A BROWN REPRESENTABILITY THEOREM VIA COHERENT FUNCTORS

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Abstract. We discuss the Brown Representability Theorem for triangulated categories having arbitrary coproducts.

In this paper we discuss the Brown Representability Theorem for triangulated categories having arbitrary coproducts. This theorem is an extremely useful tool and various versions appear in the literature. All of them require a set of objects which *generate* the category in some appropriate sense. Depending on the proof, there are essentially two types: the first type is based on the analogue of *iterated attaching of cells* which is used in the topological case; the second type is based on *solution sets* and applies a variant of Freyd's Adjoint Functor Theorem.

Motivated by recent work of Neeman [7] and Franke [2], we prove a new theorem of the first type (Theorem A) and add, as an application, a Brown Representability Theorem for covariant functors (Theorem B). The final Theorem C establishes a filtration of a triangulated category which clarifies the relation between results of the first and the second type.

The main theorem

Let $\mathcal{T}$ be a triangulated category and suppose that $\mathcal{T}$ has arbitrary coproducts.

Definition 1. A set of objects $\mathcal{S}_0$ *perfectly generates* $\mathcal{T}$ if the following holds:

(G1) an object $X \in \mathcal{T}$ is zero provided that $(S, X) = 0$ for all $S \in \mathcal{S}_0$;

(G2) for every countable set of maps $X_i \to Y_i$ in $\mathcal{T}$ the induced map
$$(S, \coprod_i X_i) \longrightarrow (S, \coprod_i Y_i)$$
is surjective for all $S \in \mathcal{S}_0$ provided that $(S, X_i) \to (S, Y_i)$ is surjective for all i and $S \in \mathcal{S}_0$.

Here, (X, Y) denotes the maps from X to Y. For example, (G2) holds if every $S \in \mathcal{S}_0$ is *small*, that is, the functor $(S, -)$ preserves arbitrary coproducts. Therefore the following result generalizes the classical Brown Representability Theorem for triangulated categories having a set of small generators [5].

Theorem A. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and suppose that $\mathcal{T}$ is perfectly generated by a set of objects. Then a functor $F \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is representable if and only if F is cohomological and sends coproducts in $\mathcal{T}$ to products.*

In [7], Neeman proves a Brown Representability Theorem for triangulated categories which are well generated. A well generated triangulated category has a set of perfect generators so that one gets a quick proof for Neeman's result.

Examples of perfectly generated categories arise very naturally from triangulated categories having a set of small generators. Take for instance the stable homotopy category of CW-spectra or the unbounded derived category of modules over an associative ring. Let $\mathcal{T}_0$ be a set of objects and let $\mathcal{T}$ be the smallest full triangulated subcategory which is closed under coproducts and contains $\mathcal{T}_0$. Then $\mathcal{T}$ is perfectly generated by a set of objects [7].

The proof of Theorem A is based on a reformulation of condition (G2) which is given in Lemma 3. Let us start with some preparations. We fix an additive category $\mathcal{T}$. Following Auslander [1], a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups is called *coherent* if there exists an exact sequence

$$(-,X) \longrightarrow (-,Y) \longrightarrow F \longrightarrow 0.$$

The natural transformations between two coherent functors form a set, and the coherent functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ form an additive category with cokernels which we denote by $\widehat{\mathcal{T}}$ (see also [3] for this concept). A basic tool is the *Yoneda functor*

$$\mathcal{T} \longrightarrow \widehat{\mathcal{T}}, \quad X \mapsto (-,X).$$

Given an additive functor $f\colon \mathcal{S} \to \mathcal{T}$, we denote by $f^*\colon \widehat{\mathcal{S}} \to \widehat{\mathcal{T}}$ the right exact functor which sends $(-,X)$ to $(-,fX)$.

