Simplicial n-fold Monoidal Categories Model All n-fold Loop Spaces

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1 Introduction

In recent years there has been an increasing interest in algebraic structures on a category motivated by coherence problems arising from topological quantum field theory. E.g. the categories of representations of quantum groups are braided monoidal categories (c.f. [5], [6]). Another motivation comes from new developments in stable homotopy theory, in particular new models for the stable homotopy category. E.g. Thomason showed that for the subcategory of connective spectra the category of small symmetric monoidal categories and lax symmetric monoidal functors is such a model with nice properties [13]. More precisely, the group completion of the classifying space of a symmetric monoidal category is an infinite loop space and hence represents a connective spectrum. Conversely, each connective spectrum arises that way up to weak equivalence.

In a similar way, monoidal categories correspond to loop spaces, and the group completion of the classifying space of a braided monoidal category is a two-fold loop space.

It has been an open question for some time which type of structure on a category corresponds to n-fold loop spaces. In [2] we introduced the notion of an n-fold monoidal category, and we could show that the classifying space of such a category is homotopy equivalent to a $\mathcal{C}_n$-space, where $\mathcal{C}_n$ is the little n-cubes operad of [1]; hence its group completion is an n-fold loop space. This poses the question whether a result similar to the connection between symmetric monoidal categories and connective spectra holds for n-fold monoidal categories and n-fold loop spaces. Since n-fold loop spaces are group completions of $\mathcal{C}_n$-spaces, this question is reduced to proving that up to weak equivalence each $\mathcal{C}_n$-space is the classifying space of an n-fold monoidal category. In the present paper we prove a slightly weaker result: up to weak equivalences the categories of $\mathcal{C}_n$-spaces and of simplicial n-fold monoidal categories are equivalent. The main technical ingredient of the proof is a new description of lax functors and of a variant of Street's second rectification, an approach which translates directly to homotopical coherence theory of [15] by applying the classifying space functor. Since this description can be extended to define lax functors and their rectifications in higher order categories, we believe that it is of independent interest. This part will be developed in Section 3. Our main results are listed in Section 2 and proved in Sections 3 to 6.

2 Definitions and main results

2.1 Definition: An *n-fold monoidal category*, $1 \leq n \leq \infty$, is a category $\mathcal{C}$ with the following structure.

1. There are n distinct multiplications
$$\Box_1, \Box_2, \ldots, \Box_n : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
which are strictly associative and $\mathcal{C}$ has an object 0 which is a strict unit for all the multiplications.

2. For each pair (i,j) such that $1 \leq i < j \leq n$ there is a natural transformation
$$\eta^{ij}_{A,B,C,D} : (A\Box_j B)\Box_i(C\Box_j D) \to (A\Box_i C)\Box_j(B\Box_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{A,B,0,0} = \eta^{ij}_{0,0,A,B} = id_{A\Box_j B}$

(b) External unit condition: $\eta^{ij}_{A,0,B,0} = \eta^{ij}_{0,A,0,B} = id_{A\Box_i B}$

(c) Internal associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U\Box_j V)\Box_i(W\Box_j X)\Box_i(Y\Box_j Z) \arrow[r, "\eta^{ij}_{U,V,W,X}\Box_i id_{Y\Box_j Z}"] \arrow[d, "id_{U\Box_j V}\Box_i \eta^{ij}_{W,X,Y,Z}"] & \big((U\Box_i W)\Box_j(V\Box_i X)\big)\Box_i(Y\Box_j Z) \arrow[d, "\eta^{ij}_{U\Box_i W, V\Box_i X, Y, Z}"] \\
(U\Box_j V)\Box_i\big((W\Box_i Y)\Box_j(X\Box_i Z)\big) \arrow[r, "\eta^{ij}_{U,V,W\Box_i Y, X\Box_i Z}"] & (U\Box_i W\Box_i Y)\Box_j(V\Box_i X\Box_i Z)
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U\Box_j V\Box_j W)\Box_i(X\Box_j Y\Box_j Z) \arrow[r, "\eta^{ij}_{U\Box_j V,W,X\Box_j Y,Z}"] \arrow[d, "\eta^{ij}_{U,V\Box_j W,X,Y\Box_j Z}"] & \big((U\Box_j V)\Box_i(X\Box_j Y)\big)\Box_j(W\Box_i Z) \arrow[d, "\eta^{ij}_{U,V,X,Y}\Box_j id_{W\Box_i Z}"] \\
(U\Box_i X)\Box_j\big((V\Box_j W)\Box_i(Y\Box_j Z)\big) \arrow[r, "id_{U\Box_i X}\Box_j\eta^{ij}_{V,W,Y,Z}"] & (U\Box_i X)\Box_j(V\Box_i Y)\Box_j(W\Box_i Z)
\end{tikzcd}$$

Finally it is required that for each triple (i,j,k) satisfying $1\leq i<j<k\leq n$ the (big!) hexagonal interchange diagram commutes.

$$\begin{tikzcd}[column sep=small]
 & ((A_1\Box_k A_2)\Box_j(B_1\Box_k B_2))\Box_i((C_1\Box_k C_2)\Box_j(D_1\Box_k D_2)) \arrow[dl, "\eta^{jk}_{A_1,A_2,B_1,B_2}\Box_i\eta^{jk}_{C_1,C_2,D_1,D_2}"'] \arrow[dr, "\eta^{ij}_{A_1\Box_k A_2,B_1\Box_k B_2,C_1\Box_k C_2,D_1\Box_k D_2}"] & \\
((A_1\Box_j B_1)\Box_k(A_2\Box_j B_2))\Box_i((C_1\Box_j D_1)\Box_k(C_2\Box_j D_2)) \arrow[d, "\eta^{ik}_{A_1\Box_j B_1,A_2\Box_j B_2,C_1\Box_j D_1,C_2\Box_j D_2}"] & & ((A_1\Box_k A_2)\Box_i(C_1\Box_k C_2))\Box_j((B_1\Box_k B_2)\Box_i(D_1\Box_k D_2)) \arrow[d, "\eta^{ik}_{A_1,A_2,C_1,C_2}\Box_j\eta^{ik}_{B_1,B_2,D_1,D_2}"'] \\
((A_1\Box_j B_1)\Box_i(C_1\Box_j D_1))\Box_k((A_2\Box_j B_2)\Box_i(C_2\Box_j D_2)) \arrow[dr, "\eta^{ij}_{A_1,B_1,C_1,D_1}\Box_k\eta^{ij}_{A_2,B_2,C_2,D_2}"] & & ((A_1\Box_i C_1)\Box_k(A_2\Box_i C_2))\Box_j((B_1\Box_i D_1)\Box_k(B_2\Box_i D_2)) \arrow[dl, "\eta^{jk}_{A_1\Box_i C_1,A_2\Box_i C_2,B_1\Box_i D_1,B_2\Box_i D_2}"'] \\
 & ((A_1\Box_i C_1)\Box_j(B_1\Box_i D_1))\Box_k((A_2\Box_i C_2)\Box_j(B_2\Box_i D_2)) &
\end{tikzcd}$$

Let $\mathcal{C}$ be a small category and $\mathbb{M}_n\mathcal{C}$ the free n-fold monoidal category generated by $\mathcal{C}$. The objects of $\mathbb{M}_n\mathcal{C}$ are all finite expressions generated by the objects of $\mathcal{C}$ using associative operations $\Box_1, \Box_2, \dots, \Box_n$, for example

$$(((C_1\Box_1C_2\Box_1C_3)\Box_2C_4\Box_2(C_5\Box_3C_6))\Box_2C_7)\Box_3(C_8\Box_2C_9)$$

We allow the vacuous expression, denoted 0, which serves as the unit object. The morphisms of $\mathbb{M}_n\mathcal{C}$ are finite composites of all possible finite formal expressions generated by the morphisms of $\mathcal{C}$ and symbols $\eta^{ij}_{A,B,C,D}$ with $1 \leq i < j \leq n$ and A, B, C, D objects of $\mathbb{M}_n\mathcal{C}$, using the associative operations $\Box_1, \Box_2, \dots$, and $\Box_n$. Two such composites of formal expressions are identified if and only if one can be converted to the other by repeated use of various functoriality, naturality and associativity diagrams. (This is a special case of forming a colimit in theories, cf. [1, p. 33 Prop.2.5].) We consider the set $\{1, \dots, k\}$ as category with objects $1, \dots, k$ and identity morphisms. Let $\mathcal{M}_n(k)$ be the full subcategory of $\mathbb{M}_n\{1, \dots, k\}$ whose objects are expressions in which each element $1, 2, \dots, k$ occurs exactly once. The symmetric group Σ_k acts freely on $\mathcal{M}_n(k)$ via functors by permuting labels on both objects and morphisms. We have maps

$$\mathcal{M}_n(k) \times \mathcal{M}_n(i_1) \times \dots \times \mathcal{M}_n(i_k) \longrightarrow \mathcal{M}_n(i_1 + \dots + i_k)$$

defined by replacing the label $r \in \{i_1, \dots, i_j\}$ in $\mathcal{M}_n(i_j)$ with the label $i_1 + \dots + i_{j-1} + r$ and substituting an element j in $\mathcal{M}_n(k)$ by the expression in $\mathcal{M}_n(i_j)$. This gives $\{\mathcal{M}_n(k)\}_{k\geq 0}$ the structure of a Σ-free operad in the category $\mathcal{C}at$ of small categories, and $\mathbb{M}_n$ is its associated monad

$$\mathbb{M}_n : \mathcal{C}at \longrightarrow \mathcal{C}at,\ \mathcal{C} \mapsto \mathbb{M}_n\mathcal{C} = \coprod_{k\geq 0} \mathcal{M}_n(k) \times_{\Sigma_k} \mathcal{C}^k.$$

Let $\mathbb{M} : \mathcal{T} \longrightarrow \mathcal{T}$ be a monad on an arbitrary category $\mathcal{T}$ with composition $\mu : \mathbb{M}\circ\mathbb{M} \longrightarrow \mathbb{M}$ and unit $\eta : Id \longrightarrow \mathbb{M}$. Recall that an $\mathbb{M}$-algebra consists of an object $X \in \mathcal{T}$ and a structure map $\xi : \mathbb{M}X \longrightarrow X$ such that the diagrams

$$\begin{tikzcd}
\mathbb{M}\mathbb{M}X \arrow[r, "\mu X"] \arrow[d, "\mathbb{M}\xi"] & \mathbb{M}X \arrow[d, "\xi"] \\
\mathbb{M}X \arrow[r, "\xi"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[dr, equal] & \mathbb{M}X \arrow[d, "\xi"] \\
 & X
\end{tikzcd}$$

commute. A homomorphism of $\mathbb{M}$-algebras $(X, \xi_X) \longrightarrow (Y, \xi_Y)$ is morphism $f : X \longrightarrow Y$ in $\mathcal{T}$ such that

$$\begin{tikzcd}
\mathbb{M}X \arrow[r, "\mathbb{M}f"] \arrow[d, "\xi_X"] & \mathbb{M}Y \arrow[d, "\xi_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes. The category of $\mathbb{M}$-algebras and homomorphisms in $\mathcal{T}$ is denoted by $\mathcal{T}^{\mathbb{M}}$. The following result is clear from the definitions (for further information see [2, Chap. 3]).

2.2 Lemma: The category of n-fold monoidal categories and strict n-fold monoidal functors is isomorphic to the category of $\mathbb{M}_n$-algebras in $\mathcal{C}at$ and homomorphisms.

2.3 Notations:

- $\mathcal{T}op$ denotes the category of compactly generated spaces [14, Expl. 5 (ii)].
- $\mathcal{SS}ets$ and $\mathcal{ST}op$ etc. denote the categores of simplicial sets, simplicial spaces etc. respectively.
- $N : \mathcal{C}at \longrightarrow \mathcal{SS}ets$ is the nerve functor.
- $|-| : \mathcal{SS}ets \longrightarrow \mathcal{T}op$ is the topological realization functor.
- $B = |-| \circ N : \mathcal{C}at \longrightarrow \mathcal{T}op$ is the classifying space functor.
- A map $f : X \longrightarrow Y$ in $\mathcal{T}op$ is a *weak equivalence* if $f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each $n \geq 0$ and each choice of base point $x \in X$.
- A map $f : X_\bullet \longrightarrow Y_\bullet$ in $\mathcal{ST}op$ is called a *weak equivalence* if $|f| : |X_\bullet| \longrightarrow |Y_\bullet|$ is a weak equivalence in $\mathcal{T}op$.
- A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ of small categories and a simplicial functor $F : \mathcal{C}_\bullet \longrightarrow \mathcal{D}_\bullet$ of simplicial categories is called a *weak equivalence* if $B(F)$ is a weak equivalence in $\mathcal{T}op$ respectively in $\mathcal{ST}op$.
- $\mathcal{C}_n$ denotes the little n-cube operad of Boardman and Vogt [1].
- If $\mathcal{M}$ is an operad in any of our categories, we denote its associated monad by $\mathbb{M}$.

