REVISITING THE SIEGEL UPPER HALF PLANE I

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Abstract

In the first part of the paper we study p-metrics on $\mathbf{X}_n = \mathbf{GL}(n,\mathbb{C})/\mathbf{U}(n,\mathbb{C})$ for $p \in [1,\infty]$. We give a complete description of p-Busemann compactifications of $\mathbf{X}_n$ for $p \in [1,\infty)$. For the Siegel upper half plane of rank n: $\mathbf{SH}_n = \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ we show that the 1-Busemann compactification is the compactification of $\mathbf{SH}_n$ as the bounded domain. In the second part of the paper we study certain properties of discrete groups Γ of biholomorphisms of $\mathbf{SH}_n$. We show that the the set of accumulation points of the orbit $\Gamma(Z)$ on the Shilov boundary of $\mathbf{SH}_n$ is independent of Z, and denote this set by $\Lambda(\Gamma)$. We associate with Γ the standard class of p-Patterson-Sullivan measures. For p-regular Γ these measures are supported on $\Lambda(\Gamma)$. For 1-regular Γ 1-Patterson-Sullivan measures are conformal densities. For Γ, with $\Lambda(\Gamma) \neq \emptyset$, we give a modified version of the class of Patterson-Sullivan measures, which are always supported on $\Lambda(\Gamma)$.

Contents

1 Introduction

In the past thirty years there have been a great deal of mathematical activity on Fuchsian and Kleinian groups. One of the most important notions is that of Patterson-Sullivan (PS) measures. This class of conformal measures was introduced by Patterson [Pat] for Fuchsian groups. The construction of conformal measures were extended by Sullivan [Sul] to hyperbolic groups acting on n-dimensional hyperbolic spaces $\mathbf{H}^n$ in general and in particular for Kleinian groups ($n = 3$). A good summary of the ideas and results on this subject are in [Nic]. See Bishop [Bis] and Bishop-Jones [BJ] for additonal recent results. There are several ways to extend the results on PS measures. One way is to consider more general hyperbolic spaces, for example hyperbolic spaces in the sense of Gromov [Coo]. Another way to consider PS measures on manifolds of negative curvature spaces and to obtain rigidity results [BCG], [Yue] and [BM]. In all the above cases, the boundary of the negatively curved spaces is the visual boundary which is diffeomorphic to a sphere of the corresponding dimension.

Another approach, which is taken in this paper, is to consider PS measures for discrete groups in higher rank symmetric spaces. It seems that the first work in this direction was done by Burger [Bur], who considered discrete groups of isometries acting on a special symmetric space of rank two, namely $\mathbf{H}^2 \times \mathbf{H}^2$. Here the boundary of $\mathbf{H}^2 \times \mathbf{H}^2$ consists of three strata. The important stratum is the Shilov boundary $S^1 \times S^1$. The general theory of PS measures for discrete groups in higher rank symmetric spaces G/K was studied by Albuquerque [Alb]. In that case the boundary of G/K consists of several strata. The important stratum is the Furstenberg boundary [Fur]. (For $\mathbf{H}^2 \times \mathbf{H}^2$ the Furstenberg boundary is equal to the Shilov boundary.) One of the main problem in this setting is to show that the PS measures are supported on the Furstenberg boundary. Albuquerque shows that for lattices the PS measure is situated on the Furstenberg boundary. Using recent results of Benoist [Be2], Albuquerque shows that there are families of Zariski dense groups for which the PS measures are supported on the Furstenberg boundary.

In his fundamental paper [Sie] Siegel introduced a special symmetric space $\mathbf{SH}_n$ of rank n for $n = 1, ...,$, which is called now the $(n-th)$ Siegel upper half plane. $\mathbf{SH}_1$ is the hyperbolic upper half plane $\mathbf{H}^2$. $\mathbf{SH}_n$ is formally defined as the subset of $n \times n$ complex symmetric matrices $\mathbf{Sym}(n, \mathbb{C})$ whose imaginary part is a positive definite matrix. In fact, the origin of $\mathbf{SH}_n$ can be traced to Riemann, who defined the Riemann matrix $A \in \mathbf{SH}_n$ corresponding to a compact Riemann surface of genus n, endowed with a specific complex structure. $\mathbf{SH}_n$ is the homogenuous space corresponding to the symplectic group $\mathbf{Sp}(n, \mathbb{R}) \leq \mathbf{SL}(2n, \mathbb{R})$ quotient by the maximal compact subgroup

$$\mathbf{K}_n := \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{SO}(2n, \mathbb{R}).$$

$\mathbf{SH}_n$ is a complex manifold of complex dimension $\frac{n(n+1)}{2}$. $\mathbf{Sp}(n, \mathbb{R})$ is the biholomorphism group of $\mathbf{SH}_n$. Of special interest is the lattice $\mathbf{Sp}(n, \mathbb{Z})$, which is called the Siegel modular group. Note that $\mathbf{Sp}(1, \mathbb{Z})$ is the classical modular group. It is known that the complex structure of any compact Riemann surface of genus n is characterized by a point in $\mathbf{SH}_n/\mathbf{Sp}(n, \mathbb{Z})$ [Nag]. Siegel upper half plane and Siegel modular group have many applications to modular forms [F]. The natural compactification of $\mathbf{SH}_n$ is the compacitfication as a bounded domain $\mathbf{SD}_n$ (the $n-th$ Siegel disk). Recall that $\mathbf{SD}_n$ is biholomorphic to $\mathbf{SH}_n$. The most important stratum of the boundary of $\mathbf{SD}_n$ is the Shilov boundary of $\mathbf{SD}_n$. It is the set of $n \times n$ unitary symmetric matrices $\mathbf{USym}(n)$, which is a manifold of real dimension $\frac{n(n+1)}{2}$. $\mathbf{USym}(2)$ is S^2 circle bundle glued by the antipodal map. The Furstenberg boundary of $\mathbf{SH}_n$ has real dimension n^2. The Satake compactifications of $\mathbf{SH}_n$ with respect to different representations give the compactification of $\mathbf{SH}_n$ as a bounded domain and the Furstenberg compactification respectively [Sat], [Moo].

The object of this paper is to study certain problems for a discrete groups $\Gamma < \mathbf{Sp}(n, \mathbb{R})$: the appropriate definitions of the limit set of Γ and the appropriate constructions of the PS measures. (Some results in this direction were given in [FrH].) These problems are closely related to $\frac{n(n+1)}{2}$ dimensional complex manifolds whose universal cover is $\mathbf{SH}_n$. As we show, there are many common features of discrete groups $\Gamma < \mathbf{Sp}(n, \mathbb{R})$ for $n > 1$ with the classical Fuchsian groups $(n = 1)$. Of course there are still many differences with Fuchsian groups, and more generally with discrete subgroups in rank one symmetric spaces. It will be apparent to the reader that the discrete groups $\Gamma < \mathbf{Sp}(n, \mathbb{R})$ posses remarkable properties, some of which we were able to expose. The most promising case is $n = 2$. Here a discrete group $\Gamma < \mathbf{Sp}(2,\mathbb{R})$ acts on $\mathbf{USym}(2)$, which seems to be a natural generalization of the action of the Kleinian group on the Riemann sphere. That is why we study in detail various compactifications of $\mathbf{SH}_2$ and the action of a single element $\gamma \in \mathbf{Sp}(2,\mathbb{R})$ on $\mathbf{SH}_2$ in our second paper.

We now outline briefly the main results of our paper. On the space $\mathbf{X}_n = \mathbf{GL}(n,\mathbb{C})/\mathbf{U}(n,\mathbb{C})$ we define a metric $d_p(A,B)$, which is a variant of p-Schatten norm of $A^{-1}B$, for any $p \in [1,\infty]$. These metrics can be viewed as Finsler metrics. d_2 is the classical Riemannian metric on the homogeneous space $\mathbf{X}_n$. All p-metrics are uniformly Lipschitz equivalent for a fixed value of n. $\mathbf{GL}(n,\mathbb{C})$ acts (from the left) as a subgroup of isometries for each $p \in [1,\infty]$. Next we consider the Busemann functions and the Busemann compactifications for d_p as in [Bal]. For $p \in (1,\infty)$ the p-Busemann compactification is the visual compactification, i.e. the end of geodesic rays from a fixed point $o \in \mathbf{X}_n$. For $p = 1,\infty$ the p-Busemann compactification is different from the visiual compactification. We analyze completetly the 1- Busemann compactification. Let $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$ be a nontrivial orthonormal decomposition of $\mathbb{C}^n$, i.e. every subspace is a strict subspace of $\mathbb{C}^n$. Denote by $\mathbf{H}(U_0)$ the real subspace of hermitian operators $T : U_0 \to U_0$. Then the 1-Busemann compactification is a union of the strata corresponding to $(U_+, \mathbf{H}(U_0), U_-)$ for all possible nontirivial decompositions of $\mathbb{C}^n$. We define the p-Busemann boundary of $\mathbf{X}_n$ to be the boundary points corresponding to the open Weyl chambers. We identify the p-Busemann boundary for all $p \in [1,\infty]$. The 1-Busemann boundary consists of $n-1$ strata, where each stratum corresponds to the orthonormal decomposition $\mathbb{C}^n = U_+ \oplus U_-$ and $\dim U_+ = k$, for $k = 1, ..., n-1$.

Next we consider $\mathbf{SH}_n$ and its various models. Observe that $\mathbf{SH}_n = \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ is a geodesic submanifold of $\mathbf{X}_{2n}$. Our first main result is that the 1-Busemann compactification of $\mathbf{SH}_n$ is the compactification of $\mathbf{SH}_n$ as a bounded domain $\mathbf{SD}_n$. More precisely the 1-Busemann boundary is the Shilov boundary of $\mathbf{SD}_n$. Here U_0 is trivial and U_+, U_- are the Lagrangian subspaces of $\mathbb{R}^{2n}$ induced by the canonical 2-form on the tangent space of $\mathbf{Sp}(n,\mathbb{R})$. The other strata of $\partial\mathbf{SD}_n$ correspond to the strata $(U_+, \mathbf{H}(U_0), U_-)$, where dim U_+ =dim $U_- = k$ and $k = 1, ..., n-1$. It is well known that each stratum of the boundary of $\mathbf{SH}_n$ is one orbit of $\mathbf{Sp}(n,\mathbb{R})$. Let Γ be a discrete subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then $\Lambda(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(Z)$ on the Shilov boundary for some $Z \in \mathbf{SH}_n$. ($\Lambda(\Gamma)$ can be an empty set for some infinite Γ.) Our second main result is that $\Lambda(\Gamma)$ is independent of the choice of $Z \in \mathbf{SH}_n$, as in the classical case of Fuchsian and Kleinian groups. (The accumulation points of $\Gamma(Z)$ on other part of the boundary of $\mathbf{SH}_n$ can depend on Z.) Somewhat similar result is given in [Be2, Thm 6.4]. $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ is called hyperbolic if all its eigenvalues are situated outside of the unit circle. For a hyperbolic γ we have

$$\begin{aligned}&\Lambda(<\gamma>) = \{\xi_+(\gamma), \xi_-(\gamma)\}, \quad \xi_+(\gamma) \neq \xi_-(\gamma),\\ &\lim_{m\to\infty} \gamma^m(Z) = \xi_+(\gamma), \quad \lim_{m\to-\infty} \gamma^m(Z) = \xi_-(\gamma), \quad \text{for all } Z \in \mathbf{SH}_n.\end{aligned}$$

It is easy to produce families of hyperbolic elements in $\mathbf{Sp}(n,\mathbb{R})$ which have at least 2^n fixed points on the Shilov boundary. We next consider the p-Poincaré series for Γ

$$\sum_{\gamma\in\Gamma} e^{-s d_p(A,\gamma(B))}, \quad s > 0.$$

Denote by $\delta_p(\Gamma)$ the Poincaré (critical) exponents of the above series. With the p-Poincaré series one associate the family of PS measures $\mathcal{M}_{\Gamma,A,p}$ as in [Nic] and [Alb]. In the rest of the introduction we assume that $\Lambda(\Gamma) \neq \emptyset$ and $n > 1$. One of the main problem here to show that the support of each $\mu \in \mathcal{M}_{\Gamma,A,p}$ is located on $\Lambda(\Gamma)$. Γ is called p-regular if the support of each $\mu \in \mathcal{M}_{\Gamma,A,p}$ is located on $\Lambda(\Gamma)$. We show that lattices are p-regular for any $p \in [1,\infty]$. This is an analog of Albuquerque's result [Alb]. Using the results of [Be2] as in [Alb] we show that there exist many discrete Zariski dense p-regular Γ. As 1-Busemann compactification is the compactification of $\mathbf{SH}_n$ as a bounded domain, we show that for a discrete 1-regular group Γ the set $\mathcal{M}_{\Gamma,A,1}$ is the set of conformal measures:

$$\frac{d\mu_X}{d\mu_{X_0}}(\xi) = e^{-\delta_1(\Gamma) b_1(\xi,X,X_0)}, \quad \xi \in \Lambda(\Gamma), \quad X_0, X \in \mathbf{SH}_n.$$

Here $b_1(\xi, X_0, X)$ is the 1-Busemann function, for which we have a simple explicit formula. Toward the end of this paper we construct a modified version of PS measures denoted by $\tilde{\mathcal{M}}_{\Gamma,A,p}$, such that the support of each $\mu \in \tilde{\mathcal{M}}_{\Gamma,A,p}$ is located on $\Lambda(\Gamma)$. The critical exponents of the corresponding series for measures in $\tilde{\mathcal{M}}_{\Gamma,A,p}$ range in the interval $[\underline{\delta}_p(\Gamma), \overline{\delta}_p(\Gamma)]$, where $\delta_p(\Gamma) \geq \overline{\delta}_p(\Gamma) \geq \underline{\delta}_p(\Gamma)$. Using the recent results in [Fr4] we show that $\underline{\delta}_p(\Gamma) > 0$ for any discrete Zariski dense subgroup Γ and any $p \in [1,\infty]$. We conjecture that $\tilde{\mathcal{M}}_{\Gamma,A,1}$ are families of conformal measures with the densities $\beta \in [\underline{\delta}_p(\Gamma), \overline{\delta}_p(\Gamma)]$. We note that our account on the PS is concise, since we did not want to repeat the standard arguments as presented in [Nic] and [Alb].

Some of the results here were obtained by the second author in his Ph.D. thesis [Fre] under the direction of the first author. The second author was supported by an FCT-Praxis XXI scholarship during his studies at UIC.

2 Metrics on certain matrix spaces

2.1 p-metrics

Let $\mathbf{M}(n, F)$ be the algebra of $n \times n$ matrices and $\mathbf{GL}(n, F)$ be the group of all invertible matrices, with the entries in a field F. We will assume that F is either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. Denote by $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{O}(n)$ and $\mathbf{SO}(n)$ the groups of $n \times n$ unitary, special unitary, real orthogonal and special real orthogonal matrices respectively. Let $\mathbf{H}(n,\mathbb{C})$, $\mathbf{H}^+(n,\mathbb{C})$ be the real linear space of $n \times n$ Hermitian matrices and the cone of positive definite $n \times n$ Hermitian matrices respectively. Let $A = (a_{pq})_1^n \in \mathbf{M}(n,\mathbb{C})$. Then $\overline{A} = (\overline{a}_{pq})_1^n$, A^{T} is transpose of A and $A^* = \overline{A}^{\mathrm{T}}$. By the spectrum of A we mean the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$ counted with their multiplicities and arranged in the following order:

$$|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|.$$

The singular values of A are given by

$$\sigma_i(A) = \sqrt{\lambda_i(AA^*)} = \sqrt{\lambda_i(A^*A)}, \quad i = 1, \ldots, n.$$

Set $\sigma(A) := (\sigma_1(A), ..., \sigma_n(A))$. For $(x_1, ..., x_n) \in F^n$ let

$$D((x_1, ..., x_n)) := \mathrm{diag}(x_1, ..., x_n) := \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n \end{pmatrix}$$

Then

$$A = U\Sigma(A)V, \quad U, V \in \mathbf{U}(n, \mathbb{C}), \quad \Sigma(A) = \mathrm{diag}(\sigma(A)) \tag{2.1.1}$$

is called the singular value decomposition (SVD). (It is also called the Cartan decomposition.) If $A \in M(n, \mathbb{R})$ then the unitary matrices U, V in can be chosen to be orthogonal matrices. Note that $||A||_2 = \sigma_1(A)$ is the l_2 norm of A viewed as a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$, where $A(x) = Ax$ and

$$x = (x_1, \ldots, x_n)^{\mathrm{T}},$$
$$||x||_p = (\textstyle\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty.$$

Use the singular value decomposition of A to deduce

$$\sigma_{n-i+1}(A^{-1}) = \sigma_i(A)^{-1}, \quad i = 1, \ldots, n, \quad A \in \mathbf{GL}(n, \mathbb{C}). \tag{2.1.2}$$

Observe next that $\sigma_i(A) = 1, \ i = 1, \ldots, n$ iff A is a unitary matrix.

In this paper we will use the notion of the compound matrices. Let $\mathbf{M}(m, n, F)$ denote the vector space of $m \times n$ matrices with entries in a field F. For $A \in \mathbf{M}(m, n, F)$ and $1 \leq k \leq \min(m, n)$ we denote by $\wedge_k A$ the $k-th$ compound matrix. Note that $\wedge_k A \in \mathbf{M}(\binom{m}{k}, \binom{n}{k}, F)$ and the entries of A are all the $k \times k$ minors of A. ($\wedge_k A$ is the representation matrix of the linear transformation from the k exterior product $\wedge_k F^n$ to $\wedge_k F^m$ induced by $A : F^n \to F^m$.) The map

$$\wedge_k : \mathbf{GL}(n, F) \to \mathbf{GL}(\binom{n}{k}, F)$$

is a homomorphism which commutes with the $*$ involution. If $A \in \mathbf{M}(n, \mathbb{C})$ has complex eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$ then $\wedge_k A$ has the following eigenvalues and singular values, and $\wedge_k e^{A}$ has the following eigenvalues respectively:

$$\begin{gathered} \lambda_{i_1}(A)\lambda_{i_2}(A) \cdots \lambda_{i_k}(A), \\ \sigma_{i_1}(A)\sigma_{i_2}(A) \cdots \sigma_{i_k}(A), \\ e^{\lambda_{i_1}(A)+\lambda_{i_2}(A)+\cdots+\lambda_{i_k}(A)}, \\ 1 \leq i_1 < \cdots < i_k \leq n. \end{gathered} \tag{2.1.3}$$

If $A \in \mathbf{H}(n, \mathbb{C})$ $(\mathbf{H}^+(n, \mathbb{C}))$ then $\wedge_k A \in \mathbf{H}(\binom{n}{k}, \mathbb{C})$ $(\mathbf{H}^+(\binom{n}{k}, \mathbb{C}))$. See for example [HJ].

Lemma 2.1.1 *The space* $\mathbf{X}_n := \mathbf{GL}(n,\mathbb{C})/\mathbf{U}(n,\mathbb{C})$ *can be identified with* $\mathbf{H}^+(n,\mathbb{C})$.

Proof. Let $A \in \mathbf{GL}(n,\mathbb{C})$. From (2.1.1) we obtain $B = AV^*U^* \in \mathbf{H}^+(n,\mathbb{C})$. It is left to show that

$$A\mathbf{U}(n,\mathbb{C}) \cap \mathbf{H}^+(n,\mathbb{C}) = \{B\}. \tag{2.1.4}$$

Note that $AA^* = BB^* = B^2$. Hence B is a positive definite square root of AA^*. Therefore the eigenspaces of AA^* and B coincide. Since B is a hermitian positive definite we deduce that B is the unique positive definite square root of AA^* and (2.1.4) follows. □

$\mathbf{GL}(n,\mathbb{C})$ acts from the left on $\mathbf{X}_n$. Clearly, this action is transitive.

Lemma 2.1.2 *Let* $(A,B),(C,D) \in \mathbf{X}_n \times \mathbf{X}_n$. *Then there exists* $T \in \mathbf{GL}(n,\mathbb{C})$ *such that*

$$T(A,B) := (TA,TB) = (C,D) \tag{2.1.5}$$

iff

$$\Sigma(A^{-1}B) = \Sigma(C^{-1}D). \tag{2.1.6}$$

Proof. Clearly, (2.1.5) implies (2.1.6). Assume that (2.1.6) hold. Clearly $A^{-1}(A,B) = (I, A^{-1}B)$. Use the SVD for $A^{-1}B$ to deduce that $W(I,A^{-1}B) = (I,\Sigma(A^{-1}B)$ for some $W \in \mathbf{U}(n,\mathbb{C})$ (as a pair of points in $\mathbf{X}_n \times \mathbf{X}_n$). Apply the same argument to the pair (C,D). Then (2.1.6) yields (2.1.5). □

For $(x_1,...,x_n) \in \mathbb{R}^n$ we use the notation

$$e^{(x_1,\ldots,x_n)} = (e^{x_1}, e^{x_2}, ..., e^{x_n}), \quad \log e^{(x_1,\ldots,x_n)} = (x_1,...,x_n).$$

Lemma 2.1.3 *Let* $1 \le p \le \infty$ *and assume that* $A,B \in \mathbf{GL}(n,\mathbb{C})$. *Let*

$$d_p(A,B) := \left(\sum_{i=1}^n |\log \sigma_i(A^{-1}B)|^p\right)^{\frac{1}{p}} = ||\log \sigma(A^{-1}B)||_p.$$

Then d_p *is a metric on the homogeneous space* $\mathbf{X}_n$. $\mathbf{X}_n$ *is complete and locally compact with respect to* d_p. *Moreover,* $\mathbf{GL}(n,\mathbb{C})$ *acts (from the left) on* $\mathbf{X}_n$ *as a subgroup of isometries for* d_p.

Proof. As

$$\sigma_i(A) = \sigma_i(AU) = \sigma_i(UA), \quad A \in \mathbf{M}(n,\mathbb{C}), \quad U \in \mathbf{U}(n),$$

we deduce that $d_p(\cdot,\cdot)$ is a nonegative continuous function defined on $\mathbf{X}_n \times \mathbf{X}_n$. It is straightforward to see that A,B belong to the same left coset of $\mathbf{U}(n)$ iff $d_p(A,B) = 0$. It is easy to check that $d_p(A,B) = d_p(B,A)$, since $\sigma_j(A^{-1}B) = \sigma_{n-j+1}(B^{-1}A)^{-1}$. We now prove the triangle inequality. As $\sigma_1(A) = ||A||_2, A \in \mathbf{M}(n)$ we get

$$\sigma_1(PQ) \le \sigma_1(P)\sigma_1(Q), \quad P,Q \in \mathbf{M}(n,\mathbb{C}).$$

Apply the norm inequality to the k-th compound matrix $\wedge_k(PQ)$ to deduce

$$\begin{aligned}
&\prod_{i=1}^{k}\sigma_i(PQ) \le \prod_{i=1}^{k}\sigma_i(P)\prod_{i=1}^{k}\sigma_i(Q), \quad k=1,\dots,n-1,\\
&\prod_{i=1}^{n}\sigma_i(PQ) = \prod_{i=1}^{n}\sigma_i(P)\prod_{i=1}^{k}\sigma_i(Q).
\end{aligned} \tag{2.1.7}$$

The last equality follows from $|\mathrm{detP}| = \prod_{\mathrm{i}=1}^{\mathrm{n}}\sigma_{\mathrm{i}}(\mathrm{P})$. As $A^{-1}C=(A^{-1}B)(B^{-1}C)$ from the above inequalities we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\log\sigma_i(A^{-1}C) \le \sum_{i=1}^{k}\big(\log\sigma_i(A^{-1}B)+\log\sigma_i(B^{-1}C)\big), \quad k=1,\dots,n-1,\\
&\sum_{i=1}^{n}\log\sigma_i(A^{-1}C) = \sum_{i=1}^{n}\big(\log\sigma_i(A^{-1}B)+\log\sigma_i(B^{-1}C)\big).
\end{aligned} \tag{2.1.8}$$

Thus $\log\sigma(A^{-1}C)$ is majorized by $\log\sigma(A^{-1}B)+\log\sigma(B^{-1}C)$. As $f(t)=|t|^p$ is a convex function on $\mathbb{R}$ for $1\le p<\infty$, the majorization principle [HLP] yields that

$$||\log\sigma(A^{-1}C)||_p^p \le ||\log\sigma(A^{-1}B)+\log\sigma(B^{-1}C)||_p^p, \quad p\in[1,\infty). \tag{2.1.9}$$

Hence

$$\begin{aligned}
&d_p(A,C) \le ||\log\sigma(A^{-1}B)+\log\sigma(B^{-1}C)||_p \le\\
&||\log\sigma(A^{-1}B)||_p + ||\log\sigma(B^{-1}C)||_p = d_p(A,B)+d_p(B,C), \quad p\in[1,\infty).
\end{aligned} \tag{2.1.10}$$

Use the continuity of p at ∞ to obtain the triangle inequality for $p\in[1,\infty]$. It is straightforward to show that $\mathbf{X}_n$ is complete and locally compact for each $d_p, 1\le p\le\infty$. Clearly, $(CA)^{-1}(CB)=A^{-1}B$. Hence $\mathbf{GL}(n,\mathbb{C})$ acts as a subgroup of isometries on $\mathbf{X}_n$. □

Note that $d_p(A,B)$ can be considered as a variant of the classical p-Schatten norm of $A^{-1}B$.