Lemma 1. *Let $\mathcal{T}$ be an additive category.*

1. *If $\mathcal{T}$ has weak kernels, then $\widehat{\mathcal{T}}$ is an abelian category.*
2. *If $\mathcal{T}$ has arbitrary coproducts, then $\widehat{\mathcal{T}}$ has arbitrary coproducts and the Yoneda functor preserves all coproducts.*

Proof. Recall that a map $X \to Y$ is a *weak kernel* for $Y \to Z$ if the induced sequence

$$(-,X) \longrightarrow (-,Y) \longrightarrow (-,Z)$$

is exact. The proof of (1) and (2) is straightforward. Note that for every family of functors F_i having a presentation

$$(-,X_i) \xrightarrow{(-,\phi_i)} (-,Y_i) \longrightarrow F_i \longrightarrow 0$$

the coproduct $F = \coprod_i F_i$ has a presentation

$$(-,\coprod_i X_i) \xrightarrow{(-,\coprod \phi_i)} (-,\coprod_i Y_i) \longrightarrow F \longrightarrow 0.$$

□

Given a class $\mathcal{S}$ of objects in an additive category $\mathcal{T}$, we denote by $\operatorname{Add}\mathcal{S}$ the closure of $\mathcal{S}$ in $\mathcal{T}$ under all coproducts and direct factors.

Lemma 2. *Let $\mathcal{T}$ be an additive category with arbitrary coproducts and weak kernels. Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and denote by $f\colon \mathcal{S} \to \mathcal{T}$ the inclusion for $\mathcal{S} = \operatorname{Add}\mathcal{S}_0$.*

1. *$\mathcal{S}$ has weak kernels and $\widehat{\mathcal{S}}$ is an abelian category.*
2. *The assignment $F \mapsto F|_{\mathcal{S}}$ induces an exact functor $f_*\colon \widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{S}}$.*
3. *The functor $f^*\colon \widehat{\mathcal{S}} \longrightarrow \widehat{\mathcal{T}}$ is a left adjoint for f_*.*
4. *$f_* \circ f^* \cong \mathrm{id}$ and f_* induces an equivalence $\widehat{\mathcal{T}}/\operatorname{Ker} f_* \to \widehat{\mathcal{S}}$.*

Proof. First observe that for every $X \in \mathcal{T}$, there exists an *approximation* $X' \to X$ such that $X' \in \mathcal{S}$ and $(S,X') \to (S,X)$ is surjective for all $S \in \mathcal{S}$. This follows from Yoneda's lemma if we take $X' = \coprod_{S \in \mathcal{S}_0} X_S$ where $X_S = \coprod_{(S,X)} S$.

(1) To prove that $\widehat{\mathcal{S}}$ is abelian it is sufficient to show that every map in $\mathcal{S}$ has a weak kernel. To obtain a weak kernel of a map $Y \to Z$ in $\mathcal{S}$, take the composite of a weak kernel $X \to Y$ in $\mathcal{T}$ and an approximation $X' \to X$.

(2) We need to check that for $F \in \widehat{\mathcal{T}}$ the restriction $F|_{\mathcal{S}}$ belongs to $\widehat{\mathcal{S}}$. It is sufficient to prove this for $F = (-, Y)$. To obtain a presentation, let $X \to Y'$ be a weak kernel of an approximation $Y' \to Y$. The composite $X' \to Y'$ with an approximation $X' \to X$ gives an exact sequence

$$(-, X')|_{\mathcal{S}} \longrightarrow (-, Y')|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0.$$

Clearly, $F \mapsto F|_{\mathcal{S}}$ is exact.

(3) Let $F \in \widehat{\mathcal{S}}$ and $G \in \widehat{\mathcal{T}}$. Suppose first that $F = (-, X)$. Then

$$(f^*F, G) = ((-, fX), G) \cong G(fX) \cong (F, f_*G).$$

This implies the adjointness isomorphism for an arbitrary F since f^* is right exact.

