INFINITESIMAL UNIPOTENT GROUP SCHEMES OF COMPLEXITY 1

ROLF FARNSTEINER, GERHARD RÖHRLE, AND DETLEF VOIGT

0. Introduction

One basic notion within the representation theory of Artin algebras is that of the representation type. According to a fundamental theorem by Drozd [6] a finite-dimensional algebra over an algebraically closed field has either finite, tame, or wild type. In this paper we are primarily concerned with the former type in case of the distribution algebra $H(\mathcal{U})$ of a unipotent infinitesimal group $\mathcal{U}$.

Every finite algebraic group $\mathcal{G}$, defined over an algebraically closed field k of positive characteristic p, is the semidirect product of an infinitesimal normal subgroup $\mathcal{G}^0$ and a reduced group $\mathcal{G}_{\text{red}}$, that is

$$\mathcal{G} \cong \mathcal{G}^0 \rtimes \mathcal{G}_{\text{red}},$$

see [17, Thm. 6.8]. When studying representation-finite algebraic groups of dimension zero one is thus led to two questions: the classification of the finite groups of finite representation type, and the solution of this problem for infinitesimal groups. The first case was addressed in [13]: a finite group G is representation-finite if and only if all its Sylow p-subgroups are cyclic. While this condition does not determine the structure of G, representation-finite infinitesimal groups are to a large extent governed by certain unipotent subquotients. More specifically, an infinitesimal group $\mathcal{G}$ has finite representation type if and only if the quotient group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ of $\mathcal{G}$ by its multiplicative center $\mathcal{M}(\mathcal{G})$ is isomorphic to a semidirect product of a $\mathcal{V}$-uniserial unipotent normal subgroup $\mathcal{U}$, and a multiplicative subgroup of type μ_{p^n}, that is

$$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mu_{p^n},$$

see [10, Thm. 2.7]. Accordingly, a description of all infinitesimal groups of finite representation type entails the classification of the $\mathcal{V}$-uniserial groups.

In this paper we determine the slightly larger class of uniserial unipotent commutative infinitesimal groups, see Theorem 1.2. Thanks to [5, (V, §1, n° 4.3)] the unipotent commutative groups correspond to Dieudonné modules. Accordingly, we first classify the uniserial Dieudonné modules in Section 2. The proof of Theorem 1.2, which we present in Section 3, interprets this classification within the category of infinitesimal commutative uniserial groups.

Our classification also illustrates some of the subtle differences between the representation theory of infinitesimal groups and its classical precursor for finite groups: although there exist more representation-finite infinitesimal unipotent groups than representation-finite p-groups, our results from Section 4 show that the class of representation-finite infinitesimal groups is better understood in general.

Date: October 12, 2000.

1991 *Mathematics Subject Classification.* Primary 16G70, Secondary 17B50.

The second application of Theorem 1.2 shows that, contrary to finite groups, representation-finite infinitesimal groups can be described via Alperin's notion of complexity. More specifically, we characterize $\mathcal{V}$-uniserial groups via the complexity of their second Frobenius kernels. This affords an approach to the main results of [10] that avoids the use of the Friedlander-Suslin theorem concerning the finite generation of the cohomology ring of cocommutative Hopf algebras.

1. $\mathcal{V}$-Uniserial Groups and Hopf Structures

1.1. We consider affine group schemes $\mathcal{G}$ defined over the algebraically closed field k of positive characteristic p. By definition, these are representable group valued functors $\mathcal{G} : \mathbb{M}_k \longrightarrow \mathbb{G}$ from the category $\mathbb{M}_k$ of commutative k-algebras into the category $\mathbb{G}$ of groups. Concerning general facts and properties of affine group schemes we refer to [5], [14], and [17].

An infinitesimal k-group $\mathcal{U}$ is called *uniserial* provided $\mathcal{U}$ has a unique composition series. If $\mathcal{U}$ is unipotent and commutative, then it is called *$\mathcal{V}$-uniserial* if the cokernel of the *Verschiebung* $\mathcal{V} : \mathcal{U}^{(p)} \longrightarrow \mathcal{U}$ is simple, that is when $\operatorname{coker} \mathcal{V} \cong \alpha_p$. Likewise, a unipotent infinitesimal group $\mathcal{U}$ is called *$\mathcal{F}$-uniserial* provided the kernel of the *Frobenius morphism* $\mathcal{F} : \mathcal{U} \longrightarrow \mathcal{U}^{(p)}$ is simple, that is $\mathcal{U}_1 \cong \alpha_p$. For the definitions of the morphisms $\mathcal{F}$ and $\mathcal{V}$, see [5, (II, §7, n° 1),(IV, §3, n° 4.4)]. Let $\mathbb{U}$ denote the category of infinitesimal unipotent commutative groups over k. We denote by $\mathcal{D} : \mathbb{U} \longrightarrow \mathbb{U}$ *Cartier duality* on $\mathbb{U}$ which is given by

$$\mathcal{D}(\mathcal{U})(R) := \operatorname{Hom}(\mathcal{U}_R, \mu_R),$$

for every $R \in \mathbb{M}_k$. Note that $\mathcal{D}$ is an involutary anti-equivalence on $\mathbb{U}$, see [5, (II, §1, n° 2.10)]. According to [5, (IV, §3, n° 4.9)], Cartier duality on $\mathbb{U}$ sends $\mathcal{V}$-uniserial groups to $\mathcal{F}$-uniserial ones and vice versa.

1.2. By $\mathcal{W} : \mathbb{M}_k \longrightarrow \mathbb{M}_\mathbb{Z}$ we denote the affine commutative group scheme (in rings) of *Witt vectors*. For $m \in \mathbb{N}$ let $\mathcal{W}_m : \mathbb{M}_k \longrightarrow \mathbb{M}_\mathbb{Z}$ be the affine commutative group scheme of *Witt vectors of length* m, see [4, (IX, §1)], or [5, (V, §1, n° 1.6)]. Since $\mathcal{W}_m$ is defined over $\mathbb{Z}$, the natural transformations $\mathcal{F}$ and $\mathcal{V}$ can be viewed as endomorphisms of $\mathcal{W}_m$, cf. [14, (I.9.4)]. It follows from [5, (V, §1, n° 1.9)] and [5, (IV, §3, n° 4.11)] that $\mathcal{W}_m$ is unipotent. We define certain infinitesimal subgroups of $\mathcal{W}_m$ as follows. Let $d, j, n \in \mathbb{N}$. For $n \geq 1$ and $d \geq 2$ we denote by $\mathcal{U}_{n,d}$ the kernel of the endomorphism $\mathcal{V}^{d-1} - \mathcal{F} : \mathcal{W}_m \longrightarrow \mathcal{W}_m$ with $m = nd$. For $n, d \geq 2$ and $1 \leq j \leq d-1$ let $\mathcal{U}^j_{n,d}$ be the intersection of $\mathcal{U}_{n,d}$ with the kernel of the endomorphism $\mathcal{V}^{(n-1)d+j} : \mathcal{W}_m \longrightarrow \mathcal{W}_m$.