Let $\mathcal{M}$ be a Σ-free operad in $\mathcal{C}at$, such as $\mathcal{M}_n$. Since the classifying space functor preserves products, $B\mathcal{M}$ is a Σ-free operad in $\mathcal{T}op$. If $B\mathbb{M}$ denotes its associated monad, then there is a natural isomorphism

2.4 $B(\mathbb{M}\mathcal{C}) \cong B\mathbb{M}(B\mathcal{C})$

In particular B defines a functor

$$B : \mathcal{C}at^{\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$$

and by prolongation a functor

$$B : \mathcal{SC}at^{\mathbb{M}} \longrightarrow \mathcal{ST}op^{B\mathbb{M}}.$$

It is well-known that the topological realization $|Z_\bullet|$ of a simplicial $B\mathbb{M}$-space $Z_\bullet$ is again a $B\mathbb{M}$-space [9, Thm. 12,21]. Hence we obtain a functor

$$|B(-)| : \mathcal{SC}at^{\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$$

The main aim of this paper is to prove

2.5 Theorem: For each n, $1 \leq n \leq \infty$ the functor $|B|$ induces an equivalence of categories

$$\mathcal{SC}at^{M_n}[we^{-1}] \longrightarrow \mathcal{T}op^{\mathbb{C}_n}[we^{-1}]$$

where the weak equivalences are those algebra homomorphisms whose underlying morphisms are weak equivalences in $\mathcal{SC}at$ respectively in $\mathcal{T}op$.

The theorem is a consequence of the following three results, the first of which is our main result of [2] (for $n = \infty$ we can take $\mathcal{D}_\infty = B\mathcal{M}_\infty \times \mathcal{C}_\infty$).

2.6 Theorem: For each n, $1 \leq n \leq \infty$, there is a proper Σ-free topological operad $\mathcal{D}_n$ and maps of operads

$$B\mathcal{M}_n \longleftarrow \mathcal{D}_n \longrightarrow \mathcal{C}_n$$

which are homotopy equivalences on underlying spaces.

Here we call a topological operad $\mathcal{C}$ *proper* if the inclusion of the identity $\{id\} \hookrightarrow \mathcal{C}(1)$ is a closed cofibration. Each $B\mathcal{M}$ for a $\mathcal{C}at$ operad $\mathcal{M}$ and each $\mathcal{C}_n$ is proper.

For a topological operad $\mathcal{M}$ let $cw\mathcal{ST}op^{\mathbb{M}}$ denote the full subcategory of the category of simplicial $\mathcal{M}$-spaces whose underlying spaces are simplicial CW-complexes with cellular structure maps.

2.7 Theorem: Let $\mathcal{M}$ be a Σ-free operad in $\mathcal{C}at$. Then $B : \mathcal{SC}at^{\mathbb{M}} \longrightarrow cw\mathcal{ST}op^{B\mathbb{M}}$ and $|-| : cw\mathcal{ST}op^{B\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$ induce equivalences of categories

$$\mathcal{SC}at^{\mathbb{M}}[we^{-1}] \simeq cw\mathcal{ST}op^{B\mathbb{M}}[we^{-1}] \simeq \mathcal{T}op^{B\mathbb{M}}[we^{-1}].$$

2.8 Proposition: Let $\alpha : \mathcal{D} \longrightarrow \mathcal{E}$ be a map of proper Σ-free topological operads which is a homotopy equivalence on underlying spaces. Then α induces an equivalence of categories

$$\mathcal{T}op^{\mathbb{E}}[we^{-1}] \simeq \mathcal{T}op^{\mathbb{D}}[we^{-1}].$$

The proof of the proposition is standard (e.g. see [9]). The monadic twosided bar construction defines a functor

$$B(\mathbb{E}, \mathbb{D}, -) : \mathcal{T}op^{\mathbb{D}} \longrightarrow \mathcal{T}op^{\mathbb{E}}.$$

If X is a $\mathbb{D}$-space, there is a diagram of weak equivalences of $\mathbb{D}$-spaces

$$B(\mathbb{E}, \mathbb{D}, X) \leftarrow B(\mathbb{D}, \mathbb{D}, X) \rightarrow X.$$

If Y is an $\mathbb{E}$-space, there are weak equivalences of $\mathbb{E}$-spaces

$$B(\mathbb{E}, \mathbb{D}, Y) \rightarrow B(\mathbb{E}, \mathbb{E}, Y) \rightarrow Y.$$

This establishes the result. $\square$

As to the existence of the localizations we adopt Thomason's approach. We assume Grothendieck's axiom of universes. For details see [13, (1.5)] and [3, p. 185 ff].

3 Lax functors and lax natural transformations

In this section we study a variant of Street's second rectification [11] in a setting which translates nicely into the topological situation. Our approach can be dualized to provide a similar variant of Street's first rectification. Although we treat only the case we need, we want to point out that our approach generalizes to define and rectify lax functors between higher order categories.

3.1 Definition: Let $\mathcal{A}$ be a category and $\underline{\mathcal{B}}$ a 2-category. A *lax functor* $\Phi : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ (with strict units) assigns to each $A \in ob\,\mathcal{A}$ an object ΦA of $\underline{\mathcal{B}}$, to each morphism $f : A_1 \longrightarrow A_2$ in $\mathcal{A}$ a morphism $\Phi f : \Phi A_1 \longrightarrow \Phi A_2$ in $\underline{\mathcal{B}}$, and to each pair of morphisms $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ in $\mathcal{A}$ a 2-cell $\varphi(g, f) : \Phi(g \circ f) \Longrightarrow \Phi(g) \circ \Phi(f)$ such that

1.

$$\begin{tikzcd}
\Phi(h\circ g\circ f) \arrow[r, Rightarrow, "\varphi(h,g\circ f)"] \arrow[d, Rightarrow, "\varphi(h\circ g,f)"'] & \Phi(h)\circ\Phi(g\circ f) \arrow[d, Rightarrow, "\Phi(h)\varphi(g,f)"] \\
\Phi(h\circ g)\circ\Phi(f) \arrow[r, Rightarrow, "\varphi(h,g)\Phi(f)"] & \Phi(h)\circ\Phi(g)\circ\Phi(f)
\end{tikzcd}$$

commutes

2. $\Phi(id_A) = id_{\Phi A}$ and $\varphi(id_{A_2}, f) = \varphi(f, id_{A_1}) = id_{\Phi(f)}$.

If $F : \mathcal{A}' \longrightarrow \mathcal{A}$ is a functor and $\underline{G} : \underline{\mathcal{B}} \longrightarrow \underline{\mathcal{B}}'$ is a 2-functor the composite lax functor $\underline{G} \circ \Phi \circ F$ is defined in the obvious way.

3.2 Construction: We define a functor

$$W : \mathcal{C}at \longrightarrow 2\text{-}\mathcal{C}at$$

from the category of small categories to the category of small 2-categories as follows. Let $\mathcal{A}$ be a category. Then $ob\, W\mathcal{A} = ob\mathcal{A}$ and

$$W\mathcal{A}(A,B) = \Big(\coprod_{n\geq 0} \mathcal{A}_{n+1}(A,B) \times \mathcal{L}^n\Big)/\sim$$

where $\mathcal{L}$ is the category $0 \xrightarrow{\iota} 1$ and

$$\mathcal{A}_{n+1}(A,B) = \{(f_n, \ldots, f_0) \in (\text{mor}\,\mathcal{A})^{n+1};\ f_n \circ \ldots \circ f_0 : A \longrightarrow B\}$$

considered as trivial category. Let

$$\max : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$$

be the unique functor sending the object (i,j) to $\max(i,j)$. The relations are given by

3.3 $(f_n, \ldots, f_0;\ \varepsilon_n, \ldots, \varepsilon_1)$

$$\begin{array}{ll} (1) & \\ (2) & \\ (3) & \\ (4) & \end{array} \sim \begin{cases} (f_n, \ldots, f_k \circ f_{k-1}, \ldots, f_0;\ \varepsilon_n, \ldots, \hat{\varepsilon}_k, \ldots, \varepsilon_1) & \text{if } \varepsilon_k = id_0 \\ (f_{n-1}, \ldots, f_0;\ \varepsilon_{n-1}, \ldots, \varepsilon_1) & \text{if } f_n = id \\ (f_n, \ldots, \hat{f}_k, \ldots, f_0;\ \varepsilon_n, \ldots, \max(\varepsilon_{k+1}, \varepsilon_k), \ldots, \varepsilon_1) & \text{if } f_k = id \\ (f_n, \ldots, f_1;\ \varepsilon_n, \ldots, \varepsilon_2) & \text{if } f_0 = id \end{cases}$$

where $(\varepsilon_n, \dots, \varepsilon_1) \in \operatorname{mor} \mathcal{L}^n$, and $\wedge$ means delete.

The composition functors

$$W\mathcal{A}(B, C) \times W\mathcal{A}(A, B) \longrightarrow W\mathcal{A}(A, C)$$

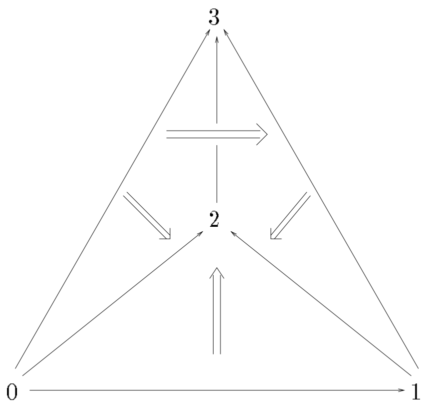
are defined by

$$\begin{aligned}(f_n, \dots, f_0;\ \varepsilon_n, \dots, \varepsilon_1) \circ (g_k, \dots, g_0;\ \delta_k \dots, \delta_1)& \\ = (f_n, \dots, f_0, g_k, \dots, g_0;\ \varepsilon_n, \dots, \varepsilon_1, id_1, \delta_k \dots, \delta_1)&.\end{aligned}$$

For a functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ the 2-functor $WF : W\mathcal{A} \longrightarrow W\mathcal{A}'$ is defined by

$$(f_n, \dots, f_0;\ \varepsilon_n, \dots, \varepsilon_1) \longmapsto (Ff_n, \dots, Ff_0; \varepsilon_n, \dots, \varepsilon_1).$$

3.4 Example: Let $\mathcal{A}$ be the poset $0 < 1 < 2 < 3$. Then $W\mathcal{A}$ is the following 2-category



If $(j > i)$ denotes the 1-morphism $i \to j$ then each triangle with vertices $i < j < k$ represents a 2-cell

$$(k > i) \xRightarrow{(k>j>i)} (k > j) \circ (j > i)$$

The four 2-cells form a commutative square giving rise to a fifth diagonal 2-cell. So the category $W\mathcal{A}(0, 3)$ is given by the commutative diagram

$$\begin{tikzcd}
(3 > 0) \arrow[r, Rightarrow, "(3>1>0)"] \arrow[d, Rightarrow, "(3>2>0)"'] \arrow[dr, Rightarrow] & (3.1) \circ (1 > 0) \arrow[d, Rightarrow, "(3>2>1)\circ(1>0)"] \\
(3 > 2) \circ (2 > 0) \arrow[r, Rightarrow, "(3>2)\circ(2>1>0)"] & (3 > 2) \circ (2 > 1) \circ (1 > 0)
\end{tikzcd}$$

3.5 Remark: Note that a 1-morphism in $W\mathcal{A}$ is uniquely represented by an element

$$(f_n, \dots, f_0;\ id_1, \dots, id_1)$$

with all $f_k \neq id$, which uniquely decomposes into

$$(f_n) \circ (f_{n-1}) \circ \dots \circ (f_0).$$

3.6 Definition: We define a lax functor

$$\eta = \eta(\mathcal{A}) : \mathcal{A} \longrightarrow W\mathcal{A},$$

natural in $\mathcal{A}$, as follows: η is the identity on objects, sends $f : A_1 \longrightarrow A_2$ to (f) and the pair $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ to the 2-cell $(g, f;\ \iota) \in W\mathcal{A}(A_1, A_3)$.

3.7 Proposition: Considering $\mathcal{A}$ as 2-category with trivial 2-cells there is a 2-functor natural in $\mathcal{A}$

$$\varepsilon : W\mathcal{A} \longrightarrow \mathcal{A}.$$

For each pair of objects A, B of $\mathcal{A}$ the 2-functor ε and the lax functor η define functors ($\mathcal{A}(A, B)$ is considered a trivial category)

$$\mathcal{A}(A, B) \rightleftarrows W\mathcal{A}(A, B)$$

such that $\varepsilon \circ \eta = id$ and there is a natural transformation $\eta \circ \varepsilon \longrightarrow id$.