(4) We have $(f_* \circ f^*)(-, X) = (-, X)$ for all $X \in \mathcal{S}$, and $f_* \circ f^* \cong \mathrm{id}$ follows since $f_* \circ f^*$ is right exact. For the rest we refer to Proposition III.5 in [4]. □

Lemma 3. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Let $\mathcal{S}_0$ be a set of objects in $\mathcal{T}$ and let $\mathcal{S} = \operatorname{Add} \mathcal{S}_0$. Then the functor*

$$h \colon \mathcal{T} \longrightarrow \widehat{\mathcal{S}}, \quad X \mapsto (-, X)|_{\mathcal{S}},$$

is cohomological. It preserves countable coproducts if and only if (G2) *holds for $\mathcal{S}_0$.*

Proof. We apply Lemma 2. To this end write h as composite

$$h \colon \mathcal{T} \longrightarrow \widehat{\mathcal{T}} \xrightarrow{f_*} \widehat{\mathcal{S}}.$$

The Yoneda functor is cohomological and f_* is exact. Therefore h is cohomological. It is clear that h preserves coproducts if and only if f_* preserves coproducts. We know that $f_* \colon \widehat{\mathcal{T}} \to \widehat{\mathcal{S}}$ induces an equivalence $\widehat{\mathcal{T}}/\operatorname{Ker} f_* \to \widehat{\mathcal{S}}$, and it is not hard to see that f_* preserves coproducts if and only if $\operatorname{Ker} f_*$ is closed under taking coproducts. We fix a coproduct $F = \coprod_i F_i$ in $\widehat{\mathcal{T}}$ and for each F_i a presentation

$$(-, X_i) \xrightarrow{(-,\phi_i)} (-, Y_i) \longrightarrow F_i \longrightarrow 0.$$

Now suppose that $F_i \in \operatorname{Ker} f_*$ for all i. Thus (S, ϕ_i) is surjective for all $S \in \mathcal{S}$ and all i. We have $F \in \operatorname{Ker} f_*$ if and only if the induced map

$$(S, \coprod_i X_i) \longrightarrow (S, \coprod_i Y_i)$$

is surjective for all $S \in \mathcal{S}$. Clearly, it is sufficient to have this for all $S \in \mathcal{S}_0$, and we conclude that h preserves countable coproducts if and only if (G2) holds for $\mathcal{S}_0$. □

Proof of the Theorem A. We fix a perfectly generating set $\mathcal{S}_0$ of objects in $\mathcal{T}$ and put $\mathcal{S} = \operatorname{Add} \mathcal{S}_0$. Replacing $\mathcal{S}_0$ by $\{\Sigma^n S \mid n \in \mathbb{Z},\, S \in \mathcal{S}_0\}$, we may assume that $\Sigma(\mathcal{S}_0) = \mathcal{S}_0$. Let $F \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ be a cohomological functor which sends coproducts in $\mathcal{T}$ to products. We construct inductively a sequence

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

of maps in $\mathcal{T}$ and a set of maps $\pi_i \colon (-, X_i) \to F$ for $i \geq 0$ as follows. Let $U = \bigcup_{S \in \mathcal{S}_0} FS$. Each $x \in U$ corresponds to an element in FS_x and we put $X_0 = \coprod_{x \in U} S_x$. We get an element in

$$\prod_{x\in U} FS_x \cong FX_0,$$

and using Yoneda's lemma, this gives a map $\pi_0\colon (-,X_0)\to F$. Suppose we have already constructed $\pi_i\colon (-,X_i)\to F$ for some $i\geq 0$. Let $K_i=\operatorname{Ker}\pi_i$ and let $U_i=\bigcup_{S\in\mathcal{S}_0}K_iS$. We define $T_i=\coprod_{x\in U_i}S_x$ and apply again Yoneda's lemma to obtain a map $T_i\to X_i$. We complete this to a triangle

$$T_i\xrightarrow{\upsilon_i}X_i\xrightarrow{\phi_i}X_{i+1}\xrightarrow{\chi_i}\Sigma T_i$$

and get an exact sequence

$$F(\Sigma T_i)\xrightarrow{F\chi_i}FX_{i+1}\xrightarrow{F\phi_i}FX_i\xrightarrow{F\upsilon_i}FT_i$$

since F is cohomological. The construction implies $(F\upsilon_i)\pi_i=0$ and this gives an element $\pi_{i+1}\in FX_{i+1}$ such that $(F\phi_i)\pi_{i+1}=\pi_i$. Thus we have a factorization

$$\pi_i\colon (-,X_i)\xrightarrow{(-,\phi_i)}(-,X_{i+1})\xrightarrow{\pi_{i+1}}F.$$