We can now state our principal result, the classification of infinitesimal unipotent commutative uniserial group schemes.

Theorem. *The following is a complete list of representatives of isomorphism classes of non-trivial infinitesimal unipotent commutative uniserial k-groups:*

(i) $(\mathcal{W}_d)_1$ *for* $d \geq 1$*;*
(i) $\mathcal{U}_{n,d}$ *for* $n \geq 1, d \geq 2$*;*
(iii) $\mathcal{U}^j_{n,d}$ *for* $n \geq 2, d \geq 2, 1 \leq j \leq d-1$*;*
(iv) $\mathcal{D}((\mathcal{W}_d)_1) \cong \alpha_{p^d}, d \geq 2$*;*
(v) $\mathcal{D}(\mathcal{U}_{n,d})$ *for* $n \geq 1, d \geq 3$*;*

(vi) $\mathcal{D}(\mathcal{U}_{n,d}^j)$ *for* $n \geq 2, d \geq 3, 1 \leq j \leq d-1$.

The groups labeled (i) - (iii) *are* $\mathcal{V}$*-uniserial, and those in* (iv) - (vi) *are* $\mathcal{F}$*-uniserial. Moreover, the groups* $\mathcal{U}_{n,2}$ *for* $n \geq 1$, $(\mathcal{W}_1)_1 \cong \alpha_p$, *and* $\mathcal{U}_{n,2}^1$ *for* $n \geq 2$ *are self-dual.*

1.3. The following result shows that Theorem 1.2 also gives a complete understanding of the cocommutative Hopf algebras whose underlying associative k-algebras are truncated polynomial rings of dimension a power of p.

Theorem. *Let* $\mathcal{G}$ *be a finite algebraic* k*-group whose algebra of measures* $H(\mathcal{G})$ *is local and of finite representation type. Then* $\mathcal{G}$ *is either a cyclic* p*-group, or* $\mathcal{G}$ *is* $\mathcal{V}$*-uniserial.*

Proof. We decompose $\mathcal{G}$ into its infinitesimal and reduced parts $\mathcal{G} \cong \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$. Thanks to [10, (4.1)] both constituents are representation-finite, with at least one of them being linearly reductive.

As $H(\mathcal{G})$ is local, its augmentation ideal $H(\mathcal{G})^\dagger$ is nilpotent. Consequently, every Hopf subalgebra of $H(\mathcal{G})$ is also local. Hence, if $\mathcal{G}^0$ is linearly reductive, then $H(\mathcal{G}^0)$ is semisimple and local, so that $\mathcal{G}^0 = e_k$. Accordingly, $H(\mathcal{G}) \cong k[\mathcal{G}(k)]$ is the group algebra of the finite group of k-rational points of $\mathcal{G}$. Owing to [13], this implies that $\mathcal{G}(k)$ is a cyclic group of order a p-power.

Alternatively, $H(\mathcal{G}_{\mathrm{red}}) \cong H(\mathcal{G}/\mathcal{G}^0)$ is local and $\mathcal{G}_{\mathrm{red}}$ is linearly reductive, so that $\mathcal{G}_{\mathrm{red}} = e_k$. Consequently, $\mathcal{G} = \mathcal{G}^0$ is infinitesimal and we may apply [10, (2.7)] to see that $\mathcal{G}$ is $\mathcal{V}$-uniserial. $\square$

Remark. Observe that Theorems 1.2 and 1.3 classify the isoclasses of the representation-finite local, cocommutative Hopf algebras. Two such are isomorphic as algebras if and only if their underlying groups have the same length.

2. $\mathcal{V}$-uniserial Dieudonné Modules

2.1. As k is perfect, the ring $\mathcal{W}(k)$ is a complete discrete valuation domain with maximal ideal $(p) = p\mathcal{W}(k)$. Its residue field is $\mathcal{W}(k)/p\mathcal{W}(k) \cong k$ (cf. [5, (V, §1, nº 1.8)]). Moreover, for each $m \in \mathbb{N}$ we have the canonical isomorphism

$$\mathcal{W}_m(k) \cong \mathcal{W}(k)/p^m\mathcal{W}(k).$$

The Frobenius morphism $\mathcal{F} : \mathcal{W} \longrightarrow \mathcal{W}$ induces an automorphism on $\mathcal{W}(k)$ which is denoted by $w \mapsto w^{(p)}$, [5, (V, §1, nº 3.1)]. In the same fashion we obtain an automorphism on $\mathcal{W}_n(k)$ afforded by $\mathcal{F}$, also denoted by $w \mapsto w^{(p)}$. Note that $\mathcal{W}_n$ is a reduced algebraic k-group with affine n-space as underlying scheme.

2.2. Let $\mathbf{D}$ be the skew polynomial ring $\mathbf{D} := \mathcal{W}(k)[F, V]$ subject to the relations

(i) $FV = VF = p \in \mathcal{W}(k)$,
(ii) $Fw = w^{(p)}F$ for $w \in \mathcal{W}(k)$, and
(iii) $wV = Vw^{(p)}$ for $w \in \mathcal{W}(k)$.

The ring $\mathbf{D}$ is customarily called the *Dieudonné ring*, see [5, (V, §1, nº 3.1)]. Note that $\mathbf{D}V^n$ and $\mathbf{D}F^n$ are two-sided ideals of $\mathbf{D}$ for each $n \in \mathbb{N}$.