Corollary 2.1.4 *Let the assumptions of Lemma 2.1.3 hold. Then*

$$\begin{aligned}
&d_\infty(A,B) = \max(|\log\sigma_1(A^{-1}B)|, |\log\sigma_1(B^{-1}A)|),\\
&d_\infty(A,B) \le d_p(A,B) \le (n)^{\frac{1}{p}} d_\infty(A,B).
\end{aligned}$$

Thus, all the metrics d_p are Lipschitz equivalent. It is straightforward to show that $d_2(A,B)$ is a Riemannian metric on $\mathbf{X}_n$. Let

$$\begin{aligned}
&\mathbf{D}(n,F) := \{A\in\mathbf{M}(n,F): \quad A=D(x), \quad x\in F^n\},\\
&\mathbf{D}^+(n,\mathbb{R}) := \mathbf{D}(n,\mathbb{R})\cap\mathbf{H}^+(n,\mathbb{C}).
\end{aligned} \tag{2.1.11}$$

Note that the exponential map

$$A \mapsto e^A, \quad A \in \mathbf{M}(n, F) \tag{2.1.12}$$

induces a diffeomorphism of $\mathbf{D}(n, \mathbb{R})$ and $\mathbf{D}^+(n, \mathbb{R})$.

2.2 Geodesics

Recall that $|| \cdot ||_p$ induces the $p - distance$

$$\delta_p(x, y) := ||x - y||_p, \quad x, y \in \mathbb{R}^n, \quad 1 \le p \le \infty.$$

The straight segment connecting the points x, y is a geodesic for all δ_p. As the norm $|| \cdot ||_p$ is strictly convex for $1 < p < \infty$ it follows that the straight segment connecting x, y is the unique geodesic for $1 < p < \infty$. Clearly, δ_2 is induced by the standard flat Riemannian metric on $\mathbb{R}^n$. It is straightforward to see that for $p = 1, \infty$ the straight segment connecting x, y is not a unique geodesic for most pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

A metric space such that there is a geodesic between any two points is called a geodesic space. A geodesic space for which the geodesic between any two points is unique is called a unique geodesic space.

The equality

$$d_p(e^{D(x)}, e^{D(y)}) = ||x - y||_p, \quad x, y \in \mathbb{R}^n, \quad 1 \le p \le \infty, \tag{2.2.1}$$

implies that the diffeomorphism between $\mathbb{R}^n$ equipped with the metric δ_p and $\mathbf{D}^+(n, \mathbb{R})$ equipped with the metrics d_p is an isometry for $1 \le p \le \infty$.

Lemma 2.2.1 *For each $1 \le p \le \infty$ the space $\mathbf{X}_n$ is a geodesic space. Furthermore, for $1 < p < \infty$ $\mathbf{X}_n$ is a unique geodesic space.*

Proof. Use Lemma 2.1.2 and the fact that $\mathbf{GL}(n, \mathbb{C})$ acts as the group of isometries to deduce that it is enough to show the existence of a geodesic between the points $A = I, C = e^{D(x)}$ where

$$x = (x_1, ..., x_n)^T \in \mathbb{R}^n, \quad x_1 \ge x_2 \cdots \ge x_n. \tag{2.2.2}$$

Use the isometry between $\mathbb{R}^n$ and $\mathbf{D}^+(n, \mathbb{R})$ to obtain that $e^{tD(x)}, 0 \le t \le 1$ is a geodesic between I and $e^{D(x)}$.

Assume that $1 < p < \infty$. Let $B \in \mathbf{H}^+(n, \mathbb{C})$. Then $\Sigma(B) = e^{D(\log \sigma(B))}$. Suppose that

$$d_p(I, B) + d_p(B, e^{D(x)}) = d_p(I, e^{D(x)}) = ||x||_p. \tag{2.2.3}$$

Clearly, $d_p(I, B) = ||\log \sigma(B)||_p$. As $|x|^p$ is a strictly convex function for $1 < p < \infty$ (2.2.3) yields equalities in all inequalities in (2.1.8) [HLP] and equalities in all inequalities in (2.1.10). Since $\mathbb{R}^n$ is a unique geodesic space for $1 < p < \infty$, the second equality in (2.1.10) yields that $\log \sigma(B) = tx$ for some $t \in [0, 1]$. Clearly, we have equalities in (2.1.7) for all k and $P = B, Q = B^{-1}e^{D(x)}$. Consider first the equality for $k = 1$:

$$||e^{D(x)}||_2 = ||B||_2 ||B^{-1}e^{D(x)}||_2. \tag{2.2.4}$$

Let $f^i = (\delta_{1i}, \ldots, \delta_{ni})^T$ for $i = 1, ..., n$. Then

$$||e^{D(x)}||_2 = ||e^{D(x)}f^1||_2 = ||B(B^{-1}e^{D(x)}f^1)||_2 \leq$$
$$||B||_2||B^{-1}e^{D(x)}f^1||_2 \leq ||B||_2||B^{-1}e^{D(x)}||_2.$$

(2.2.4) yields

$$||B(B^{-1}e^{D(x)}f^1)||_2 = ||B||_2||B^{-1}e^{D(x)}f^1||_2,$$
$$||B^{-1}e^{D(x)}f^1||_2 = ||B^{-1}e^{D(x)}||_2.$$

Since $B \in \mathbf{H}^+(n, \mathbb{C})$, the first equality implies that $B^{-1}e^{D(x)}f^1 = e^{x_1}B^{-1}f^1$ is an eigenvector of B corresponding to the largest eigenvalue $\lambda_1(B) = \sigma_1(B)$. A straightforward calculation shows that $Bf^1 = \lambda_1(B)f^1$. Repeat the same argument for $k = 2$ in the equality in (2.1.7) to deduce that $f^1 \wedge f^2$ is an eigenvector of $B \wedge B$ for the eigenvalue $\lambda_1(B)\lambda_2(B)$. That is, the subspace spanned by f^1, f^2 spanned by by the two eigenvectors of B corresponding to the eigenvalues $\lambda_1(B), \lambda_2(B)$. Hence $Bf^2 = \lambda_2(B)f^2$. Repeat this argument for $k = 3, ..., n$ to deduce that

$$Bf^i = \lambda_i(B)f^i, \quad i = 1, ..., n.$$

Since $\log \sigma(B) = tx, t \in [0, 1]$ we deduce that $B = e^{tD(x)}$, i.e. C is a point on the unique geodesic given above. □

Let $B \in \mathbf{H}^+(n, \mathbb{C})$. Then there exists a unique $A \in \mathbf{H}(n, \mathbb{C})$ such that $e^A = B$. Thus the exponential map (2.1.12) is a diffeomorphism between $\mathbf{H}(n, \mathbb{C})$ and $\mathbf{H}^+(n, \mathbb{C})$.

Corollary 2.2.2 *Let $0 \neq A \in \mathbf{H}(n, \mathbb{C})$. Then e^{tA} is a biinfinite geodesic $\mathbf{X}_n$ which passes through I and e^A for any metric $d_p, 1 \leq p \leq \infty$. The involution $B \mapsto B^{-1}$ is an isometry on $\mathbf{H}^+(n, \mathbb{C}) = \mathbf{X}_n$ for any d_p, which reverses the above geodesic.*

Proof. There exists a unitary U so that $A = UD(x)U^*$ and $x \neq 0$ is of the form (2.2.2). From the proof of Lemma 2.2.1 it follows that $e^{tA} = Ue^{tD(x)}U^*$ is a geodesic for any d_p.

Let $B, C \in \mathbf{H}^+(n, \mathbb{C})$. Then the singular values of $B^{-1}C$ are the positive square roots of the eigenvalues of either $B^{-2}C^2$ or C^2B^{-2}. Hence the singular values of BC^{-1} are equal to the inverse of the singular eigenvalues $B^{-1}C$. Thus $B \mapsto B^{-1}$ is an isometry for each d_p. Clearly, it reverses the geodesic $e^{tA}, t \in \mathbb{R}$. □

For the metric $d_p, 1 < p < \infty$ the space $\mathbf{X}_n$ is unique geodesic, geodesically complete [Bal, p. 3] and a symmetric space. For d_1, d_∞ $\mathbf{X}_n$ is not a unique geodesic space since $\mathbb{R}^n$ is not a unique geodesic space for the metrics δ_1, δ_∞. Let $\mathbf{H}_1^+(n, \mathbb{C})$ be the set of positive Hermitian matrices with determinant 1. Then $\mathbf{H}^+(n, \mathbb{C}) = \mathbf{H}_1^+(n, \mathbb{C}) \times \mathbb{R}_+$. Use Lemma 2.1.1 to identify $\mathbf{SL}(n, \mathbb{C})/\mathbf{SU}(n, \mathbb{C})$ with $\mathbf{H}_1^+(n, \mathbb{C})$. Note that the restriction of the Riemannian metric d_2 on $\mathbf{SL}(n, \mathbb{C})/\mathbf{SU}(n, \mathbb{C})$ gives a simple symmetric space of rank $n-1$, which has a nonpositive curvature. Hence $\mathbf{X}_n$, equipped with the Riemannian metric d_2, is a symmetric space of nonpositive curvature of rank n. It is possible to define a space of nonnegative curvature for a geodesic space [Bal]. It is easy to show that $\mathbb{R}^2$ is does not have a nonpositive curvature for the metric $\delta_p, p \neq 2$.

Indeed, let $1 \leq p \leq \infty$. As in [Bal, p. 1] choose arbitrary $x, y \in \mathbb{R}^2$ and $z = 0$. Let U be a small enough convex neighborhood of 0 for which the inequality (denoted by (*)) [Bal, p. 1] holds. Take a small enough positive ϵ so that $\epsilon x, \epsilon y \in U$. Then the straight segment connecting the points $\epsilon x, \epsilon y$ is a geodesic. The middle point of this geodesic is $m = \frac{\epsilon(x+y)}{2}$. After dividing (*) by ϵ we obtain:

$$||x+y||_p^2 + ||x-y||_p^2 \leq 2||x||_p + 2||y||_p. \tag{2.2.5}$$

Assume that $1 \leq p < 2$. Choose $x = (1,0)^T, y = (0,1)^T$. Then

$$||x+y||_p^2 + ||x-y||_p^2 = 2^{\frac{2}{p}} + 2^{\frac{2}{p}} > 2 + 2$$

and (2.2.5) does not hold. Assume that $p > 2$. Choose $x = (1,1), y = (1,-1)$. Then

$$||x+y||_p^2 + ||x-y||_p^2 = 2^2 + 2^2 > 2 \cdot 2^{\frac{2}{p}} + 2 \cdot 2^{\frac{2}{p}}$$

and (2.2.5) does not hold.

Hence $\mathbf{X}_n$ is not a space of nonpositive curvature for the metric $d_p, p \neq 2$, i.e. $\mathbf{X}_n$ is a Hadamard space only for d_2 [Bal, §I.5].

3 Busemann functions and compactifications

3.1 General setting

Let $\mathbf{X}$ be a complete geodesic space and locally compact with respect to a metric d. Then it is possible to compactify $\mathbf{X}$ by adding the boundary at infinity $\mathbf{X}(\infty)$ using Busemann functions [Bal, §II.1]. We call this compactification the Busemann compactification. Recall the definition of the Busemann functions [Bal, §II.1]. Let

$$b(x,y,z) = d(x,z) - d(x,y), \quad x, y, z \in \mathbf{X}. \tag{3.1.1}$$

Then for a fixed $x, y \in \mathbf{X}$, the function $b(x,y,\cdot) : \mathbf{X} \to \mathbb{R}$ is a Lipschitz function with the Lipschitz constant 1, i.e. nonexpansive function. For a fixed y let

$$b_y : \mathbf{X} \to C(\mathbf{X}), \quad b_y(x) = b(x,y,\cdot). \tag{3.1.2}$$

The assumption that $\mathbf{X}$ is a complete geodesic space implies that the map (3.1.2) is an embedding. In $C(X)$ one introduces the topology of uniform convergence on bounded subsets. Then the Busemann compactification is the compactification of $b_y(\mathbf{X})$ with respect to the topology of uniform convergence on bounded on $C(\mathbf{X})$. This compactification is independent of the point y. As

$$b(x,y,y) = 0, \tag{3.1.3}$$

the assumption that $\mathbf{X}$ is locally compact implies that any sequence of nonexpansive functions $b_y(x_k), k = 1, \ldots$ has a convergent subsequence. That is the Busemann compactification of $\mathbf{X}$, which is denoted by

$$\overline{\mathbf{X}} := \mathbf{X} \cup \mathbf{X}(\infty),$$

is a compact metric space. The compact subspace $\mathbf{X}(\infty)$ is called the Busemann boundary at infinity. An unbounded sequence $x_k, k = 1, \ldots$ is said to converge to $\xi \in \mathbf{X}(\infty)$ if the sequence of functions $b_y(x_k)$ is uniformly converges on the bounded subsets of X. We denote the limit function by $b_y(\xi) = b(\xi, y, \cdot)$ and call it the *Busemann function* at ξ. Note that $b_y(\xi)$ is a nonexpansive function which vanishes at y. Two unbounded sequences $\{x_k\}_{k=1}^{\infty}, \{x'_k\}_{k=1}^{\infty}$ are called equivalent, if the corresponding sequence functions $\{b_y(x_k)\}_{k=1}^{\infty}, \{b_y(x'_k)\}_{k=1}^{\infty}$, converge to the same $b_y(\xi)$. The level set

$$\{z \in \mathbf{X} : \quad b(\xi, y, z) = a\},$$

is called a horosphere (through z) and the set

$$\{z \in \mathbf{X} : \quad b(\xi, y, z) \leq a\},$$

is called a horoball, (both centered at ξ). Then $\mathbf{X}(\infty)$ consists of all points ξ as above.

3.2 $\mathbb{R}^n$

In the next two subsections we consider the Busemann compactification for $\mathbf{X}_n$ for the metrics $d_p, p \in [1, \infty)$. To understand these compactifications, it is helpful to consider the Busemann compactification for $\mathbf{X} = \mathbb{R}^n$ with respect to the Hölder metrics $\delta_p, p \in [1, \infty)$. We denote by $\mathbb{R}^n_p(\infty)$ the Busemann boundary at infinity for $\delta_p, p \in [1, \infty]$. For $\xi \in \mathbb{R}^n_p(\infty)$ let $b_{y,p}(\xi) = b_p(\xi, y, \cdot)$ be the Busemann function for the distance δ_p. It is probably well known that for $1 < p < \infty$ $\mathbb{R}^n_p(\infty)$ is diffeomorphic to S^{n-1} and can be identified with:

$$S^{n-1}_p := \{\xi \in \mathbb{R}^n : \quad ||\xi||_p = 1\}.$$

In what follows we have the obvious convention

$$a|a|^{p-2} = 0 \quad \text{for} \quad a = 0 \in \mathbb{R}, \quad 1 < p < \infty.$$

Lemma 3.2.1 *Let $1 < p < \infty$. Then $\mathbb{R}^n_p(\infty)$ can be identified with S^{n-1}_p. For $\xi = (\xi_1, ..., \xi_n)^T \in S^{n-1}_p$ let*

$$Q_p(\xi, y) := -\sum_{i=1}^{n} y_i \xi_i |\xi_i|^{p-2}, \quad y = (y_1, ..., y_n)^T \in \mathbb{R}^n. \tag{3.2.1}$$

Then

$$b_p(\xi, y, z) = Q_p(\xi, z) - Q_p(\xi, y), \quad y, z \in \mathbb{R}^n. \tag{3.2.2}$$

Proof. Fix $\xi =\in S_p^{n-1}, y \in \mathbb{R}^n$. Set $x(t) = t\xi$. Then for $t >> 1$ the Taylor series of $||x(t) - y||_p$ in t^{-1} yield

$$||x(t) - y||_p = t||\xi - \frac{y}{t}||_p = t + Q_p(\xi, y) + O(\frac{1}{t}).$$

Hence, for a fixed $y, z \in \mathbb{R}^n$ and $||x||_p >> 1$

$$b_p(x, y, z) = ||x - z||_p - ||x - y||_p = Q_p(\xi, z) - Q_p(\xi, y) + O(\frac{1}{||x||_p}),$$

where $\xi = \frac{x}{||x||_p}$. (Here $t = ||x||_p$.) Assume that we have a sequence $0 \neq x_k \in \mathbb{R}^n, k = 1, ...,$ such that $||x_k||_p \to \infty$. Let $\xi_k = \frac{x_k}{||x||_p}, k = 1, ...,$. By taking a subsequence if needed, we may assume that $\lim_{k\to\infty} \xi_k = \xi \in S_p^{n-1}$. Clearly

$$\lim_{k\to\infty} b_p(x_k, y, z) = Q_p(\xi, z) - Q_p(\xi, y).$$

Hence the sequence $x_k, k = 1, \ldots,$ converges to a point in $\mathbb{R}_p^n(\infty)$ which is denoted by ξ. Thus (3.2.2) holds. Let $\xi, \eta \in S_p^{n-1}$ be fixed. Assume that $Q(\xi, z) = Q(\eta, z)$ for all $z \in \mathbb{R}^n$. It is straightforward to show that $\xi = \eta$. Hence $\mathbb{R}_p^n(\infty)$ can be identified with S_p^{n-1}. $\square$

The Busemann boundary at infinity of $\mathbb{R}^n$ for $p = 1$ is more complicated. Let $< n >:= \{1, 2, \ldots, n\}$ and denote by $2^{<n>}$ all *nonempty* subsets of $< n >$. Fix $\alpha \in 2^{<n>}$. Then $\{1, -1\}^\alpha$ denotes the set of all possible maps of α to $\{1, -1\}$. This set has cardinality $2^{|\alpha|}$, where $|\alpha|$ is the cardinality of the set α. Thus an element $\epsilon \in \{1, -1\}^\alpha$ is a set $\{\epsilon_j\}_{j\in\alpha}$ where $\epsilon_j = \pm 1, j \in \alpha$. We agree that $\mathbb{R}^0$ is a set consisting of one element and $|\emptyset| = 0$.

Lemma 3.2.2 *The Busemann boundary at infinity of $\mathbb{R}^n$ with respect to δ_1 has the stratification*

$$\mathbb{R}_1^n(\infty) = \cup_{\alpha\in 2^{<n>}} \{1, -1\}^\alpha \times \mathbb{R}^{|<n>\setminus\alpha|} \tag{3.2.3}$$

That is, a sequence $x_k = (x_{1,k}, ..., x_{n,k})^T, k = 1, \ldots$ converges to $\xi = \{\epsilon_j\}_{j\in\alpha} \times (u_1, ..., u_m)^T$ if the following conditions hold:

$$\begin{aligned}
&\alpha = \{\alpha_1, ..., \alpha_l\}, \quad 1 \le \alpha_1 < \ldots < \alpha_l \le n, \\
&< n > \setminus\alpha = \{\beta_1, \ldots, \beta_m\}, \quad 1 \le \beta_1 < \beta_2 < \ldots < \beta_m \le n, \quad m = n - l, \\
&\lim_{k\to\infty} \epsilon_{\alpha_i} x_{\alpha_i,k} = +\infty, \quad i = 1, ..., l, \\
&\lim_{k\to\infty} x_{\beta_j,k} = u_j, \quad j = 1, ..., m.
\end{aligned} \tag{3.2.4}$$

For $y = (y_1, ..., y_n)^T \in \mathbb{R}^n$ and ξ as above let

$$Q_1(\xi, y) := -\sum_{i=1}^{l} y_{\alpha_i}\epsilon_{\alpha_i} + \sum_{j=1}^{m} |u_j - y_{\beta_j}|. \tag{3.2.5}$$

Then (3.2.2) holds for $p = 1$.

The proof of the Lemma is straightforward and is left to the reader. Note that (3.2.4) implies that the component $\{\rho_j\}_{j\in\gamma}\times\mathbb{R}^{|<n>\setminus\gamma|}$ of the strata $\{1,-1\}^\gamma\times\mathbb{R}^{|<n>\setminus\gamma|}$ is a boundary of $\{\epsilon_j\}_{j\in\alpha}\times\mathbb{R}^{|<n>\setminus\alpha|}$ iff α is a strict subset of γ and $\epsilon_i=\rho_i$ for $i\in\alpha$. It is straightforward to see that $b_1(\xi,y,z)$ is a continuous for on $\mathbb{R}^n_1(\infty)\times U\times V$ where U,V are any bounded compact sets in $\mathbb{R}^n$. However, $Q_1(\xi,y)$ is continuous only on $W\times U$, where W is a compact set of form $\{\epsilon_i\}_{i\in\alpha}\times W'$ for some compact set $W'\subset\mathbb{R}^{|<n>\setminus\alpha|}$.

The stratification of $\mathbb{R}^n_\infty$ is similar to the stratification of $\mathbb{R}^n_1$. One can also define the function $Q_\infty(\xi,y)$ on each strata of $\mathbb{R}^n_\infty$ so that (3.2.2) holds for $p=\infty$. Since we are mainly interested in the metric δ_1 we omit the details of $p=\infty$.

3.3 $\mathbf{X}_n$

In this subsection we identify $\mathbf{X}_n$ with $\mathbf{H}^+(n,\mathbb{C})$. Let

$$b_p(E,B,C)=d_p(E,C)-d_p(E,B),\quad E,B,C\in\mathbf{H}^+(n,\mathbb{C}),\quad 1\le p\le\infty. \tag{3.3.1}$$

be the p-Busemann function. Then

$$b_{B,p}:\mathbf{H}^+(n,\mathbb{C})\to C(\mathbf{H}^+(n,\mathbb{C})),\quad b_{B,p}(E)=b_p(E,B,\cdot).$$

The p-Busemann compactification of $\mathbf{H}^+(n,\mathbb{C})$ is denoted by

$$\overline{\mathbf{X}}_{n,p}:=\mathbf{X}_n\cup\mathbf{X}_{n,p}(\infty)=\mathbf{H}^+(n,\mathbb{C})\cup\mathbf{X}_{n,p}(\infty).$$

Fix $0\ne A\in\mathbf{H}(n,\mathbb{C})$. Recall the spectral decomposition of A

$$\begin{aligned}&Ax^i=\lambda_i(A)x^i,\quad (x^i)^*x^j=\delta_{ij},\quad i,j=1,...,n,\\&\lambda_1(A)\ge\lambda_2(A)\ge\cdots\ge\lambda_n(A),\\&\lambda(A)=(\lambda_1(A),...,\lambda_n(A)),\quad U=(x^1,\dots,x^n)\in\mathbf{U}(n,\mathbb{C})\\&A=U\Lambda(A)U^*,\quad \Lambda(A)=D(\lambda(A)).\end{aligned} \tag{3.3.2}$$

As the exponential map is a diffeomorphism between $\mathbf{H}(n,\mathbb{C})$ and $\mathbf{H}^+(n,\mathbb{C})$ we deduce that for any $E\in\mathbf{H}^+(n,\mathbb{C}),E\ne I$ there exists a unique $A=A(E,p)\in\mathbf{H}(n,\mathbb{C}),A\ne0$ such that

$$E=e^{tA(E,p)},\quad t>0,\quad ||\lambda(A)||_p=1.$$

We first show that any sequence

$$E_k=e^{t_kA},\quad ||\lambda(A)||_p=1,\quad \lim_{k\to\infty}t_k=+\infty$$

converges to boundary point $\xi_p(A)$ with respect to the metric d_p.

To compute explicitly the Busemann function $b_{B,p}(\xi)$ for $\xi\in\mathbf{X}_{n,p}(\infty)$ we need good asymptotic expansions for the singular values of $C^{-1}e^{tA}$ which are the positive square roots of the eigenvalues of $e^{tA}C^{-2}e^{tA}$. These expansions are obtained using the standard perturbation techniques for eigenvalues of Hermitian matrices, e.g. [Fr1].