For each $i\geq 0$ the map υ_i induces an epimorphism $(-,T_i)|_{\mathcal{S}}\to K_i|_{\mathcal{S}}$ and we get therefore an exact sequence

$$(-,T_i)|_{\mathcal{S}}\xrightarrow{(-,\upsilon_i)|_{\mathcal{S}}}(-,X_i)|_{\mathcal{S}}\xrightarrow{\pi_i|_{\mathcal{S}}}F|_{\mathcal{S}}\longrightarrow 0.$$

We obtain in $\widehat{\mathcal{S}}$ for each $i\geq 0$ the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i|_{\mathcal{S}} & \longrightarrow & (-,X_i)|_{\mathcal{S}} & \xrightarrow{\pi_i|_{\mathcal{S}}} & F|_{\mathcal{S}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & K_{i+1}|_{\mathcal{S}} & \longrightarrow & (-,X_{i+1})|_{\mathcal{S}} & \xrightarrow{\pi_{i+1}|_{\mathcal{S}}} & F|_{\mathcal{S}} & \longrightarrow & 0
\end{array}$$

where $\psi_i=(-,\phi_i)|_{\mathcal{S}}$. Each ψ_i has a factorization

$$\psi_i\colon (-,X_i)|_{\mathcal{S}}\xrightarrow{\pi_i|_{\mathcal{S}}}F|_{\mathcal{S}}\xrightarrow{\iota_i}(-,X_{i+1})|_{\mathcal{S}}$$

and therefore $\pi_{i+1}|_{\mathcal{S}}\circ\iota_i=\mathrm{id}$. This gives the following commutative diagram

$$\begin{array}{ccccccc}
(-,X_1)|_{\mathcal{S}} & \xrightarrow{\psi_1} & (-,X_2)|_{\mathcal{S}} & \xrightarrow{\psi_2} & (-,X_3)|_{\mathcal{S}} & \xrightarrow{\psi_3} & \cdots\\
\downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} & & \\
F|_{\mathcal{S}}\coprod K_1|_{\mathcal{S}} & \xrightarrow{\mathrm{id}\coprod 0} & F|_{\mathcal{S}}\coprod K_2|_{\mathcal{S}} & \xrightarrow{\mathrm{id}\coprod 0} & F|_{\mathcal{S}}\coprod K_3|_{\mathcal{S}} & \xrightarrow{\mathrm{id}\coprod 0} & \cdots
\end{array}$$

and taking colimits in $\widehat{\mathcal{S}}$, we get an exact sequence

$$(*)\qquad 0\longrightarrow\coprod_i(-,X_i)|_{\mathcal{S}}\xrightarrow{\mathrm{id}-\psi_i}\coprod_i(-,X_i)|_{\mathcal{S}}\xrightarrow{(\pi_i|_{\mathcal{S}})}F|_{\mathcal{S}}\longrightarrow 0.$$

Now consider the triangle

$$\coprod_i X_i\xrightarrow{\mathrm{id}-\phi_i}\coprod_i X_i\longrightarrow X\longrightarrow\Sigma(\coprod_i X_i)$$

and observe that

$$(\pi_i)\in\prod_i FX_i\cong F(\coprod_i X_i)$$

induces a map $\pi\colon(-,X)\to F$. We apply the functor

$$\mathcal{T}\longrightarrow\widehat{\mathcal{S}},\quad X\mapsto(-,X)|_{\mathcal{S}},$$

which is cohomological and preserves countable coproducts by Lemma 3. This gives an exact sequence

$$\coprod_i (-,X_i)|_{\mathcal{S}} \xrightarrow{\mathrm{id}-\psi_i} \coprod_i (-,X_i)|_{\mathcal{S}} \longrightarrow (-,X)|_{\mathcal{S}} \longrightarrow \coprod_i (-,\Sigma X_i)|_{\mathcal{S}} \xrightarrow{\mathrm{id}-\Sigma\psi_i} \coprod_i (-,\Sigma X_i)|_{\mathcal{S}}.$$