2.3. Let $\mathrm{mod}_{\mathbf{D}}$ be the category of $\mathbf{D}$-modules of finite length on which F and V operate nilpotently. Consequently, a module M in $\mathrm{mod}_{\mathbf{D}}$ can be viewed as a module over the ring $\mathbf{D}/(\mathbf{D}F^n+\mathbf{D}V^n)$ for a suitable $n \in \mathbb{N}$. Moreover, since $p^n = (FV)^n = F^nV^n$, the module M is also a $\mathcal{W}_n(k)$-module. In particular, the residue field k of $\mathcal{W}(k)$ with V and F operating trivially, is the unique simple module in $\mathrm{mod}_{\mathbf{D}}$. This implies that the length of M in $\mathrm{mod}_{\mathbf{D}}$ coincides with the length of M viewed as a $\mathcal{W}(k)$-module. In the sequel we refer to the members of $\mathrm{mod}_{\mathbf{D}}$ as *Dieudonné modules*.

2.4. According to [5, (IV, §3, n° 4.11), (V, §1, n° 4.3, 4.6)] there is an anti-equivalence of categories

$$\Upsilon : \mathbb{U} \longrightarrow \mathrm{mod}_{\mathbf{D}}\,.$$

The functor Υ induces, via Cartier duality, an anti-equivalence of categories $D : \mathrm{mod}_{\mathbf{D}} \longrightarrow \mathrm{mod}_{\mathbf{D}}$, satisfying

$$\Upsilon(\mathcal{D}(\mathcal{U})) \cong D(\Upsilon(\mathcal{U})),$$

where $\mathcal{U} \in \mathbb{U}$, see [5, (V, §4, n° 5.6)]. We refer to D as *duality* on $\mathrm{mod}_{\mathbf{D}}$.

2.5. A Dieudonné module M is called *uniserial* if it has a unique composition series; M is called *V-uniserial* if $M \supset VM \supset V^2M \supset \ldots \supset (0)$ is a composition series of M. Observe that M is V-uniserial if and only if M/VM is a simple $\mathbf{D}$-module. Since $\ell(M) = \ell(VM)+\ell(\ker V)$, this is equivalent to $\ker V$ being a simple Dieudonné module. Analogously, we define *F-uniserial* Dieudonné modules.

The duality on $\mathrm{mod}_{\mathbf{D}}$ sends F-uniserial modules to V-uniserial ones and vice versa [5, (V, §4, n° 5.2)].

The anti-equivalence $\Upsilon : \mathbb{U} \longrightarrow \mathrm{mod}_{\mathbf{D}}$ takes uniserial groups to uniserial Dieudonné modules, and accordingly $\mathcal{V}$-uniserial groups to V-uniserial $\mathbf{D}$-modules, likewise for $\mathcal{F}$-uniserial groups.

Lemma. *Let M be in $\mathrm{mod}_{\mathbf{D}}$. Then the following statements hold:*

(i) *M is uniserial if and only if it is V-uniserial or F-uniserial.*
(ii) *M is V-uniserial if and only if M/F^2M is V-uniserial.*
(iii) *If M is V-uniserial, then $M = \mathcal{W}(k)m + VM$ for any $m \in M \setminus VM$.*

Proof. (i). If M is V-uniserial or F-uniserial, then it is uniserial, since the simple $\mathbf{D}$-modules are annihilated by F and V. For the other implication, let M be uniserial of length $\ell(M) \geq 2$, and $N \subseteq M$ its unique submodule of length $\ell(N) = \ell(M) - 2$. If M is neither V-uniserial, nor F-uniserial, then $VM + FM \subseteq N$. Thus, M/N is a uniserial module of length 2 for $\mathbf{D}/(\mathbf{D}F + \mathbf{D}V) \cong k$, a contradiction.

(ii). Suppose that M/F^2M is V-uniserial. Then $M/(VM + F^2M)$ is simple. Let $J = (F, V)$ be the Jacobson radical of $\mathbf{D}$. If $M \neq (0)$, then $M \neq JM = VM+FM$. Consequently, $VM+FM = VM+F^2M$, so that $F\cdot M/VM = F^2\cdot M/VM$. This implies $F\cdot M/VM = (0)$, whence $FM \subset VM$. As a result, $M/VM = M/(VM+F^2M)$ is simple, and M is V-uniserial.

(iii). Since M is V-uniserial, the proper submodule FM is contained in VM. Consequently, $\mathcal{W}(k)m + VM$ is a $\mathbf{D}$-submodule of M, which is not contained in the unique maximal submodule VM. □

Remark. Let $\mathcal{U}$ be an infinitesimal commutative unipotent k-group. Since $\Upsilon(\ker \mathcal{F}^2_{\mathcal{U}}) \cong \operatorname{coker} F^2_{\Upsilon(\mathcal{U})}$ (cf. [5, (V, §1, n° 4.6)]), Lemma 2.5(ii) shows that the group $\mathcal{U}$ is $\mathcal{V}$-uniserial if and only if its second Frobenius kernel $\mathcal{U}_2$ has this feature.

2.6. Given $l \in \mathbb{Z}$ we denote the l-th iterate of the automorphim $w \mapsto w^{(p)}$ of $\mathcal{W}(k)$ by $w \mapsto w^{(p^l)}$. The induced automorphism on $\mathcal{W}_n(k)$ is labeled in the same fashion. Let M be a $\mathcal{W}_n(k)$-module. By $M^{(l)}$ we denote the $\mathcal{W}_n(k)$-module with underlying abelian group M and module structure given by $w \cdot m = w^{(p^l)}m$, for $w \in \mathcal{W}_n(k)$ and $m \in M$. Let $F_M : M \longrightarrow M^{(1)}$ and $V_M : M \longrightarrow M^{(-1)}$ be two $\mathcal{W}_n(k)$-module maps. Then the two morphisms F_M and V_M define the structure of a $\mathbf{D}$-module on M that is compatible with the $\mathcal{W}_n(k)$-module structure if and only if $F_M \circ V_M = p \cdot id_M = V_M \circ F_M$. If M is free, then it suffices to define F_M and V_M on a basis. In the following certain Dieudonné modules are defined in this fashion.

Definition. Let $d, j, n \in \mathbb{N}$. Let M be a free $\mathcal{W}_n(k)$-module with basis $\{e_1, \dots , e_d\}$.

(i) For $d \geq 2$ we denote by $E_{n,d}$ the Dieudonné module with underlying $\mathcal{W}_n(k)$-space M and operations given by
  (a) $V_M(e_i) = e_{i+1}$ for $1 \leq i \leq d-1$, and $V_M(e_d) = pe_1$;
  (b) $F_M(e_1) = e_d$, $F_M(e_i) = pe_{i-1}$ for $2 \leq i \leq d$.