Lemma 3.3.1 *Let $0 \neq A \in \mathbf{H}(n,\mathbb{C})$ satisfy (3.3.2). Suppose furthermore*

$$\begin{aligned}
&\lambda_1(A) = \lambda_2(A) = \cdots = \lambda_{j_1}(A) > \lambda_{j_1+1}(A) = \cdots \\
&\cdots = \lambda_{j_2}(A) > \cdots > \lambda_{j_{p-1}+1}(A) = \cdots = \lambda_n(A), \\
&j_0 = 0 < j_1 < \cdots < j_p = n.
\end{aligned} \tag{3.3.3}$$

Assume that $C \in \mathbf{H}^+(n,\mathbb{C})$. Let $\mu_1(A,C) \geq \ldots \geq \mu_{j_1}(A,C)$ be the eigenvalues of the positive definite matrix F_1:

$$\begin{aligned}
&F_1 := ((x^l)^* C^{-2} x^m)_{l,m=1}^{j_1} \in \mathbf{H}^+(j_1,\mathbb{C}) \\
&\lambda(F_1) = (\mu_1(A,C),\ldots,\mu_{j_1}(A,C)).
\end{aligned} \tag{3.3.4}$$

Then for $t >> 1$

$$\begin{aligned}
&\log \sigma_i(C^{-1}e^{At}) = t\lambda_i(A) + \frac{1}{2}\log \mu_i(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}) = \\
&t\lambda_1(A) + \frac{1}{2}\log \mu_i(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}), \quad i = 1,\ldots,j_1.
\end{aligned} \tag{3.3.5}$$

Proof. Consider the positive definite matrix $e^{tA}C^{-2}e^{tA}$. By considering the similar Hermitian matrix $U^*e^{tA}U(U^*CU)^{-2}U^*e^{tA}U$ w.l.o.g. (without loss of generality) we may assume that $A = \Lambda(A)$. Let

$$\begin{aligned}
&E(t) = e^{-2\lambda_1(A)t}e^{tA}C^{-2}e^{tA}, \\
&\lim_{t\to\infty} E(t) = E(\infty).
\end{aligned}$$

Then $E(\infty)$ is a nonnegative definite matrix of rank j_1, which has a block diagonal form $F_1 \oplus 0$. Hence $\mu_1(A,C),\ldots,\mu_{j_1}(A,C)$ are the nonzero eigenvalues of $E(\infty)$. Clearly

$$E(t) = E(\infty) + O(e^{-at}), \quad a = \lambda_1(A) - \lambda_{j_1+1}(A), \quad t >> 1.$$

Weyl's inequalities [HJ] yield

$$|\lambda_i(E(t)) - \lambda_i(E(\infty))| \leq ||E(t) - E(\infty)||_2 = O(e^{-at}), \quad i = 1,\ldots,n.$$

Clearly

$$\lambda_i(e^{At}C^{-2}e^{At}) = e^{2\lambda_1(A)t}\lambda_i(E(t)), \quad i = 1,\ldots,n.$$

As singular values of $C^{-1}e^{tA}$ are the positive square roots of the eigenvalues of $e^{tA}C^{-2}e^{tA}$, from the above arguments we deduce (3.3.5). □

Corollary 3.3.2 *Let the assumptions of Lemma 3.3.1 hold. Let*

$$\alpha_1(A,C) := \log \sqrt{||F_1||_2}. \tag{3.3.6}$$

Then

$$\log \sigma_1(C^{-1}e^{At}) = t\lambda_1(A) + \alpha_1(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}) \quad \text{for} \quad t >> 1,$$
$$\sum_{i=1}^{j_1} \log \sigma_i(C^{-1}e^{At}) = t\sum_{i=1}^{j_1} \lambda_i(A) + \frac{1}{2}\log\det(F_1) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}), \quad \text{for} \quad t >> 1. \tag{3.3.7}$$

It is possible to give similar formulas to (3.3.5) for each $\log \sigma_i(C^{-1}e^{At})$ using Schur complements of C^{-2}. We prefer to give more transparent formulas using the wedge (exterior) products. In addition to what is said about the wedge products in §2.1, we recall that for a field F the k exterior product $\wedge_k F^n$ is spanned by all wedge products of the form $u = y^1 \wedge y^2 \wedge \ldots \wedge y^k$. u is nonzero vector iff $y^1, ..., y^k$ are linearly independent. Further discussion of the wedge products is given in §4.1.

Theorem 3.3.3 *Let $A \in \mathbf{H}(n,\mathbb{C})$ satisfy (3.3.2) and (3.3.3). Assume that $i \in [1,p] \cap \mathbb{Z}$ and $k \in [j_{i-1}+1, j_i] \cap \mathbb{Z}$. Let $V_k \subset \mathbb{C}^{\binom{n}{k}}$ be the subspace spanned by*

$$x^1 \wedge x^2 \wedge \ldots \wedge x^{j_{i-1}} \wedge x^{l_1} \wedge x^{l_2} \wedge \ldots \wedge x^{l_{k-j_{i-1}}},$$

where $l_1, ..., l_{k-j_{i-1}}$ range over all indices satisfying

$$j_{i-1} + 1 \le l_1 < \ldots < l_{k-j_{i-1}} \le j_i.$$

Denote by $P_k \in \mathbf{M}(\binom{n}{k}, \mathbb{C})$ the unitary projection on V_k for $k = 1, \ldots, n$. Let $C \in \mathbf{H}^+(n,\mathbb{C})$. Set

$$\begin{aligned}
&\alpha_0(A,C) = 0,\\
&\alpha_k(A,C) := \log ||(\wedge_k C^{-1})P_k||_2, \quad k = 1, ..., n-1\\
&\alpha_n(A,C) = \log\det(\mathrm{C}^{-1})
\end{aligned} \tag{3.3.8}$$

Then for $t >> 1$

$$\begin{aligned}
&\log \sigma_k(C^{-1}e^{tA}) = t\lambda_k(A) + \alpha_k(A,C) - \alpha_{k-1}(A,C) + E_k(t),\\
&\lim_{t\to\infty} E_k(t) = 0,\\
&k = 1, \ldots, n.
\end{aligned} \tag{3.3.9}$$

Proof. Observe first that P_n is the identity operator on $\mathbb{C}$ and $\wedge_n C^{-1} = \det(\mathrm{C}^{-1})$. Hence

$$\alpha_n(A,C) = \log ||(\wedge_n C^{-1})P_n||_2 = \log\det(\mathrm{C}^{-1}).$$

As in the proof of Lemma 3.3.1 we may assume that $A = \Lambda(A)$ and

$$x^i = (\delta_{i1}, ..., \delta_{in})^T, \quad i = 1, ..., n.$$

Then P_1 is a diagonal matrix whose first j_1 diagonal are equal to 1 and all other diagonal entry are equal to 0. Assume first that $k = 1$. Then

$$\lambda_1(F_1) = ||P_1 C^{-1} C^{-1} P_1||_2 = ||C^{-1}P_1||_2^2.$$

Thus the definitions of α_1 given in (3.3.6) and (3.3.8) coincide. Hence for k=1 (3.3.9) follows from (3.3.7).

Let $k \in [\max(j_{i-1}, 1) + 1, j_i] \cap \mathbb{Z}$. Consider $\wedge_k e^{tA}$ for $t > 0$. Use (2.1.3) to deduce that V_k is the eigenspace corresponding to the maximal eigenvalue $e^{t\sum_{l=1}^k \lambda_l(A)}$ of $\wedge_k e^{tA}$. As $A = \Lambda(A)$ we deduce that

$$\lim_{t\to\infty} e^{-t\sum_{l=1}^k \lambda_l(A)} \wedge_k e^{tA} = P_k.$$

Apply Corollary 3.3.2 to $\wedge_k C^{-1} \wedge_k e^{tA}$ to obtain

$$\log ||\wedge_k C^{-1} \wedge_k e^{tA}|| = \sum_{l=1}^{k} \log \sigma_l(C^{-1}e^{tA}) = t\sum_{l=1}^{k} \lambda_i(A) + \alpha_k(A,C) + E^{(k)}(t), \quad \lim_{t\to\infty} E^{(k)}(t) = 0.$$

Subtract from the above expression the similar expression for $k-1$ to deduce (3.3.9). $\square$

Theorem 3.3.4 *Let $0 \neq A \in \mathbf{H}(n,\mathbb{C})$, $B \in \mathbf{H}^+(n,\mathbb{C})$. Let t_m, $m = 1, \ldots,$ be a sequence of real numbers converging to ∞. Then $b_p(e^{t_m A}, B, \cdot)$ converges to the Busemann function $b_p(\xi, B, \cdot)$ for any $1 \le p \le \infty$ on $\mathbf{H}^+(n,\mathbb{C})$. More precisely, let $C \in \mathbf{H}^+(n,\mathbb{C})$. Then*
for $p = \infty$

$$b_\infty(\xi, B, C) = \alpha_1(A,C) - \alpha_1(A,B), \text{ if } \lambda_1(A) > -\lambda_n(A),$$

$$\begin{aligned} b_\infty(\xi, B, C) &= \alpha_{n-1}(A,C) - \alpha_n(A,C) - \alpha_{n-1}(A,B) + \alpha_n(A,B), \\ &\quad \text{if } \lambda_1(A) < -\lambda_n(A), \end{aligned}$$

$$\begin{aligned} b_\infty(\xi, B, C) &= \max(\alpha_1(A,C), \alpha_{n-1}(A,C) - \alpha_n(A,C)) \\ &\quad - \max(\alpha_1(A,B), \alpha_{n-1}(A,B) - \alpha_n(A,B)), \text{ if } \lambda_1(A) = -\lambda_n(A). \end{aligned} \quad (3.3.10)$$

For $p = 1$,

$$b_1(\xi, B, C) = \alpha_n(A, C) - \alpha_n(A, B), \text{ if } \lambda_n(A) > 0,$$

$$b_1(\xi, B, C) = -\alpha_n(A, C) + \alpha_n(A, B), \text{ if } \lambda_1(A) < 0,$$

$$b_1(\xi, B, C) =$$

$$\alpha_{j_{k-1}}(A, C) + \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A, C) - \alpha_{i-1}(A, C)| + \alpha_{j_k}(A, C) - \alpha_n(A, C)$$

$$-\alpha_{j_{k-1}}(A, B) - \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A, B) - \alpha_{i-1}(A, B)| - \alpha_{j_k}(A, B) + \alpha_n(A, B)$$

$$\text{if } \lambda_{j_k}(A) = 0,$$

$$b_1(\xi, B, C) = 2\alpha_{j_k}(A, C) - \alpha_n(A, C)$$
$$-2\alpha_{j_k}(A, B) + \alpha_n(A, B), \text{ if } \lambda_{j_k}(A) > 0 > \lambda_{j_{k+1}}(A). \tag{3.3.11}$$

For $1 < p < \infty$

$$b_p(\xi, B, C) = \tag{3.3.12}$$
$$(\sum_{i=1}^{n} |\lambda_i(A)|^p)^{\frac{1-p}{p}} \sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A, C) - \alpha_{i-1}(A, C) - \alpha_i(A, B) + \alpha_{i-1}(A, B)).$$

Proof. Recall that

$$d_p(e^{t_m A}, B) = (\sum_{i=1}^{n} |\log \sigma_i(B^{-1}e^{t_m A})|^p)^{\frac{1}{p}}.$$

We first consider $p = \infty$. Note that

$$d_\infty(e^{t_m A}, B) = \max(|\log \sigma_1(B^{-1}e^{t_m A})|, |\log \sigma_n(B^{-1}e^{t_m A})|).$$

We use Theorem 3.3.3. If $-\lambda_n(A) < \lambda_1(A)$, which implies that $\lambda_1(A) > 0$, then for $t_m >> 1$

$$d_\infty(e^{t_m A}, B) = \log \sigma_1(B^{-1}e^{t_m A}) = t_m \lambda_1(A) + \alpha_1(A, B) + E_1(t_m).$$

Replacing in the above formula B by C we deduce the first case of formula (3.3.10). Similary, the assumption that $\lambda_1(A) < -\lambda_n(A)$ yields that $\lambda_n(A) < 0$ and

$$d_\infty(e^{t_m A}, B) = -\log \sigma_n(B^{-1}e^{t_m A}) = -t_m \lambda_n(A) - \alpha_n(A, B) + \alpha_{n-1}(A, B) - E_n(t_m).$$

Replace B by C to deduce the second equality in (3.3.10). Suppose finally that $\lambda_1(A) = -\lambda_n(A)$. Then

$$\begin{aligned}
&d_\infty(e^{t_m A}, B) = \\
&\max(\log\sigma_1(B^{-1}e^{t_m A}), -\log\sigma_n(B^{-1}e^{t_m A})) = \\
&t_m\lambda_1(A) + \max(\alpha_1(A,B), -\alpha_n(A,B) + \alpha_{n-1}(A,B)) + E(t_m),
\end{aligned}$$

and the last equality of (3.3.10) follows.

Assume now that $p = 1$. Suppose first that $\lambda_n(A) > 0$. Then Theorem 3.3.3 yields that all singular values of $e^{t_m A}B$ tend to ∞. Hence

$$d_1(e^{t_m A}, B) = t_m\big(\sum_{i=1}^n \lambda_i(A)\big) + \alpha_n(A,B) + E^{(n)}(t_m).$$

Replace B by C and deduce the first equality of (3.3.11). If $\lambda_1(A) < 0$ then Theorem 3.3.3 yields that all singular values of $B^{-1}e^{t_m A}$ tend to -∞. Hence

$$d_1(e^{t_m A}, B) = -(t_m\big(\sum_{i=1}^n \lambda_i(A)\big) + \alpha_n(A,B) + E^{(n)}(t_m)),$$

and the second equality of (3.3.11) follows. Suppose next that $\lambda_{j_k}(A) = 0$. Then all $\sigma_i(B^{-1}e^{t_n A})$ tend to ∞ for $i \le j_{k-1}$ (if $j_{k-1} > 0$), all $\sigma_i(B^{-1}e^{t_m A})$ tend to $-\infty$ for $i > j_k$ (if $j_k < n$), and all $\sigma_i(B^{-1}e^{t_m A})$ are bounded for $j_{k-1} < i \le j_k$. Hence

$$\begin{aligned}
&d_1(e^{t_m A}, B) = t_m\sum_{i=1}^n |\lambda_i(A)| \\
&+\alpha_{j_{k-1}}(A,B) + \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A,B) - \alpha_{i-1}(A,B)| + \alpha_{j_k}(A,B) - \alpha_n(A,B).
\end{aligned}$$

Use the same formula for $d_1(e^{t_m A}, C)$ to deduce the third formula of (3.3.11). Similarly one deduces the last formula of (3.3.11).

Assume now that $1 < p < \infty$. If $\lambda_i(A) \ne 0$ then Theorem 3.3.3 yields:

$$|\log\sigma_i(B^{-1}e^{t_m A})|^p = t_m^p|\lambda_i(A)|^p + pt_m^{p-1}\frac{|\lambda_i(A)|^p}{\lambda_i(A)}(\alpha_i(A,B) - \alpha_{i-1}(A,B)) + o(t_m^{p-1}).$$

If $\lambda_i(A) = 0$ Theorem 3.3.3 yields that

$$|\log\sigma_i(B^{-1}e^{t_m A})|^p = O(1).$$

Hence

$$d_p(e^{t_m A}, B) =$$

$$
\begin{aligned}
&(t_m^p \sum_{i=1}^n |\lambda_i(A)|^p + p t_m^{p-1} \sum_{i=1}^n \frac{|\lambda_i(A)|^p}{\lambda_i(A)} (\alpha_i(A,B) - \alpha_{i-1}(A,B)) + o(t_m^{p-1}))^{\frac{1}{p}} = \\
&t_m (\sum_{i=1}^n |\lambda_i(A)|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |\lambda_i(A)|^p)^{\frac{1-p}{p}} \sum_{i=1}^n \frac{|\lambda_i(A)|^p}{\lambda_i(A)} (\alpha_i(A,B) - \alpha_{i-1}(A,B)) + o(1).
\end{aligned} \tag{3.3.13}
$$

Replace B by C and deduce (3.3.12). $\square$

It is possible to introduce a new metric depending on $k \in [1,n] \cap \mathbb{Z}$ which will interpolate between d_∞ and d_1 as follows

$$
\begin{aligned}
&\rho_k(A,B) = \max(\sum_{i=1}^k |\log \sigma_i(A^{-1}B)|, \sum_{i=1}^k |\log \sigma_i(B^{-1}A)|) = \\
&\max(\sum_{i=1}^k |\log \sigma_i(A^{-1}B)|, \sum_{i=n-k+1}^n |\log \sigma_i(A^{-1}B)|), \\
&k = 1, \ldots, n.
\end{aligned} \tag{3.3.14}
$$

Clearly,

$$
d_\infty(A,B) = \rho_1(A,B) \le \rho_2(A,B) \le \cdots \le \rho_n(A,B) = d_1(A,B).
$$

One can study the Busemann compactification with respect to the metric ρ_k for $k = 2, ..., n-1$. This will not be done here.

Recall that the hyperbolic space $\mathbf{H}^n$ can be viewed as the subspace of $\mathbf{X}_{n+1}$ given by $\mathbf{SO}(n,1)/(\mathbf{SO}(n+1) \cap \mathbf{SO}(n,1))$. As any $A \in \mathbf{SO}(n,1)$ has at least $n-1$ singular values equal to 1, it follows that

$$
d_p(\cdot,\cdot) = 2^{\frac{1}{p}} d_\infty(\cdot,\cdot), \quad p \in [1,\infty].
$$

3.4 $\mathbf{X}_{n,p}(\infty)$ for $p \in (1,\infty)$

Let

$$
|||A|||_p := ||\sigma(A)||_p, \quad A \in \mathbf{M}(n,\mathbb{C}), \quad p \in [1,\infty]
$$

be the p-Schatten norm of A. Note that $|||A|||_\infty = ||A||_2$. Fix $0 \ne A \in \mathbf{H}^+(n,\mathbb{C})$. Let $t_m, m = 1, \ldots,$ be a sequence of real numbers converging to ∞. Theorem 3.3.4 yields that $e^{t_m A}$ converges to $\xi \in \mathbf{X}_{n,p}(\infty)$, independently on the sequence of $t_m, m = 1, \ldots$. If we replace A by $A' = aA, a > 0$ the sequence $e^{t_m A'}$ will still converge to the same limit ξ. That is, ξ corresponds to the end of the geodesic ray eminating from the point $I \in \mathbf{H}^+(n,\mathbb{C})$. In the direction of the tangent vector A. Fix $p \in [0,\infty]$. Then the *visual* boundary $\partial_v \mathbf{X}_{n,p}$ of $\mathbf{X}_{n,p}$ is given by the end of the geodesic ray represented by $A \in S_{n,p}$, where

$$
S_{n,p} := \{A \in \mathbf{H}(n,\mathbb{C}) : \quad |||A|||_p = 1\}.
$$

The topology on $\partial_v \mathbf{X}_n$ is equivalent to the standard topology on $S_{n,p}$. Furthermore, given a sequence $\{t_m\}_1^\infty$ which converges to ∞ and a sequence $\{A_m\}_1^\infty \subset S_{n,p}$ then the sequence $e^{t_m A_m}$ converges to a point in $\partial_v \mathbf{X}_{n,p}$ corresponding to $A \in S_{n,p}$ iff

$$\lim_{m\to\infty} A_m = A.$$

See for example Karpelivich [Kar] for the Riemannian case $p = 2$. Clearly, all $S_{n,p}$ are homeomorphic to $S_{n,2}$. Hence, we will refer to $\partial_v \mathbf{X}_{n,p}$ as the visual boundary of $\mathbf{X}_n$ and we will denote it by $\partial_v \mathbf{X}_n$.

Theorem 3.4.1 *Let $1 < p < \infty$. Then the Busemann p-boundary at infinity of $\mathbf{X}_n$ can be identified with the visual boundary of $\mathbf{X}_n$.*

To prove this theorem we need the following results:

Lemma 3.4.2 *Let $0 \neq A \in \mathbf{H}(n,\mathbb{C})$ satisfy* (3.3.2) *and* (3.3.3). *Then for any $C \in \mathbf{H}^+(n,\mathbb{C})$ the following inequalities hold:*

$$\begin{aligned}
&\sum_{i=1}^{k} \log \lambda_{n-i+1}(C^{-1}) \le \alpha_k(A,C) \le \sum_{i=1}^{k} \log \lambda_i(C^{-1}),\\
&k = 1, \ldots, n-1,\\
&\alpha_n(A,C) = \sum_{i=1}^{n} \log \lambda_i(C^{-1}).
\end{aligned} \tag{3.4.1}$$

Let $k \in [1, n-1] \cap \mathbb{Z}$ be a fixed integer that satisfies $j_{i-1} < k \le j_i$. Then equality in the right-hand side inequality of (3.4.1) *holds iff the subspace U_{j_i} spanned by $x^1, \ldots, x^{j_i}$ contains k linearly independent eigenvectors of of C^{-1} corresponding to the first k-eigenvalues of C^{-1}. Equality in the left-hand side of* (3.4.1) *holds iff any k-dimensional subspace of U_{j_i} is a subspace that spanned by last k-eigenvalues of C^{-1}. Furthermore,*

$$\begin{aligned}
&\alpha_{j_{k-1}+1}(A,C) - \alpha_{j_{k-1}}(A,C) \ge \alpha_{j_{k-1}+2}(A,C) - \alpha_{j_{k-1}+1}(A,C) \ge \ldots\\
&\ge \alpha_{j_k}(A,C) - \alpha_{j_k-1}(A,C),\\
&k = 1, \ldots, p.
\end{aligned} \tag{3.4.2}$$

Proof. Assume that $k = 1$. The maximal characterization of $\lambda_1(C^{-2})$ and the minimal characterization of $\lambda_n(C^{-2})$ and the definition of F_1 in (3.3.4) yield [Fr1]

$$\lambda_n(C^{-2}) \le \mu_{j_1}(A,C) = \lambda_{j_1}(F_1) \le \mu_1(A,C) = \lambda_1(F_1) \le \lambda_1(C^{-2}).$$

Equality in the right-hand side of the above inequality holds iff U_{j_1} contains an eigenvector of C^{-2} corresponding to $\lambda_1(C^{-2})$. Equality $\lambda_n(C^{-2}) = \lambda_1(F)$ yields the equalities

$$\lambda_n(C^{-2}) = \lambda_{j_1}(F_1) = \ldots = \lambda_1(F).$$

These equalities hold iff any nonzero vector in U_{j_1} is an eigenvector of C^{-2} corresponding to $\lambda_n(C^{-2})$. As C^{-1} is a positive definite matrix we deduce that

$$\lambda_i(C^{-2}) = \lambda_i(C^{-1})^2, \quad i = 1, ..., n.$$

Use (3.3.6) and the above arguments to deduce the lemma for $k = 1$. To deduce the lemma for $1 < k < n$ one repeats the above arguments for $\wedge_k C^{-2} = (\wedge_k C^{-1})^2$. To deduce the formula for $\alpha_n(A, C)$ observe that $\wedge_n C^{-2}$ is a positive number equal $det(C^{-2})$.

The inequalities (3.4.2) follow from (3.3.9), (3.3.3) and the fact that the singular values of any matrix are arranged in a decreasing order. □

Corollary 3.4.3 *Let $0 \neq A \in \mathbf{H}^+(n, \mathbb{C})$. Then*

$$\alpha_k(A, I) = 0, \quad k = 1, ..., n.$$

Note that (3.4.2) can be used to simplify slightly the third formula of (3.3.11).