Proof: $\varepsilon(f_n, \dots, f_0;\ \varepsilon_n, \dots, \varepsilon_1) = f_n \circ \dots \circ f_0$. Let $(f_n, \dots, f_0;\ 1, \dots, 1)$ be an object in $W\mathcal{A}(A, B)$. To it the natural transformation assigns the morphism

$$(f_n \circ \dots \circ f_0) \xRightarrow{(f_n, \dots, f_0;\ \iota, \dots, \iota)} (f_n, \dots, f_0; 1, \dots, 1)$$

in $W\mathcal{A}(A, B)$. $\square$

3.8 Proposition: The correspondence $F \longmapsto F \circ \eta$ defines a bijection

$$\{\text{2-functors } W\mathcal{A} \longrightarrow \underline{\mathcal{B}}\} \longrightarrow \{\text{ lax functors } \mathcal{A} \longrightarrow \underline{\mathcal{B}}\}$$

Proof: We construct the inverse map. Let $\Phi : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be a lax functor (with strict units). By (3.1.1) there is a unique 2-cell

$$\varphi(f_n, \dots, f_0) : \Phi(f_n \circ \dots \circ f_0) \Longrightarrow \Phi(f_n) \circ \dots \circ \Phi(f_0).$$

Let $F : W\mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be the 2-functor sending an object A to $\Phi(A)$, a morphism f to $\Phi(f)$ and the 2-cell $(f_n, \dots, f_0;\ \iota, \dots, \iota)$ to $\varphi(f_n, \dots, f_0)$. Note that 2-cells $(f_n, \dots, f_0;\ \varepsilon_n, \dots, \varepsilon_1)$ with some $\varepsilon_i = id_0$ can be reduced by relation (3.3.1). If $\varepsilon_k = id_1$ this 2-cell is the horizontal composite

$$(f_n, \dots, f_k;\ \varepsilon_n, \dots, \varepsilon_{k+1}) \circ (f_{k-1} \dots, f_0;\ \varepsilon_{k-1}, \dots, \varepsilon_1).$$

Hence F is completely determined by the given data. It is easy to check that F is a 2-functor and that the assignment $\Phi \longmapsto F$ is the required inverse map. $\square$

3.9 Remark: 1) The W-construction (3.2) can easily be extended to a 2-functor 2-$\mathcal{C}at \longrightarrow$ 2-$\mathcal{C}at$ to study lax functors between 2-categories.
2) Usually lax functors are not assumed to satisfy our strict unit condition (3.1.2). Instead, one has a 2-cell $\Phi(id_A) \Longrightarrow id_{\Phi A}$ satisfying certain coherence conditions. We can cover this case by first whiskering $\mathcal{A}$ to obtain a 2-category $\tilde{\mathcal{A}}$ and a canonical 2-functor $\tilde{\mathcal{A}} \longrightarrow \mathcal{A}$ and then applying the W-construction for 2-categories to $\tilde{\mathcal{A}}$.

Our next aim is the rectification of lax functors $\mathcal{A} \longrightarrow \mathcal{C}at$.

3.10 Definition: Let $F : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be a strict and $\Phi : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ a lax functor into a 2-category $\underline{\mathcal{B}}$. A *reduced lax natural transformation*

$$\theta : F \longrightarrow \Phi$$

assigns to each object A of $\mathcal{A}$ a morphism $\theta_A : FA \longrightarrow \Phi A$ and to each $f : A_1 \longrightarrow A_2$ a 2-cell $\theta_f : \theta_{A_2} \circ Ff \Longrightarrow \Phi f \circ \theta_{A_1}$ such that for each pair $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ the diagram

$$\begin{tikzcd}
\theta_{A_3} \circ F(g) \circ F(f) = \theta_{A_3} \circ F(g \circ f) \arrow[r, Rightarrow, "\theta_{g \circ f}"] \arrow[d, Rightarrow, "\theta_g Ff"'] & \theta(g \circ f) \circ \theta_{A_1} \arrow[d, Rightarrow, "\varphi(f,g)\theta_{A_1}"] \\
\Phi(g) \circ \theta_{A_2} \circ F(f) \arrow[r, Rightarrow, "\Phi(g)\theta_f"] & \Phi(g) \circ \Phi(f) \circ \theta_{A_1}
\end{tikzcd}$$

commutes, and $\theta_{id_A} = id_{\theta_A}$.

3.11 Remark: A strict functor $\mathcal{A} \longrightarrow \underline{\mathcal{B}}$ is a 2-functor, where $\mathcal{A}$ has trivial 2-cells. We use the term reduced lax natural transformation to emphasize that F is strict.

3.12 Let $\theta : F \longrightarrow \Phi$ be a reduced lax natural transformation and $\sigma : G \to F$ a strict natural transformation of strict functors. Then we have a canonical composite reduced lax natural transformation $\theta \circ \sigma : G \to \theta$ defined by

$$(\theta \circ \sigma)_A = \theta_A \circ \sigma_A : GA \xrightarrow{\sigma_A} FA \xrightarrow{\theta_A} \Phi A$$
$$(\theta \circ \sigma)_f = \theta_f \circ \sigma_f : (\theta \circ \sigma)_{A_2} \circ Gf \Longrightarrow \theta f \circ (\theta \circ \sigma)_{A_1}$$

where σ_f is the identity 2-cell $\sigma_{A_2} \circ Gf = Ff \circ \sigma_{A_1}$.

3.13 Construction: Let $\mathcal{A}$ be a category. For each pair of objects A, B in $\mathcal{A}$ let $\widetilde{W}\mathcal{A}(A,B)$ be the category

$$\widetilde{W}\mathcal{A}(A,B) = \Big(\coprod_{n \geq 0} \mathcal{A}_{n+1}(A,B) \times \mathcal{L}^n\Big) / \sim$$

with relations (3.3.1),...,(3.3.3) but NOT relation (3.3.4).

$W\mathcal{A}(B,B')$ operates from the left on $\widetilde{W}\mathcal{A}(A,B)$ and $\mathcal{A}(A',A)$ operates from the right. The two operations are given by

$$W\mathcal{A}(B,B') \times \widetilde{W}\mathcal{A}(A,B) \longrightarrow \widetilde{W}\mathcal{A}(A,B')$$

$$((g_k,\dots,g_0;\delta_k,\dots,\delta_1),(f_n,\dots,f_0;\varepsilon_n,\dots,\varepsilon_1))$$
$$\mapsto (g_k,\dots,g_0,f_n,\dots,f_0;\delta_k,\dots,\delta_1,1,\varepsilon_n,\dots,\varepsilon_1)$$

and by

$$\begin{array}{ccc} \widetilde{W}\mathcal{A}(A,B) \times \mathcal{A}(A',A) & \longrightarrow & \widetilde{W}\mathcal{A}(A',B) \\ ((f_n,\dots,f_0;\varepsilon_n,\dots,\varepsilon_1),g) & \longmapsto & (f_n,\dots,f_0 \circ g;\varepsilon_n,\dots,\varepsilon_1). \end{array}$$

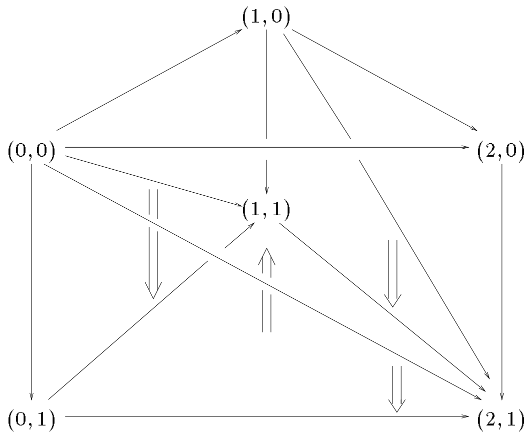
The two operations commute.

We note that these data define a 2-category $\hat{W}\mathcal{A}$ with $ob\,\hat{W}\mathcal{A} = ob(\mathcal{A} \times \mathcal{L})$ and

$$\hat{W}\mathcal{A}((A',0),(A,0)) = \mathcal{A}(A',A)$$
$$\hat{W}\mathcal{A}((A,0),(B,1)) = \widetilde{W}\mathcal{A}(A,B)$$
$$\hat{W}\mathcal{A}((B,1),(B',1)) = W\mathcal{A}(B,B').$$

The operations define composition. In particular, $\mathcal{A}$ and $W\mathcal{A}$ are full sub 2-categories of $\hat{W}\mathcal{A}$.

3.14 Example: Let $\mathcal{A}$ be the poset $0 < 1 < 2$. Then $\hat{W}\mathcal{A}$ looks like



We have a copy of $\mathcal{A}$ at level 0, i.e a commuting triangle of 1-morphisms $(j > i)_0$. There is a copy of $W\mathcal{A}$ at level 1, i.e. a triangle of 1-morphisms $(j > i)_1$ commuting up to a 2-cell

$$(2 > 1 > 0)_1 : (2 > 0)_1 \Longrightarrow (2 > 1)_1 \circ (1 > 0)_1$$

There are six additional 1-morphisms

$$(i = i)_{01} : (i, 0) \to (i, 1) \quad \text{and} \quad (j > i)_{01} : (i, 0) \to (j, 1)$$

represented by (id_i) and $(j > i)$ in $\widetilde{W}\mathcal{A}$ respectively. The upper triangles of the vertical faces all commute while the lower triangles commute up to a 2-cell

$$(j > i = i)_{01} : (j > i)_{01} = (j = j)_{01} \circ (j > i)_0 \Longrightarrow (j > i)_1 \circ (i = i)_{01}$$

represented by $(j > i, id_i) \times \mathcal{L}^1$ in $\widetilde{W}\mathcal{A}$.

The four 2-cells give rise to a commutative square obtained from $(2 > 1, 1 > 0, id_0) \times \mathcal{L}^2$ in $\widetilde{W}\mathcal{A}$. Hence there is a fifth diagonal 2-cell. In particular, the category $\hat{W}\mathcal{A}((0,0),(2,1))$ is given by the commutative diagram

$$\begin{tikzcd}
(2 > 0)_{01} = (2 > 1)_{01} \circ (1 > 0)_0 \arrow[r, Rightarrow, "(2>1=1)_{01} \circ (1>0)_0"] \arrow[d, Rightarrow, "(2>0=0)_{01}"'] \arrow[dr, Rightarrow] & (2 > 1)_1 \circ (1 > 0)_{01} \arrow[d, Rightarrow, "(2>1)_1 \circ (1>0=0)_{01}"] \\
(2 > 0)_1 \circ (0 = 0)_{01} \arrow[r, Rightarrow, "(2>1>0)_1 \circ (0=0)_{01}"] & (2 > 1)_1 \circ (1 > 0)_1 \circ (0 = 0)_{01}
\end{tikzcd}$$

3.15 Proposition: The reduced lax natural transformations $\theta : F \Longrightarrow \Phi$ correspond bijectively to 2-functors

$$H : \hat{W}\mathcal{A} \longrightarrow \underline{\mathcal{B}}$$

such that $H|\mathcal{A} = F$ and $H|W\mathcal{A} = \Phi$.

Proof: Suppose we are given θ. Then H is determined on $\mathcal{A}$ and $W\mathcal{A}$ and hence on all objects. We have to specify the functors

$$H : \widetilde{W}\mathcal{A}(A,B) \longrightarrow \underline{\mathcal{B}}(FA, \Phi B)$$

On objects define $H(f_n, \ldots, f_0; 1, \ldots, 1) = \Phi(f_n) \circ \ldots \circ \Phi(f_1) \circ \theta_{A_1} \circ F(f_0)$, where A_1 is the target of f_0. We set

$$\begin{aligned} H(f, id_A; \iota) &= \theta_f \\ H(f_n, \ldots, f_0; \iota, \ldots, \iota) &= \varphi(f_n, \ldots, f_1)\theta_{A_1} \circ \theta_{f_n \circ \ldots \circ f_1} F(f_0). \end{aligned}$$

This determines H completely because $(f_n, \ldots, f_0; \varepsilon_n, \ldots, \varepsilon_1)$ reduces by (3.3.1) if some $\varepsilon_k = id_0$ and decomposes into

$$(f_n, \ldots, f_k; \varepsilon_n, \ldots, \varepsilon_{k+1}) \circ (f_{k-1}, \ldots, f_0; \varepsilon_{k-1}, \ldots, \varepsilon_1)$$

if $\varepsilon_k = id_1$.

The conditions on θ ensure that H is a 2-functor. Conversely, given H we obtain a reduced lax natural transformation by setting

$$\theta_A = H(id_A), \qquad \theta_f = H(f, id; \iota)$$

with $id_A \in ob\,\widetilde{W}\mathcal{A}(A,A)$ and $(f, id; \iota) \in \operatorname{mor} \widetilde{W}\mathcal{A}(A,B)$. $\square$

3.16 Rectification construction: Let $\Phi : \mathcal{A} \longrightarrow \mathcal{C}at^{\mathbb{M}}$ be a lax functor into the 2-category $\mathcal{C}at^{\mathbb{M}}$ of $\mathbb{M}$-algebras, where $\mathcal{M}$ is any $\mathcal{C}at$-operad. Using (3.8) we think of Φ as a 2-functor $W\mathcal{A} \longrightarrow \mathcal{C}at^{\mathbb{M}}$. For each $A \in \mathcal{A}$ let

$$\hat{\Phi}(A) \subset \prod_{B \in \mathcal{A}} \operatorname{Funct}(\widetilde{W}\mathcal{A}(A,B), \Phi(B))$$

be the full subcategory of all tuples $(\alpha_B;\ B \in \mathcal{A})$ of functors

$$\alpha_B : \widetilde{W}\mathcal{A}(A,B) \longrightarrow \Phi(B)$$

such that

$$\alpha_B(f_n, \dots, f_0; \varepsilon_n, \dots, \varepsilon_1)$$
$$= \Phi(f_n, \dots, f_k; \varepsilon_n, \dots, \varepsilon_{k+1})(\alpha_C(f_{k-1}, \dots, f_0; \varepsilon_{k-1}, \dots, \varepsilon_0))$$

if $\varepsilon_k = id_1$ and $C = \text{target}\, f_{k-1}$. Note that $\varphi = \Phi(f_n, \dots, f_k;\, \varepsilon_n, \dots, \varepsilon_{k+1})$ is a natural transformation and $g = \alpha_C(f_{k-1}, \dots, f_0;\, \varepsilon_{k-1}, \dots, \varepsilon_0)$ is a morphism. The symbol $\varphi(g)$ for $g : c_1 \longrightarrow c_2$ in $\Phi(C)$ and $\varphi : G_1 \longrightarrow G_2$ stands for the diagonal in

$$\begin{tikzcd}
G_1(c_1) \arrow[r, "\varphi"] \arrow[d, "G_1(g)"'] & G_2(c_1) \arrow[d, "G_2(g)"] \\
G_1(c_2) \arrow[r, "\varphi"] & G_2(c_2)
\end{tikzcd}$$

Since Φ takes values in $\mathcal{C}at^{\mathbb{M}}$ the coordinatewise operation of $\mathcal{M}$ on the $\Phi(B)$ gives $\hat{\Phi}(A)$ the structure of an $\mathcal{M}$-category.