(ii) For $n \geq 2$, $d \geq 2$ and $1 \leq j \leq d-1$ we define the $\mathbf{D}$-module $E^j_{n,d}$ as the quotient
$$E^j_{n,d} := E_{n,d}/V^{(n-1)d+j}E_{n,d}.$$

(iii) For $d \geq 1$ we write E_d for the Dieudonné module with underlying $\mathcal{W}_1(k)$-space M and operations given by
  (a) $V_M(e_i) = e_{i+1}$ for $1 \leq i \leq d-1$, and $V_M(e_d) = 0$;
  (b) $F_M = 0$.

Lemma. *The Dieudonné modules $E_{n,d}$ for $n \geq 1, d \geq 2$, $E^j_{n,d}$ for $n, d \geq 2, 1 \leq j \leq d-1$, and E_d for $d \geq 1$ are V-uniserial and pairwise distinct.*

Proof. Observe that V, F and thus $FV = p$ operate trivially on $E_{n,d}/VE_{n,d}$. Therefore, $E_{n,d}/VE_{n,d} \cong k$ is a simple $\mathbf{D}$-module, and $E_{n,d}$ is V-uniserial. Hence, its factor modules $E^j_{n,d}$ have the same property. Similarly, we have $E_d/VE_d \cong k$, so that E_d is also V-uniserial.

By definition, $E_{n,d}$ is a free $\mathcal{W}_n(k)$-module of rank d and thus has length dn (viewed as a module for $\mathcal{W}_n(k)$ or $\mathbf{D}$). Since $E_{n,d}$ is V-uniserial, the module $E^j_{n,d}$ has length $(n-1)d+j$. Because the Loewy length of a $\mathcal{W}_n(k)$-module is the minimal power of p that annihilates it, n is the Loewy length of the $\mathcal{W}_n(k)$-modules $E_{n,d}$ and $E^j_{n,d} = E_{n,d}/p^{n-1}V^jE_{n,d}$. Moreover, the length of the quotients $E_{n,d}/pE_{n,d} \cong E_{1,d} \cong E^j_{n,d}/pE^j_{n,d}$ is d.

By the above, the modules $E_{n,d}$, $E^j_{n,d}$ are pairwise non-isomorphic. Since $\operatorname{rank} E_d = d$, no two modules of this type are isomorphic. As F acts trivially on E_d, this module is not isomorphic to one of the form $E_{n,d}$ or $E^j_{n,d}$. □

2.7. We are now in the position to prove the chief result of this section which is a revision of the approach in [16].

Theorem. *The following provides a complete list of representatives of isomorphism classes of non-zero uniserial Dieudonné modules:*

(i) *E_d for $d \geq 1$;*

(ii) $E_{n,d}$ *for* $n \geq 1, d \geq 2$*;*
(iii) $E^j_{n,d}$ *for* $n \geq 2, d \geq 2, 1 \leq j \leq d-1$*;*
(iv) DE_d *for* $d \geq 2$*;*
(v) $DE_{n,d}$ *for* $n \geq 1, d \geq 3$*;*
(vi) $DE^j_{n,d}$ *for* $n \geq 2, d \geq 3, 1 \leq j \leq d-1$.

The modules labeled (i) - (iii) *are* V*-uniserial, and the ones listed in* (iv) - (vi) *are* F*-uniserial. Moreover, the modules* E_1*,* $E_{n,2}$ *for* $n \geq 1$*, and* $E^1_{n,2}$ *for* $n \geq 2$ *are self-dual.*

Proof. Proceeding in several steps, we first determine the list of V-uniserial modules.

(a) *Let* M *be a* V*-uniserial Dieudonné module of length* d *such that* $pM = (0)$*. Then* M *is isomorphic to either* $E_{1,d}$ *or* E_d*.*

Since $pM = (0)$, M is a k-vector space of dimension d. Moreover, V acts on M as a semi-linear nilpotent operator of order d. If $m \in M$ satisfies $V^{d-1}m \neq 0$, then $\{m, Vm, V^2m, \dots, V^{d-1}m\}$ is a k-basis of M. As $VFm = pm = 0$, we get $Fm \in \ker V = kV^{d-1}m$, so that $Fm = \lambda V^{d-1}m$, for some $\lambda \in k$.

If $\lambda = 0$, then $M \cong E_d$. Alternatively, let $\xi \in k$ be a solution of the equation $\lambda^{p^{d-1}}X^{p^d} - X = 0$. Replacing m by ξm, we may assume that $\lambda = 1$. Consequently, $e_i \mapsto V^{i-1}m$ defines an isomorphism $E_{1,d} \cong M$.

(b) *Let* M *be a* V*-uniserial Dieudonné module such that* $pM \neq (0)$ *and* $\ell(M/pM) = d$*. Then there exists an element* $m \in M \setminus VM$ *so that* $Fm = V^{d-1}m$*.*

By (a) M/pM is isomorphic to $E_{1,d}$ or E_d. In the latter case we have $FM \subseteq pM$, so that $pM = VFM \subseteq VpM = pVM \subseteq pM$, whence $VpM = pM$. Since V operates nilpotently on M, we have $pM = (0)$, a contradiction. Hence $M/pM \cong E_{1,d}$ and there exists an element $m_0 \in M \setminus VM$ with $Fm_0 \equiv V^{d-1}m_0 \mod pM$. Since $\ell(M/pM) = d$, we have $V^dM = pM$.

We are going to construct inductively a sequence $(m_j)_{j\in\mathbb{N}_0}$ of elements in M, satisfying the following properties:

(i) $m_{j+1} \equiv m_j \mod V^{j+1}M$, and
(ii) $Fm_j \equiv V^{d-1}m_j \mod V^{d+j}M$.