Proof of Theorem 3.4.1. Fix $p \in (1, \infty)$. We first show that if A and A' are two distinct points in $S_{n,p}$ then the corresponding induced points $\xi, \xi' \in \mathbf{X}_{n,p}(\infty)$ are distinct. Assume to the contrary that $\xi = \xi'$. W.l.o.g. we may assume that $B = I$. The assumption that $\xi = \xi'$ combined with (3.3.12) and Corollary 3.4.3 yields

$$\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A, C) - \alpha_{i-1}(A, C)) =$$
$$\sum_{i=1}^{n} \lambda_i(A')|\lambda_i(A')|^{p-2}(\alpha_i(A', C) - \alpha_{i-1}(A', C)), \tag{3.4.3}$$

Observe that the sequence $\{\lambda_i(A)|\lambda_i(A)|^{p-2}\}_1^n$ is a decresing sequence. Furthermore

$$\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A, C) - \alpha_{i-1}(A, C)) =$$
$$\sum_{i=1}^{n-1} \alpha_i(A, C)(\lambda_i(A)|\lambda_i(A)|^{p-2} - \lambda_{i+1}(A)|\lambda_{i+1}(A)|^{p-2}) + \alpha_n(A, C)\lambda_n(A)|\lambda_n(A)|^{p-2}. \tag{3.4.4}$$

In (3.4.3) choose $C = e^{-A'}$. Then Lemma 3.4.2 yields

$$\alpha_i(A', C) = \sum_{k=1}^{i} \lambda_k(A'), \quad k = 1, ..., n.$$

Since $A' \in S_{n,p}$ the right-hand side of (3.4.3) is equal to 1. Use Lemma 3.4.2 and (3.4.4) to deduce that the left-hand side of (3.4.3) is bounded above by

$$\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}\lambda_i(A').$$

Use the Hölder p-inequality to deduce that the above expression is bounded above by $||A||_p||A'||_p = 1$. Hence $\lambda(A) = \lambda(A')$. Furthermore, the right-hand side inequalities in (3.4.1) are equalities for $C = e^{-A'}$ whenever $\lambda_i(A) > \lambda_{i+1}(A)$. Lemma 3.4.2 for $k = j_i$ yields that U_{j_i} is spanned by the eigenvectors of $e^{A'}$ corresponding to the first j_i eigenvalues of $e^{A'}$ for $i = 1, ..., p-1$. As $\lambda(A) = \lambda(A')$ we deduce that for each eigenvalue $\lambda = \lambda_{j_i}(A) = \lambda_{j_i}(A')$ the eigenspaces of A and A' coincide. Hence $A = A'$ contrary to our assumption.

Let $\{A_m\}_1^\infty \subset S_{n,p}$ be a convergent sequence $\lim_{m\to\infty} A_m = A \in S_{n,p}$. Clearly

$$\lim_{m\to\infty} \lambda(A_m) = \lambda(A). \tag{3.4.5}$$

As A may have multiple eigenvalues, the similar statement for the eigenspaces of $\{A_m\}_1^\infty$ is as follows. Assume that A satisfies (3.3.3). Then the eigenspace $U_{j_i,m}$, corresponding to the first j_i eigenvalues of A_m, converges to the eigenspace subspace U_{j_i}, corresponding to the first j_i eigenvalues of A, for $i = 1, ..., p$. Hence

$$\lim_{m\to\infty} \alpha_{j_i}(A_m, C) = \alpha_{j_i}(A, C), \quad i = 1, ..., p. \tag{3.4.6}$$

Let $\lim_{m\to\infty} t_m = \infty$. We have to show that

$$\lim_{m\to\infty} b_p(e^{t_m A_m}, I, C) = b_p(\xi, I, C), \tag{3.4.7}$$

where ξ is the limit point of the geodesic ray induced by A. Use (3.3.9), (3.4.4), the last equality of (3.3.8) and the assumption that $B = I$ to obtain

$$\begin{aligned} b_p(e^{t_m A_m}, I, C) = &\sum_{l=1}^{n-1} \alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2}) \\ &+ \log det(C)\lambda_n(A_m)|\lambda_n(A_m)|^{p-2} + o(\frac{1}{t}). \end{aligned} \tag{3.4.8}$$

Observe that all the numbers $\alpha_l(A_m, C)$ are uniformaly bounded for a fixed $C \in \mathbf{H}^+(n, \mathbb{C})$. Consider a summand

$$\alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2}) \tag{3.4.9}$$

appearing in (3.4.8). We claim that this summand converges to

$$\alpha_l(A, C)(\lambda_l(A)|\lambda_l(A)|^{p-2} - \lambda_{l+1}(A)|\lambda_{l+1}(A)|^{p-2}).$$

For $l = j_i$ this claim follows from (3.4.6) and (3.4.5). For $l \in (j_{i-1}, j_i) \cap \mathbb{Z}$ (3.4.9) converges to 0. Hence (3.4.7) holds. □

Thus, the structure of $\mathbf{X}_{n,p}(\infty)$ is similar to the structure of $\mathbb{R}_{n,p}(\infty)$ for $1 < p < \infty$.

3.5 $\mathbf{X}_{n,1}(\infty)$

In this subsection we show that the structure of $\mathbf{X}_{n,1}(\infty)$ is similar in principle to that of $\mathbb{R}_{n,1}(\infty)$, but more complicated. Given $A \in \mathbf{H}(n, \mathbb{C})$ and a set $T \subset \mathbb{R}$ we denote by $U_T(A)$ the subspace spanned by all eigenvectors of A corresponding to the eigenvalues in T. Let $P_T(A)$ be the orthogonal projection on $U_T(A)$. Set

$$U_+(A) := U_{(0,\infty)}(A), \quad U_0(A) := U_{\{0\}}(A), \quad U_-(A) := U_{(-\infty,0)}(A).$$

Then

$$\mathbb{C}^n = U_+(A) \oplus U_0(A) \oplus U_-(A) \tag{3.5.1}$$

is an orthonormal decomposition of $\mathbb{C}^n$, with some of the factors may be trivial. Note that $U_-(A)$ is determined by $U_+(A), U_0(A)$. For $A \neq 0$ we denote the above orthonormal decomposition simply as

$$\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-, \quad U_0 \neq \mathbb{C}^n. \tag{3.5.2}$$

Lemma 3.5.1 *The Busemann compactification of the geodesic rays of the form $e^{tA}, A \in S_{n,1}, t > 0$ with respect to the metric d_1 depends only on the eigenspaces $U_+(A), U_0(A), U_-(A)$. Moreover $A, A' \in S_{n,1}$ induce the same point $\xi \in \mathbf{X}_{n,1}(\infty)$ if and only if the eigenspaces of A, A' corresponding to positive, zero and negative eigenvalues coincide respectively.*

Proof. Consider the formulas for $b_1(\xi, B, C)$ in (3.3.11). Recall that

$$\alpha_n(A, C) = \log det(C^{-1}).$$

Assume first that $U_0(A) = \{0\}$, i.e. A is nonsingular. Then it is straightforward to see that the Busemann function depends only on $U_+(A)$. Assume now that $U_0(A)$ is a nontrivial subspace. Then $b_1(\xi, B, C)$ is given by the third formula of (3.3.11). Clearly, $\alpha_{j_{k-1}}(A, C)$ depends only on $U_+(A)$. The definition of $\alpha_l(A, C)$ (in Theorem 3.3.3) for $l \in (j_{k-1}, j_k] \cap \mathbb{Z}$ depends on the choice of an orthonormal basis in $U_+(A)$ and $U_0(A)$. It is straightforward to show that the values of $\alpha_l(A, C), l \in (j_{k-1}, j_k] \cap \mathbb{Z}$ are independent of the choice of these orthonormal bases. (Suffices to note that $x^1 \wedge \ldots \wedge x^{j_{i-1}} = \wedge_{j_{i-1}} U_{j_{i-1}}$.) Hence $b_1(\xi, B, C)$ depends only on $U_+(A), U_0(A)$. It is straightforward to show, using the formulas (3.3.11), that different decompositions (3.5.2) induce different Busemann functions. (One may take the convenient choice $B = I$.) Hence A, A' induce the same point ξ iff the orthogonal decomposition $\mathbb{C}^n$ to the eigenspaces corresponding to positive, zero and negative eigenvalues of A, A' are identical. □

Proposition 3.5.2 *Let $A \in \mathbf{H}(n, \mathbb{C})$, $B \in \mathbf{GL}(n, \mathbb{C})$. Then*

$$\frac{1}{2} \log \lambda_n(BB^*) \leq \log \sigma_1(Be^A) - \lambda_1(A) \leq \frac{1}{2} \log \lambda_1(BB^*).$$

Proof. Consider the matrix

$$E = e^{-\lambda_1(A)} B e^{A} = B e^{A - \lambda_1(A) I}.$$

Then

$$BPB^* \leq EE^* \leq BB^*,$$

where $P := P_{\lambda_1(A)}(A)$. Clearly

$$\sigma_1(E)^2 = ||EE^*||_2 \leq ||BB^*||_2 = \lambda_1(BB^*).$$

Assume that $Pu = u, ||u||_2 = 1$. Then

$$\sigma_1(E)^2 \geq ||BPB^*||_2 = ||BP||_2^2 \geq ||BPu||_2^2 = ||Bu||_2^2 = u^* B^* B u \geq \lambda_n(B^* B) = \lambda_n(BB^*).$$

□

Let U be a vector space with an inner product. Denote by $\mathbf{H}(U)$ the real space of selfadjoint (Hermitian) operators on U. We agree that $\mathbf{H}(\{0\}) = 0$.

Theorem 3.5.3 *To each orthogonal decomposition of $\mathbb{C}^n$ of the form (3.5.2) associate the space $(U_+, \mathbf{H}(U_0), U_-)$. Then the union of all these spaces with respect to all orthogonal decomposition of $\mathbb{C}^n$ of the form (3.5.2) can be identified with the Busemann 1-boundary at infinity of $\mathbf{X}_n$. Let $\{A_m\}_1^\infty \subset \mathbf{H}(n, \mathbb{C})$ be an unbounded sequence. Then $\{e^{A_m}\}_1^\infty$ converges to the point $(U_+, E, U_-), E \in \mathbf{H}(U_0)$ if and only if the following conditions hold:*

$$\begin{aligned}
&\lim_{m\to\infty} \lambda_i(A_m) = \infty, \quad i = 1, ..., l_1, \\
&\lim_{m\to\infty} \lambda_i(A_m) = \theta_i \in (-\infty, \infty), \quad i = l_1 + 1, ..., l_2, \\
&\lim_{m\to\infty} \lambda_i(A_m) = -\infty, \quad i = l_2 + 1, ..., n, \\
&a = \min(-1, \theta_{l_2} - 1), \quad b = \max(1, \theta_{l_1+1} + 1), \\
&\lim_{m\to\infty} U_{(b,\infty)}(A_m) = U_+, \\
&\lim_{m\to\infty} U_{(a,b)}(A_m) = U_0, \\
&\lim_{m\to\infty} U_{(-\infty,a)}(A_m) = U_-, \\
&\lim_{m\to\infty} P_{(a,b)}(A_m) A_m P_{(a,b)} = E.
\end{aligned} \tag{3.5.3}$$

We remark that the above choice of the interval $(a, b) \supset (-1, 1)$ is somewhat arbitrary. Any other choice of a finite interval (a, b), which contains the points $\theta_{l_1+1}, ..., \theta_{l_2}$ (if such points exist), is a valid choice.

Proof. For simplicity of the exposition we assume that

$$dim\, U_+ = l_1 > 0, \quad dim\, U_0 = l_2 - l_1 > 0, \quad dim\, U_- = n - l_2 > 0.$$

We claim that for any $C \in \mathbf{H}^+(n, \mathbb{C})$

$$\log \sigma_i(C^{-1} e^{A_m}) = \lambda_i(A_m) + O(1), \quad i = 1, ..., n. \tag{3.5.4}$$

The case $i = 1$ follows straighforward from Proposition 3.5.2. Apply Proposition 3.5.2 to $\wedge_k(Be^A)$ for $k > 1$ to deduce

$$\sum_1^k \log \sigma_i(C^{-1} e^{A_m}) = \sum_{i=1}^k \lambda_i(A_m) + O(1).$$

Hence (3.5.4) holds for any sequence $\{A_m\}_1^\infty \in \mathbf{H}(n, \mathbb{C})$. Assume that (3.5.3) holds. Then

$$\lim_{m\to\infty} \sigma_i(C^{-1} e^{A_m}) = \infty, \quad i = 1, ..., l_1,$$
$$\lim_{m\to\infty} \sigma_i(C^{-1} e^{A_m}) = -\infty, \quad i = l_2 + 1, ..., n.$$

Let $A \in S_{n,1}$ and assume that the decompositions (3.5.1) and (3.5.2) coincide. We claim that

$$\lim_{m\to\infty} \sum_1^{l_1} |\log \sigma_i(C^{-1} e^{A_m})| - \sum_1^{l_1} \lambda_i(A_m) = \alpha_{l_1}(A, C),$$
$$\lim_{m\to\infty} \sum_{l_2+1}^{n} |\log \sigma_i(C^{-1} e^{A_m})| + \sum_{i=l_2+1}^{n} \lambda_i(A_m) = \alpha_n(A, C) - \alpha_{l_2}(A, C). \tag{3.5.5}$$

The first formula of (3.5.5) is deduced by considering the norm $|| \wedge_{l_1} C^{-1} \wedge_{l_1} A_m||_2$, as in the proof of Theorem 3.3.3. One has to notice that the ratio of a nonmaximal eigenvalue of $\wedge_{l_1} e^{A_m}$ to the maximal eigenvalue $e^{\lambda_1(A_m) + \cdots + \lambda_{l_1}(A_m)}$ of $\wedge_{l_1} e^{A_m}$ converges to 0. The second formula of (3.5.5) is deduced by using the same arguments for the sequence of the inverse matrices $e^{-A_m} C$.

Assume in addition that for a big enough N

$$\lambda_i(A_m) = 0 \quad \text{for} \quad i = l_1 + 1, ..., l_2 \quad \text{and} \quad m > N. \tag{3.5.6}$$

Repeat the arguments of the proof of Theorem 3.3.4 for $p = 1$ to deduce that $\{e^{A_m}\}_1^\infty$ converges to ξ, the end of the ray $e^{At}, t > 0$. Note that $E = 0$.

We now consider the general case. Let

$$E_m := P_{(a,b)}(A_m) A_m P_{(a,b)}(A_m)$$
$$E_m \in \mathbf{H}(P_{(a,b)}(A_m)\mathbb{C}^n)$$
$$\lim_{m\to\infty} E_m = E \in \mathbf{H}(U_0),$$
$$A'_m := A_m - E_m,$$
$$m = 1, ..., .$$

Then the sequence $\{A'_m\}_1^\infty$ satisfies (3.5.6). From the definition of E_m it follows that $A_m E_m = E_m A_m$. Hence

$$d_1(e^{A_m}, B) = d_1(e^{A'_m}, e^{-E_m}B),$$
$$b_1(e^{A_m}, B, C) = b_1(e^{A'_m}, e^{-E_m}B, e^{-E_m}C),$$
$$m = 1, ..., .$$

Here $e^{-E_m}B$ is viewed as a point in $\mathbf{X}_n$. The above arguments show that

$$\lim_{m\to\infty} b_1(e^{A_m}, B, C) = b_1(\xi, e^{-E}B, e^{-E}C). \tag{3.5.7}$$

This shows that any sequence $\{A_m\}_1^\infty \subset \mathbf{H}(n, \mathbb{C})$ satisfying the conditions (3.5.3) converges to a boundary point (U_+, E, U_-). A straightforward argument shows that two different elements $(U_+, E, U_-), (U'_+, E', U'_-)$ induce two different Busemann functions. Hence the above two points in $\mathbf{X}_{n,1}(\infty)$ are distinct. Given a decomposition (3.5.2) and $E \in \mathbf{H}(U_0)$ it is straightforward to construct a sequence $\{A_m\} \in \mathbf{H}(n, \mathbb{C})$ which satisfies the conditions (3.5.3) for the given triple (U_+, E, U_-). Hence any allowed triple (U_+, E, U_0) is in $\mathbf{X}_{n,1}(\infty)$. Finally, for a given unbounded sequence $\{G_l\}_1^\infty \subset \mathbf{H}(n, \mathbb{C})$ there exists a subsequence $\{A_m\}_1^\infty$ satisfying the conditions (3.5.3). Hence all allowable triples (U_+, E, U_-) form $\mathbf{X}_{n,1}(\infty)$. $\square$

The conditions (3.5.3) are not the minimal conditions. For example, the convergence of any two sequences of subspaces $\{U_{+,m}\}_{m=1}^\infty, \{U_{0,m}\}_{m=1}^\infty, \{U_{-,m}\}_{m=1}^\infty$, where $\mathbb{C}^n = U_{+,m} \oplus U_{0,m} \oplus U_{-,m}$ is an orthogonal decomposition of $\mathbb{C}^n$ for each m, implies the convergence of the third sequence of subspaces. Also, the convergence to E implies the convergence of the eigenvalue sequences $\{\lambda_i(A_m)\}_{m=1}^\infty$ for $i = l_1+1, ..., l_2$. We tried to make these conditions more transparent. The equality (3.5.7) shows how to obtain the Busemann function corresponding to (U_+, E, U_-) from the basic case given by the flag $U_+ \subset U_+ \oplus U_0 \subset \mathbb{C}^n$, which corresponds to the end of the ray $e^{tA}, t > 0$.

3.6 Remarks

Let SR be a symmetric Riemannian space SR. Thus $SR = G/K$, where G a connected semisimple Lie group, with finite center such that all its simple factors are noncompact, and K is a maximal compact subgroup of G. Let g be the Lie algebra of G, and let $\rho : g \to gl(V)$ be a faithful irreducible representation on a finite dimensional complex Lie algebra over the vector space V of dimension m. Then one obtains the corresponding irreducible representation $\rho : SR \to \mathbf{H}_1^+(m, \mathbb{C})$, where

$$\mathbf{H}_1^+(m, \mathbb{C}) := \mathbf{H}^+(m, \mathbb{C}) \cap \mathbf{SL}(m, \mathbb{C}).$$

p-Busemann compactification of SR is the compactification of $\rho(SR)$ in $\mathbf{X}_{m,p}$. Nonvisual compactifications arise only for $p = 1, \infty$. The most natural compactification is for the adjoint representation of SR. We will study later the 1-Busemann compactification of the Siegel upper half plane.

Recall the classical Satake compactification [Sat]. Let $\mathbf{PH}(m, C)$ be the projective space of all nonzero $m \times m$ Hermitian matrices. That is $\mathbf{PH}(m, C)$ are the real rays in $\mathbf{H}(m, \mathbb{C}) - \{0\}$. Note that $\mathbf{H}_1^+(m, \mathbb{C}) \subset \mathbf{PH}(m, C)$. Equvalently, $\mathbf{PH}(m, \mathbb{C})$ can be identified with the p-unit ball $S_{m,p}$ for any $p \in [1, \infty]$. A sequence $\{A_i\}_1^\infty \subset \mathbf{H}_1^+(m, \mathbb{C})$ converges in $\mathbf{PH}(m, C)$ iff the sequence $\{\frac{A_i}{||A_i||_p}\}_1^\infty$ converges. The boundary of $\mathbf{H}_1^+(m, \mathbb{C})$ are all singular nonnegative definite matrices of (p) norm one. The classical Satake compactification of RS is obtained by taking the compactification of $\rho(S)$ in $\mathbf{PH}(m, C)$. A special case of the Satake compactification is the Furstenberg compactification [Fur], which was given originally in terms of certain probability measures. It was shown by Moore [Moo] that Furstenberg compactification is the Satake compactification for a suitable choice of the representation ρ.

In what follows we are interested in the smallest part of the above compactifications, which can be described as follows. Fix a point $0 \in SR$ which corresponds to the coset K in G/K. Consider a Weyl chamber through 0. It corresponds to an element $A \in \mathbf{H}(m, \mathbb{C})$ given by the adjoint representation ρ. Consider a geodesic $e^{tA} \in \mathbf{H}_1^+(m, \mathbb{C})$ and let ξ be boundary point with respect to one of the above compactifications (Satake or p-Busemann for $p = 1, \infty$). It can be shown that ξ is independent of the choice of an element A in the Weyl chamber. We call the above boundary points as Satake or p-Busemann boundaries $(p = 1, \infty)$ respectively:

Definition 3.6.1 *The $1(\infty)$-Busemann boundary is the set points ξ obtained as the limit points of the geodesic rays e^{tA} with respect to the metric $d_1(d_\infty)$, where A ranges over all elements in the interior of the Weyl chambers in the Cartan subalgebra of the Lie aglebra of G.*

It can be shown that these boundaries are closed sets in the closure of SR with respect to Satake or p-Busemann compactifications respectively. These boundaries are invariant under the action of G. In the special case of the Furstenbeg compactification we obtain the Furstenberg boundary. The Furstenberg boundary can be given as the quotient G/P. Here $P = MAN$ is the standard minimal parabolic subgroup of G determined by the Iwasawa decomposition $G = KAN$, where M is the centrilizer of A in K [Kai], [Alb].

It is useful to consider the example $\mathbf{H}_1^+(n, \mathbb{C}) = \mathbf{SL}(n, \mathbb{C})/\mathbf{SU}(n, \mathbb{C})$. The Satake boundary for the adjoint representation is the projective variety $\mathbb{P}^{n-1}$, which correspond to all one dimensional subspaces of $\mathbb{C}^n$. (Note that $\lim_{t\to\infty} \frac{e^{tA}}{||e^{tA}||_2}$ converges to a rank one matrix Hermitian matrix, for a $A \in \mathbf{H}(n, \mathbb{C})$ representing a Weyl chamber.) The ∞-Busemann boundary corresponds to two copies of $\mathbb{P}^{n-1}$. (Use the first two cases of (3.3.10) in Theorem 3.3.4.) The 1-Busemann boundary corresponds to all nontrivial 2 flags $U_+(A) \subset \mathbb{C}^n = U_+(A) \oplus U_-(A)$. The Furstenberg boundary corresponds to the space of all full flags

$$U_1 \subset \ldots \subset U_n = \mathbb{C}^n .$$

4 The Siegel upper half plane

4.1 Models

Let $F = \mathbb{R}, \mathbb{C}$. Denote by $\mathbf{Sym}(n, F) \subset \mathbf{M}(n, F)$ the subspace of $n \times n$ symmetric matrices. Recall the definition of the *symplectic group* $\mathbf{Sp}(n, F)$ as a subgroup of $\mathbf{GL}(2n, F)$. Let

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbf{SL}(2n, \mathbb{R}).$$

Then

$$\mathbf{Sp}(n, F) := \{M \in \mathbf{GL}(2n, F) : \quad M^{\mathrm{T}} J_n M = J_n\}.$$

We will use J for J_n when no ambiguity arises. Partition $M \in \mathbf{M}(2n, F)$ into 2×2 block matrices as J:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4.1.1}$$

Then $M \in \mathbf{Sp}(n, F)$ if and only if

$$A^{\mathrm{T}} C \text{ and } B^{\mathrm{T}} D \text{ are symmetric and } A^{\mathrm{T}} D - C^{\mathrm{T}} B = I_n. \tag{4.1.2}$$

Furthermore, if $M \in \mathbf{Sp}(n, F)$ then $M^{\mathrm{T}} \in \mathbf{Sp}(n, F)$ and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^{\mathrm{T}} & -B^{\mathrm{T}} \\ -C^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix}. \tag{4.1.3}$$

More arguments are needed to show that $\mathbf{Sp}(n, F) \leq \mathbf{SL}(2n, F)$ (see [Sie], [FuH] or [Fre]). Clearly, $\mathbf{Sp}(1, F) = \mathbf{SL}(2, F)$. In what follows we restrict ourselves to the case $F = \mathbb{R}$. It is well known that

$$\mathbf{K}_n := \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{SO}(2n, \mathbb{R})$$

is a maximal compact group in $\mathbf{Sp}(n, \mathbb{R})$ [Hel]. Then

$$\mathbf{Y}_n := \mathbf{Sp}(n, \mathbf{R}) / \mathbf{K}_n$$

is a symmetric space. Clearly $\mathbf{Y}_n$ is a subspace of $\mathbf{X}_{2n}$ discussed in previous sections. By restricting the metric d_p to $\mathbf{Y}_n$ we obtain the complete metric space $\mathbf{Y}_{n,p}$. In particular, $\mathbf{Y}_{n,2}$ is a Riemannian manifold. Obviously,

$$\mathbf{Y}_1 = \mathbf{SL}(2, \mathbb{R}) / \mathbf{SO}(2, \mathbb{R}).$$

Let $\mathbf{W}_1, \mathbf{W}_2$ be two metric spaces with the metrics δ_1, δ_2 respectively. We call $\mathbf{W}_1, \mathbf{W}_2$ *scaled* isometrically, if there exists a bijection $\iota : \mathbf{W}_1, \mathbf{W}_2$ so that

$$\delta_1(u, v) = c\, \delta_2(\iota(u), \iota(v)), \quad \forall u, v \in \mathbf{W}_1$$

and some $c>0$. We call ι a scaled isometry. It is known that $\mathbf{Y}_{1,2}$ is scaled isometrically to the standard hyperbolic upper half plane $\mathbf{H}^2$. Similarly we will show that $\mathbf{Y}_{n,2}$ is scaled isometrically to $\mathbf{SH}_n$, the $n-th$ Siegel upper half plane:

$$\mathbf{SH}_n := \{X+\sqrt{-1}Y \in \mathbf{Sym}(n,\mathbb{C}) : \quad X,Y \in \mathbf{Sym}(n,\mathbb{R}) \quad \text{and} \quad Y>0\}.$$