The correspondence $A \longmapsto \hat{\Phi}(A)$ extends to a strict functor

$$\hat{\Phi} : \mathcal{A} \longrightarrow \mathcal{C}at^{\mathbb{M}}.$$

Let $f : A_1 \longrightarrow A_2$ be a morphism in $\mathcal{A}$. We define

$$\hat{\Phi}(f) : \hat{\Phi}(A_1) \longrightarrow \hat{\Phi}(A_2), \qquad (\alpha_B) \longmapsto (\bar{\alpha}_B)$$

with $\bar{\alpha}_B(f_n, \dots, f_0;\, \varepsilon_n, \dots, \varepsilon_1) = \alpha_B(f_n, \dots, f_1, f_0 \circ f;\, \varepsilon_n, \dots, \varepsilon_1)$.

3.17 Proposition: There is a reduced lax natural transformation $\nu : \hat{\Phi} \longrightarrow \Phi$ such that any reduced lax natural transformation $\theta : G \longrightarrow \Phi$ factors uniquely as $\theta = \nu \circ \hat{\theta}$ where $\hat{\theta} : G \longrightarrow \hat{\Phi}$ is a strict natural transformation of strict functors and $\nu \circ \hat{\theta}$ is the canonical composite of (3.12).

Proof: We define ν as a 2-functor $\hat{W}\mathcal{A} \longrightarrow \mathcal{C}at^{\mathbb{M}}$. On the subcategories $\mathcal{A}$ and $W\mathcal{A}$ it is determined by $\hat{\Phi}$ and Φ. It remains to define

$$\nu : \widetilde{W}\mathcal{A}(A, B) \longrightarrow \mathcal{C}at^{\mathbb{M}}(\hat{\Phi}(A), \Phi(B)).$$

This functor is the adjoint of the evaluation functor

$$\widetilde{W}\mathcal{A}(A, B) \times \hat{\Phi}A \longrightarrow \Phi B$$

$$((f_n, \dots, f_0;\, \varepsilon_n, \dots, \varepsilon_1)) \times (\alpha_C) \longmapsto \alpha_B(f_n, \dots, f_0;\, \varepsilon_n, \dots, \varepsilon_1).$$

Given any reduced lax natural transformation $\theta : G \longrightarrow \Phi$ the only correspondence

$$\hat{\theta}_A : GA \longrightarrow \hat{\Phi}A$$

satisfying $\nu \circ \hat{\theta} = \theta$ is induced by the adjoint of

$$\widetilde{W}\mathcal{A}(A, B) \times GA \longrightarrow \Phi B$$

obtained from the adjoint of

$$\theta : \widetilde{W}\mathcal{A}(A, B) \longrightarrow \mathcal{C}at^{\mathbb{M}}(GA, \Phi B).$$

It is easy to check that $\hat{\theta}$ is a natural transformation. □

3.18 Proposition: For each object A of $\mathcal{A}$ there is a functor

$$\rho_A : \Phi A \longrightarrow \hat{\Phi} A$$

such that $\nu_A \circ \rho_A = id_{\Phi A}$, and a natural transformation

$$\tau_A : id_{FA} \Longrightarrow \rho_A \circ \nu_A.$$

If Φ is a strict functor, the ρ_A combine to a natural transformation $\rho : \Phi \Longrightarrow \hat{\Phi}$.

Proof: Note that $\nu_A : \hat{\Phi} A \longrightarrow \Phi A$ sends $(\alpha_B;\, B \in \mathcal{A})$ to $\alpha_A(id_A)$. The functor $\rho_A : \Phi A \longrightarrow \hat{\Phi} A$ sends an object a in ΦA to $(\alpha_B;\, B \in \mathcal{A}) \in \hat{\Phi} A$ given by $\alpha_B(f_n, \ldots, f_0; \varepsilon_n, \ldots, \varepsilon_1) = \Phi(f_n \circ \ldots \circ f_0)(id_a)$ and a morphism $g : a \to b$ to

$$(\alpha_B;\, B \in \mathcal{A}) \longrightarrow (\beta_B;\, B \in \mathcal{A})$$

given by $\Phi(f_n \circ \ldots \circ f_0)(g)$. Since $\Phi(id) = id$ we have $\nu_A \circ \rho_A = id$.

We note that $\rho_A \circ \nu_A(\alpha_B;\, B \in \mathcal{A}) = (\beta_B;\; B \in \mathcal{A})$ is defined by

$$\begin{aligned}\beta_B(f_n, \ldots, f_0; \varepsilon_n, \ldots, \varepsilon_1) &= \Phi(f_n \circ \ldots \circ f_0)(\alpha_A(id_A)) \\ &= \alpha_B(f_n \circ \ldots \circ f_0, id_A; 1).\end{aligned}$$

The natural transformation τ_A is induced by natural transformations

$$\tau_{A,B} : \alpha_B \Longrightarrow \beta_B$$

defined by $\tau_{A,B}(f_n, \ldots, f_0; 1, \ldots, 1) = \alpha_B(f_n, \ldots, f_0, id_A; 1, \ldots, 1, \iota)$. By definition, the ρ_A define a natural transformation $\Phi \Rightarrow \hat{\Phi}$ if Φ is a strict functor. □

3.19 Definition: Let $\Phi, \Psi : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be lax functors. A *strict natural transformation of lax functors*

$$\tau : \Phi \longrightarrow \Psi$$

is a strict natural transformation of the associated 2-functors $W\mathcal{A} \longrightarrow \underline{\mathcal{B}}$. In other words, τ assigns to each object A of $\mathcal{A}$ a morphism $\tau_A : \Phi A \to \Psi A$ such that
(1) $\tau_{A_2} \circ \Phi f = \Psi f \circ \tau_{A_1}$ for each $f : A_1 \to A_2$ in $\mathcal{A}$.
(2) $\tau_{A_3} \circ \phi(g,f) = \psi(g,f) \circ \tau_{A_1} : \tau_{A_3} \circ \Phi(g \circ f) \Rightarrow \Psi(g) \circ \Psi(f) \circ \tau_{A_1}$ for each pair $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ in $\mathcal{A}$.

3.20 Let $F : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be a strict and $\Phi, \Psi : \mathcal{A} \longrightarrow \underline{\mathcal{B}}$ be lax functors. Let $\theta : F \longrightarrow \Phi$ be a reduced lax natural transformation and $\tau : \Phi \longrightarrow \Psi$ a strict natural transformation of lax functors. There is a canonical composite $\tau \circ \theta : F \longrightarrow \Psi$ defined by
(1) $(\tau \circ \theta)_A = \tau_A \circ \theta_A : FA \longrightarrow \Phi A \longrightarrow \Psi A$
(2) $(\tau \circ \theta)_f = \tau_{A_2} \circ \theta_f : \tau_{A_2} \circ \theta_{A_2} \circ Ff \Longrightarrow \tau_{A_2} \circ \Phi f \circ \theta_{A_1} = \Psi f \circ \tau_{A_1} \circ \theta_{A_1}$.

3.21 Proposition: If $\mathcal{L}ax\mathcal{F}unct(\mathcal{A}, \underline{\mathcal{B}})$ denotes the category of lax functors $\mathcal{A} \to \underline{\mathcal{B}}$ and strict natural transformations and $\mathcal{F}unct(\mathcal{A}, \underline{\mathcal{B}})$ the category of strict functors and natural transformations there is a rectification functor

$$R : \mathcal{L}ax\mathcal{F}unct(\mathcal{A}, \underline{\mathcal{B}}) \longrightarrow \mathcal{F}unct(\mathcal{A}, \underline{\mathcal{B}})$$

Proof: R is defined on objects by sending Φ to $\hat{\Phi}$. Consider the diagram

$$\begin{tikzcd}
\hat{\Phi} \arrow[r, "\widehat{\tau \circ \nu_\Phi}"] \arrow[d, "\nu_\Phi"] & \hat{\Psi} \arrow[d, "\nu_\Psi"] \\
\Phi \arrow[r, "\tau"] & \Psi
\end{tikzcd}$$

where $\tau : \Phi \to \Psi$ is a strict natural transformation and $\widehat{\tau \circ \nu_\Phi}$ the unique strict natural transformation of strict functors of (3.17) making the square commute. We take $R(\tau) = \widehat{\tau \circ \nu_\Phi}$. The functor axioms follow from the uniqueness part of (3.17). $\square$

4 A lax action of a lax operad

Throughout this section let $\mathcal{M}$ be a Σ-free operad in $\mathcal{C}at$ and X be a $B\mathbb{M}$-space.

We construct a "lax" operad $\mathcal{P} = \mathcal{P}_{\mathcal{M}}$ (we usually drop the suffix $\mathcal{M}$ from the notation) and a category $\mathcal{C}_X$, on which $\mathcal{P}$ acts in a "lax" way, together with a natural weak equivalence of "lax" operads

$$\tau : \mathcal{P}_{\mathcal{M}} \to \mathcal{M}$$

and a natural weak equivalence

$$\eta : B\mathcal{C}_X \to X$$

compatible with the action of $B\mathcal{P}$ on $B\mathcal{C}_X$ and of $B\mathcal{M}$ on X.

"Lax" means that all axioms hold apart from the associativity axiom which holds up to natural isomorphisms satisfying MacLane's pentagon coherence condition [8, VII.1].

This is the first step of the construction of the functor from $\mathcal{T}op^{B\mathbb{M}}$ to $\mathcal{C}at^{\mathbb{M}}$.

Let $[n]$ denote the ordered set $\{0 < 1 < \cdots < n\}$. We frequently identify $[n]$ with its associated category

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n.$$

Let $\mathcal{P}ris \subset \mathcal{C}at$ denote the full subcategory of prisms, i.e. of products

$$[n_1, n_2, \ldots, n_k] = [n_1] \times [n_2] \times \cdots \times [n_k], \text{ all } n_i > 0.$$

We allow the empty product, denoted by $[0]$. The category $\mathcal{P}ris$ is strictly monoidal with structure functor

$$\begin{array}{rcl} \mathcal{P}ris \times \mathcal{P}ris & \xrightarrow{\times} & \mathcal{P}ris \\ ([m_1, \ldots, m_k],\ [n_1, \ldots, n_l]) & \longmapsto & [m_1, \ldots, m_k,\ n_1, \ldots, n_l] \end{array}$$

and unit $[0]$.