We compare this sequence with $(*)$. The map $\mathrm{id}-\Sigma\psi_i$ is a monomorphism, since $\Sigma(\mathcal{S})=\mathcal{S}$, and it follows that $\pi|_{\mathcal{S}}\colon (-,X)|_{\mathcal{S}} \to F|_{\mathcal{S}}$ is an isomorphism. Moreover, the subcategory of all $Y\in\mathcal{T}$ such that π_Y is an isomorphism is triangulated, contains $\mathcal{S}_0$, and is closed under arbitrary coproducts.

Now let $\mathcal{T}'$ be the localizing subcategory of $\mathcal{T}$ which is generated by $\mathcal{S}_0$. Thus $\mathcal{T}'$ is the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}_0$ and is closed under coproducts. We claim that $\mathcal{T}'=\mathcal{T}$. To see this let $Y\in\mathcal{T}$ and apply the construction in the first part of this proof to $F=(-,Y)$. The corresponding map $\pi\colon(-,X)\to(-,Y)$ is induced by a map $X\to Y$ since $X\in\mathcal{T}'$. We complete this map to a triangle

$$W \longrightarrow X \longrightarrow Y \longrightarrow \Sigma W$$

and use (G1) to obtain $W=0=\Sigma W$ since $(S,X)\to(S,Y)$ is an isomorphism for all $S\in\mathcal{S}_0$, and $\Sigma(\mathcal{S}_0)=\mathcal{S}_0$ by our assumption. Thus $\mathcal{T}'=\mathcal{T}$ and we obtain $(-,X)\cong F$ in the first part of the proof. □

Corollary. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts which is perfectly generated by a set of objects $\mathcal{S}_0$. Suppose that $\mathcal{T}'$ is a full triangulated subcategory which is closed under countable coproducts and contains all coproducts of objects in $\mathcal{S}_0$. Then $\mathcal{T}'=\mathcal{T}$.*

Thus perfect generators are strong generators in the sense of [2]. Note that for every cardinal $\beta>\aleph_0$ a β-perfect generating set in the sense of [7] is automatically perfect as in Definition 1.

Remark. If $\mathcal{T}$ has a set $\mathcal{S}_0$ of perfect generators, then the construction of each $X\in\mathcal{T}$ implies that the functor $X\mapsto(-,X)|_{\mathcal{S}}$ preserves arbitrary coproducts. Therefore the following stronger condition holds for $\mathcal{S}_0$:

(G2′) for every set of maps $X_i\to Y_i$ in $\mathcal{T}$ the induced map
$$(S,\coprod_i X_i)\longrightarrow(S,\coprod_i Y_i)$$
is surjective for all $S\in\mathcal{S}_0$ provided that $(S,X_i)\to(S,Y_i)$ is surjective for all i and $S\in\mathcal{S}_0$.

Brown representability for the dual

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. In this section we prove a Brown Representability Theorem for $\mathcal{T}^{\mathrm{op}}$. The first result of this type is due to Neeman [6] and requires $\mathcal{T}$ to be generated by a set of small objects. This has been generalized in [7]. The concept which is used here stresses the symmetry between $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$.