We denote by $ds(Z,W)$ the Siegel distance between $Z,W \in \mathbf{SH}_n$. Note note that $ds(z,w)$ is the hyperbolic distance $dh(z,w)$ between $z,w \in \mathbf{H}^2$. Recall that $\mathbf{Sp}(n,\mathbb{R})$ acts on $\mathbf{SH}_n$ as follows [Sie]: For

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(n,\mathbb{R}) Z \in \mathbf{SH}_n$$

$$M(Z) := (AZ+B)(CZ+D)^{-1}. \tag{4.1.4}$$

We will call these maps *generalized Möbius transformations*. Here, like in the 2-dimensional upper half plane, the matrices M and $-M$ have the same action. It is shown in [Sie] that the projective symplectic group

$$\mathbf{PSp}(n,\mathbb{R}) := \mathbf{Sp}(n,\mathbb{R})/\{\pm I_{2n}\}$$

is equal to the group of of biholomorphisms of $\mathbf{SH}_n$. Furthermore, $\mathbf{PSp}(n,\mathbb{R})$ acts as a subgroup of isometries with respect to the Siegel metric [Sie]. Let

$$\mathbf{DH}_n := \mathbf{D}(n,\mathbb{C}) \cap \mathbf{SH}_n = \mathbf{D}(n,\mathbb{R}) + \sqrt{-1}\mathbf{D}^+(n,\mathbb{R}).$$

Clearly

$$D(z) \in \mathbf{DH}_n \qquad \Longleftrightarrow z=(z_1,...,z_n) \in \mathbf{H}^2 \times \cdots \times \mathbf{H}^2.$$

Thus $\mathbf{DH}_n \sim (\mathbf{H}^2)^n$. Then

$$ds(D(z),D(w)) = (\sum_{j=1}^n dh(z_j,w_j)^2)^{\frac{1}{2}}, \quad D(z),D(w) \in \mathbf{DH}_n.$$

Hence

$$ds(\sqrt{-1}I_n, \sqrt{-1}D(x)) = (\sum_{j=1}^n \log^2 x_j)^{\frac{1}{2}}, \quad D(x) \in \mathbf{D}^+(n,\mathbb{R}). \tag{4.1.5}$$

Theorem 4.1.1 *$\mathbf{SH}_n$ and $\mathbf{Y}_{n,2}$ are scaled isometrically Riemannian manifolds. More precisely, Let $\phi_1 : \mathbf{SH}_n \to \mathbf{Sp}(n,\mathbb{R})$ be the map*

$$\phi_1(X+\sqrt{-1}Y) := \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y^{-1}} \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}. \tag{4.1.6}$$

Then the map

$$\begin{array}{rccc} \Phi_1 & : & \mathbf{SH}_n & \to & \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n \\ & & Z & \mapsto & \phi_1(Z)\mathbf{K}_n \end{array}$$

is a bijection. Furthermore,

$$ds(Z,W) = \sqrt{2} d_2(\Phi_1(Z), \Phi_1(B)), \quad Z,W \in \mathbf{SH}_n. \tag{4.1.7}$$

Proof. According to Siegel [Sie] $\mathbf{Sp}(n,\mathbb{R})$ acts transitively on $\mathbf{SH}_n$. A straightforward computation shows that

$$\mathrm{Stab}(\sqrt{-1}I_n) := \{M \in \mathbf{Sp}(n,\mathbb{R}): \quad M(\sqrt{-1}I_n) = \sqrt{-1}I_n\}$$

is equal to $\mathbf{K}_n$:

$$\mathrm{Stab}(\sqrt{-1}I_n) = \mathbf{K}_n. \tag{4.1.8}$$

Hence $\mathbf{Y}_n$ is isomorphic to $\mathbf{SH}_n$. Note that $\psi_1(X+\sqrt{-1}Y)$ given as a product of two symplectic matrices. Hence $\phi_1(X+\sqrt{-1}Y) \in \mathbf{Sp}(n,\mathbb{R})$. Clearly, $\Phi_1(\sqrt{-1}I_n) = \mathbf{K}_n$. To show that Φ_1 is bijection, it is enough to show that Φ_1 is an injection. Suppose that

$$\Phi_1(X_1+\sqrt{-1}Y_1) = \Phi_1(X_2+\sqrt{-1}Y_2).$$

That is

$$M := \phi_1(X_1+\sqrt{-1}Y_1)^{-1}\phi_1(X_2+\sqrt{-1}Y_2) \in \mathbf{K}_n.$$

Clearly M an upper block diagonal symplectic matrix. Hence M^{-1} is also a block upper triangular matrix. As M is an orthogonal matrix we deduce that $M^{-1} = M^{\mathrm{T}}$ is a lower block triangular matrix. Hence M must be block diagonal matrix, whose each diagonal block is a product of two positive definite matrices

$$\sqrt{Y_1^{-1}}\sqrt{Y_2}, \quad \sqrt{Y_1}\sqrt{Y_2^{-1}}.$$

Note that the eigenvalues of each of the matrices above are real positive numbers. Since M is orthogonal we deduce that $M = I_{2n}$. This implies that $X_1 = X_2, Y_1 = Y_2$. Furthermore, Φ_1 commutes with the action of $\mathbf{Sp}(n,\mathbb{R})$ on $\mathbf{SH}_n$. (For more details see [F].) To prove (4.1.7) we consider the following the Siegel distance between $\sqrt{-1}I_n, \sqrt{-1}D(x)$ given by (4.1.5). Then

$$\phi_1(\sqrt{-1}I_n) = I_{2n}, \quad \phi_1(\sqrt{-1}D(x)) = \mathrm{diag}(\sqrt{D(x)}, \sqrt{D(x)^{-1}}).$$

Clearly, the singular valus of $\phi_1(\sqrt{-1}D(x))$ are $\sqrt{x_j}, \sqrt{x_j^{-1}}, j = 1,...,n$. Use the definition of d_2 (Lemma 2.1.3) to deduce that (4.1.7) holds for $Z = \sqrt{-1}I_n, W = \sqrt{-1}D(x)$. To deduce (4.1.7) for any pair Z, W we recall that $\mathbf{Sp}(n,\mathbb{R})$ acts a subgroup of isometries on $\mathbf{Y}_{n,2}$ and $\mathbf{SH}_n$ respectively. Accoriding to Siegel any pair of points $Z_1, Z_2 \in \mathbf{SH}_n$ there exists $M \in \mathbf{Sp}(n,\mathbb{R})$ so that

$$M(Z_1) = \sqrt{-1}I_n, \quad M(Z_2) = \sqrt{-1}D(x) \tag{4.1.9}$$

for some $x \in \mathbb{R}^n_+$. (We give an independent proof of this fact later on.) Hence (4.1.7) holds. $\square$

The next model for $\mathbf{Y}_n$ is the Siegel disk

$$\mathbf{SD}_n := \{Z \in \mathbf{Sym}(n,\mathbb{C}): \quad I - Z\overline{Z} > 0\}.$$

This is a generalization of the unit disk $\mathbf{D}$, since the condition $I - Z\overline{Z} > 0$ can be rewritten as $||Z||_2 < 1$. There are two complex symplectic maps connecting these two models, namely

$$\begin{array}{rcccc} \Phi_2 & : & \mathbf{SH}_n & \to & \mathbf{SD}_n \\ & & Z & \mapsto & (Z - \sqrt{-1}I_n)(Z + \sqrt{-1}I_n)^{-1} \end{array}$$

and

$$\begin{array}{rcccc} \Phi_2^{-1} & : & \mathbf{SD}_n & \to & \mathbf{SH}_n. \\ & & Z & \mapsto & \sqrt{-1}(I_n + Z)(I_n - Z)^{-1} \end{array}$$

These maps can be expressed as rational transformations given by (4.1.4), associated with the matrices

$$\left(\begin{array}{cc} I_n & -\sqrt{-1}I_n \\ I_n & \sqrt{-1}I_n \end{array}\right), \left(\begin{array}{cc} \sqrt{-1}I_n & \sqrt{-1}I_n \\ -I_n & I_n \end{array}\right) \in \mathbf{Sp}(n, \mathbb{C}), \tag{4.1.10}$$

respectively. Recall the definition of $\mathbf{SU}(n,n)$:

$$\mathbf{SU}(n,n) := \{M \in \mathbf{SL}(2n, \mathbb{C}) : \quad M^* \operatorname{diag}(I_n, -I_n)M = \operatorname{diag}(I_n, -I_n)\}.$$

Then all biholomorphisms of $\mathbf{SD}_n$ are of the form (4.1.4) where M belongs to the subgroup

$$\mathbf{Sp}(n, \mathbb{R})' := \mathbf{Sp}(n, \mathbb{C}) \cap \mathbf{SU}(n,n). \tag{4.1.11}$$

See [Sie] and [Hel]. Let

$$\operatorname{Stab}(0) := \{M \in \mathbf{Sp}(n, \mathbb{R})' : \quad M(0) = 0\}.$$

A straightforward computation shows that Stab(0) is isomorphic to $\mathbf{U}(n)$:

$$M \in \operatorname{Stab}(0) \quad \iff \quad M(Z) = UZU^{\mathrm{T}}, \quad U \in \mathbf{U}(n), \quad Z \in \mathbf{SD}_n. \tag{4.1.12}$$

The classical result of Schur [Sch] (see also [Fr2]) states:

Lemma 4.1.2 *Let $Z \in \mathbf{Sym}(n, \mathbb{C})$. Then there exists a unitary $U \in \mathbf{U}(n)$ so that $Z = U\Sigma(Z)U^{\mathrm{T}}$.*

Corollary 4.1.3 *Let $W_1, W_2 \in \mathbf{SD}_n$. Then there exists $M' \in \mathbf{Sp}(n, \mathbb{R})'$ such that*

$$M'(W_1) = 0, \quad M'(W_2) = D(y), \quad y = (y_1, ..., y_n), \quad 1 > y_1 \geq \ldots \geq y_n \geq 0.$$

Suppose furthermore that $Z_1, Z_2 \in \mathbf{SH}_n$. Then there exists $M \in \mathbf{Sp}(n, \mathbb{R})$ such that (4.1.9) *holds.*

Proof. Since $\mathbf{Sp}(n, \mathbb{R})'$ acts trasitively on $\mathbf{SD}_n$ there exists $M_1 \in \mathbf{Sp}(n, \mathbb{R})'$ so that $M_1(W_1) = 0$. Use Schur's lemma to deduce that there exits $M_2 \in \operatorname{Stab}(0)$ so that $M_2M_1(W_2) = \Sigma(M_1(W_2)) = D(y)$. Use the biholomorphisms Φ_2, Φ_2^{-1} to deduce the corollary for Z_1, Z_2. $\square$

Siegel metric on $\mathbf{SD}_n$ is defined uniquely by assuming that Φ_2 is an isometry (with respect to the Siegel metric defined on $\mathbf{SH}_n$). Hence $\mathbf{Sp}(n,\mathbb{R})'$ is a subgroup of the group of isometries with respect to Siegel metric on $\mathbf{SD}_n$. As in the case $\mathbf{SH}_n$ Siegel metric on $\mathbf{SD}_n$ can be constructed as follows. Clearly, $\mathbf{SD}_n \cap \mathbf{D}(n,\mathbb{C})$ is isomorphic to $\mathbf{D}^n$. Then Siegel metric on $\mathbf{SD}_n \cap \mathbf{D}(n,\mathbb{C})$ is given by the the Riemannian metric on $\mathbf{D}^n$ induced (in the standard way) by the hyperbolic metric on $\mathbf{D}$. The assumption that $\mathbf{Sp}(n,\mathbb{R})'$ acts isometrically determine uniquely Siegel metric on $\mathbf{SD}_n$.

We now consider the projective model of $\mathbf{Y}_n$. Let $F = \mathbb{R},\mathbb{C}$. Consider the Grassmannian $G_{2n,n}F$, which is the variety of all n-dimensional subspaces of F^{2n}. Denote by $\mathbf{M}(2n,n;n;F)$ all $2n \times n$ matrices of maximal rank n. Let $A \in \mathbf{M}(2n,n;n;F)$ and view the columns of A as a basis of a subspace of F^{2n}. Denote by $[A]$ the n-dimesnional subspace spanned by the columns of A. Note $[B] = [A]$ if and only if $B \in A\mathbf{GL}(n,F)$. Hence

$$G_{2n,n}F = \mathbf{M}(2n,n;n,F)/\mathbf{GL}(n,F).$$

Let $S^o_{2n,n}F$ be the following quasiprojective variety in $G_{2n,n}F$:

$$S^o_{2n,n}F : \{[A] : \quad A = \begin{pmatrix} Z \\ I_n \end{pmatrix}, \quad Z \in \mathbf{Sym}(n,F).\} \tag{4.1.13}$$

The model for $\mathbf{Y}_n$ is be the set of all n dimensional subspaces of $S^o_{2n,n}\mathbb{C}$ that admit as a representative a matrix A of the above type with $Z \in \mathbf{SH}_n$. We denote this set by $\mathbf{SPH}_n$. Then $\mathbf{Sp}(n,\mathbb{R})$ acts on $\mathbf{SPH}_n$ by a left matrix multiplication:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z \\ I_n \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} = \begin{bmatrix} (AZ+B)(CZ+D)^{-1} \\ I_n \end{bmatrix}.$$

It's trivial to see that the action is well defined. The map connecting $\mathbf{SH}_n$ to $\mathbf{SPH}_n$ is clearly

$$\begin{array}{rccc} \Phi_3 : & \mathbf{SH}_n & \to & \mathbf{SPH}_n \\ & Z & \mapsto & \begin{bmatrix} Z \\ I_n \end{bmatrix} \end{array}$$

which is a 1-1 map. This model and the action are studied in a more general setting in [SZ]. (Note that the projective model explains the form of the birational transformation of (4.1.4).)

It is also useful to consider another projective model related to this one. Take the set $\wedge_n\mathbf{SPH}_n := \{\wedge_n W : [W] \in \mathrm{SPH}_n\}$ with the identification $v = u$ if and only if there exists a nonzero complex number z such that $v = uz$. This is a subset of the projective space $\mathbb{CP}^{N-1}$, $N = \binom{2n}{n}$. The action is defined as left multiplication by $\wedge_n M$: for $M \in \mathbf{Sp}(n,\mathbb{R})$ and $v \in \wedge_n\mathbf{SPH}_n$, the action is $[v] \mapsto [\wedge_n Mv]$. Notice that if V and V' are two representatives of the same class in $G_{2n,n}\mathbb{C}$, then $V' = VU$, for some $U \in \mathbf{GL}(n,\mathbb{C})$. Then $\wedge_n V' = (\wedge_n V) \cdot \mathrm{detU}$, since $\wedge_n U = \mathrm{detU}$. This allows us to write $[\wedge_n V] = \wedge_n[V]$, and we have a well defined map from $\mathbf{SPH}_n$ to $\wedge_n\mathbf{SPH}_n$ given by $[V] \mapsto [\wedge_n V]$.

We now see that this map gives a 1-1 correspondence between these last two models. A class in $\mathbf{SPH}_n$ is determined by the span of the columns of any of its representatives, so if $[V] \neq [W]$, $W, V \in \mathbf{M}(2n,n,\mathbb{C})$, then the column spans of V and W are not the same, and in this case it is well known that $<\wedge_n V> \neq <\wedge_n W>$, and $[\wedge_n V] \neq [\wedge_n W]$ in $\mathbb{CP}^{N-1}$.

4.2 Compactifications

We start by compactifying the bounded domain model, by taking the closure of $\mathbf{SD}_n$:

$$\mathrm{Cl}(\mathbf{SD}_n) = \{Z \in \mathbf{M}(n, \mathbb{C}) : \quad I - Z\overline{Z} \geq 0\}.$$

Our first remark is that this space has a stratification of the boundary. The strata are

$$\partial_k \mathbf{SD}_n = \{Z \in \partial \mathbf{SD}_n : \quad \mathrm{rank}(I - Z\overline{Z}) = n - k\}.$$

This can be written in terms of singular values as:

$$\partial_k \mathbf{SD}_n = \{Z \in \partial \mathbf{SD}_n : \quad \sigma_1(Z) = \ldots = \sigma_k(Z) = 1 > \sigma_{k+1}(Z)\}$$

for $k \leq n - 1$, and

$$\partial_n \mathbf{SD}_n = \{Z \in \partial \mathbf{SD}_n : \quad \sigma_1(Z) = \cdots = \sigma_n(Z) = 1\}.$$

Let

$$\mathbf{USym}(n) := \partial_n \mathbf{SD}_n = \mathbf{U}(n) \cap \mathbf{Sym}(n, \mathbb{C}).$$

The group acting on $\mathbf{SD}_n$ is $\mathbf{Sp}(n, \mathbb{R})'$ which is a conjugate of $\mathbf{Sp}(n, \mathbb{R})$ in $\mathbf{Sp}(n, \mathbb{C})$. The quotient of this group by the subgroup $\{\pm I_{2n}\}$ is the biholomorphism group of $\mathbf{SD}_n$. The action of $\mathbf{Sp}(n, \mathbb{R})'$ extends to $\mathrm{Cl}(\mathbf{SD}_n)$. The following result is a particular case of the general result about boundary components of bounded symmetric domains as described in [Bai, p. 200].

Proposition 4.2.1 *Each stratum of $\partial \mathbf{SD}_n$ is an orbit for the action of $\mathbf{Sp}(n, \mathbb{R})'$.*

We will bring a short proof of the above proposition later in this section. It is useful to consider a similar compactification $\mathbf{SH}_n$. Let

$$\mathrm{Cl}(\mathbf{SH}_n) = \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \mathrm{Im}(Z) \geq 0\},$$

be the closure of the Siegel upper half plane in $\mathbf{Sym}(n, \mathbb{C})$. We call the boundary of $\mathrm{Cl}(\mathbf{SH}_n)$ as a *finite* boundary of $\mathbf{SH}_n$ and denote it by

$$\mathrm{fin}(\partial \mathbf{SH}_n) := \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \quad \mathrm{Im}(Z) \geq 0 \textit{ and } \mathrm{rank}(\mathrm{Im}\, Z) < n\}.$$

Clearly, we have the following stratification of the finite boundary:

$$\mathrm{fin}(\partial_k \mathbf{SH}_n) = \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \mathrm{Im}(Z) \geq 0 \textit{ and } \mathrm{rank}(\mathrm{Im}\, Z) = n - k\}, \quad k = 1, ..., n-1.$$

A straightforward calculation [Fre] shows:

Proposition 4.2.2

$$\Phi_2(\mathrm{fin}(\partial_k \mathbf{SH}_n)) \subset \partial_k \mathbf{SD}_n, \quad k = 1, ..., n-1.$$

The complete compactification of $\mathbf{SH}_n$ is achieved as follows. Consider the projective model $\mathbf{SPH}_n \subset G_{2n,n}\mathbb{C}$. Since $G_{2n,n}\mathbb{C}$ is a compact projective variety we can compactify $\mathbf{SPH}_n$ by considering $\mathrm{Cl}(\mathbf{SPH}_n)$. We identify $\partial\mathbf{SPH}_n$ with $\partial\mathbf{SH}_n$.

Lemma 4.2.3 *The compactification of* $\mathbf{SPH}_n$ *is equivalent to the compactification of* $\mathbf{SD}_n$ *as a bounded domain. Furthermore, the finite boundary of* $\mathbf{SH}_n$ *correspond to the set of all equivalence classes that admit a representative of the type*

$$\begin{pmatrix} Z \\ I \end{pmatrix} \text{ with } Z \text{ symmetric and } \operatorname{Im} Z \geq 0,$$

and such a representative is unique. Moreover, let Z_1, Z_2 *be points in the finite boundary of* $\mathbf{SH}_n$ *such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z_1 \\ I \end{bmatrix} = \begin{bmatrix} Z_2 \\ I \end{bmatrix}.$$

Then $CZ_1 + D$ *is invertible.*

Proof. Let $\mathbf{SPD}_n \subset S^o_{2n,n}\mathbb{C}$ be the projective model of $\mathbf{SD}_n$ given by representatives A in (4.1.13) with $Z \in \mathbf{SD}_n$. Clearly, $\mathrm{Cl}(\mathbf{SD}_n)$ is represented by $\mathrm{Cl}(\mathbf{SPD}_n)$, which is given by representatives A in (4.1.13) with $Z \in \mathrm{Cl}(\mathbf{SD}_n)$. Let P_2, P_2^{-1} be the complex symplectic matrices which are given in (4.1.10). Recall that P_2 is a complex automorphism of $G_{2n,n}\mathbb{C}$. Furthermore, P_2 is a biholomorphism of $\mathbf{SPH}_n$ and $\mathbf{SPD}_n$. Hence P_2 extends to a homeomorphism of $\partial\mathbf{SPH}_n$ and $\partial\mathbf{SPD}_n$. Other claims of the lemma are straightforward. $\square$

This compactification of $\mathbf{SH}_n$ is called the compactification of $\mathbf{SH}_n$ as a bounded domain. For simplicity of the exposition we view this compactification of $\mathbf{SH}_n$ obtained by extending the biholomorphism Φ_2^{-1} to $\partial\mathbf{SD}_n$. This is done by adding an additional boundary part to $\mathrm{fin}(\partial\mathbf{SH}_n)$. We call this part of the compactification of $\mathbf{SH}_n$ as the *infinite* boundary of $\mathbf{SH}_n$. Clearly, the infinite boundary of $\mathbf{SH}_n$ corresponds to the set

$$\{Z \in \partial\mathbf{SD}_n : \quad \det(\mathrm{Z} - \mathrm{I}) = 0\}, \tag{4.2.1}$$

where Φ_2^{-1} is not defined.

Proposition 4.2.4 *Let* G *be a subgroup of* 2×2 *block upper triangular matrices in* $\mathbf{Sp}(n, \mathbb{R})$. *Then* G *is generated by translations and congruencies:*

$$\begin{aligned} Z \mapsto T(Z) = Z + B, \quad & B \in \mathbf{Sym}(\mathbb{R}, n), \\ T = & \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \\ Z \mapsto Q(Z) = AZA^{\mathrm{T}}, \quad & A \in \mathbf{GL}(n, \mathbb{R}), \\ Q = & \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix}. \end{aligned}$$

G stablizes each $\mathrm{fin}(\partial_k\mathbf{SH}_n)$. *Furthermore,* G *acts transitively on* $\mathbf{SH}_n$ *and on each* $\mathrm{fin}(\partial_k\mathbf{SH}_n)$.

Proof. Clearly, translations and congruencies are block upper triangular elements of $\mathbf{Sp}(n,\mathbb{R})$. Let M be a block upper upper triangular matrix in $\mathbf{Sp}(n,\mathbb{R})$. Then M is of the form (4.1.1) with $C=0$. The third condition of (4.1.2) yields that $D=(A^{\mathrm{T}})^{-1}$. Let $Q\in\mathbf{Sp}(n,\mathbb{R})$ be defined as above. Then MQ^{-1} is equal to a translation T. Clearly, G fixes each $\mathrm{fin}(\partial_k\mathbf{SH}_n)$.

Let $Z=X+\sqrt{-1}Y\in\mathrm{Cl}(\mathbf{SH}_n)$. Then there exists a translation T so that $T(Z)=\sqrt{-1}Y$. The Sylvester law of inertia implies that any $Y\in\mathbf{Sym}(n,\mathbb{R}), Y\geq 0$ is congruent to the unique matrix

$$AYA^{\mathrm{T}}=\mathrm{diag}(I_k,0),\quad k=\mathrm{rank}(Y).$$

Hence G acts transitively on $\mathbf{SH}_n$ and on each $\mathrm{fin}(\partial_k\mathbf{SH}_n)$. $\square$

Proof of Proposition 4.2.1. Use (4.1.12) to deduce that the action of $Stab(0)$ stabilizes each $\partial_k\mathbf{SD}_n$. Let $W_1,W_2\in\partial_k\mathbf{SD}_n$. Use Lemma 4.1.2 to deduce the existence of $M'\in Stab(0)$ so that $M'(W_1)$ and $M'(W_2)$ do not satisfy (4.2.1). Proposition 4.2.2 implies

$$Z_1:=\Phi_2^{-1}M'(W_1),\quad Z_2:=\Phi_2^{-1}M'(W_2)\in\partial_k\mathbf{SH}_n.$$

Proposition 4.2.4 implies the existence of $M\in G$ so that $M(Z_1)=Z_2$. Use Proposition 4.2.2 to deduce the proposition. $\square$

Recall that $\mathbf{SD}_n$ is a complex manifold. The Shilov boundary of $\mathbf{SD}_n$ is the minimal closed subset of $S\subset\partial\mathbf{SD}_n$ with the following property: The maximum modulus of any continuous complex valued function f on $\mathrm{Cl}(\mathbf{SD}_n)$, which is analytic on $\mathbf{SD}_n$, is achieved on S. The following result is well known and we bring its short proof for completeness.