4.1 Construction: We define $\mathcal{P}(n)$ to be the over category $\big(\mathcal{P}ris \downarrow \mathcal{M}(n)\big)$. The unit $1 \in \mathcal{P}(1)$ is the object

$$[0] \longrightarrow \mathcal{M}(1), \qquad 0 \longmapsto 1.$$

The composition map

$$\gamma : \mathcal{P}(k) \times \mathcal{P}(i_1) \times \cdots \times \mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1 + \cdots + i_k)$$

sends $\big((f, C); (g_1, D_1), \ldots , (g_k, D_k)\big)$ to

$$C \times D_1 \times \cdots \times D_k \xrightarrow{f \times g_1 \times \cdots \times g_k} \mathcal{M}(k) \times \mathcal{M}(i_1) \times \cdots \times \mathcal{M}(i_k) \xrightarrow{\gamma_{\mathcal{M}}} \mathcal{M}(i_1 + \cdots + i_k).$$

Clearly, $1 \in \mathcal{P}(1)$ is a strict unit, but composition is associative only up to coherent isomorphisms. E.g. $\gamma \circ (\gamma \times id)$ maps the element

$$x = \Big((f, C); (g_1, D_1), (g_2, D_2); (h_{11}, E_{11}), (h_{12}, E_{12}), (h_{21}, E_{21}), (h_{22}, E_{22})\Big)$$

in $\mathcal{P}(2) \times \mathcal{P}(2) \times \mathcal{P}(2) \times \mathcal{P}(k_1) \times \mathcal{P}(k_2) \times \mathcal{P}(l_1) \times \mathcal{P}(l_2)$ to the functor

$$\begin{array}{c}
C \times D_1 \times D_2 \times E_{11} \times E_{12} \times E_{21} \times E_{22} \\
\Big\downarrow {\scriptstyle f \times g_1 \times g_2 \times h_{11} \times \cdots \times h_{22}} \\
\mathcal{M}(2) \times \mathcal{M}(2) \times \mathcal{M}(2) \times \mathcal{M}(k_1) \times \mathcal{M}(k_2) \times \mathcal{M}(l_1) \times \mathcal{M}(l_2) \\
\Big\downarrow {\scriptstyle \gamma_{\mathcal{M}} \times id} \\
\mathcal{M}(4) \times \mathcal{M}(k_1) \times \mathcal{M}(k_2) \times \mathcal{M}(l_1) \times \mathcal{M}(l_2) \\
\Big\downarrow {\scriptstyle \gamma_{\mathcal{M}}} \\
\mathcal{M}(k_1 + k_2 + l_1 + l_2)
\end{array}$$

while $\gamma(id \times \gamma)$ maps it to the functor

$$\begin{array}{c}
C \times D_1 \times E_{11} \times E_{12} \times D_2 \times E_{21} \times E_{22} \\
\Big\downarrow {\scriptstyle f \times g_1 \times h_{11} \times h_{12} \times g_2 \times h_{21} \times h_{22}} \\
\mathcal{M}(2) \times \mathcal{M}(2) \times \mathcal{M}(k_1) \times \mathcal{M}(k_2) \times \mathcal{M}(2) \times \mathcal{M}(l_1) \times \mathcal{M}(l_2) \\
\Big\downarrow {\scriptstyle id \times \gamma_{\mathcal{M}} \times \gamma_{\mathcal{M}}} \\
\mathcal{M}(2) \times \mathcal{M}(k_1 + k_2) \times \mathcal{M}(l_1 + l_2) \\
\Big\downarrow {\scriptstyle \gamma_{\mathcal{M}}} \\
\mathcal{M}(k_1 + k_2 + l_1 + l_2)
\end{array}$$

Since $\mathcal{M}$ is a genuine operad the following diagram with the obvious permutation σ commutes

$$\begin{tikzcd}
\mathcal{M}(2)\times\mathcal{M}(2)\times\mathcal{M}(2)\times\mathcal{M}(k_1)\times\mathcal{M}(k_2)\times\mathcal{M}(l_1)\times\mathcal{M}(l_2) \arrow[r, "\gamma_{\mathcal{M}}\times id"] \arrow[d, "\sigma"] & \mathcal{M}(4)\times\mathcal{M}(k_1)\times\cdots\times M(l_2) \arrow[dd, "\gamma_{\mathcal{M}}"] \\
\mathcal{M}(2)\times\mathcal{M}(2)\times\mathcal{M}(k_1)\times\mathcal{M}(k_2)\times\mathcal{M}(2)\times\mathcal{M}(l_1)\times\mathcal{M}(l_2) \arrow[d, "id\times\gamma_{\mathcal{M}}\times\gamma_{\mathcal{M}}"] & \\
\mathcal{M}(2)\times\mathcal{M}(k_1+k_2)\times\mathcal{M}(l_1+l_2) \arrow[r, "\gamma_{\mathcal{M}}"] & \mathcal{M}(k_1+k_2+l_1+l_2)
\end{tikzcd}$$

Hence the same permutation

$$\sigma : C\times D_1\times D_2\times E_{11}\times\cdots\times E_{22}\longrightarrow C\times D_1\times E_{11}\times E_{12}\times D_2\times E_{21}\times E_{22}$$

is an isomorphism in $\big(\mathcal{P}ris\downarrow\mathcal{M}(k_1+k_2+l_1+l_2)\big)$.

In the general case we have a similar picture and it is easy to check that the associating isomorphisms satisfy the coherence pentagon condition [8, VII.1]. The action of the symmetric group Σ_n on $\mathcal{P}(n)$ is defined by

$$(f\cdot\sigma)(x)=f(x)\cdot\sigma$$

for $x\in C$. Here we encounter a similar problem, which we again illustrate by an example. Let $\sigma=(1,2,3)\in\Sigma_3$ be a cycle. Then the following diagram commutes

4.2

$$\begin{tikzcd}
C\times D_1\times D_2\times D_3 \arrow[r, "id\times\sigma"] \arrow[d, "f\times g_1\times g_2\times g_3"] & C\times D_3\times D_1\times D_2 \arrow[d, "f\times g_3\times g_1\times g_2"] \\
\mathcal{M}(3)\times\mathcal{M}(j_1)\times\mathcal{M}(j_2)\times\mathcal{M}(j_3) \arrow[r, "id\times\sigma"] \arrow[dd, "\cdot\sigma\times id"] & \mathcal{M}(3)\times\mathcal{M}(j_3)\times\mathcal{M}(j_1)\times\mathcal{M}(j_2) \arrow[d, "\gamma_{\mathcal{M}}"] \\
 & \mathcal{M}(j_1+j_2+j_3) \arrow[d, "\cdot\sigma(j_1+j_2+j_3)"] \\
\mathcal{M}(3)\times\mathcal{M}(j_1)\times\mathcal{M}(j_2)\times\mathcal{M}(j_3) \arrow[r, "\gamma_{\mathcal{M}}"] & \mathcal{M}(j_1+j_2+j_3)
\end{tikzcd}$$

Now

$$\begin{aligned}
&\gamma((f\cdot\sigma;g_1,g_2,g_3))=\gamma_{\mathcal{M}}\circ(\sigma\times id)\circ(f\times g_1\times g_2\times g_3)\\
&\gamma((f;g_1,g_2,g_3))\cdot\sigma(j_1,j_2,j_3)\\
&\qquad\qquad\qquad=(-\cdot\sigma(j_1,j_2,j_3))\circ\gamma_{\mathcal{M}}\circ(f\times g_1\times g_2\times g_3).
\end{aligned}$$

Hence the axiom for an operad concerning the right Σ-action only holds up to a natural isomorphism, which is given by

$$id \times \sigma : C \times D_1 \times D_2 \times D_3 \to C \times D_3 \times D_1 \times D_3$$

A map $\alpha : \mathcal{M} \to \mathcal{N}$ of operads induces a map $\mathcal{P}_{\mathcal{M}} \to \mathcal{P}_{\mathcal{N}}$ of lax operads by sending (f, C) to $(\alpha \circ f, C)$.

We define

4.3 $\quad \tau : \mathcal{P} \longrightarrow \mathcal{M}$

by sending the object (f, C) to $f(T)$, where T is the unique terminal object of C. This correspondence extends uniquely to a functor called *last vertex functor*. It is a unit preserving strict functor of lax operads.

4.4 Construction: For $X \in \mathcal{T}op^{B\mathbb{M}}$ let $\mathcal{C}_X$ denote the category whose objects are pairs (f, C) with $C \in \mathcal{P}ris$ and $f : BC \longrightarrow X$ a map. The morphisms from (f, C) to (g, D) are morphisms $h : C \longrightarrow D$ in $\mathcal{P}ris$ such that

$$\begin{tikzcd} BC \arrow[rr, "Bh"] \arrow[dr, "f"'] & & BD \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes.

We have a lax action of the lax categorical operad $\mathcal{P}$ on the category $\mathcal{C}_X$ defined by the functors

$$\alpha_k : \mathcal{P}(k) \times (\mathcal{C}_X)^k \longrightarrow \mathcal{C}_X$$

sending the element $\big((f, C); (g_1, D_1), \ldots, (g_k, D_k)\big)$ to the object

$$B(C \times D_1 \times \cdots \times D_k) = BC \times BD_1 \times \cdots \times BD_k \xrightarrow{Bf \times g_1 \times \cdots \times g_k} B\mathcal{M}(k) \times X^k \xrightarrow{\beta_k} X$$

where β is the structure of X. The unit condition of an operad action holds but the associativity condition and the condition for permutations only hold up to natural isomorphisms for the same reason as in Construction (4.1). For later reference we include the two relevant diagrams in special cases: to illustrate associativity take $(f, C) \in \mathcal{P}(2)$, $(g_1, D_1), (g_2, D_2) \in \mathcal{P}(1)$ and $(h_i, E_i) \in \mathcal{C}_X$. We have a commutative diagram

4.5

$$\begin{array}{ccccc}
BC \times BD_1 \times BD_2 \times BE_1 \times BE_2 & \overset{\mu}{\cong} & BC \times BD_1 \times BE_1 \times BD_2 \times BE_2 \\
\Big\downarrow{\scriptstyle Bf\times Bg_1\times Bg_2\times h_1\times h_2} & & \Big\downarrow{\scriptstyle Bf\times Bg_1\times h_1\times Bg_2\times h_2} \\
B\mathcal{M}(2) \times B\mathcal{M}(1) \times B\mathcal{M}(1) \times X \times X & \underset{\mu}{\cong} & B\mathcal{M}(2) \times B\mathcal{M}(1) \times X \times B\mathcal{M}(1) \times X \\
\Big\downarrow{\scriptstyle B\gamma_{\mathcal{M}}\times id} & & \Big\downarrow{\scriptstyle id\times\beta_1\times\beta_1} \\
B\mathcal{M}(2) \times X^2 & \xrightarrow{\beta_2} X \xleftarrow{\beta_2} & BM(2) \times X^2
\end{array}$$

Observe that

$$\alpha_2(\gamma(f; g_1, g_2); h_1, h_2,) = \beta_2 \circ (B\gamma_{\mathcal{M}} \times id) \circ (Bf \times Bg_1 \times Bg_2 \times h_1 \times h_2)$$

while

$$\begin{aligned}
&\alpha_2(f; \alpha_1(g_1; h_1), \alpha_1(g_2; h_2)) \\
&\qquad = \beta_2 \circ (id \times \beta_1 \times \beta_1) \circ (Bf \times Bg_1 \times h_1 \times Bg_2 \times h_2).
\end{aligned}$$

The evident shuffle μ defines a natural isomorphism between the two functors involved.

We have a similar picture with respect to the permutations (we use the notation of (4.2)). We have a commutative diagram

4.6

$$\begin{array}{ccc}
BC \times BE_1 \times BE_2 \times BE_3 & \xrightarrow{id\times\sigma} & BC \times BE_3 \times BE_1 \times BE_2 \\
\Big\downarrow{\scriptstyle Bf\times h_1\times h_2\times h_3} & & \Big\downarrow{\scriptstyle Bf\times h_3\times h_1\times h_2} \\
B\mathcal{M}(3) \times X^3 & \xrightarrow{id\times\sigma} & B\mathcal{M}(3) \times X^3 \\
\Big\downarrow{\scriptstyle (\cdot\sigma)\times id} & & \Big\downarrow{\scriptstyle \beta^3} \\
B\mathcal{M}(3) \times X^3 & \xrightarrow{\beta_3} & X
\end{array}$$

Note that

$$\alpha_3(f\cdot\sigma;h_1,h_2,h_3)=\beta_3\circ(id\times\sigma)\circ(Bf\times h_1\times h_2\times h_3)$$

while

$$\alpha_3(f;\sigma(h_1,h_2,h_3))=\beta_3\circ(Bf\times h_1\times h_2\times h_3)$$

and the two maps differ by the isomorphism $(id\times\sigma)$.

A homomorphism $\alpha : X\to Y$ of $B\mathbb{M}$-spaces induces a functor

$$\mathcal{C}_\alpha:\mathcal{C}_X\longrightarrow\mathcal{C}_Y,\qquad (f,C)\longmapsto(\alpha\circ f,C)$$

and $\mathcal{C}_\alpha$ is a strict homomorphism of lax $\mathcal{P}$-categories.

4.7 Lemma: There is a natural map

$$\eta: B\mathcal{C}_X\longrightarrow X$$

compatible with the lax action of $B\mathcal{P}$ on $B\mathcal{C}_X$ and the action of $B\mathcal{M}$ on X.