The initial element m_0 is the one from above. Suppose that a sequence $\{m_0, m_1, \dots, m_j\}$ satisfying (i) and (ii) has already been constructed. Since $m_j \equiv m_0 \mod VM$, it follows that $m_j \notin VM$. By (ii) there exists an element $m' \in M$ such that $Fm_j = V^{d-1}m_j + V^{d+j}m'$. Recall that M is a $\mathcal{W}_n(k)$-module for some n. Thanks to Lemma 2.5(iii) we may write $m' = am_j + Vm''$, for some $a \in \mathcal{W}_n(k)$ and $m'' \in M$. Thus, we obtain

$$\begin{aligned} Fm_j &= V^{d-1}m_j + V^{d+j}(am_j + Vm'') \\ &\equiv V^{d-1}m_j + V^{d+j}am_j \mod V^{d+j+1}M. \end{aligned} \tag{1}$$

Since $\mathcal{W}_n(k)$ is a connected affine algebraic group defined over the field with p^d elements, the Corollary [3, (16.5)] to the Theorem of Lang applies. Consequently, there is an element $\xi \in \mathcal{W}_n(k)$ satisfying

$$\xi^{(p^d)} - \xi + a = 0.$$

We set

$$m_{j+1} := m_j + V^{j+1}\xi m_j.$$

In virtue of (1) we obtain

$$\begin{aligned}
Fm_{j+1} &= Fm_j + FV^{j+1}\xi m_j \\
&= Fm_j + \xi^{(p^{-j})}V^{j+1}Fm_j \\
&\equiv V^{d-1}m_j + V^{d+j}am_j + \xi^{(p^{-j})}V^{d+j}m_j \mod V^{d+j+1}M \\
&\equiv V^{d-1}m_j + V^{d+j}am_j + V^{d+j}\xi^{(p^d)}m_j \mod V^{d+j+1}M \\
&\equiv V^{d-1}m_j + V^{d+j}\xi m_j \mod V^{d+j+1}M \\
&\equiv V^{d-1}m_{j+1} \mod V^{d+j+1}M.
\end{aligned}$$

Let j be such that $V^{d+j}M = (0)$. Then m_j has the requisite properties.

(c) *Let M be a V-uniserial Dieudonné module with $pM \neq (0)$, $\ell(M/pM) = d$, and such that n is the Loewy length of the $\mathcal{W}(k)$-module M. Then $M \cong E_{n,d}$ if $\ell(p^{n-1}M) = d$ and $M \cong E^j_{n,d}$ if $j := \ell(p^{n-1}M) < d$.*

For $m \in M \setminus VM$ as in (b) we consider the $\mathcal{W}_n(k)$-module morphism $\phi : E_{n,d} \longrightarrow M$ which maps e_i to $V^{i-1}m$ for $1 \leq i \leq d$. Direct computation shows that ϕ is in fact a map of **D**-modules. Since M is generated by m (cf. Lemma 2.5(iii)), the map ϕ is surjective.

If $\ell(p^{n-1}M) = d = \ell(M/pM)$, it follows that $\ell(M) = nd = \ell(E_{n,d})$, so that ϕ is an isomorphism in this case.

Alternatively, $\ell(p^{n-1}M) = j < d$ and $\ker \phi \neq (0)$. Note that $p^{n-1}E_{n,d} \not\subseteq \ker \phi$, because the Loewy length of M is n. Since M is uniserial, we obtain $\ker \phi \subseteq p^{n-1}E_{n,d}$. As $\phi|_{p^{n-1}E_{n,d}} : p^{n-1}E_{n,d} \longrightarrow p^{n-1}M$ is surjective, we have $\ell(\ker \phi) = d - j$, so that $\ker \phi = p^{n-1}V^jE_{n,d} = V^{(n-1)d+j}E_{n,d}$.

By combining Lemma 2.6 with (a) and (c) we obtain

(d) *The modules in* (i) - (iii) *form a complete list of representatives of the V-uniserial Dieudonné modules.*

(e) *The modules E_1, $E_{n,2}$ for $n \geq 1$, or $E^1_{n,2}$ for $n \geq 2$ are precisely the self-dual uniserial Dieudonné modules.*

Let M be a self-dual, uniserial Dieudonné module. According to Lemma 2.5, M is both, V- and F-uniserial. By inspection we find that $E_{n,2}$, $E^1_{n,2}$, and E_1 are the only F-uniserial modules listed in (i) - (iii).

Clearly, $E_1 \cong k$ is self-dual. As $E_{n,2}$ and $E^1_{n,2}$ are V-uniserial and F-uniserial of lengths $2n$ and $2(n-1)+1$, respectively, their duals have the same property. Consequently, these modules are self-dual.

(f) *Let M be a uniserial Dieudonné module. Then M is isomorphic to exactly one of the modules listed in* (i) - (vi).

By Lemma 2.5 M is V-uniserial or F-uniserial. In the first instance the assertion follows from (d). If M is not V-uniserial, then DM is V-uniserial but not self-dual. Thus, observing (e), we see that $M \cong D^2M$ belongs to (iv) - (vi). $\square$

3. Proof of Theorem 1.2

By abuse of notation, we denote the images of F and V in the quotients $\mathbf{D}_m := \mathbf{D}/\mathbf{D}V^m$ again by F and V, respectively.

Lemma. *We have*
(i) $\Upsilon(\mathcal{U}_{n,d}) \cong E_{n,d}$,
(ii) $\Upsilon(\mathcal{U}^j_{n,d}) \cong E^j_{n,d}$, *and*
(iii) $\Upsilon((\mathcal{W}_d)_1) \cong E_d$.

Proof. (i). Let $m = nd$. By definition, the linear map $\mathbf{D}_m \longrightarrow E_{n,d}$ sending 1 to e_1 factors through to the quotient $\widehat{\mathbf{D}}_m := \mathbf{D}_m/\mathbf{D}_m(V^{d-1} - F)$. The resulting map $\psi : \widehat{\mathbf{D}}_m \longrightarrow E_{n,d}$ is obviously surjective. As $\ell(\widehat{\mathbf{D}}_m) \leq nd$, we see that ψ is in fact an isomorphism. The anti-equivalence Υ sends the exact sequence

$$e_k \longrightarrow \mathcal{U}_{n,d} \longrightarrow \mathcal{W}_m \xrightarrow{\mathcal{V}^{d-1}-\mathcal{F}} \mathcal{W}_m$$

to the exact sequence

$$\mathbf{D}_m \longrightarrow \mathbf{D}_m \longrightarrow \Upsilon(\mathcal{U}_{n,d}) \longrightarrow (0),$$

where the left-hand map is right multiplication by $V^{d-1} - F$ (cf. [5, (V, §1, n° 4.2)]). Hence we get the isomorphism $\Upsilon(\mathcal{U}_{n,d}) \cong E_{n,d}$.