Proposition 4.2.5 $\mathbf{USym}(n)$ *is the Shilov boundary of* $\mathbf{SD}_n$.

Proof. Note that $\mathbf{SD}_n\cap\mathbf{D}(n,\mathbb{C})$ is equal to $\mathbf{D}^n$. Hence the Shilov boundary of $\mathbf{SD}_n\cap\mathbf{D}(n,\mathbb{C})$ is $(S^1)^n$, which is equal to $\mathbf{USym}(n)\cap\mathbf{D}(n,\mathbb{C})$. Let $f:\mathbf{SD}_n\to\mathbb{C}$ be a holomorphic function which extends to a continuous function on $\mathrm{Cl}(\mathbf{SD}_n)$. Then

$$|f(0)|\leq\max_{Z\in\mathbf{USym}(n)\cap\mathbf{D}(n,\mathbb{C})}|f(Z)|\leq\max_{Z\in\mathbf{USym}(n)}|f(Z)|.$$

As $\mathbf{Sp}(n,\mathbb{R})'$ acts transitively on $\mathbf{SD}_n$ and preserves $\mathbf{USym}(n)$, for any $W\in\mathbf{SD}_n$ we have

$$|f(W)|\leq\max_{Z\in\mathbf{USym}(n)}|f(Z)|.$$

$\square$

The following lemma is known and can be found in [Joh]. We bring its short proof since we need its argumentns later.

Lemma 4.2.6 *The Shilov boundary* $\mathbf{USym}(n)$ *of* $\mathbf{SD}_n$ *can be presented as a homogeneous space*

$$\mathbf{U}(n)/\mathbf{O}(n,\mathbb{R})\quad\sim\quad\mathbf{K}_n/\tilde{\mathbf{O}}(n,\mathbb{R}),$$

where $\tilde{\mathbf{O}}(n,\mathbb{R})<\mathbf{Sp}(n,\mathbb{R})$ *is the group matrices of the form* $diag(Q,Q)$ *with* $Q\in\mathbf{O}(n,\mathbb{R})$.

Proof. Recall that $\mathbf{U}(n) \sim \mathrm{Stab}(0)$ and the action on $\mathbf{U}(n)$ is given by (4.1.12). Use Lemma 4.1.2 to deduce that $\mathrm{Stab}(0)$ acts transitively on $\mathbf{USym}(n)$. Consider

$$\mathrm{Stab}_1(0) := \{M \in \mathrm{Stab}(0) : \quad M(I_n) = I_n\}.$$

Then $\mathrm{Stab}_1(0)$ is presented by all $U \in \mathbf{U}(n)$ such that $UI_nU^{\mathrm{T}} = I_n$. Hence $U^{-1} = U^{\mathrm{T}}$. Since U is unitary we deduce that $U^{\mathrm{T}} = U^*$. Hence U is real, i.e. $U \in \mathbf{O}(n, \mathbb{R})$, and $\mathbf{USym}(n) \sim \mathbf{U}(n)/\mathbf{O}(n, \mathbb{R})$. Similar arguments for the subgroup of $Stab(\sqrt{-1}I_n)$ which stabilizes $0 \in \partial_n \mathbf{SH}_n$ yields the second part of the lemma. □

The main result of this section is:

Theorem 4.2.7 *The compactification of* $\mathbf{SH}_n$ *as a bounded domain is equivalent to the compactification of* $\mathbf{Y}_{n,1}$ *with respect to the Busemann function* d_1

The proof of this theorem is given at the end of the next subsection.

4.3 Properties of symplectic matrices and applications

Let F be a field of characteristic 0. Let W be a vector field over F of dimension $2n$. Let (u, v) be a skew form on W. That is $(v, u) = -(u, v)$. $(\cdot, \cdot)$ is called nondegenerate if the linear functional $f : W \to F$, given by $f(x) = (x, u)$, $0 \neq u \in W$, is a nonzero functional. A symplectic basis $(e^1, \ldots, e^n, f^1, \ldots, f^n)$ in W satisfies

$$(e^j, f^k) = -(f^k, e^j) = \delta_{jk}, \quad (e^j, e^k) = (f^j, f^k) = 0 \text{ for all } j, k = 1, \ldots, n.$$

It is known that W has a symplectic basis. (See the following lemma.) A subspace $V \subset W$ is isotropic if for all u, v in the subspace, $(u, v) = 0$. An isotropic subspace is called Lagrangean if it has the maximal dimension n. Clearly, $\mathrm{span}(e^1, ..., e^n)$ and $\mathrm{span}(f^1, ..., f^n)$ are Lagrangian subspaces. A nontrivial subspace U is called nondegenerate if the restriction of the form $(\cdot, \cdot)$ to U is nondegenerate. Two subspaces $U, V \subset W$ are called skew orthogonal if $(u, v) = 0$ for every $u \in U, v \in V$. A decomposition of

$$W = \sum_{i=1}^{k} \oplus U_i \tag{4.3.1}$$

is called to an orthoskew decomposition if any two distinct subspaces U_i, U_j are orthogonal with respect to the given skew form. The following lemma is well known and we bring its short proof for completeness:

Lemma 4.3.1 *Let* $(\cdot, \cdot)$ *be a nondegenerate skew form on a vector space* W *of dimension* $2n$. *Then the following are equivalent:*
(a) (4.3.1) *is an orthoskew decomposition with* $U_i \neq \{0\}$ *for* $i = 1, ..., k$;
(b) *There exists a symplectic basis* $(e^1, \ldots, e^n, f^1, \ldots, f^n)$ *of* W *and* $k+1$ *integers*

$$j_0 = 0, \ 1 \leq j_1 < j_2 < \ldots < j_k = n$$

such that

$$U_i = \mathrm{span}(e^{j_{i-1}+1}, \ldots, e^{j_i}, f^{j_{i-1}+1}, \ldots, f^{j_i}), \quad i = 1, \ldots, k.$$

Proof. Clearly (b) implies (a). Assume (a). Choose a nonzero element $f^1 \in U_1$. As the linear functional $f(x) = (x, f^1)$ is a nonzero functional there exists an element $e^1 \in W$ so that $f(e^1) = (e^1, f^1) = 1$. Since (4.3.1) is an orthoskew decomposition it follows that we can assume that $e^1 \in U_1$. Let U_1' be the orthoskew complement of V_1 =span(e^1, f^1) in U_1. Clearly, $U_1 = V_1 \oplus U_1'$ is a orthoskew decomposition. Let

$$W_1 = U_1' \oplus \sum_{i=2}^{k} \oplus U_i.$$

It is straightforward to show that the restriction of $(\cdot, \cdot)$ to W_1 is a nondegenerate skew form. Use the induction to deduce the lemma. $\Box$

On $F^{2n} \Theta F^{2n}$ define a skew (symplectic) form as

$$(u, v) := u^{\mathrm{T}} J v.$$

Note that this skew form is nondegenerate. Then $M \in \mathbf{GL}(2n, F)$ is symplectic if and only if $(Mu, Mv) = (u, v)$ for all $u, v \in F^{2n}$. Furthermore, $M \in \mathbf{GL}(2n, F)$ is symplectic if and only if it is a change of basis matrix from one symplectic basis to another one. Let $A \in \mathbf{M}(m, \mathbb{C})$. Denote by $\mathrm{spec}(A) \subset C$ the spectrum of A:

$$\mathrm{spec}(A) := \{\lambda \in \mathbb{C} : \quad \det(\lambda I_m - A) = 0\}.$$

In what follows we need the following subsets of $\mathrm{spec}(A)$:

$$\begin{aligned}
\mathrm{spec}_{1^+}(A) &:= \{\lambda \in \mathrm{spec}(A) : \quad |\lambda| > 1\}, \\
\mathrm{spec}_{1^-}(A) &:= \{\lambda \in \mathrm{spec}(A) : \quad |\lambda| < 1\}, \\
\mathrm{spec}_{1}(A) &:= \{\lambda \in \mathrm{spec}(A) : \quad |\lambda| = 1\}. \\
\mathrm{spec}_{q}(A) &:= \{\lambda \in \mathrm{spec}(A) : \quad |\lambda| \le 1, \ \mathrm{Im}\,\lambda \ge 0\}.
\end{aligned}$$

For any set $L \subset \mathbb{C}$ let $P_L(A) \in \mathbf{M}(m, \mathbb{C})$ be the spectral projection on the generalized eigenspace of A associated with $L \cap \mathrm{spec}(A)$. See [Kat]. Note that if $L \cap \mathrm{spec}(A) = \emptyset$ then $P_L(A) = 0$. Furthermore, if $L = \overline{L}$ and $A \in \mathbf{M}(m, R)$ then $P_L(A) \in \mathbf{M}(m, \mathbb{R})$ and $\mathbb{C} \otimes P_L(A)\mathbb{R}^m = P_L(A)\mathbb{C}^m$. Suppose that $L \subset \mathbb{C}\backslash\{0\}$. Then

$$L^{-1} := \{z \in \mathbb{C} : \quad z^{-1} \in L\}.$$

Denote by $P_{1^+}(A), P_{1^-}(A), P_1(A)$ the spectral projections on $\mathrm{spec}_{1^+}(A), \mathrm{spec}_{1^-}(A), \mathrm{spec}_1(A)$ respectively. On $\mathbb{C}^m$ define a symmetric form

$$< u, v >= u^{\mathrm{T}} v, \quad u, v \in \mathbb{C}^m.$$

Note that on $\mathbb{R}^m$ this symmetric form is positive definite. The following proposition is well known and its proof can be deduced straightforward from the Jordan canonical form of $A \in \mathbf{M}(m, \mathbb{C})$:

Proposition 4.3.2 *Let $A \in \mathbf{M}(m, \mathbb{C})$, $L, L' \in \mathbf{C}, L \cap L' = \emptyset$. Then*

$$< u, v >= 0 \quad \text{for} \quad u \in P_L(A)\mathbb{C}^m \quad \text{and} \quad v \in P_{L'}(A^{\mathrm{T}})\mathbb{C}^m .$$

Proposition 4.3.3 *Let $M \in \mathbf{Sp}(n, \mathbb{C})$. Let $L, L_1 \subset \mathbb{C}\backslash\{0\}$ such that $L_1 \cap L^{-1} = \emptyset$. Then $P_L(M)\mathbb{C}^{2n}, P_{L_1}(M)\mathbb{C}^{2n}$ are skew orthogonal. Assume furthermore that $L \cap L^{-1} = \emptyset$. Then $P_L(M)\mathbb{C}^{2n}$ is an isotropic subspace. Suppose furthermore that $M \in \mathbf{Sp}(n, \mathbb{R})$ and $L = \overline{L}$. Then $P_L(M)\mathbb{R}^{2n}$ is an isotropic subspace of $\mathbb{R}^{2n}$.*

Proof. Proposition 4.3.2 yields that $P_{L^{-1}}(M^{\mathrm{T}})\mathbb{C}^{2n}$ and $P_{L_1}(M)\mathbb{C}^{2n}$ are orthogonal with respect to the symmetric form $< \cdot, \cdot >$. Observe that

$$M^{-1} = J_n M^{\mathrm{T}} J_n, \quad M \in \mathbf{Sp}(n, \mathbb{C}).$$

Hence

$$\begin{aligned} &P_L(M) = P_{L^{-1}}(M^{-1}) = P_{L^{-1}}(J_n M^{\mathrm{T}} J_n) = J_n P_{L^{-1}}(M^{\mathrm{T}}) J_n \quad \Rightarrow \\ &P_{L^{-1}}(M^{\mathrm{T}})\mathbb{C}^{2n} = J_n P_L(M)\mathbb{C}^{2n}. \end{aligned}$$

Therefore $P_L(M)\mathbb{C}^{2n}, P_{L_1}(M)\mathbb{C}^{2n}$ are skew orthogonal. Assume that $L \cap L^{-1} = \emptyset$. Set $L_1 = L$ to deduce that $P_L(M)\mathbb{C}^{2n}$ is an isotropic subspace. The last claim of the proposition is immediate. □

Corollary 4.3.4 *Let $M \in \mathbf{Sp}(n, \mathbb{R})$. Then $P_{1^+}(M)\mathbb{R}^{2n}$ and $P_{1^-}(M)\mathbb{R}^{2n}$ are isotropic subspaces.*

In what follows we need a preciser version of the above Corollary.

Lemma 4.3.5 *Let $M \in \mathbf{Sp}(n, \mathbb{R})$. Then*

$$\mathbb{R}^{2n} = \sum_{\lambda \in \mathrm{spec}_q(M)} \oplus P_{\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} \tag{4.3.2}$$

is an orthoskew decomposition of $\mathbb{R}^{2n}$. Assume that $\lambda \in \mathrm{spec}_q(M)\backslash\mathrm{spec}_1(M)$. Then

$$P_{\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} = P_{\{\lambda, \overline{\lambda}\}}(M)\mathbb{R}^{2n} \oplus P_{\{\lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} \tag{4.3.3}$$

is a decomposition of $P_{\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n}$ to a direct sum of its two Lagrangian subspaces.

Proof. Observe first that

$$\mathrm{spec}(M) = \mathrm{spec}(M)^{-1} = \overline{\mathrm{spec}(M)}.$$

Hence (4.3.2) is a spectral decomposition of $\mathbb{R}^{2n}$. Proposition 4.3.3 yields that (4.3.2) is an orthoskew decomposition. Apply Proposition 4.3.3 again to deduce that (4.3.3) is a spectral decomposition of $P_{\{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n}$ to a direct sum of its two Lagrangian subspaces. □

Let

$$\mathbf{Sp}(n, \mathbb{R})^+ := \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{H}^+(2n, \mathbb{C})$$

Corollary 4.3.6 *Assume that $A \in \mathbf{Sp}(n,\mathbb{R}) \cap \mathbf{Sym}(2n,\mathbb{R})$. Then*

$$A = OBO^T, \quad O \in \mathbf{K}_n, \quad B \in \mathbf{Sp}(n,\mathbb{R}) \cap \mathbf{D}(2n,\mathbb{R}). \tag{4.3.4}$$

Furthermore any $A \in \mathbf{Sp}(n,\mathbb{R})$ has the SVD:

$$A = O_1(D \oplus D^{-1})O_2, \quad O_1, O_2 \in \mathbf{K}_n, \quad D = \mathrm{diag}(d_1, ..., d_n), \quad 0 < d_1 \leq \ldots \leq d_n \leq 1. \tag{4.3.5}$$

In particular any $A \in \mathbf{Sp}(n,\mathbb{R})^+$ has the above form with $O_2 = O_1^{\mathrm{T}}$.

Proof. Since A is a real symmetric matrix it follows $\mathrm{spec}(A) \subset \mathbb{R}$, and the spectral orthoskew decomposition (4.3.2) is also an orthogonal decomposition with respect to the inner product $< \cdot, \cdot >$ on $\mathbb{R}^{2n}$. We claim that each subspace $P_{\{\lambda,\lambda^{-1}\}}(A)\mathbb{R}^{2n}$ has an orthonormal basis which is also a symplectic basis. Suppose first that $\lambda \neq \pm 1$. Choose an orthonormal basis of $u^1, ..., u^m$ of $P_\lambda(A)\mathbb{R}^{2n}$. The arguments of Proposition 4.3.3 yield that $J_n u^1, ..., J_n u^m$ is an orthonormal basis for $P_{\lambda^{-1}}(A)\mathbb{R}^{2n}$. Therefore $(u^1, ..., u^m, J_n u^1, ..., J_n u^m)$ is a symplectic orthonormal basis of $P_{\{\lambda,\lambda^{-1}\}}(A)\mathbb{R}^{2n}$. Suppose that $\lambda = \pm 1$. Then $J_n P_\lambda(A)\mathbb{R}^{2n} = P_\lambda(A)\mathbb{R}^{2n}$. In that case it is straightforward to show that $P_\lambda(A)\mathbb{R}^{2n}$ has a symplectic orthonormal basis. Combine the above bases of $P_{\{\lambda,\lambda^{-1}\}}(A)\mathbb{R}^{2n}$ to a symplectic orthonormal basis of $\mathbb{R}^{2n}$ to deduce (4.3.4). Suppose furthermore that $A \in \mathbf{Sp}(n,\mathbb{R})^+$. It is straightforward to show that one can rearrange the above orthonormal basis to obtain that $B = D \oplus D^{-1}$ where D satisfies (4.3.5). For a general matrix $A \in \mathbf{Sp}(n,\mathbb{R})$ consider the decompositions (4.3.4) of AA^{T} and $A^{\mathrm{T}}A$ to deduce (4.3.5). □

Use the above result and the proof of Lemma 2.1.2 to deduce the analog of Lemma 2.1.2 for $\mathbf{Y}_n$:

Corollary 4.3.7 *Let $(A,B),(C,D) \in \mathbf{Y}_n \times \mathbf{Y}_n$. Then there exists $T \in \mathbf{Sp}(n,\mathbb{R})$ such that $T(A,B) = (C,D)$ if and only if $\Sigma(A^{-1}B) = \Sigma(C^{-1}D)$. In particular, for any pair $(A,B) \in \mathbf{Y}_n \times \mathbf{Y}_n$ there exits $T \in \mathbf{Sp}(n,\mathbb{R})$ such that $T(A,B) = (I_{2n}, D \oplus D^{-1})$, where D satisfies* (4.3.5).

Note that Corollary 4.3.7 is an equivalent version of Siegel's result that for a given two pairs $(Z_1, Z_2), (W_1, W_2) \in \mathbf{SH}_n$ there exists $M \in \mathbf{Sp}(n,\mathbb{R})$ such that $(M(Z_1), M(Z_2)) = (W_1, W_2)$ iff the two pairs $(Z_1, Z_2), (W_1, W_2)$ have the same cross ratio. In particular, Corollary 4.3.7 yields (4.1.9).

Proof of Theorem 4.2.7. Corollary 4.3.6 yields that $\mathbf{Y}_n$ can be presented by $\mathbf{Sp}(n,\mathbb{R})^+$. Let $\Phi_1^{-1} : \mathbf{Sp}(n,\mathbb{R})^+ \to \mathbf{SH}_n$ be the inverse map to $\Phi_1 : \mathbf{SH}_n \to \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. Consider a sequence of matrices $D_m \oplus D_m^{-1} \in \mathbf{Sp}(n,\mathbb{R})^+$, $m = 1, \ldots$, where each D_m is of the form given in (4.3.5). Theorem 3.5.3 yields that the sequence $\{D_m \oplus D_m^{-1}\}_1^\infty$ converges to a point in $\mathbf{Y}_{n,1}(\infty)$ iff

$$D_m = diag(d_{1,m}, \ldots, d_{n,m}), \quad 0 \leq d_{1,m} \leq \ldots \leq d_{n,m} \leq 1, \quad m = 1, \ldots,$$
$$\lim_{m\to\infty} D_m = \Delta := diag(\delta_1, \ldots \delta_n),$$

$$0 = \delta_1 = \ldots = \delta_{i_1} < \delta_{i_1+1} = \ldots = \delta_{i_2} < \ldots < \delta_{i_{l-1}+1} = \ldots = \delta_{i_l} \leq 1,$$
$$0 = i_0 < i_1 < i_2 < \ldots < i_l = n. \tag{4.3.6}$$

Note that $i_1 = n$ iff $\Delta = 0$. Thus,

$$\lim_{m\to\infty} \Phi_1^{-1}(D_m \oplus D_m^{-1}) = \sqrt{-1}\Delta^2.$$

Hence $\sqrt{-1}\Delta^2 \in \mathrm{fin}(\partial_{i_1}\mathbf{SH}_n)$. Assume that the limit point in $\mathbf{Y}_{n,1}(\infty)$, given by the sequence (4.3.6), corresponds to the boundary point $\sqrt{-1}\Delta^2$. Let $\{B_m\}_1^\infty \subset \mathbf{Sp}(n,\mathbb{R})^+$ be a sequence of points converging to a point $\eta \in \mathbf{Y}_{n,1}(\infty)$. Corollary 4.3.6 yields that $B_m = O_m(D_m \oplus D_m^{-1})O_m^{\mathrm{T}}$, where $O_m \in \mathbf{K}_n$ and D_m is of the above form. As $\{B_m\}_1^\infty$ converges to η Theorem 3.5.3 yields that (4.3.6) holds. Pick up a subsequence $\{O_{m_l}\}_{l=1}^\infty$ which converges to $O \in \mathbf{K}_n$. Let $\{B_{m_l}\}$ correspond to a boundary point $C = O(\sqrt{-1}\Delta^2)$ in the finite or infinite boundary of $\mathbf{SH}_n$. Our first claim is that C does not depend on the subsequence $\{B_{m_l}\}$, i.e. $C = C(\eta)$. By considering the sequence $\{PB_mP^{\mathrm{T}}\}_1^\infty$ for a suitable $P \in \mathbf{K}_n$, to prove the first claim we may assume that η corresponds to the limit point given by the sequence (4.3.6). Use Theorem 3.5.3 to deduce that O has the block diagonal form:

$$O = \sum_{j=1}^{2l} \oplus O_j, \quad O_j, O_{2l-j+1} \in \mathbf{O}(i_j - i_{j-1}, \mathbb{R}), \quad j = 1, ..., l. \tag{4.3.7}$$

As $O \in \mathbf{K}_n$ we have the additional equalities

$$O_{2l-j+1} = O_j, \quad j = 1, ..., l.$$

Thus $O(\sqrt{-1}\Delta^2) = \sqrt{-1}\Delta^2$ and the first claim is proved. Our second claim that $C(\eta) = O(\sqrt{-1}\Delta^2)$ gives any point on the finite or infinite boundary of $\mathbf{SH}_n$, for a suitable choice of Δ and $O \in \mathbf{K}_n$. (We can assume that $O_m = O,\ m = 1, ...$) Observe that

$$-\Phi_2(\sqrt{-1}\Delta^2) = (I - \Delta^2)(I + \Delta^2)^{-1} = \Sigma((I - \Delta^2)(I + \Delta^2)^{-1}).$$

Use Schur's lemma 4.1.2 to deduce that any $B \in \partial_p\mathbf{SD}_n$ is of the form $U(I-\Delta^2)(I+\Delta^2)U^{\mathrm{T}}$ for some $U \in \mathbf{U}(n,\mathbb{C})$, and a corresponding Δ. The second claim is established. Our third claim is that for $\xi, \eta \in \mathbf{Y}_{n,1}(\infty),\ \xi \neq \eta$ we have $C(\eta) \neq C(\xi)$. Let η be given by $\{D_m \oplus D_m^{-1}\}_1^\infty$, where each $\tilde{D}_m$ is of the form (4.3.5). Assume that

$$B_m = O_m(\tilde{D}_m \oplus \tilde{D}_m^{-1})O_m^{\mathrm{T}}, \quad O_m \in \mathbf{K}_n,$$

where each $\tilde{D}_m$ is of the form (4.3.5), converges to ξ. The above arguments show that we can assume

$$\lim_{m\to\infty} \tilde{D}_m = \tilde{\Delta} \quad \text{and} \quad \lim_{m\to\infty} O_m = O \in \mathbf{K}_n.$$

Then $C(\xi) = O(\sqrt{-1}\tilde{\Delta}^2)$. Assume to the contrary that $C(\xi) = C(\eta) = \sqrt{-1}\Delta^2$. We claim that $\tilde{\Delta} = \Delta$ and O is of the form (4.3.7). A simple way to show this claim is to consider the equality $-\Phi_2(C(\xi)) = -\Phi_2(C(\eta))$:

$$U\tilde{\Sigma}U^T = \Sigma,\ \Sigma = (I - \Delta^2)(I + \Delta^2)^{-1},\ \tilde{\Sigma} = (I - \tilde{\Delta}^2)(I + \tilde{\Delta}^2)^{-1},\ U \in \mathbf{U}(n,\mathbb{C}).$$

Schur's lemma 4.1.2 yields that $\tilde{\Sigma} = \Sigma$. Hence $\tilde{\Delta} = \Delta$. Use the original arguments of Schur [Sch] (or [Fr2, Lemma 2]) and the arguments of the proof of Lemma 4.2.6 to deduce that the above equality implies

$$U(n) = \sum_{j=1}^{l} \oplus O_j, \quad O_j \in \mathbf{O}(i_j - i_{j-1}, \mathbb{R}), \quad j = 1, ..., l.$$

It is straightforward to show that the above equality yields that O is of the form (4.3.7). From the arguments of the proof of our first claim it follows that $\eta = \xi$, contrary to our assumption. $\square$

Proposition 4.3.8 *Let ξ be a point in 1-Busemann boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. Then ξ is uniquely presented by the Lagrangian subspace $\Xi \subset \mathbf{R}^{2n}$. Identify ξ with a unit vector in the one dimensional subspace $\wedge_n \Xi$. Then*

$$b_1(\xi, X_0, X) = 2 \log ||(\wedge_n X)\xi||_2 - 2 \log ||(\wedge_n X_0)\xi||_2.$$

Proof. A point ξ in the 1-Busemann boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ corresponds to the two flag

$$P_{1^+}(A)\mathbb{R}^{2n} \subset P_{1^+}(A)\mathbb{R}^{2n} \oplus P_{1^-}(A)\mathbb{R}^{2n} = \mathbb{R}^{2n}, \quad A \in \mathbf{Sp}(n, \mathbb{R})^+.$$

Hence A is hyperbolic and $\Xi := P_{1^+}(A)\mathbb{R}^{2n}$ is a Langragian subspace. Clearly, $\Xi^\perp = P_{1^-}(A)\mathbb{R}^{2n}$. Theorem 3.5.3 yields that ξ is determined uniquely by $U_+ = \Xi$, $U_- = \Xi^\perp$ ($\mathbf{H}(U_0) = 0$). Vice versa, assume that Ξ is a Langrangian subspace. Use Corollary 4.3.6 to find $A \in \mathbf{Sym}(2n, \mathbb{R})$, $e^A \in \mathbf{Sp}(n, \mathbb{R})^+$, such that $P_{1^+}(e^A)\mathbb{R}^{2n} = \Xi$. Note that $\lambda_n(A) > 0$. Then $e^{tA} \to \xi$ as $t \to \infty$.