Proof: An n-simplex in the nerve of $\mathcal{C}_X$ is a sequence of functors of prisms

$$C_0\xrightarrow{f_1}C_1\xrightarrow{f_2}\cdots\xrightarrow{f_n}C_n$$

together with a map of spaces $g: BC_n\longrightarrow X$. Consider $[n]$ as a category and define a functor $[n]\longrightarrow C_n$ by sending k to the image of the last vertex of C_k in C_n. Realization defines a singular n-simplex

$$\Delta^n=B([n])\longrightarrow BC_n\xrightarrow{g}X$$

in X and we obtain a simplicial map

$$N\mathcal{C}_X\longrightarrow \operatorname{Sing}X$$

where $\operatorname{Sing}X$ is the singular functor on X. We define η to be the composite of the realization with the canonical weak equivalence

$$B\mathcal{C}_X\longrightarrow|\operatorname{Sing}X|\longrightarrow X.$$

Let $\mathcal{S}imp/X\subset\mathcal{C}_X$ denote the full sub category of standard simplices over X, i.e. its objects are pairs $(g,[n])$ with continuous maps $g:\Delta^n=B([n])\longrightarrow X$. It is well-known that the composite

$$N(\mathcal{S}imp/X)\longrightarrow N(\mathcal{C}_X)\longrightarrow\operatorname{Sing}X$$

is a weak equivalence (e.g. [7]). The inclusion functor $\mathcal{S}imp/X \longrightarrow \mathcal{C}_X$ is a weak equivalence by Quillen's Theorem A [10] because for each object $(f, C) \in \mathcal{C}_X$

$$B\big(\mathcal{C}_X(-,(f,C)), \mathcal{S}imp/X, *\big) \cong B(\mathcal{S}imp/BC) \simeq |\operatorname{Sing} BC| \simeq *.$$

It remains to check the compatability of η with the actions of $\mathcal{P}$ and $\mathcal{M}$. Let

$$\alpha : Id \to \operatorname{Sing} |-| \qquad \text{and} \qquad \beta : |\operatorname{Sing}(-)| \to Id$$

denote the adjunction morphisms. Since the singular functor preserves products, $\operatorname{Sing}(B\mathcal{M})$ is an operad in the category of simplicial sets operating on $\operatorname{Sing} X$. The adjunction α defines a map of operads $\alpha : N\mathcal{M} \to \operatorname{Sing} B\mathcal{M}$. Hence $N\mathcal{M}$ operates on $\operatorname{Sing} X$ and $B\mathcal{M}$ on $|\operatorname{Sing} X|$. Moreover, since the inverse of the natural map $|\operatorname{Sing} X \times \operatorname{Sing} Y| \longrightarrow |\operatorname{Sing} X| \times |\operatorname{Sing} Y|$ is cellular (e.g. see [9, Thm. 11.5]), the action of $B\mathcal{M}$ on $|\operatorname{Sing} X|$ is cellular. Given an n-simplex σ in $N\mathcal{M}(k)$ and a collection of k singular n-simplexes τ_i of X, the singular n-simplex defined by the operation of σ on the τ_i is the composite

$$\Delta^n \xrightarrow{(\sigma,\tau_1,\ldots,\tau_k)} B\mathcal{M}(k) \times X^k \longrightarrow X.$$

Since

$$\beta(B\mathcal{M}) \circ |\alpha(N\mathcal{M})| = id : B\mathcal{M} \to |\operatorname{Sing} B\mathcal{M}| \to B\mathcal{M}$$

the natural map $|\operatorname{Sing} X| \to X$ is a homomorphism of $B\mathcal{M}$-spaces.

The compatability of the map $N\mathcal{C}_X \to \operatorname{Sing} X$ with the action of $N\mathcal{P}$ and $N\mathcal{M}$ follows immediately from the definitions. $\square$

In the same way one can show

4.8 Lemma: The functor $\tau : \mathcal{P} \longrightarrow \mathcal{M}$ of lax operads is a weak equivalence.

For later use we restate part of the proof of Lemma 4.7:

4.9 Lemma: Let $\mathcal{M}$ be a $\mathcal{C}at$-operad and X a $B\mathcal{M}$-space. Then $|\operatorname{Sing} X|$ is a $B\mathcal{M}$-space with cellular action, and the natural map $|\operatorname{Sing} X| \to X$ is a weak equivalence of $B\mathcal{M}$-spaces.

5 A functor from $\mathcal{T}op^{B\mathbb{M}}$ to $\mathcal{S}\mathcal{C}at^{\mathbb{M}}$

If $\mathcal{P}$ were a genuine operad acting strictly on $\mathcal{C}_X$, we would proceed as follows: let $\mathbb{P}$ denote the monad associated with $\mathcal{P}$. We would form the functorial 2-sided bar construction $B_\bullet(\mathbb{M}, \mathbb{P}, \mathcal{C}_X)$ to obtain the required simplicial object in $\mathcal{C}at^{\mathbb{M}}$. Since $\mathcal{P}$ is only a lax operad acting in a lax way on $\mathcal{C}_X$, we have to detour:

1. We replace the monad construction $\mathbb{P}$ by the homotopy monad construction,
$$\mathbb{P}_h\mathcal{C} = \coprod_{k\geq 0} \mathcal{P}(k) \times_{h\Sigma_k} \mathcal{C}^k$$
2. For this we need the homotopy orbit construction in $\mathcal{C}at$.
3. We then mimic the 2-sided bar construction $B_\bullet(\mathbb{M}, \mathbb{P}_h, \mathcal{C}_X)$ and obtain a lax functor $Q_\bullet(X) : \Delta^{op} \longrightarrow \mathcal{C}at^{\mathbb{M}}$.
4. We finally rectify this lax functor to obtain a strict functor $\hat{Q}_\bullet(X)$: $\Delta^{op} \longrightarrow \mathcal{C}at^{\mathbb{M}}$.

The homotopy orbit construction in $\mathcal{C}at$: Given a category $\mathcal{K}$ and two functors
$$F : \mathcal{K}^{op} \to \mathcal{C}at, \quad G : \mathcal{K} \to \mathcal{C}at$$
we define
$$\mathcal{C} = F \times_{h\mathcal{K}} G$$
to be the following category: objects are triples (x, k, y) with $k \in ob\,\mathcal{K}$, $x \in ob\,F(k)$, $y \in ob\,G(k)$. A morphism
$$(f, \alpha, g) : (x_0, k_0, y_0) \to (x_1, k_1, y_1)$$
consists of a morphism $\alpha : k_0 \to k_1$ in $\mathcal{K}$, a morphism $f : x_0 \to F(\alpha)(x_1)$ in $F(k_0)$, and a morphism $g : G(\alpha)(y_0) \to y_1$ in $G(k_1)$.

Composition of two morphisms
$$(x_0, k_0, y_0) \xrightarrow{(f_1,\alpha_1,g_1)} (x_1, k_1, y_1) \xrightarrow{f_2,\alpha_2,g_2)} (x_2, k_2, y_2)$$
is defined to be
$$(F(\alpha_1)(f_2) \circ f_1, \alpha_2 \circ \alpha_1, g_2 \circ G(\alpha_2)(g_1)) : (x_0, k_0, y_0) \to (x_2, k_2, y_2).$$

5.1 Proposition: Let G be a group considered as category with one object. Let $\mathcal{X}$ be a category with right G-action and $\mathcal{Y}$ a category with left G-action. Then

$$B(\mathcal{X} \times_{hG} \mathcal{Y}) \simeq B(B\mathcal{X}, G, B\mathcal{Y})$$

Proof: Construct a functor $H : G \to \mathcal{C}at$, by sending the single object to $\mathcal{X} \times \mathcal{Y}$ and a morphism π to the functor

$$\mathcal{X} \times \mathcal{Y} \xrightarrow{(-)\cdot\pi^{-1} \times \pi\cdot(-)} \mathcal{X} \times \mathcal{Y}.$$

The Grothendieck construction $G \int H$ is the following category: $ob(G \int H) = ob\,\mathcal{X} \times ob\,\mathcal{Y}$. A morphism is a pair

$$(\pi, (f, g)) : (x_0, y_0) \to (x_1, y_1)$$

with $\pi \in G$ and $(f, g) : H(\pi)(x_0, y_0) \to (x_1, y_1)$, i.e. $f : x_0 \cdot \pi^{-1} \to x_1$ and $g : \pi y_0 \to y_1$. The composition of the two morphisms

$$(x_0, y_0) \xrightarrow{(\pi_1,(f_1,g_1))} (x_1, y_1) \xrightarrow{(\pi_2,(f_2,g_2))} (x_2, y_2)$$

is the morphism

$$(\pi_2 \circ \pi_1, (f_2, g_2) \circ H(\pi_2)(f_1, g_1)) : (x_0, y_0) \to (x_2, y_2)$$

Lemma: $\mathcal{X} \times_{hG} \mathcal{Y} \cong G \int H$

Proof: The functor $U : \mathcal{X} \times_{hG} \mathcal{Y} \to G \int H$, given by the identity on objects and by $U(f, \pi, g) = (\pi, (f \cdot \pi^{-1}, g))$ on morphisms, is an isomorphism of categories. □

Hence $B(\mathcal{X} \times_{hG} \mathcal{Y}) \cong B(G \int H)$. By [12, Thm.1.2] there are two natural homotopy equivalences connecting $B(G \int H)$ and $\mathrm{hocolim}_G(B \circ H)$, i.e. the homotopy colimit of the G-diagram sending the single object to $B\mathcal{X} \times B\mathcal{Y}$ and π to the map

$$B\mathcal{X} \times B\mathcal{Y} \to B\mathcal{X} \times B\mathcal{Y}, \quad (x, y) \mapsto (x \cdot \pi^{-1}, \pi \cdot y).$$

It is well-known that $\mathrm{hocolim}_G\, B\mathcal{X} \times B\mathcal{Y} \cong B(B\mathcal{X}, G, B\mathcal{Y})$. □

5.2 Remark: If G acts freely on $\mathcal{X}$ or $\mathcal{Y}$, then the canonical map

$$B(B\mathcal{X}, G, B\mathcal{Y}) \to B\mathcal{X} \times_G B\mathcal{Y}$$

is a homotopy equivalence. Hence the canonical functor

$$\mathcal{X} \times_{hG} \mathcal{Y} \to \mathcal{X} \times_G \mathcal{Y}$$

is a homotopy equivalence in this case.

The homotopy monad $\mathbb{P}_h$*:* For any small category $\mathcal{C}$ we define

$$\mathbb{P}_h\mathcal{C} = \coprod_{n\geq 0} \mathcal{P}(n) \times_{h\Sigma_n} \mathcal{C}^n.$$

The lax action of $\mathcal{P}$ on $\mathcal{C}_X$ induces a kind of action

$$\xi : \mathbb{P}_h\mathcal{C}_X \to \mathcal{C}_X$$

of the homotopy monad $\mathbb{P}_h$ on $\mathcal{C}_X$: Let $(f,\pi,g) : (p_0,\mathbf{x_0}) \to (p_1,\mathbf{x_1})$ be a morphism in $\mathcal{P}(n) \times_{h\Sigma_n} \mathcal{C}_X^n$ with $f : p_0 \to p_1 \cdot \pi$ in $\mathcal{P}(n)$ and $g : \pi\mathbf{x_0} \to \mathbf{x_1}$ and $\mathbf{x_i}$ in $\mathcal{C}_X^n$. Then

$$(f,\pi,g) : (p_0,\mathbf{x_0}) \xrightarrow{(f,id,id)} (p_1\pi,\mathbf{x_0}) \xrightarrow{(id,\pi,id)} (p_1,\pi\mathbf{x_0}) \xrightarrow{(id,id,g)} (p_1,\mathbf{x_1})$$

Hence it suffices to define $\xi_n : \mathcal{P}(n) \times_{h\Sigma_n} \mathcal{C}_X^n \to \mathcal{C}_X$ for the three types of morphisms (f,id,id), (id,π,id), and (id,id,g). On objects we define

$$\xi_n : ((p,C);(x_1,D_1),\ldots,(x_n,D_n)) \mapsto (\beta_n \circ (Bp \times x_1 \times \ldots \times x_n), C \times D_1 \times \ldots \times D_n)$$

where $\beta_n : B\mathcal{M}(n) \times X^n \to X$ is the $B\mathcal{M}$-structure on X:

$$\begin{tikzcd} B(C \times D_1 \times \ldots \times D_n) = BC \times BD_1 \times \ldots \times BD_n \arrow[d, "Bp\times x_1\times\ldots\times x_n"] \\ B\mathcal{M}(n) \times X^n \arrow[d, "\beta_n"] \\ X \end{tikzcd}$$

The morphisms $\xi_n(f,id,id)$ and $\xi_n(id,id,g)$ are the obvious ones. E.g. the morphism

$$\begin{tikzcd} C_0 \arrow[rr, "f"] \arrow[dr, "p_0"'] & & C_1 \arrow[dl, "p_1\pi"] \\ & \mathcal{M}(n) & \end{tikzcd}$$

is mapped to $f \times id : C_0 \times D_1 \times \ldots \times D_n \to C_1 \times D_1 \times \ldots \times D_n$ in $\mathcal{C}_X$.

$\xi_n(id, \pi, id)$ is the permutation morphism

$$C \times D_1 \times \ldots \times D_n \to C \times D_{\pi^{-1}(1)} \times \ldots \times D_{\pi^{-1}(n)}$$

which is a morphism over X by (4.6).

Clearly ξ_n preserves composition of morphisms of the same type. To prove that ξ_n is a functor, we have to show that it respects the relations

5.3

$$\begin{array}{rclcl}
(id, \pi, id) \circ (f, id, id) & = & (f, \pi, id) & = & (f \cdot \pi^{-1}, id, id) \circ (id, \pi, id) \\
(id, id, g) \circ (id, \pi, id) & = & (id, \pi, g) & = & (id, \pi, id) \circ (id, id, \pi \cdot g) \\
(id, id, g) \circ (f, id, id) & = & (f, id, g) & = & (f, id, id) \circ (id, id, g)
\end{array}$$

For the third relation this is clear. The left side of the first relation is mapped to

$$f \times \pi \times C_0 \times D_1 \times \ldots \times D_n \to C_1 \times D_{\pi^{-1}(1)} \times \ldots \times D_{\pi^{-1}(n)}$$

such that

$$\begin{tikzcd}
BC_0 \times BD_1 \times \ldots \times BD_n \arrow[r, "Bf \times id"] \arrow[d, "Bp_0 \times \mathbf{x}"] & BC_1 \times BD_1 \times \ldots \times BD_n \arrow[r, "id \times \pi"] \arrow[d, "Bp_1 \cdot \pi \times \mathbf{x}"] & BC_1 \times BD_{\pi^{-1}(1)} \times \ldots \times BD_{\pi^{-1}(n)} \arrow[d, "Bp_1 \times x_{\pi^{-1}(1)} \times \ldots \times x_{\pi^{-1}(n)}"] \\
B\mathcal{M}(n) \times X^n \arrow[r, phantom, "="] \arrow[dr, "\alpha_n"] & B\mathcal{M}(n) \times X^n \arrow[d, "\alpha_n"] & B\mathcal{M}(n) \times X^n \arrow[dl, "\alpha_n"] \\
& X &
\end{tikzcd}$$

commutes. The outer part of the diagram is the same the outer part of

$$\begin{tikzcd}
BC_0 \times BD_1 \times \ldots \times BD_n \arrow[r, "id \times \pi"] \arrow[d, "Bp_0 \times \mathbf{x}"] & BC_0 \times BD_{\pi^{-1}(1)} \times \ldots \times BD_{\pi^{-1}(n)} \arrow[r, "Bf \cdot \pi^{-1} \times id"] \arrow[d, "Bp_0 \cdot \pi^{-1} \times \mathbf{x}"] & BC_1 \times BD_{\pi^{-1}(1)} \times \ldots \times BD_{\pi^{-1}(n)} \arrow[d, "Bp_1 \times \pi \cdot \mathbf{x}"] \\
B\mathcal{M}(n) \times X^n \arrow[dr, "\alpha_n"] & B\mathcal{M}(n) \times X^n \arrow[r, equal] \arrow[d, "\alpha_n"] & B\mathcal{M}(n) \times X^n \arrow[dl, "\alpha_n"] \\
& X &
\end{tikzcd}$$

Hence the first relation holds. The second relation is shown the same way.