(ii). The endomorphisms $\mathcal{V}^{d-1} - \mathcal{F}$ and $\mathcal{V}^{(n-1)d+j}$ of $\mathcal{W}_m$ commute. Thus, $\mathcal{V}^{(n-1)d+j}$ induces an endomorphism of $\mathcal{U}_{n,d}$. This affords an exact sequence in groups

$$e_k \longrightarrow \mathcal{U}^j_{n,d} \longrightarrow \mathcal{U}_{n,d} \xrightarrow{\mathcal{V}^{(n-1)d+j}} \mathcal{U}_{n,d}.$$

In view of (i) Υ sends the above exact sequence in groups to an exact sequence of left $\mathbf{D}$-modules

$$E_{n,d} \longrightarrow E_{n,d} \longrightarrow \Upsilon(\mathcal{U}^j_{n,d}) \longrightarrow (0),$$

where the left-hand map is right multiplication by $V^{(n-1)d+j}$. Thus, we have $\Upsilon(\mathcal{U}^j_{n,d}) \cong E^j_{n,d}$, as desired.

(iii). Consider the exact sequence in groups

$$e_k \longrightarrow (\mathcal{W}_d)_1 \longrightarrow \mathcal{W}_d \xrightarrow{\mathcal{F}} \mathcal{W}_d.$$

An application of Υ yields the exact sequence in left $\mathbf{D}$-modules

$$\mathbf{D}_d \xrightarrow{F} \mathbf{D}_d \longrightarrow \Upsilon((\mathcal{W}_d)_1) \longrightarrow (0).$$

Since $\mathbf{D}_d/\mathbf{D}_dF \cong E_d$, we obtain $\Upsilon((\mathcal{W}_d)_1) \cong E_d$. □

Proof of Theorem 1.2. Except for the isomorphism $\mathcal{D}((\mathcal{W}_d)_1) \cong \alpha_{p^d}$, Theorem 1.2 now follows from Theorem 2.7, the lemma above, and the fact that the functor Υ from 2.3 is an anti-equivalence of categories.

Since α_{p^d} is an $\mathcal{F}$-uniserial group of length d that is annihilated by $\mathcal{V}$ (see [5, (IV, §3, n° 4.5)]), its dual is a $\mathcal{V}$-uniserial group of length d that is annihilated by $\mathcal{F}$. Thus, $\mathcal{D}(\alpha_{p^d}) \cong (\mathcal{W}_d)_1$, and $\mathcal{D}((\mathcal{W}_d)_1) \cong \alpha_{p^d}$. □

4. Representation-Finite Infinitesimal Groups

4.1. Let X be an indeterminate over k and for given $n \in \mathbb{N}_0$ set $x := X + (X^{p^n})$. We consider the k-functor $R \mapsto \mathcal{L}_n(R)$, where

$$\mathcal{L}_n(R) := \bigoplus_{i=0}^{n-1} Rx^{p^i} \subseteq R[X]/(X^{p^n}),$$

for every $R \in \mathbb{M}_k$. Note that $\mathcal{L}_n(R)$ is an abelian restricted R-Lie algebra with p-map given by the ordinary p-power operator. For $m, l \in \mathbb{N}_0$ satisfying $0 \leq l \leq p^m - 1$ and such that p does not divide l, we define the group scheme

$$\mathcal{T}_n^{m,l} := \mathcal{L}_n \rtimes \mu_{p^m}$$

with operation on $\mathcal{T}_n^{m,l}(R)$ given by

$$(\sum_{i=0}^{n-1} a_i x^{p^i}, r) \cdot (\sum_{i=0}^{n-1} b_i x^{p^i}, s) := (\sum_{i=0}^{n-1} (a_i + r^{lp^i} b_i) x^{p^i}, rs).$$

Note that $\mathcal{T}_n^{m,l}$ is trigonalizable with unipotent radical $\mathcal{L}_n$ on which, by choice of l, the group μ_{p^m} operates faithfully via conjugation.

4.2. We denote the principal block of the distribution algebra of an infinitesimal k-group $\mathcal{G}$ by $\mathcal{B}_0(\mathcal{G}) \subseteq H(\mathcal{G})$. The following result refines [10, (2.7)].

Theorem. *Let $\mathcal{G}$ be an infinitesimal k-group with representation-finite principal block $\mathcal{B}_0(\mathcal{G})$. Then $\mathcal{G}/\mathcal{M}(\mathcal{G})$ is isomorphic to either $\mathcal{U}_{n,d}$, $\mathcal{U}_{n,d}^j$, or $\mathcal{T}_n^{m,l}$ for a suitable choice of parameters.*

Proof. By [10, (2.7)], the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mu_{p^m}$ is a semidirect product with a $\mathcal{V}$-uniserial normal subgroup $\mathcal{U}$ on which μ_{p^m} operates faithfully via conjugation.

Suppose that $\mathcal{U}$ has height at least 2. Thanks to [9, (3.1)], the connected component of the automorphism scheme of $\mathcal{U}$ is unipotent. Accordingly, μ_{p^m} operates trivially, so that $\mu_{p^m} = e_k$. The assertion now follows from Theorem 1.2.

Alternatively, $\mathcal{U}$ has height 1 and is thus isomorphic to $(\mathcal{W}_n)_1$. The Lie algebra L_n of $(\mathcal{W}_n)_1$ is nil-cyclic of dimension n, and μ_{p^m} operates on L_n via the adjoint representation. Let $\mathcal{K}$ be the kernel of this operation. By virtue of [5, (II, §7, n° 4.3)], we have

$$\operatorname{Lie} \mathcal{K} \subseteq \operatorname{Cent}_{\operatorname{Lie} \mathcal{T}_n^{m,l}}(L_n) = \operatorname{Lie}(\mathcal{CENT}_{\mathcal{T}_n^{m,l}}((\mathcal{W}_n)_1)) = \operatorname{Lie}(\mathcal{W}_n)_1 = L_n.$$

Thus, $\operatorname{Lie} \mathcal{K} \subseteq \operatorname{Lie} \mu_{p^m} \cap L_n = (0)$. Since $\mathcal{K}$ is infinitesimal, we obtain $\mathcal{K} = e_k$ implying that μ_{p^m} operates faithfully on L_n.