We use (3.3.11) calculate $b_1(\xi, X_0, X)$. Observe that $\lambda_n(A) > 0 > \lambda_{n+1}(A)$, i.e. $j_k = n$. As $\mathbf{Sp}(n, \mathbb{R}) < \mathbf{SL}(2n, \mathbb{R})$ we deduce that

$$b_1(\xi, X_0, X) = 2\alpha_n(A, X) - 2\alpha_n(X_0).$$

Use (3.3.8) and the fact that $\wedge_n \Xi$ is a one dimensional subspace to obtain the proposition. $\square$

Corollary 4.3.9 *The 1-Busemann boundary of $\mathbf{SH}_n$ is equivalent to the Shilov boundary $\mathbf{USym}_n$.*

Proof. Let

$$\begin{aligned} &B = diag(b_1, ..., b_n) \in \mathbf{D}(n, \mathbb{R}), \quad b_1 < b_2 < \ldots < b_n < 0, \\ &C = B \oplus -B. \end{aligned} \tag{4.3.8}$$

Then C represents a Weyl chamber in $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. The geodesic ray $e^{tC}, t > 0$ converges to the point ξ on 1-Busemann boundary of $\mathbf{SH}_n$. Clearly ξ corresponds to $0 \in fin(\partial_n \mathbf{SH}_n)$.

As 1-Busemann boundary is given by the limit of the geodesic rays $Oe^{tC}O^{\mathrm{T}}$, $O \in \mathbf{K}_n$, we deduce the corollary. $\square$

We conclude this section with brief comparison of the Shilov and Furstenberg boundary of $\mathbf{SH}_n$ ($\mathbf{SD}_n$), which is well known to the experts. Let $\mathcal{D}_m$ be the group of all $m \times m$ diagonal matrices with diagonal entries equal to plus or minus 1.

Proposition 4.3.10 *The Furstenberg boundary of* $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ *can be presented as*

$$\mathbf{K}_n/(\mathcal{D}_{2n} \cap \mathbf{K}_n) \sim \mathrm{U}_n/\mathcal{D}_n.$$

Proof. Let C given by (4.3.8) be the generator of the Cartan subalgebra A appearing in the Iwasawa decomposition of $\mathbf{K}_n AN$ of $\mathbf{Sp}(n,\mathbb{R})$. Then the centralizer of A in $\mathbf{K}_n$ is equal to $\mathcal{D}_{2n} \cap \mathbf{K}_n$. Hence the Furstenberg boundary is given by $\mathbf{K}_n/(\mathcal{D}_{2n} \cap \mathbf{K}_n)$. Replace the $\mathbf{SPH}_n$ model by the $\mathbf{SD}_n$ model to deduce the second part of the proposition. $\square$

Note that Lemma 4.2.6 shows that the Shilov boundary can be obtained from the Furstenberg boundary using the action of $\mathbf{O}(n)$.

Corollary 4.3.11 *The Furstenberg boundary and the Shilov boundary of* $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ *have dimensions* n^2 *and* $n(n+1)/2$ *respectively.*

5 Discrete subgroups of $\mathbf{Sp}(n,\mathbb{R})$

5.1 Limit sets

In the rest of this paper we always assume that Γ is a discrete subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Assume that Γ is torsion free. As $\mathbf{PSp}(n,\mathbb{R})$ is the group of biholomorphisms of $\mathbf{SH}_n$ it follows that $\mathbf{SH}_n/\Gamma = \Gamma\backslash\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ is a complex manifold of dimension $\frac{n(n+1)}{2}$, whose universal cover is $\mathbf{SH}_n$. Assume that Γ has torsion. According to Selberg [Sel] Γ has a subgroup Γ_0 of finite index in such that Γ_0 is torsion free. Hence the manifold $\mathbf{SH}_n/\Gamma_0$ is a finite cover of the orbifold $\mathbf{SH}_n/\Gamma$. Therefore $\mathbf{SH}_n/\Gamma$ is a complex space [GR]. The case when Γ is a lattice in $\mathbf{Sp}(n,\mathbb{R})$ is very closely related to modular forms and algebraic geometry [Sie], [F]. (In many known cases $\mathbf{SH}_n/\Gamma$ is a quasiprojective variety.) As $\mathbf{Sp}(n,\mathbb{R})$ is a simple Lie group of rank n, for $n > 1$ the study of Γ falls into cathegory of discrete subgroups in higher rank groups. Some aspects of such discrete subgroups, in particular the Patterson-Sullivan theory, is treated in Albuquerque [Alb]. For $n = 1$ Γ is a Fuchsian group. The modern treatment of Fuchsian and Kleinian groups can be found in [Nic]. To compare the properties of Γ (for $n > 1$) with the properties of Fuchsian groups it is useful to note that $\mathbf{SL}(2,\mathbb{R})^n := \mathbf{SL}(2,\mathbb{R}) \times \ldots \times \mathbf{SL}(2,\mathbb{R})$ is isomorphic to a subgroup of $\mathbf{Sp}(n,\mathbb{R})$:

$$\Theta : \mathbf{SL}(2,\mathbb{R})^n \to \mathbf{Sp}(n,\mathbb{R}),$$
$$\Theta(M_1 \times \ldots \times M_n) = M_1 \odot \ldots \odot M_n,$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \odot \ldots \odot \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ \hline c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{array}\right).$$

Note that the action of $\mathbf{SL}(2,\mathbb{R})^n$ on $(\mathbf{H}^2)^n$ is isomorphic to the action of $\Theta(\mathbf{SL}(2,\mathbb{R})^n)$ on $\mathbf{DH}_n$:

$$M_1 \odot \ldots \odot M_m(\mathrm{diag}(z_1,...,z_n)) = \mathrm{diag}(M(z_1),...,M(z_n)), \quad z_1,...,z_n \in \mathbf{H}^2.$$

Definition 5.1.1 *For any set $T \subset \mathbf{SH}_n$ denote by $\mathrm{BCl}(T)$ the closure of T with respect to 1-Busemann compactification of $\mathbf{SH}_n$, i.e. the compactification of $\mathbf{SH}_n$ as a bounded domain. The 1-Busemann boundary of $\mathbf{SH}_n$, denoted by $\partial_n \mathbf{SH}_n$, is called the Shilov boundary of $\mathbf{SH}_n$.*

Note that $\mathrm{Cl}(\mathbf{SH}_n) \subset \mathrm{BCl}(\mathbf{SH}_n)$. To define the limit set of Γ we need the following theorem:

Theorem 5.1.2 *Let $\gamma_k \in \mathbf{Sp}(n,\mathbb{R})$, $k = 1,...,$ be a given sequence. Assume that for $Z \in \mathbf{SH}_n$ the sequence $\gamma_k(Z)$, $k = 1,...,$ converges to a point $P \in \partial_n \mathbf{SH}_n$. Then for any $W \in \mathbf{SH}_n$ the sequence $\gamma_i(W)$, $k = 1,...,$ converges to P.*

Proof. Since $\mathbf{Sp}(n,\mathbb{R})$ acts transitively on $\partial_n \mathbf{SH}_n$, there exists $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ so that $\lim_{i\to\infty} \gamma\gamma_i(Z) = 0 \in fin(\partial_n \mathbf{SH}_n)$. Hence to prove the lemma it is enough to consider the case $P = 0$. Write

$$\gamma_k(Z) = X_k + \sqrt{-1}Y_k, \quad \gamma_k(W) = U_k + \sqrt{-1}V_k, \quad k = 1,...$$

Let $A\mathbf{K}_n$ and $B\mathbf{K}_n$ be the cosets in $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ corresponding to Z and W respectively. Theorem 4.1.1 implies that $\gamma_k A\mathbf{K}_n$ and $\gamma_k B\mathbf{K}_n$ have the following representatives A_k and B_k respectively

$$A_k = \begin{pmatrix} Y_k^{\frac{1}{2}} & X_k Y_k^{-\frac{1}{2}} \\ 0 & Y_k^{-\frac{1}{2}} \end{pmatrix}, \quad B_k = \begin{pmatrix} V_k^{\frac{1}{2}} & U_k V_k^{-\frac{1}{2}} \\ 0 & V_k^{-\frac{1}{2}} \end{pmatrix}.$$

Clearly

$$||B^{-1}A||_2 = \sigma_1(B^{-1}A) = \sigma_1(B_k^{-1}A_k) = ||B_k^{-1}A_k||_2.$$

Furthermore

$$B_k^{-1}A_k = \begin{pmatrix} V_k^{-\frac{1}{2}}Y_k^{\frac{1}{2}} & V_k^{-\frac{1}{2}}X_k Y_k^{-\frac{1}{2}} - V_k^{-\frac{1}{2}}U_k Y_k^{-\frac{1}{2}} \\ 0 & V_k^{\frac{1}{2}}Y_k^{-\frac{1}{2}} \end{pmatrix},$$

We claim that

$$||B^{-1}A||_2 = ||B_k^{-1}A_k||_2 \geq ||V_k^{\frac{1}{2}}Y_k^{-\frac{1}{2}}||_2.$$

The last inequality follows from the standard inequalities on l_2 norms of matrices as follows. For any $C \in \mathbf{M}(2n, \mathbb{R})$, its operator norm is given by

$$||C||_2 = \max_{||x||_2 = ||y||_2 = 1} |y^{\mathrm{T}} C x|.$$

Hence

$$||C||_2 \geq \max_{i,j=1,2} ||C_{ij}||_2, \text{ for } C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \ C_{ij} \in \mathbf{M}(n, \mathbb{R}), \ i,j = 1,2.$$

Observe next

$$||V_k^{\frac{1}{2}}||_2 = ||(V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}) Y_k^{\frac{1}{2}}||_2 \leq ||V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}||_2 ||Y_k^{\frac{1}{2}}||_2 \Rightarrow ||V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}||_2 \geq ||V_k^{\frac{1}{2}}||_2 ||Y_k^{\frac{1}{2}}||_2^{-1}.$$

Hence

$$||B^{-1}A||_2 \geq ||V_k||_2^{\frac{1}{2}} ||Y_k||_2^{-\frac{1}{2}} \ \Rightarrow \ ||V_k||_2 \leq ||B^{-1}A||_2^2 ||Y_k||_2.$$

As $||Y_k||_2 \to 0$ we deduce that $||V_k|| \to 0$. Using the above arguments for $(1,2)$ block of $B_k^{-1} A_k$ we obtain

$$||B^{-1}A||_2 \geq ||V_k^{-\frac{1}{2}} (X_k - U_k) Y_k^{-\frac{1}{2}}||_2 \geq ||V_k^{-\frac{1}{2}} (X_k - U_k)||_2 \ ||Y_k^{\frac{1}{2}}||_2^{-1} \geq$$
$$||V_k^{\frac{1}{2}}||_2^{-1} ||(X_k - U_k)||_2 \ ||Y_k^{\frac{1}{2}}||_2^{-1}.$$

Thus

$$||X_k - U_k||_2 \leq ||B^{-1}A||_2 ||Y_k||_2^{\frac{1}{2}} ||V_k||_2^{\frac{1}{2}}, \quad k = 1, \ldots$$

Since $X_k, Y_k, V_k \to 0$, we deduce that $\gamma_k(W) = U_k + \sqrt{-1} V_k \to 0$. □

We remark that Theorem 5.1.2 does not hold if $\gamma_k(Z) \to P \in \partial_m \mathbf{SH}_n$ for any $m \in [1, n-1] \cap \mathbb{Z}$. Indeed, let $M = \mathrm{diag}(\frac{1}{2}, 2) \in \mathbf{SL}(2, \mathbb{R})$ and define

$$\gamma_k = \underbrace{M^k \odot \ldots \odot M^k}_{m \text{ times}} \odot \underbrace{I_2 \odot \ldots \odot I_2}_{n-m \text{ times}} \in \mathbf{Sp}(n, \mathbb{R}), \quad k = 1, \ldots$$

Then

$$\lim_{k \to \infty} \gamma_k(\mathrm{diag}(z_1, \ldots, z_n)) = \mathrm{diag}(0, \ldots, 0, z_{m+1}, \ldots, z_n), \quad \mathrm{diag}(z_1, \ldots, z_n) \in \mathbf{DH}_n.$$

Corollary 5.1.3 *Let* $\gamma_k \in \mathbf{Sp}(n, \mathbb{R})$, $k = 1, \ldots$, *be a given sequence. Then for any* $Z \in \mathbf{SH}_n$ *all the accumulation points of the sequence* $\{\gamma_k(Z)\}_1^\infty$ *lie in the Shilov boundary of* $\mathbf{SH}_n$ *if and only if*

$$\lim_{k \to \infty} \sigma_i(\gamma_k) = \infty, \quad i = 1, \ldots, n. \tag{5.1.1}$$

Proof. Corollary 4.3.6 implies that

$$\begin{aligned}&\gamma_k = O_{1,k}(D_k \oplus D_k^{-1})O_{2,k},\\ &O_{1,k}, O_{2,k} \in \mathbf{K}_n,\\ &D_k = \mathrm{diag}(\sigma_{2n}(\gamma_k), \ldots, \sigma_{n+1}(\gamma_k)),\\ &k = 1, \ldots\end{aligned} \tag{5.1.2}$$

Let $Z = \sqrt{-1}I_n$. Then Z is represented by the coset I_{2n} in $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ Suppose that a subsequence $\gamma_{k_i}(Z) \to P$. Pick up a subsequence $\{k_i'\}_{i=1}^{\infty}$ such that $O_{1,k_i'} \to O \in \mathbf{K}_n$. Then $O_{k_i'}\gamma_{k_i'}(Z) \to O(P)$. Clearly

$$O_{1,k}^{\mathrm{T}}\gamma_k(Z) = \sqrt{-1}\mathrm{diag}(\sigma_{2n}(\gamma_k)^2, \ldots, \sigma_{n+1}(\gamma_k)^2), \quad k = 1, \ldots$$

Thus $\gamma_{k_i'}(Z) \to P$ iff

$$\lim_{i\to\infty} \sigma_j(\gamma_{k_i'}) = \delta_{2n-j+1} \leq 1, \quad j = 1, ..., n.$$

Then $P \in \partial_n\mathbf{SH}_n$ iff $\delta_{n+1} = 0$. Thus all the accumulation points of the sequence $\{\gamma_k(Z)\}_1^{\infty}$ lie in $\partial_n\mathbf{SH}_n$ iff an only iff (5.1.1) holds. Use Theorem 5.1.2 to deduce the corollary. □

We remark that in [Fr3] one of the authors studied certain discrete groups G acting on matrix spaces. In [Fr3, Thm 2.2, Thm 3.1] it is shown that G acts properly discontinuously, if the singular values of G satisfy conditions similar to (5.1.1). These results were generalized in [Be1]. For $\mathcal{S} \subset \mathbf{Sp}(n, \mathbb{R})$ and $Z \in \mathrm{BCl}(\mathbf{SH}_n)$ we denote by $\mathcal{S}(Z)$ the $\mathcal{S}$-orbit of Z:

$$\mathcal{S}(Z) := \{\gamma(Z) : \quad \gamma \in \mathcal{S}\}.$$

Definition 5.1.4 *Let Γ be a discrete group of $\mathbf{Sp}(n, \mathbb{R})$. Then the limit set $\Lambda(\Gamma)$ is given by the set $\mathrm{BCl}(\Gamma(Z)) \cap \partial_n\mathbf{SH}_n$ for some $Z \in \mathbf{SH}_n$.*

Theorem 5.1.2 implies that the definition of $\Lambda(\Gamma)$ is independent of the choice of $Z \in \mathbf{SH}_n$ as in the case of Fuchsian groups. Corollary 5.1.3 gives a necessary and sufficient conditions for Γ so that $\Lambda(\Gamma) \neq \emptyset$. For a Fuchsian group Γ the limit set $\Lambda(\Gamma) \neq \emptyset$ iff Γ is infinite. For $n > 1$ there exist infinite Γ for which $\Lambda(\Gamma) = \emptyset$. Indeed, let $\Gamma_1, \ldots, \Gamma_n$ be Fuchsian groups, where Γ_1 is finite and $\Gamma_2, ..., \Gamma_n$ infinite. Then the above arguments show that $\Lambda(\Gamma_1 \odot \ldots \odot \Gamma_n) = \emptyset$.

An element $\gamma \in \mathbf{Sp}(n, \mathbb{R})$ is called *hyperbolic* if it does not have eigenvalues on the unit circle, i.e. $\mathrm{spec}_1(\gamma) = \emptyset$.

Proposition 5.1.5 *Let $\gamma \in \mathbf{Sp}(n, \mathbb{R})$ be hyperbolic. Then*

$$\begin{aligned}&\gamma = T\tilde{\gamma}T^{-1}, \quad T \in \mathbf{Sp}(n, \mathbb{R}),\\ &\tilde{\gamma} = \begin{pmatrix} C & 0 \\ 0 & (C^{\mathrm{T}})^{-1} \end{pmatrix}, \quad C \in \mathbf{GL}(n, \mathbb{R}),\\ &\mathrm{spec}_{1^-}(\gamma) = \mathrm{spec}(C).\end{aligned} \tag{5.1.3}$$

Proof. Corollary 4.3.4 yields that $P_{1^-}(\gamma)\mathbb{R}^{2n}$ and $P_{1^+}(\gamma)\mathbb{R}^{2n}$ are Lagrangian subspaces. Pick bases $e^1, ..., e^n$ and $f^1, ..., f^n$ in the above Lagrangian subspaces such that such that $e^1, ..., e^n, f^1, ..., f^n$ is a symplectic base of $\mathbb{R}^{2n}$. Let $e^1, ...e^n, f^1, ..., f^n$ be the columns of T. Then $T \in \mathbf{Sp}(n, \mathbb{R})$ and $\tilde{\gamma}$ has the block diagonal form $\mathrm{diag}(C, C')$, where $C, C' \in \mathbf{M}(n, \mathbb{R})$. As $\mathrm{diag}(C, C')$ is symplectic we deduce that $C' = (C^{\mathrm{T}})^{-1}$. As C represents the restriction of γ to $P_{1^-}(M)\mathbb{R}^{2n}$. Hence the last equality of (5.1.3) holds. $\square$

Lemma 5.1.6 *Let $\gamma \in \mathbf{Sp}(n, \mathbb{R})$ be hyperbolic. Then there exist two distinct fixed points $\xi_+(\gamma), \xi_-(\gamma) \in \partial_n \mathbf{SH}_n$ of γ such that*

$$\begin{aligned} &\lim_{k\to\infty} \gamma^k(Z) = \xi_+(\gamma), \\ &\lim_{k\to\infty} \gamma^{-k}(Z) = \xi_-(\gamma), \\ &\text{for all } \ Z \in \mathbf{SH}_n. \end{aligned} \tag{5.1.4}$$

Proof. Without loss of generality we may assume that γ is equal to $\tilde{\gamma}$ given in (5.1.3). Then

$$\tilde{\gamma}^k(Z) = C^k Z (C^{\mathrm{T}})^k, \quad k \in \mathbb{Z}.$$

As all the eigenvalues of C are in the open unit disk, we deduce $\lim_{k\to\infty} C^k = 0$. Hence the first equality of (5.1.4) holds with $\xi_+(\tilde{\gamma}) = 0$. Observe next that $\tilde{\gamma}^{-1} = J_n \tilde{\gamma}^{\mathrm{T}} J_n^{-1}$. Hence the second equality of (5.1.4) holds with $\xi_-(\gamma) = J_n(0)$. $\square$

For $n = 1$ the hyperbolic element $\gamma \in \mathbf{SL}(2, \mathbb{R})$ has exactly two fixed points in the closure of $\mathbf{H}^2$ which are located on the boundary. For $n > 1$ a hyperbolic element can have more then two fixed points in $\partial\,\mathrm{BCl}(\mathbf{SH}_n)$. Indeed, let $\gamma_1, ..., \gamma_n \in \mathbf{SL}(2, \mathbb{R})$ be n hyperbolic elements so that the set $\{\xi_+(\gamma_1), \xi_-(\gamma_1), \ldots, \xi_+(\gamma_n), \xi_+(\gamma_n)\}$ is a set of $2n$ distinct real points. Let $\gamma = \gamma_1 \odot \ldots \odot \gamma_n \in \mathbf{Sp}(n, \mathbb{R})$. Then the following 2^n points are fixed points of γ:

$$\mathrm{diag}(\xi_\pm(\gamma_1), \ldots, \xi_\pm(\gamma_n)) \in fin(\partial_n \mathbf{SH}_n).$$

With some effort one can show that that such γ has exactly 2^n fixed points in $\mathrm{BCl}(\mathbf{SH}_n)$. Note that

$$\xi_+(\gamma) = \mathrm{diag}(\xi_+(\gamma_1), \ldots, \xi_+(\gamma_n)), \quad \xi_-(\gamma) = \mathrm{diag}(\xi_-(\gamma_1), \ldots, \xi_-(\gamma_n)).$$

It is possible to show that a hyperbolic transformation has at most 2^n isolated fixed points in $\mathrm{BCl}(\mathbf{SH}_n)$. It may happen that a hyperbolic transformation has less than 2^n isolated points. In our second paper we show that for $n = 2$ any hyperbolic transformation has either 2, 3 or 4 isolated fixed points in $\partial_2 \mathbf{SH}_2$ or two isolated fixed points and a closed connected real 1-dimensional variety of fixed points $\sim S^1$ in $\partial_2 \mathbf{SH}_2$.

Denote by Γ_h the set of all hyperbolic elements in Γ. Assume that $\gamma \in \Gamma_h$. Then $\xi_\pm(\gamma) \in \Lambda(\Gamma)$. As $\alpha\gamma\alpha^{-1} \in \Gamma_h$ for any $\alpha \in \Gamma$ it follows that $\alpha(\xi_\pm(\gamma)) \in \Lambda(\Gamma)$.