The multiplication $\mathbb{P}_h\mathbb{P}_h\mathcal{C} \xrightarrow{\mu} \mathbb{P}_h\mathcal{C}$ of the homotopy monad is defined on objects by

$$\begin{array}{l}
\mu((p, C); ((q_1, D_1); x_{11}, \ldots, x_{1r_1}), \ldots, ((q_n, D_n); x_{n1}, \ldots, x_{nr_n})) \\
= ((q, E); x_{11}, \ldots, x_{1r_1}, \ldots, x_{nr_1} \ldots, x_{nr_n})
\end{array}$$

with (q, E) given by

$$\begin{array}{c} C \times D_1 \times \ldots \times D_n \\ \Big\downarrow{\scriptstyle p \times q_1 \times \ldots \times q_n} \\ \mathcal{M}(n) \times \mathcal{M}(r_1) \times \ldots \times M(r_n) \\ \Big\downarrow{\scriptstyle \gamma_{\mathcal{M}}} \\ \mathcal{M}(r_1 + \ldots + r_n) \end{array}$$

On morphisms (f, π, g) with $f : (p_0, C_0) \to (p_1\pi, C_1)$ in $\mathcal{P}(n)$, $\pi \in \Sigma_n$, and $\mathbf{g} \in (\mathbb{P}_h\mathcal{C})^n$, $\mathbf{g} = (g_1, \ldots, g_n)$, we define μ in a manner analogous to ξ_n: $\mu(f, id, id)$ is given by

$$C_0 \times D_1 \times \ldots \times D_n \xrightarrow{f \times id} C_1 \times D_1 \times \ldots \times D_n$$

and the identity on the other components.

$\mu(id, \pi, id)$ is given by the permutations

$$\begin{array}{c} (C \times D_1 \times \ldots \times D_n,\ p \cdot \pi \times q_1 \times \ldots \times q_n,\ \mathbf{x_1}, \ldots, \mathbf{x_n}) \\ \downarrow \\ (C \times D_{\pi^{-1}(1)} \times \ldots \times D_{\pi^{-1}(n)},\ p \times q_{\pi^{-1}(1)} \times \ldots \times q_{-1(n)},\ \mathbf{x}_{\pi^{-1}(\mathbf{1})}, \ldots, \mathbf{x}_{\pi^{-1}(\mathbf{n})}) \end{array}$$

and $\mu(id, id, g)$ is the obvious morphism.

It is straight forward to check that this is a functor. The "multiplication functor" μ is associative up to natural isomorphisms, because of the lax operad structure of $\mathcal{P}$. It admits a strict unit

$$\eta : \mathcal{C} \to \mathbb{P}_h\mathcal{C}, \quad x \mapsto (1, x) \in \mathcal{P}(1) \times_{h\Sigma_1} \mathcal{C} = \mathcal{P}(1) \times \mathcal{C}$$

The functor $\tau : \mathcal{P} \to \mathcal{M}$ induces a right action of $\mathbb{P}_h$ on $\mathbb{M}$:

$$\mathcal{P}(n) \times_{h\Sigma_n} \mathcal{C}^n \xrightarrow{\tau \times id} \mathcal{M}(n) \times_{h\Sigma_n} \mathcal{C}^n \longrightarrow \mathcal{M}(n) \times_{\Sigma_n} \mathcal{C}^n$$

Since the permutation functors $D_1 \times \ldots \times D_n \xrightarrow{\pi} D_{\pi^{-1}(1)} \times \ldots \times D_{\pi^{-1}(n)}$ preserve the last vertex, we obtain a natural transformation $\rho : \mathbb{M}\mathbb{P}_h \to \mathbb{M}$ such that the following diagrams commute:

$$
\begin{tikzcd}
\mathbb{M}{\mathbb{P}_h}^2 \arrow[r, "\mathbb{M}\mu"] \arrow[d, "\rho\mathbb{P}_h"] & \mathbb{M}\mathbb{P}_h \arrow[d, "\rho"] \\
\mathbb{M}\mathbb{P}_h \arrow[r, "\rho"] & \mathbb{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{M}{\mathbb{P}_h}^3 \arrow[r, "\mathbb{M}\mathbb{P}_h\mu"] \arrow[d, "\mathbb{M}\mu\mathbb{P}_h"] & \mathbb{M}{\mathbb{P}_h}^2 \arrow[d, "\rho\mathbb{M}\mu"] \\
\mathbb{M}{\mathbb{P}_h}^2 \arrow[r, "\rho\mathbb{M}\mu"] & \mathbb{M}
\end{tikzcd}
$$

and $\mathbb{M} \xrightarrow{\mathbb{M}\eta} \mathbb{M}\mathbb{P}_h \xrightarrow{\rho} \mathbb{M}$ is $id_{\mathbb{M}}$.

These data allow us to define the lax functor

$$Q_\bullet(X) : \Delta^{op} \longrightarrow \mathcal{C}at^{\mathbb{M}}$$

of the next step in our program. We set $Q_n(X) = \mathbb{M}{\mathbb{P}_h}^n\mathcal{C}_X$, The structure maps are given by

5.4
$$
\begin{aligned}
d^i \quad &= \begin{cases} \mathbb{M}{\mathbb{P}_h}^{n-1}\xi & i = 0 \\ \mathbb{M}{\mathbb{P}_h}^{n-i-1}\mu{\mathbb{P}_h}^{i-1}\mathcal{C}_X & 0 < i < n \\ \overline{\mu}{\mathbb{P}_h}^{n-1}\mathcal{C}_X & i = n \end{cases} \\
s^i \quad &= \mathbb{M}{\mathbb{P}_h}^{n-i}\eta{\mathbb{P}_h}^{i}\mathcal{C}_X \qquad\qquad 0 \le i \le n
\end{aligned}
$$

with $\overline{\mu} : \mathbb{M}\mathbb{P}_h \xrightarrow{\mathbb{M}\tau} \mathbb{M}\mathbb{M} \xrightarrow{\mu} \mathbb{M}$.

Finally we apply the rectification (3.16) to $Q_\bullet(X)$ to obtain a strict functor

$$\hat{Q}_\bullet(X) : \Delta^{op} \longrightarrow \mathcal{C}at^{\mathbb{M}}.$$

Since all our constructions are functorial in $\mathcal{M}$ and X, this defines a functor

$$\hat{Q}_\bullet : \mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{S}\mathcal{C}at^{\mathbb{M}}.$$

6 Proof of Theorem 2.7

Throughout this section let $\mathcal{M}$ be a Σ-free $\mathcal{C}at$-operad.

6.1 Proposition: $|-|: cw\mathcal{S}\mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$ induces an equivalence of categories

$$cw\mathcal{S}\mathcal{T}op^{B\mathbb{M}}[we^{-1}] \simeq \mathcal{T}op^{B\mathbb{M}}[we^{-1}].$$

Proof: Let $T = |\operatorname{Sing}(-)| : \mathcal{T}op \longrightarrow \mathcal{T}op$ be the functorial CW-approximation and

$$\beta(X) : TX \longrightarrow X$$

the associated natural weak equivalence. Let

$$\mathcal{T}op(\Delta^\bullet, -) : \mathcal{T}op \longrightarrow \mathcal{S}\mathcal{T}op$$

be the continuous singular functor right adjoint to the realization functor, and let

$$\alpha^{top}(Y_\bullet) : Y_\bullet \longrightarrow \mathcal{T}op(\Delta^\bullet, |Y_\bullet|), \quad \beta^{top}(X) : |\mathcal{T}op(\Delta^\bullet, X)| \longrightarrow X$$

be the natural continuous adjunction maps. Since $\mathcal{T}op(\Delta^\bullet, -)$ is a continuous product preserving functor, it defines a functor $\mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{S}\mathcal{T}op^{B\mathbb{M}}$ right adjoint to the realization functor $|-| : \mathcal{S}\mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$. By (4.9) the functor T also induces a functor $T : \mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{T}op^{B\mathbb{M}}$ and $\beta(X) : TX \longrightarrow X$ is a morphism of $B\mathcal{M}$-spaces for $B\mathcal{M}$-spaces X. Let $T_\bullet : \mathcal{S}\mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{S}\mathcal{T}op^{B\mathbb{M}}$ be the prolongation of T. Now define

$$Q = T_\bullet \circ \mathcal{T}op(\Delta^\bullet, -) : \mathcal{T}op^{B\mathbb{M}} \longrightarrow cw\mathcal{S}\mathcal{T}op^{B\mathbb{M}}$$

Claim 1: Let X be a $B\mathcal{M}$-space. Then the composite

$$\beta^{top}(X) \circ |\beta(\mathcal{T}op(\Delta^\bullet, X))| : |QX| \longrightarrow |\mathcal{T}op(\Delta^\bullet, X))| \longrightarrow X$$

is a weak equivalence in $\mathcal{T}op^{B\mathbb{M}}$.

Proof: Let $X_\bullet$ denote the constant simplicial space on X so that $|X_\bullet| = X$. Consider the commutative diagram

$$\begin{tikzcd}
T(X) = |T_\bullet X_\bullet| \arrow[r, "|T_\bullet \alpha^{top}(X_\bullet)|"] \arrow[d, "\beta(X) = |\beta(X_\bullet)|"] & |T_\bullet \mathcal{T}op(\Delta^\bullet, X)| = |QX| \arrow[d, "\beta(|\mathcal{T}op(\Delta^\bullet, X)|)"] \\
X = |X_\bullet| \arrow[r, "|\alpha^{top}(X_\bullet)|"] \arrow[dr, equal] & |\mathcal{T}op(\Delta^\bullet, X)| \arrow[d, "\beta^{top}(X)"] \\
& X
\end{tikzcd}$$

Since $\alpha^{top}(X_\bullet)$ is dimensionwise a homotopy equivalence, so is $T_\bullet\alpha^{top}(X_\bullet)$. Since $T_\bullet X_\bullet$ and QX are proper as simplicial spaces (i.e. the inclusion of the space of degenerate n-simplices into the space of all n-simplices is a closed cofibration for each n), realization gives a weak equivalence $TX \longrightarrow |QX|$. Since $\beta(X)$ is a weak equivalence, the claim follows.

Claim 2: Let $Y_\bullet$ be in $cw\mathcal{STop}^{B\mathbb{M}}$. Then there are weak equivalences in $cw\mathcal{STop}^{B\mathbb{M}}$

$$Y_\bullet \xleftarrow[\beta(Y_\bullet)]{} T_\bullet Y_\bullet \xrightarrow{T_\bullet\alpha^{top}(Y_\bullet)} Q|Y_\bullet|$$

Proof: The left equivalence is induced by the homotopy equivalences $\beta(Y_n) : TY_n \to Y_n$. For the right equivalence consider the commutative diagram

$$\begin{tikzcd}
{|T_\bullet Y_\bullet|} \arrow[r, "|T_\bullet\alpha^{top}(Y_\bullet)|"] \arrow[d, "|\beta(Y_\bullet)|"] & {|Q(|Y_\bullet|)| = |T_\bullet\mathcal{T}op(\Delta^\bullet, |Y_\bullet|)|} \arrow[d, "|\beta(\mathcal{T}op(\Delta^\bullet,|Y_\bullet|))|"] \\
{|Y_\bullet|} \arrow[r, "|\alpha^{top}(Y_\bullet)|"] \arrow[dr, equal] & {|\mathcal{T}op(\Delta^\bullet, |Y_\bullet|)|} \arrow[d, "\beta^{top}(|Y_\bullet|)"] \\
 & {|Y_\bullet|}
\end{tikzcd}$$

Since $\beta(Y_\bullet)$ is a weak equivalence, $T_\bullet\alpha^{top}(Y_\bullet) : T_\bullet Y_\bullet \longrightarrow Q(|Y_\bullet|)$ is a weak equivalence by Claim 1. $\square$

6.2 Proposition: The functor $B : \mathcal{SC}at^{\mathbb{M}} \longrightarrow cw\mathcal{STop}^{B\mathbb{M}}$ induces an equivalence of categories

$$\mathcal{SC}at^{\mathbb{M}}[we^{-1}] \simeq cw\mathcal{STop}^{B\mathbb{M}}[we^{-1}].$$

Recall the functor $\hat{Q}_\bullet : \mathcal{T}op^{B\mathbb{M}} \longrightarrow \mathcal{SC}at^{\mathbb{M}}$ of the previous section. Define

$$D : cw\mathcal{STop}^{B\mathbb{M}} \longrightarrow \mathcal{SC}at^{\mathbb{M}}$$

by $D(X_*) = d\hat{Q}_\bullet(X_*)$, where d stands for the diagonal.