Definition 2. A set $\mathcal{S}_0$ of objects is a set of *symmetric generators* for $\mathcal{T}$ if the following holds:

(G1) an object $X\in\mathcal{T}$ is zero provided that $(S,X)=0$ for all $S\in\mathcal{S}_0$;

(G3) there exists a set $\mathcal{T}_0$ of objects in $\mathcal{T}$ such that for every map $X\to Y$ in $\mathcal{T}$ the induced map $(S,X)\to(S,Y)$ is surjective for all $S\in\mathcal{S}_0$ if and only if $(Y,T)\to(X,T)$ is injective for all $T\in\mathcal{T}_0$.

It is clear that (G3) implies (G2), and that $\mathcal{T}$ has a set of symmetric generators if and only if $\mathcal{T}^{\mathrm{op}}$ has a set of symmetric generators. Therefore the following Brown Representability Theorem for $\mathcal{T}^{\mathrm{op}}$ is an immediate consequence of Theorem A.

Theorem B. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and suppose that $\mathcal{T}$ has a set of symmetric generators. Then $\mathcal{T}$ has arbitrary products, and a functor $F\colon \mathcal{T} \to \mathrm{Ab}$ is representable if and only if F is cohomological and preserves products.*

Proof. We have a set of perfect generators for $\mathcal{T}$ and therefore arbitrary products in $\mathcal{T}$. In fact, Theorem A implies that for every family $(X_i)_{i\in I}$ of objects the functor $\prod_i(-,X_i)$ is represented by an object which is $\prod_i X_i$. The set $\mathcal{T}_0$ which arises in (G3) is a set of perfect generators for $\mathcal{T}^{\mathrm{op}}$, and it follows from Theorem A that a functor $F\colon \mathcal{T} \to \mathrm{Ab}$ is representable if and only if F is cohomological and preserves products. $\square$

An example for a set of symmetric generators is any set $\mathcal{S}_0$ of small objects satisfying (G1). To see this, take for $\mathcal{T}_0$ the set of objects representing the functors

$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad X \mapsto ((S,X), \mathbb{Q}/\mathbb{Z})$$

where $S \in \mathcal{S}_0$. This shows that the stable homotopy category of CW-spectra or the unbounded derived category of modules over an associative ring have sets of symmetric generators.

Remark. Let $\mathcal{S}_0$ be a set of perfect generators and let $\mathcal{S} = \operatorname{Add}\mathcal{S}_0$. Each injective object $I \in \widehat{\mathcal{S}}$ gives rise to an object in $\mathcal{T}$ representing

$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad X \mapsto ((-,X)|_{\mathcal{S}}, I).$$

Therefore (G3) holds for $\mathcal{S}_0$ if and only if $\widehat{\mathcal{S}}$ has an injective cogenerator.

Neeman's Brown Representability Theorem for the dual in [7] involves the existence of an injective cogenerator for a category which is equivalent to some $\widehat{\mathcal{S}}$; it is therefore a consequence of Theorem B.

A filtration

Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. In this section we study a filtration $\mathcal{T} = \bigcup_\kappa \mathcal{S}_\kappa$ which is defined in terms of a set $\mathcal{S}_0$ of appropriate generators.

Let α be a cardinal. Recall that an object S is *α-small* if every map $S \to \coprod_{i\in I} X_i$ factors through $\coprod_{i\in J} X_i$ for some $J \subseteq I$ with $\operatorname{card} J < \alpha$.

Theorem C. *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts, and suppose that $\mathcal{T}$ is perfectly generated by a set $\mathcal{S}_0$ of α-small objects. Let κ be the successor of λ^α for some cardinal*

$$\lambda \geq \sup\{\operatorname{card}(S, \coprod_{i\in I} S_i) \mid S, S_i \in \mathcal{S}_0 \text{ and } \operatorname{card} I < \alpha\} + \operatorname{card}\mathcal{S}_0.$$

Then the objects $X \in \mathcal{T}$ satisfying $\operatorname{card}(S,X) < \kappa$ for all $S \in \mathcal{S}_0$ form a subcategory $\mathcal{S}_\kappa$ having the following properties:

1. *$\mathcal{S}_\kappa$ is a triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}_0$;*
2. *$\mathcal{S}_\kappa$ is closed under taking coproducts of less than κ objects;*
3. *the isomorphism classes of objects in $\mathcal{S}_\kappa$ form a set;*
4. *every subcategory $\mathcal{T}'$ of $\mathcal{T}$ which satisfies* (1) *and* (2) *contains $\mathcal{S}_\kappa$;*
5. *every object in $\mathcal{S}_\kappa$ is κ-small.*

A triangulated category which is well generated in the sense of Neeman [7] satisfies the assumption of the preceding theorem. The conclusion of this theorem implies the condition on a triangulated category which Franke assumes in [2] for his proof of the Brown Representability Theorem. Note that the proof in [2] is based on a variant of Freyd's Adjoint Functor Theorem. Thus Theorem C provides a link between results having completely different proofs.

Proof of the Theorem C. (1) is clear. To prove (2), let $S \in \mathcal{S}_0$ and $(X_i)_{i\in I}$ be a family of less than κ objects in $\mathcal{S}_\kappa$. Suppose first that $X_i \in \mathcal{S}_0$ for all i. Every map $S \to \coprod_{i\in I} X_i$ factors through $\coprod_{i\in J} X_i$ for some $J \subseteq I$ with $\operatorname{card} J < \alpha$. We have $\operatorname{card}(S, \coprod_{i\in J} X_i) \leq \lambda$, and I has at most $(\lambda^\alpha)^\alpha = \lambda^\alpha$ subsets of cardinality less than α. Therefore

$$\operatorname{card}(S, \coprod_i X_i) \leq \lambda^\alpha \cdot \lambda = \lambda^\alpha < \kappa.$$

Now let each $X_i \in \mathcal{S}_\kappa$ be arbitrary. We have for each $i \in I$ a map $T_i \to X_i$ such that T_i is a coproduct of less than κ objects from $\mathcal{S}_0$ and the induced map $(S, T_i) \to (S, X_i)$ is surjective for all $S \in \mathcal{S}_0$. Using (G2′), it follows that the induced map

$$(S, \coprod_i T_i) \longrightarrow (S, \coprod_i X_i)$$

is surjective for all $S \in \mathcal{S}_0$. Thus $\coprod_i X_i$ belongs to $\mathcal{S}_\kappa$ since $\coprod_i T_i \in \mathcal{S}_\kappa$ by the first part of this proof.

(3) and (4) follow from the proof of Theorem A where it is shown that each object in $\mathcal{S}_\kappa$ can be constructed in countably many steps from objects in $\mathcal{S}_0$ by taking coproducts of less than κ factors and cofibers. Note that in each step there is only a set of possible choices.

(5) follows from (4) since the κ-small objects form a triangulated subcategory which satisfies (1) and (2). $\square$

As an example take the stable homotopy category $\mathcal{S}$ of CW-spectra. The set $\mathcal{S}_0 = \{\Sigma^n S \mid n \in \mathbb{Z}\}$ of suspensions of the sphere spectrum $S = S^0$ is a set of perfect generators where $\alpha = \aleph_0$. For every regular cardinal $\kappa > \aleph_0$ the subcategory

$$\mathcal{S}_\kappa = \{X \in \mathcal{S} \mid \operatorname{card} \pi_*(X) < \kappa\}$$

has the properties (1) – (5) of the preceding theorem.

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References

[1] M. Auslander, Coherent functors, in: Proceedings of the conference on categorical algebra (La Jolla, 1965), Springer-Verlag (1966), 189–231.
[2] J. Franke, On the Brown representability theorem for triangulated categories, Topology, to appear.
[3] P. Freyd, Stable homotopy, in: Proceedings of the conference on categorical algebra (La Jolla, 1965), Springer-Verlag (1966), 121–172.
[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France **90** (1962), 323–448.
[5] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. **9** (1996), 205–236.
[6] A. Neeman, Brown representability for the dual, Invent. Math. **133** (1998), 97–105.
[7] A. Neeman, Triangulated categories, Princeton Univ. Press, to appear.

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