Note that the subspace V generated by $L_n^{[p]}$ is a μ_{p^m}-submodule of codimension 1. As μ_{p^m} is multiplicative, there is a one-dimensional μ_{p^m}-stable complement ky to V in L_n. The group μ_{p^m} operates on ky via a character $\lambda : \mu_{p^m} \longrightarrow \mu_k$. Thus $r \cdot (y \otimes 1)^{[p]^i} = \lambda(r)^{p^i}(y \otimes 1)$ for $r \in R$ and $R \in \mathbb{M}_k$. Since $L_n = \oplus_{i=0}^{n-1} ky^{[p]^i}$, it follows that λ is injective. This readily implies the existence of an integer $0 \leq l \leq p^m - 1$ not divisible by p such that $\lambda_R(r) = r^l$. Consequently, the map sending y to x and fixing μ_{p^m} pointwise defines an isomorphism $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{T}_n^{m,l}$. □

Corollary. *Let $\mathcal{G}$ be an infinitesimal k-group and $\mathcal{B} \subseteq H(\mathcal{G})$ a representation-finite block admitting a one-dimensional module. Then either $\mathcal{B} \cong k[X]/(X^{p^n})$, or $\mathcal{B} \cong H(\mathcal{T}_n^{m,l})$ for a suitable choice of parameters. In particular, $\mathcal{B}$ is a Nakayama algebra.*

Proof. Let $\lambda : H(\mathcal{G}) \longrightarrow k$ be the character defining the one-dimensional $\mathcal{B}$-module k_λ. The convolution $\lambda * id_{H(\mathcal{G})}$ is an automorphism ψ_λ of $H(\mathcal{G})$ whose composition $\varepsilon \circ \psi_\lambda$ with the counit ε of $H(\mathcal{G})$ coincides with λ. Thus, $\varepsilon(\psi_\lambda(\mathcal{B})) \neq (0)$, so that ψ_λ sends $\mathcal{B}$ onto $\mathcal{B}_0(\mathcal{G})$. In particular, $\mathcal{B}_0(\mathcal{G})$ is representation-finite and Theorem 4.2 determines the structure of $\mathcal{G}/\mathcal{M}(\mathcal{G})$. Thanks to [10, (2.4)], we have $\mathcal{B}_0(\mathcal{G}) \cong H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$, and the assertion follows. □

Remark. The foregoing result suggests that representation-finite blocks of distribution algebras are Nakayama algebras. This is known to be true for groups of height at most 1 and supersolvable groups of arbitrary height, see [7, (3.2)] and [8, (5.3)].

5. Unipotent Groups of Complexity 1

5.1. The notion of the complexity of a module, first introduced by Alperin in [1] and then systematically elaborated on by Alperin and Evens in [2], plays an important rôle in the representation theory of self-injective algebras. For instance, modules belonging to representation-finite and tame algebras have complexities bounded by 1 and 2, respectively.

Given a finite-dimensional module M over a finite-dimensional k-algebra Λ, the *complexity* $c_\Lambda(M)$ of M is defined to be the rate of growth of a minimal projective resolution $(P_n)_{n\in\mathbb{N}_0}$ of M. Thus, we have

$$c_\Lambda(M) := \min\{c \in \mathbb{N}_0 \cup \{\infty\} \ ; \exists\ \lambda > 0 \text{ such that } \dim_k P_n \leq \lambda n^{c-1} \ \ \forall\ n \geq 1\}.$$

If $\mathcal{G}$ is a finite algebraic k-group, we let $c_\mathcal{G} := c_{H(\mathcal{G})}(k)$ be the complexity of the trivial module of the algebra of measures on $\mathcal{G}$. Thus, $c_\mathcal{G} = 0$ if and only if $H(\mathcal{G})$ is semisimple and $c_\mathcal{G} \leq 1$, whenever $\mathcal{G}$ is representation-finite. By Nagata's Theorem (cf. [5, (IV, §3, n° 3.6)]), the infinitesimal groups of complexity 0 are just the multiplicative groups.

Let $\mathcal{H} \subset \mathcal{G}$ be a subgroup. Since $H(\mathcal{G})$ is a free $H(\mathcal{H})$-module (cf. [15, (2.6)]), we readily obtain $c_\mathcal{H} \leq c_\mathcal{G}$. The Künneth formula implies $c_{\alpha_{p^r}} = r$. If $\mathcal{U}$ is a $\mathcal{V}$-uniserial group of length n, then $H(\mathcal{U}) \cong k[X]/(X^{p^n})$, so that $c_\mathcal{U} = 1$.

5.2. In this subsection we characterize $\mathcal{V}$-uniserial groups in terms of subgroups of α_{p^2}.

Lemma. *Let $\mathcal{U}$ be an infinitesimal unipotent k-group such that $\mathcal{U}$ contains exactly one copy of α_p and no copy of α_{p^2}. Then $\mathcal{U}$ is $\mathcal{V}$-uniserial.*

Proof. We proceed by induction on the length $\ell(\mathcal{U})$ of $\mathcal{U}$, and denote by $\mathcal{U}'$ the unique subgroup of $\mathcal{U}$ that is isomorphic to α_p. Note that $\mathcal{U}'$ lies in the center of $\mathcal{U}$, and consider the factor group $\mathcal{U}'' := \mathcal{U}/\mathcal{U}'$.

If $\mathcal{U}'' = e_k$, then $\mathcal{U} \cong \alpha_p$ is $\mathcal{V}$-uniserial. Alternatively, $\mathcal{U}''$ contains a central subgroup $\mathcal{Z}$ that is isomorphic to α_p (cf. [5, (IV, §4, n° 1.3)]).

Suppose $\mathcal{U}''$ contains a subgroup $\mathcal{X} \neq \mathcal{Z}$ isomorphic to α_{p^2} or α_p. We put $\mathcal{N}'' := \mathcal{X}$ in the former case, and $\mathcal{N}'' := \mathcal{X}\mathcal{Z}$ in the latter. Let $\mathcal{N} \subset \mathcal{U}$ be the preimage of $\mathcal{N}''$ under the canonical projection $\mathcal{U} \longrightarrow \mathcal{U}''$. Then we have $\ell(\mathcal{N}) = 3$ and $\mathcal{N}/\mathcal{U}' \cong \alpha_{p^2}$, or $\mathcal{N}/\mathcal{U}' \cong \alpha_p \times \alpha_p$. Owing to [5, (IV, §3, n° 4.5)] the group $\mathcal{N}/\mathcal{U}'$ is annihilated by $\mathcal{V}_{\mathcal{N}/\mathcal{U}'}$. Thus, [10, (2.5)] and [11, (1.2)] apply, and $\mathcal{N}$ contains a copy of α_{p^2}, a contradiction.