6.3 Lemma: Let X be a CW-complex. Then there is a sequence of weak equivalences in $cw\mathcal{STop}^{B\mathbb{M}}$ natural in X, which join $B\hat{Q}_\bullet(X)$ and the constant simplicial space $X_\bullet$ on X.

The proof is based on an analysis of the rectification $\nu : \hat{Q}_X \Longrightarrow Q_X$.

There is a topological version W_{top} of the W-construction (3.2): just replace $\mathcal{L}$ by the unit interval. Let $\mathcal{A}$ be a small indexing category. A *coherently homotopy commutative* $\mathcal{A}$-diagram in $\mathcal{T}op$ is just a continuous functor

$$D : W_{top}\mathcal{A} \longrightarrow \mathcal{T}op.$$

This is the topological version of Proposition 3.8. A lax natural transformation corresponds to the concept of a (coherent) homotopy homomorphism, and a reduced lax natural transformation to the concept of a source reduced homotopy homomorphism in the terminology of [15, p. 18]. There is a rectification $\hat{D} : \mathcal{A} \longrightarrow \mathcal{T}op$ of $D : W_{top}\mathcal{A} \longrightarrow \mathcal{T}op$ together with a source reduced homotopy homomorphism

$$\nu_{top} : \hat{D} \longrightarrow D$$

such that each $\nu_{top}(A) : \hat{D}(A) \longrightarrow D(A)$ is a homotopy equivalence. ν_{top} has the following universal property: given an $\mathcal{A}$-diagram $E : \mathcal{A} \longrightarrow \mathcal{T}op$ and a source reduced homotopy homomorphism $\beta : E \longrightarrow D$ there is a unique homomorphism of $\mathcal{A}$-diagrams $\hat{\beta} : E \longrightarrow \hat{D}$ making the following diagram commute

$$\begin{tikzcd} & \hat{D} \arrow[d, "\nu_{top}"] \\ E \arrow[ur, "\hat{\beta}"] \arrow[r, "\beta"'] & D \end{tikzcd}$$

This is the topological version of Propositions 3.17 and 3.18. If we start with a strict $\mathcal{A}$-diagram D and take $\beta = id_D$ we obtain a homomorphism of $\mathcal{A}$-diagrams $\rho = \hat{id} : D \longrightarrow \hat{D}$ such that $\nu_{top} \circ \rho = id_D$ as source reduced homotopy homomorphisms. Since each each $\nu_{top}(A) : \hat{D}(A) \longrightarrow D(A)$ is a homotopy equivalence, ρ is a weak equivalence of $\mathcal{A}$-diagrams.

If we consider diagrams in $\mathcal{T}op^{\mathbb{C}}$ where $\mathcal{C}$ is any topological operad then the morphisms ν_{top}, $\hat{\beta}$, and ρ are homomorphisms of diagrams in $\mathcal{T}op^{\mathbb{C}}$ and not just in $\mathcal{T}op$. For more details see [15].

By construction, the classifying space functor maps the categorical situation onto the topological one. In particular,

$$B(\nu) : B\hat{Q}_\bullet(X) \longrightarrow BQ_\bullet(X)$$

is a source reduced homotopy homomorphism of Δ^{op}-diagrams of $B\mathbb{M}$-spaces. The map $\tau : \mathcal{P} \to \mathcal{M}$ of lax operads together with the weak equivalence

$\eta : B\mathcal{C}_X \to X$ induce a weak equivalence of $W_{top}\Delta^{op}$-diagrams in $\mathcal{T}op^{B\mathbb{M}}$ by (4.7), (4.8), and (5.2)

$$BQ_\bullet(X) \longrightarrow B_\bullet(B\mathbb{M}, B\mathbb{M}, X).$$

Composing the two we obtain a source reduced homotopy homomorphism

$$\beta : B\hat{Q}_\bullet(X) \longrightarrow B_\bullet(B\mathbb{M}, B\mathbb{M}, X).$$

of simplicial $B\mathbb{M}$-spaces, such that each $\beta_n : BQ_n(X) \longrightarrow B_n(B\mathbb{M}, B\mathbb{M}, X)$ is a weak equivalence. Since $B_\bullet(B\mathbb{M}, B\mathbb{M}, X)$ is a strict Δ^{op}-diagram of $B\mathbb{M}$-spaces we obtain a sequence of weak equivalences of simplicial $B\mathbb{M}$-spaces defining the botton row of the following diagram.

6.4

$$\begin{tikzcd}
T(B\hat{Q}_\bullet(X)) \arrow[r, "T\hat{\beta}"] \arrow[d] & T((B_\bullet(B\mathbb{M}, B\mathbb{M}, X))^\wedge) \arrow[d] & T(B_\bullet(B\mathbb{M}, B\mathbb{M}, X)) \arrow[l, "T\rho"'] \arrow[d] & \\
B\hat{Q}_\bullet(X) \arrow[r, "\hat{\beta}"] & (B_\bullet(B\mathbb{M}, B\mathbb{M}, X))^\wedge & B_\bullet(B\mathbb{M}, B\mathbb{M}, X) \arrow[l, "\rho"'] \arrow[r] & X_\bullet
\end{tikzcd}$$

Unfortunately, $(B_\bullet(B\mathbb{M}, B\mathbb{M}, X))^\wedge$ is not a simplicial CW-complex. We resolve this by applying the standard functorial CW-approximation $T = |\operatorname{Sing}(-)|$, which defines the rest of the diagram. By Lemma 4.9 the vertical maps are weak equivalence of simplicial $B\mathbb{M}$-spaces. All spaces of the diagram apart from $(B_\bullet(B\mathbb{M}, B\mathbb{M}, X))^\wedge$ live in $cw\mathcal{ST}op^{B\mathbb{M}}$ and the maps joining them are weak equivalences in $cw\mathcal{ST}op^{B\mathbb{M}}$. This proves Lemma 6.3.

6.5 Lemma: Let X_* be an object in $cw\mathcal{ST}op^{B\mathbb{M}}$. Then there is a sequence of natural weak equivalences in $cw\mathcal{ST}op^{B\mathbb{M}}$ joining $BD(X_*)$ and X_*.

Proof: Diagram (6.4) gives rise to a diagram of bisimplicial $B\mathcal{M}$-spaces

$$\begin{tikzcd}
T(B\hat{Q}_\bullet(X_*)) \arrow[r, "T\hat{\beta}"] \arrow[d] & T((B_\bullet(B\mathbb{M}, B\mathbb{M}, X_*))^\wedge) & T(B_\bullet(B\mathbb{M}, B\mathbb{M}, X_*)) \arrow[l, "T\rho"'] \arrow[d] \\
B\hat{Q}_\bullet(X_*) \arrow[rr, "\hat{\beta}"] & & X_{\bullet *}
\end{tikzcd}$$

where $X_{\bullet *}$ is constant in the $\bullet$-direction. The maps are weak equivalences in each dimension $*$. The diagonals are objects in $cw\mathcal{ST}op^{B\mathbb{M}}$, and we obtain the required sequence joining $BD(X_*)$ and X_*. $\square$

6.6 Lemma: Let $\mathcal{A}$ be an $\mathcal{M}$-category. Then there is a sequence of weak equivalences in $\mathcal{S}\mathcal{C}at^{\mathbb{M}}$ natural in $\mathcal{A}$, which joins $\hat{Q}_\bullet(B\mathcal{A})$ with the constant simplicial $\mathcal{M}$-category $\mathcal{A}_\bullet$ on $\mathcal{A}$.

Proof: Let $\mathcal{C}_\mathcal{A} = \mathcal{P}ris \downarrow \mathcal{A}$ denote the category of prisms over $\mathcal{A}$. The lax operad $\mathcal{P}$ acts on $\mathcal{C}_\mathcal{A}$ analogous to Construction 4.4, and the last vertex maps $\tau : \mathcal{P} \longrightarrow \mathcal{M}$ and $\eta : \mathcal{C}_\mathcal{A} \longrightarrow \mathcal{A}$ are compatible with the lax action of $\mathcal{P}$ on $\mathcal{C}_\mathcal{A}$ and the strict action of $\mathcal{M}$ on $\mathcal{A}$. Let

$$Q_\bullet(\mathcal{A}) : \Delta^{op} \longrightarrow \mathcal{C}at^{\mathbb{M}}$$

be the lax functor defined as $Q_\bullet(X)$ but with $\mathcal{C}_X$ replaced by $\mathcal{C}_\mathcal{A}$. The last vertex map defines a strict natural transformation of lax Δ^{op}-diagrams in $\mathcal{C}at^{\mathbb{M}}$

$$Q_\bullet(\mathcal{A}) \Longrightarrow B_\bullet(\mathbb{M}, \mathbb{M}, \mathcal{A})$$

such that each functor $Q_k(\mathcal{A}) \to B_k(\mathbb{M}, \mathbb{M}, \mathcal{A})$ is a weak equivalence. We note that $B_\bullet(\mathbb{M}, \mathbb{M}, \mathcal{A})$ is a strict Δ^{op}-diagram. Furthermore there is a strict natural transformation of lax Δ^{op}-diagrams in $\mathcal{C}at^{\mathbb{M}}$

$$F(\mathcal{A}) : Q_\bullet(\mathcal{A}) \Longrightarrow Q_\bullet(B\mathcal{A})$$

induced by the functor

$$\mathcal{C}_\mathcal{A} \longrightarrow \mathcal{C}_{B\mathcal{A}}, \qquad (f, C) \mapsto (Bf, C)$$

6.7 Lemma: The functor $\mathcal{C}_\mathcal{A} \longrightarrow \mathcal{C}_{B\mathcal{A}}$ is a weak equivalence.

Proof: By (4.8) the last vertex map defines a homotopy equivalence

$$B\mathcal{C}_\mathcal{A} \longrightarrow B\mathcal{A}.$$

By (4.7) the last vertex map defines a homotopy equivalence $B(\mathcal{C}_{B\mathcal{A}}) \longrightarrow B\mathcal{A}$. Hence the result follows. □

We have strict natural transformations of lax Δ^{op}-diagrams in $\mathcal{C}at^{\mathbb{M}}$

$$Q_\bullet(B\mathcal{A}) \Longleftarrow Q_\bullet(\mathcal{A}) \Longrightarrow B_\bullet(\mathbb{M}, \mathbb{M}, \mathcal{A})$$

such that the functors

$$Q_k(B\mathcal{A}) \longleftarrow Q_k(\mathcal{A}) \longrightarrow B_k(\mathbb{M}, \mathbb{M}, \mathcal{A})$$

are weak equivalences. Applying the rectification functor (3.16) we obtain natural maps of strict Δ^{op}-diagrams in $\mathcal{C}at^{\mathbb{M}}$

$$\hat{Q}_\bullet(B\mathcal{A}) \Longleftarrow \hat{Q}_\bullet(\mathcal{A}) \Longrightarrow B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A})^\wedge$$

which are dimensionwise weak equivalences. Since $B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A})$ is a strict simplicial $\mathcal{M}$-category, there is a natural map

$$B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A})^\wedge \overset{\rho}{\Longleftarrow} B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A})$$

which is dimensionwise a weak equivalence. Supplementing the diagram with the evaluation to the constant simplicial $\mathcal{M}$-category $\mathcal{A}_\bullet$ we obtain a sequence of weak equivalences in $\mathcal{SC}at^{\mathbb{M}}$

6.8

$$\hat{Q}_\bullet(B\mathcal{A}) \Longleftarrow \hat{Q}_\bullet(\mathcal{A}) \Longrightarrow B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A})^\wedge \overset{\rho}{\Longleftarrow} B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A}) \Longrightarrow \mathcal{A}.$$

natural in $\mathcal{A}$. □

6.9 Lemma: Let $\mathcal{A}_*$ be a simplicial $\mathcal{M}$-category. There is a natural chain of weak equivalences in $\mathcal{SC}at^{\mathbb{M}}$ joining $DB(\mathcal{A}_*)$ and $\mathcal{A}_*$.

Proof: Diagram (6.8) gives rise to a diagram of bisimplicial $\mathcal{M}$-categories

$$\hat{Q}_\bullet(B\mathcal{A}_*) \Longleftarrow \hat{Q}_\bullet(\mathcal{A}_*) \Longrightarrow B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A}_*)^\wedge \Longleftarrow B_\bullet(\mathbb{M},\mathbb{M},\mathcal{A}_*) \Longrightarrow \mathcal{A}_*.$$

consisting of weak equivalences in $\mathcal{SC}at^{\mathbb{M}}$ for each fixed dimension $*$. The diagonals are the required chain of weak equivalences. □

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