi(h^+)^2\,d\overline{\mu}\ .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{A}^0(\chi h^+,\chi h^+) + \alpha_0\int\chi^2(h^+)^2\,d\overline{\mu} &\le \int\langle A\nabla\chi,\nabla\chi\rangle(h^+)^2\,d\overline{\mu}\\
&\quad + \int\langle B-B^0,\nabla\chi\rangle\chi(h^+)^2\,d\overline{\mu} + \int\chi(\partial_t\chi)(h^+)^2\,d\overline{\mu}\ . \quad\square
\end{aligned}
$$

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