By inductive hypothesis the group $\mathcal{U}''$ is $\mathcal{V}$-uniserial and [12, (2.3)] (which also holds for $p = 2$) ensures the commutativity of $\mathcal{U}$. Note that $\mathcal{U}$ is uniserial, so that $\mathcal{U}_2$ also has this property. If $\mathcal{U}_2$ is not $\mathcal{V}$-uniserial, then Lemma 2.5 implies that $\mathcal{U}_2$ is $\mathcal{F}$-uniserial, whence

$\ell(\mathcal{U}_2) = 2$. According to Theorem 1.2 this readily implies, $\mathcal{U}_2 \cong \alpha_{p^2}$, a contradiction. Our result now follows from Remark 2.5. $\square$

5.3. We now establish the main result of this section which characterizes $\mathcal{V}$-uniseriality in terms of the structure and the complexity of the second Frobenius kernel.

Theorem. *Let $\mathcal{U} \neq e_k$ be an infinitesimal unipotent k-group. Then the following statements are equivalent:*
(i) *$\mathcal{U}$ is $\mathcal{V}$-uniserial.*
(ii) *$\mathcal{U}_2$ is $\mathcal{V}$-uniserial.*
(iii) $c_{\mathcal{U}_2} = 1$.

Proof. Since the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial, we only verify (iii) $\Rightarrow$ (i). Suppose that $\mathcal{N} \subset \mathcal{U}$ is a subgroup such that $\mathcal{N} \cong \alpha_{p^2}$. Then $c_{\mathcal{N}} = 2$ and $\mathcal{N} \subset \mathcal{U}_2$, a contradiction. Let $\mathcal{Z}$ be a subgroup of the center of $\mathcal{U}$ that is isomorphic to α_p. If $\mathcal{N} \subset \mathcal{U}$ is another subgroup of type α_p, then $\mathcal{M} := \mathcal{N}\mathcal{Z} \subset \mathcal{U}_1$ is isomorphic to $\alpha_p \times \alpha_p$. As $c_{\mathcal{M}} = 2$, this contradicts (iii). Consequently, Lemma 5.2 yields the $\mathcal{V}$-uniseriality of $\mathcal{U}$. $\square$

Corollary. *Let $\mathcal{U} \neq e_k$ be an infinitesimal unipotent uniserial group of height ≤ 1. Then $\mathcal{U} \cong (\mathcal{W}_d)_1$ for some $d \geq 1$.*

Proof. This follows directly from Theorems 1.2 and 5.3. $\square$

Remark. Lemma 5.2 affords the following approach towards the classification of infinitesimal groups of complexity 1. Let $\mathcal{G}$ be an infinitesimal group with $c_{\mathcal{G}_2} = 1$. Then $\mathcal{G}$ is supersolvable (cf. [10, (2.1)]), so that $\mathcal{G}/\mathcal{M}(\mathcal{G})$ is a semidirect product of a unipotent normal subgroup $\mathcal{U}$ and a multiplicative group (cf. [10, (2.3)]). The assumption $c_{\mathcal{G}_2} = 1$ implies that $\mathcal{U}$ satisfies the conditions of Lemma 5.2. Hence $\mathcal{U}$ is $\mathcal{V}$-uniserial.

Acknowledgments: Work on this paper began during a stay of the authors at the Mathematics Research Institute Oberwolfach supported by the Volkswagen Stiftung ("Research in Pairs" at Oberwolfach). It is a great pleasure to thank the members of the Institute for their hospitality. Part of this paper was written while the first author was visiting the SFB 343 "Diskrete Strukturen in der Mathematik" at the University of Bielefeld. We are grateful to C.M. Ringel for his continued support.

The first author was also supported in part by NSA grant MDA904-00-1-0039.

References

[1] J. Alperin, *Periodicity in Groups*, Illinois J. Math. **21**, (1977), 776–783.
[2] J. Alperin and L. Evens, *Representations, Resolutions and Quillen's Dimension Theorem*, J. Pure Appl. Algebra **22**, (1981), 1–9.
[3] A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, **126**, Springer-Verlag, (1991).
[4] N. Bourbaki, *Éléments de mathématique. Algèbre commutative.* Chapitre 8 - 9. Masson, Paris, (1983).
[5] M. Demazure and P. Gabriel, *Groupes algébriques*, Paris; North-Holland Publishing Co., Amsterdam, (1970).
[6] Yu. Drozd, *Tame and Wild Matrix Problems*, In: Representation Theory II. Lecture Notes in Mathematics **832**, (1980), 242–258.

[7] R. Farnsteiner, *Periodicity and Representation Type of Modular Lie Algebras*, J. reine angew. Math. **464**, (1995), 47–65.
[8] ______, *On the Auslander-Reiten Quiver of an Infinitesimal Group*, Nagoya Math. J. **160**, (2000), 1–19.
[9] R. Farnsteiner and D. Voigt, *Modules of Solvable Infinitesimal Groups and the Structure of Representation-Finite Cocommutative Hopf Algebras*, Math. Proc. Camb. Phil. Soc. **127**, (1999), 441–459.
[10] ______, *On Cocommutative Hopf Algebras of Finite Representation Type*, Adv. in Math. **155**, (2000), 1–22.
[11] ______, *Representations of Infinitesimal Groups of Characteristic 2*, Arch. Math. (Basel) (to appear).
[12] ______, *On Infinitesimal Groups of Tame Representation Type*, SFB-Preprint 99–120, Universität Bielefeld.
[13] D.G. Higman, *Indecomposable Representations at Characteristic p*. Duke Math. J. **21**, (1954), 377–381.
[14] J.C. Jantzen, *Representations of Algebraic Groups*, Pure and Applied Mathematics, **131**, Academic Press (1987).
[15] U. Oberst and H. Schneider, *Über Untergruppen endlicher algebraischer Gruppen*, Manuscripta math., **8** (1973), 217–241.
[16] W. Pfautsch and D. Voigt, *The Representation-Finite Algebraic Groups of Dimension Zero*, C. R. Acad. Sci. Paris Sr. I Math. **306** (1988), no. 16, 685–689.
[17] W.C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics, **66**, Springer-Verlag, New York-Berlin, (1979).

Department of Mathematics, University of Wisconsin, Milwaukee, WI 53201, USA.
E-mail address: `rolf@uwm.edu`

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.
E-mail address: `roehrle@mathematik.uni-bielefeld.de`

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.