Definition 5.1.7 *Let* Γ *be a discrete subgroup of* $\mathbf{Sp}(n,\mathbb{R})$*. Then*

$$\Lambda_h(\Gamma) = \mathrm{BCl}(\cup_{\gamma\in\Gamma_h}\{\xi_+(\gamma),\xi_-(\gamma)\}).$$

$\Lambda_h(\Gamma)$ is a closed Γ-invariant subset of $\Lambda(\Gamma)$. If Γ is a nonelementary Fuchsian group then $\Lambda_h(\Gamma) = \Lambda(\Gamma)$. Moreover $\Lambda_h(\Gamma)$ is an uncountable perfect set [B2]. An analog of a nonelementary Fuchsian group is a discrete Zariski dense subgroup. $\Gamma < \mathbf{Sp}(n,\mathbb{R})$. Since $\mathbf{Sp}(n,\mathbb{R})$ is a simple Lie group, the results of Goldsheid-Margulis [GM2] yields that any Zariski dense subgroup of $\mathbf{Sp}(n,\mathbb{R})$ contains hyperbolic elements. The following theorem is closely related to the Lemma in [Be2, 3.6]:

Theorem 5.1.8 *Let* $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ *be a discrete Zariski dense subgroup. Let* T *be a closed* Γ*-invariant subset of* $\mathrm{BCl}(\mathbf{SH}_n)$*. Then* T *contains* $\Lambda_h(\Gamma)$*. Furthermore,* $\Lambda_h(\Gamma)$ *is a perfect set.*

Proof. By conjugating Γ by an element in $\mathbf{Sp}(n,\mathbb{R})$, we may assume that Γ has an element $\tilde{\gamma}$ of the form (5.1.3). Let $W \in \mathrm{BCl}(\mathbf{SH}_n)\backslash\mathrm{fin}(\partial\mathbf{SH}_n)$. Consider the projective model $\mathbf{SPH}_n$. Then W is presented by the following representative:

$$W_1 = \begin{pmatrix} A \\ B \end{pmatrix}, \quad \det B = 0.$$

That is, W_1 is located on algebraic variety of $G_{2n,n}\mathbb{R}$. As Γ is Zariski dense in $\mathbf{Sp}(n,\mathbb{R})$, there exists $\alpha \in \Gamma$ such that $V = \alpha(W) \in \mathrm{Cl}(\mathbf{SH}_n)$. Asume that T is Γ-invariant set. The above argument show that there exists $V \in T\cap\mathrm{Cl}(\mathbf{SH}_n)$. Then $\tilde{\gamma}^k(V) \to 0 = \xi_+(\tilde{\gamma})$. Since T is closed $0 \in T$. Hence $\xi_\pm(\beta) \in T$ for any $\beta \in \Gamma_h$. Thus $T \supset \Lambda_h(\Gamma)$. To show that $\Lambda_h(\Gamma)$ is a perfect set we must show that $\Lambda_h(\Gamma)$ does not contain isolated points. Assume to the contrary that $\eta \in \Lambda_h(\Gamma)$ is an isolated point. From the definition of $\Lambda_h(\Gamma)$ it follows that $\eta = \xi_+(\alpha)$ for some $\alpha \in \Gamma_h$. Without a loss of generality we may assume that $\eta = \xi_+(\tilde{\gamma}) = 0$. As Γ Zariski dense, there exists $\beta \in \Gamma$ such that $0 \neq \beta(0) \in \mathrm{fin}(\partial_n\mathbf{SH}_n)$. Then $\tilde{\gamma}^k(\beta(0)), k = 1, ...,$ is a sequence of pairwise distinct points in $\Lambda_h(\Gamma)$ which converges to 0, contrary to our assumption. □

We do not know if $\Lambda(\Gamma) = \Lambda_h(\Gamma)$ for any Zariski dense subgroup Γ and $n > 1$. Let $\Omega(\Gamma)$ be the open set of the Shilov boundary of $\mathbf{SH}_n$ on which Γ acts properly discontinuously. ($\Omega(\Gamma)$ may be an empty set.)

Definition 5.1.9 *Let* Γ *be a discrete subgroup of* $\mathbf{Sp}(n,\mathbb{R})$*. Denote by* $\Lambda_d(\Gamma)$ *the smallest closed set in the Shilov boundary of* $\mathbf{SH}_n$ *such that* Γ *acts properly discontinuously on the complement of* $\Lambda_d(\Gamma)$ *in the Shilov boundary of* $\mathbf{SH}_n$ *(*$\Omega(\Gamma)$*).*

$\Lambda_d(\Gamma)$ is a closed Γ-invariant set of $\partial_n\mathbf{SH}_n$. For a Fuchsian (Kleinian) group $\Lambda_d(\Gamma) = \Lambda(\Gamma)$.

Lemma 5.1.10 *Let* $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ *be a discrete group. Then* $\Lambda(\Gamma) \subset \Lambda_d(\Gamma)$*.*

Proof. . Clearly, it is enough to consider the case where $\Lambda(\Gamma) \neq \emptyset$ Assume that $\gamma_k(Z) \to P \in \Lambda(\Gamma)$, where $\gamma_k \in \Gamma$, $k = 1, ...$ and $Z \in \mathbf{SH}_n$. Assume that γ_k is of the form (5.1.2).

By chosing a subsequence of γ_k, $k = 1, ...,$ we may assume that $O_{1,k} \to O_1, O_{2,k} \to O_2$. Use the proof of Corollary 5.1.3 to deduce that $P = O_1(0)$. Let $W = O_2^{-1}(0) \in \partial_n \mathbf{SH}_n$. Use the proof of Corollary 5.1.3 again to conclude that $\gamma_k(O_2^{-1}(0)) \to O_1(0)$. $\square$

It is not difficult to find simple examples for which $\Lambda(\Gamma) \neq \Lambda_d(\Gamma)$. Let $\alpha = \mathrm{diag}(\frac{1}{2}, 2) \in \mathbf{SL}(2, \mathbb{R})$, $\gamma = \alpha \odot \alpha \in \mathbf{Sp}(2, \mathbb{R})$ and $\Gamma =< \gamma >$. Then $\Lambda(\Gamma) = \{0, J_2(0)\}$. We will show in the next paper that γ has a curve of fixed points $F \subset \partial_2 \mathbf{SH}_2$, which belongs to $\Lambda_d(\Gamma)$. Hence $\Lambda(\Gamma)$ is strictly contained in $\Lambda_d(\Gamma)$.

The structure of $\Omega(\Gamma)$ is closely related to the fundamental domains of Γ in $\mathbf{Y}_n = \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. We consider here the Dirichlet domains. Fix $p \in [0, \infty]$ and $A \in \mathbf{Y}_n$. Let

$$D_p(A, \Gamma) := \{B \in \mathbf{Y}_n : \quad d_p(B, \gamma A) - d_p(B, A) \geq 0, \quad \gamma \in \Gamma\}. \tag{5.1.5}$$

Let

$$\tilde{D}_p(A, \Gamma) := \mathrm{BCl}(D_p(A, \Gamma)) \cap \partial_n \mathbf{SH}_n.$$

If $\tilde{D}_p(A, \Gamma)$ has an open interior (relative to the Shilov boundary) then it belongs to $\Omega(\Gamma)$. Since the 1-Busemann boundary is the Shilov boundary it is natural to choose $p = 1$. In (5.1.5) let B converge to ξ in 1-Busemann boundary. Use the definition of 1-Busemann function to obtain

$$\tilde{D}_1(A, \Gamma) = \{\xi \in \partial_n \mathbf{SH}_n : \quad b_1(\xi, A, \gamma A) \geq 0, \quad \gamma \in \Gamma\}. \tag{5.1.6}$$

See Proposition 4.3.8 for the simple formula for $b_1(\xi, A, B)$.

5.2 Patterson-Sullivan measures

Let $\mathcal{S} \subset \mathbf{Sp}(n, \mathbb{R})$ be a countable discrete set. ($\mathcal{S}$ has no accumulation points in $\mathbf{Sp}(n, \mathbb{R})$.) Assume furhtermore that $\mathcal{S}$ is symmetric, i.e.

$$\gamma \in \mathcal{S} \quad \Longleftrightarrow \quad \gamma^{-1} \in \mathcal{S}.$$

(We assume that an empty set is a symmetric set.) Fix $A, B \in \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ and $p \in [1, \infty]$. For $r > 0$ let

$$N_p(\mathcal{S}, r, A, B) = \#\{\gamma \in \mathcal{S} : \quad d_p(A, \gamma B) < r\}$$

be the p-orbital counting function [Nic]. Taking in account the properties of $d_p(\cdot, \cdot)$ given in §2.1, the fact that $||x||_p$, $x \in \mathbf{R}^m$ is a decreasing function of $p, p \geq 1$, we deduce in a straightforward manner that

$$\begin{aligned}
&N_p(\mathcal{S}, r, A, B) = N_p(\mathcal{S}, r, B, A), \\
&N_p(\mathcal{S}, r, A, B) \leq N_p(\mathcal{S}, r + d_p(A, C), C, B), \\
&N_{\mathcal{S}, p_1}(r, A, B) \leq N_{p_2}(\mathcal{S}, r, A, B), \quad 1 \leq p_1 \leq p_2, \\
&N_\infty(\mathcal{S}, r, A, B) \leq N_p(\mathcal{S}, (2n)^{\frac{1}{p}} r, A, B).
\end{aligned}$$

Hence, the p-Poincaré exponent:

$$\delta_p(\mathcal{S}) := \limsup_{r\to\infty} \frac{\log N_p(\mathcal{S}, r, A, B)}{r},$$

is independent of the choices A, B. Note that

$$\begin{aligned}&\delta_p(\emptyset) = -\infty,\\ &\delta_p(\mathcal{S}) = 0 \quad \text{if} \quad \mathcal{S} \ \text{is a nonempty finite set},\\ &\text{for all} \quad p \in [1, \infty].\end{aligned}$$

The associated Poincaré series is

$$g_{s,p}(\mathcal{S}, A, B) := \sum_{\gamma\in\mathcal{S}} e^{-s\, d_p(A,\gamma B)}, \quad s > 0. \tag{5.2.1}$$

Assume that $\mathcal{S}$ is infinite. Then $\delta_p(\mathcal{S}) \geq 0$ and $g_{0,p}(\mathcal{S}, A, B) = \infty$. Assume that $0 < \delta_p(\mathcal{S}) < \infty$. It is straightforward to show that the Poincaré series converges for $s > \delta_p(\mathcal{S})$ and diverges for $s < \delta_p(\mathcal{S})$ [Nic]. If the Poincaré series diverges for $s = \delta_p(\mathcal{S})$ then $\mathcal{S}$ is called of p-divergence type. Otherwise, $\mathcal{S}$ is called of p-convergence type. (The divergence (convergence) type of $\mathcal{S}$ dependends only on the value of p.) The construction of the family of PS measures is straightforward for infinite discrete symmetric sets $\mathcal{S}$ of divergence type. In what follows B is kept fixed while A may vary. Let

$$\mu_{\mathcal{S},s,A,p} = \frac{1}{g_{s,p}(\mathcal{S}, B, B)} \sum_{\gamma\in\mathcal{S}} e^{-s\, d_p(A,\gamma B)} \Delta_{\gamma B}, \quad s > 0. \tag{5.2.2}$$

Here Δ_B denote the Dirac measure on $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ at the point B. Then $\mu_{\mathcal{S},s,A,p}$ is a finite measure on $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. Identify $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ with $\mathbf{SH}_n$. We view $\mu_{\mathcal{S},s,A,p}$ as a finite measure on on $\mathrm{BCl}(\mathbf{SH}_n)$. Let $\{s_m\}_1^\infty$ be a strictly decreasing sequence which converges to $\delta_p(\Gamma)$. The Helly selection principle states that we can find a subsequence $\{m_k\}_{k=1}^\infty$ so that the sequence of measures $\mu_{\mathcal{S},s_{m_k},A,p}$ converges weakly to a finite measure $\mu_{\mathcal{S},A,p}$. The assumption that $\mathcal{S}$ was of divergence type implies straightforward

$$\operatorname{supp} \mu_{\mathcal{S},A,p} \subset \mathrm{BCl}(\mathcal{S}(B)) \cap \partial\, \mathrm{BCl}(\mathbf{SH}_n). \tag{5.2.3}$$

Let $\mathcal{M}_{\mathcal{S},A,p}$ be the family of all measures $\mu_{\mathcal{S},A,p}$ obtained by considering all weakly convergent subsequences of $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^\infty$. If $\mathcal{S}$ is of p-convergent type, then one has to modify the definition of the Poincaré series (5.2.1) and induced measures (5.2.2) as in [Pat].

Lemma 5.2.1 *Let $\mathcal{S} \subset \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ be an infinite discrete set. Assume that $0 < \delta_p(\mathcal{S}) < \infty$ and $\mathcal{S}$ be of p-convergent type. Then there exists a continuous nondecreasing function $h_p : [0, \infty) \to [0, \infty)$ with the following properties:*
(a) *For any $A, B \in \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ the series $\sum_{\gamma\in\mathcal{S}} e^{-s\, d_p(A,\gamma B)} h_p(e^{d_p(A,\gamma B)})$ converges for $s > \delta_p(\mathcal{S})$ and diverges for $s = \delta_p(\mathcal{S})$.*
(b) *For a given $\epsilon > 0$ there exists $r_\epsilon > 0$ so that for $r > r_\epsilon,\ t > 1$ $h_p(rt) < t^\epsilon h_p(r)$.*

Proof. Fix $A, B \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. Use the construction of h in [Nic, Lemma 3.1.1] to construct h_p, using the metric $d_p(\cdot,\cdot)$, which satisfies properties (a) and (b). Use property (b) to deduce that the convergence and the divergence of the series in (a) do not depend on the choice of $A, B \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. $\square$

Let $\mathcal{S}$ be an infinite discrete set of p-convergent type. Let $h_p(\cdot)$ be the function defined in Lemma 5.2.1. Set

$$
\begin{aligned}
g^*_{s,p}(\mathcal{S}, A, B) &:= \sum_{\gamma\in\mathcal{S}} e^{-s d_p(A,\gamma B)} h_p(d_p(A,\gamma B)), \quad s > 0,\\
\mu_{\mathcal{S},s,A,p} &= \frac{1}{g^*_{s,p}(\mathcal{S}, B, B)} \sum_{\gamma\in\mathcal{S}} e^{-s d_p(A,\gamma B)} h_p(d_p(A,\gamma B))\Delta_{\gamma B}, \quad s > 0.
\end{aligned}
\tag{5.2.4}
$$

Then $\mathcal{M}_{\mathcal{S},A,p}$ is the family of all measures $\mu_{\mathcal{S},A,p}$ obtained by considering all weakly convergent subsequences $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^{\infty}$, where $\{s_m\}_1^{\infty}$ is a strictly decreasing sequence which converges to $\delta_p(\Gamma)$. Clearly, (5.2.3) holds. Note that if a subsequence $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^{\infty}$ converges weakly for $A = A_0$ then this sequence converges weakly for any $A \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ [Nic]. Hence each $\mu_{\mathcal{S},A,p}$ represents a family of measures, which depends on a parameter A.

Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be an infinite discrete group. Then $\mathcal{M}_{\Gamma,A,p}$ is the set of PS measures. We claim that $\delta_p(\Gamma) \le v_{n,p} < \infty$ for any $p \in [1,\infty]$. The constant $v_{n,p}$ is the *volume growth* of p-balls in $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$, induced by the Siegel metric on $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. Let

$$B_{n,p}(A,r) = \{B \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n : \quad d_p(A,B) < r\}$$

be the open p-ball of radius $r > 0$ for any $p \in [1,\infty]$. For a measurable set $T \subset \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$, denote by $\mathrm{vol}(T)$ the volume of T with respect to the Haar measure on $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$, induced by the Siegel metric. Then

$$v_{n,p} := \limsup_{r\to\infty} \frac{\log \mathrm{vol}(B_{n,p}(A,r))}{r}, \quad p \in [1,\infty].$$

Clearly, $v_{n,p}$ is independent of $A \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. In what follows we use the standard notation $f \asymp g$, for two positive functions $f(r), g(r)$ defined on (t,∞), if

$$0 < \liminf_{r\to\infty} \frac{f(r)}{g(r)} \le \limsup_{r\to\infty} \frac{f(r)}{g(r)} < \infty.$$

Proposition 5.2.2 *For $n > 1$, there exists a constant $\kappa_n > 0$, such that for $p \in [1,\infty]$ and $r > 0$*

$$
\begin{aligned}
&\mathrm{vol}(B_{n,p}(I,r)) = \\
&\kappa_n \int_{\log y\in\Theta_{n,p}(r)} \prod_{1\le i<j\le n} \frac{(y_iy_j - 1)(y_i - y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{(y_i^2-1)}{y_i^2} dy_1 \dots dy_n,\\
&\Theta_{n,p}(r) := \{x = (x_1,...,x_n) \in \mathbb{R}^n : \quad 0 \le x_n \le \ldots \le x_1, \quad ||x||_p < 2^{\frac{p-1}{p}} r\}.
\end{aligned}
\tag{5.2.5}
$$

In particular

$$v_{n,1} = n \le v_{n,p} \le v_{n,\infty} = n(n+1). \tag{5.2.6}$$

Proof. Recall that $\mathrm{Stab}(\sqrt{-1}I_n) = \mathbf{K}_n$. Furthermore, for each $Z \in \mathbf{SH}_n$ there exists $O \in \mathbf{K}_n$ such that

$$O(Z) = \sqrt{-1}\mathrm{diag}(y_1, ..., y_n), \quad y_1 \ge y_2 \ge \ldots \ge y_n \ge 1,$$
$$\sigma_i(\phi_1(O(Z))) = \sigma_i(\phi_1(Z)) = \sqrt{y_i}, \quad i = 1, ..., n.$$

Assume that $y_1 > y_2 \ldots > y_n > 1$. Then the stabilizer of $\phi_1(\sqrt{-1}\mathrm{diag}(y_1, ..., y_n))$ is a finite group of diagonal matrices $\mathcal{D}_{2n} \cap \mathbf{K}_n$. Let

$$\mathbb{R}^n_{\ge} := \{y = (y_1, ..., y_n) \in \mathbb{R}^n : \quad y_1 \ge y_2 \ge \ldots \ge y_n \ge 1\}.$$

Then up to a zero measure we have the decomposition

$$\begin{aligned} &\mathbf{SH}_n \sim \mathbf{K}_n/(\mathcal{D}_{2n} \cap \mathbf{K}_n) \times \mathbb{R}^n_{\ge}, \\ &U + \sqrt{-1}V = O(\mathrm{diag}(y_1, ..., y_n)), \quad O \in \mathbf{K}_n, \quad (y_1, ..., y_n) \in \mathbb{R}^n_{\ge}. \end{aligned} \tag{5.2.7}$$

With respect to the above decomposition the ball $B_{n,p}(I, r)$ is identified with

$$\{y = e^x : \quad x \in \Theta_{n,p}(r)\}.$$

Recall [Sie] that Siegel metric on $\mathbf{SH}_n$ is given by the quadratic form

$$ds^2 = \mathrm{trace}(V^{-1}dUV^{-1}dU + V^{-1}dVV^{-1}dV), \quad U, V \in \mathbf{Sym}(n, \mathbb{R}),\ U + \sqrt{-1}V \in \mathbf{SH}_n.$$

We compute ds^2 for $U = 0, V = Y = \mathrm{diag}(y_1, ..., y_n)$ using (5.2.7). Recall that the Lie algebra of $\mathbf{K}_n$ is given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad -A^{\mathrm{T}} = A = (a_{ij})_1^n,\ B^{\mathrm{T}} = B = (b_{ij})_1^n.$$

A straightforward computation shows

$$dU = dB - Y(dB)Y, \quad dV = (dA)Y - YdA + dY,$$
$$ds^2 = 2\sum_{1 \le i < j \le n} \frac{(y_iy_j - 1)^2(db_{ij})^2 + (y_i - y_j)^2(da_{ij})^2}{y_iy_j} + \sum_{1 \le i \le n} \frac{(y_i^2 - 1)^2(db_{ii})^2 + (dy_i)^2}{y_i^2}.$$

Hence the volume element for the Siegel metric is

$$d\omega = 2^{n(n+1)} \prod_{1 \le i < j \le n} \frac{(y_iy_j - 1)(y_i - y_j)}{y_iy_j} \prod_{1 \le i \le n} \frac{y_i^2 - 1}{y_i^2} \prod_{1 \le i < j \le n} da_{ij} \prod_{1 \le i \le j \le n} db_{ij} \prod_{1 \le i \le n} dy_i.$$

Integrate the above expression over $\mathbf{K}_n/(\mathcal{D}_{2n} \cap \mathbf{K}_n) \times \Theta_{n,p}(r)$ to deduce (5.2.5). As $||x||_p$ is a decreasing function of p we deduce that $B_{n,p}(I, r)$ are increasing set in p for any fixed values of n and r. Hence $v_{n,p}$ are increasing functions in $p \in [1,\infty]$ for any integer $n > 1$. We first estimate $v_{n,\infty}$ from above. Clearly

$$\prod_{1\le i<j\le n} \frac{(y_iy_j-1)(y_i-y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{y_i^2-1}{y_i^2} < \prod_{1\le i<j\le n} (y_i-y_j) < \prod_{1\le i\le n} y_i^{n-i}, \quad y \in \mathbb{R}^n_{\ge},$$
$$\int_{1\le y_n\le y_{n-1}\le\ldots\le y_1\le e^{2r}} \prod_{1\le i\le n} y_i^{n-i} dy_1\ldots dy_n \asymp e^{n(n+1)r}.$$

The definition of $v_{n,\infty}$, (5.2.5) and the above inequalities yield $v_{n,\infty} \le n(n+1)$. Fix $\epsilon > 0$ and let

$$\Theta_{n,p,\epsilon}(r) := \{x \in \Theta_{n,\infty}(r) : \quad \epsilon \le x_n, \quad x_i + \epsilon \le x_{i+1}, \quad i = 1,...,n-1\}.$$

A straightforward argument show that

$$\int_{\log y\in\Theta_{n,\infty,\epsilon}(r)} \prod_{1\le i<j\le n} \frac{(y_iy_j-1)(y_i-y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{(y_i^2-1)}{y_i^2} dy_1\ldots dy_n \asymp$$
$$\int_{\log y\in\Theta_{n,\infty,\epsilon}(r)} \prod_{1\le i\le n} y_i^{n-i} dy_1\ldots dy_n \asymp e^{n(n+1)r} \asymp e^{n(n+1)r}.$$

Hence $v_{n,\infty} = n(n+1)$. Similar arguments show that $v_{n,1} = n$. $\square$

Recall that $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ is called a lattice if Γ is discrete and $\mathrm{vol}(\mathbf{SH}_n/\Gamma) < \infty$. Siegel modular group $\mathbf{Sp}(n,\mathbb{Z})$ is a lattice. The volume estimate for discrete groups and lattices [Nic], [EM], [Alb] combined with Proposition 5.2.2 yield:

Theorem 5.2.3 *Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be a discrete group. Then $\delta_p(\Gamma) \le v_{n,p}$. Assume that Γ is a lattice. Then $\delta_p(\Gamma) = v_{n,p}$ and Γ is of divergence type.*

Definition 5.2.4 *Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be a discrete group and $n > 1$. Then Γ is called p-regular if for any $\mu \in \mathcal{M}_{\Gamma,A,p}$* supp $\mu \subset \Lambda(\Gamma)$.

We are interested in conditions which insure that for a given $p \in [1,\infty]$ Γ is p-regular. For a fixed $t \ge 0$ let

$$\mathbf{Sp}(n,\mathbb{R})_t := \{\gamma \in \mathbf{Sp}(n,\mathbb{R}) : \quad \sigma_n(\gamma) \le e^t\},$$
$$\Gamma_t := \Gamma \cap \mathbf{Sp}(n,\mathbb{R})_t.$$

Definition 5.2.5 *Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be a discrete group. Then Γ is called p-strongly regular if for any $t \ge 0$:*

$$\delta_p(\Gamma_t) < \delta_p(\Gamma).$$

Lemma 5.2.6 *Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be a p-strongly regular discrete subgroup of $\mathbf{Sp}(n,\mathbb{R})$ for some $p \in [1,\infty]$. Then $\delta_p(\Gamma) > 0$, $\Lambda(\Gamma) \ne \emptyset$, and* supp $\mu \subset \Lambda(\Gamma)$ *for every $\mu \in \mathcal{M}_{\Gamma,A,p}$.*

Proof. Since $I \in \Gamma$ it follows that $\Gamma_t \neq \emptyset$. Hence $0 \leq \delta_p(\Gamma_t) < \delta_p(\Gamma)$. Moreover, Γ contains a sequence $\{\gamma_k\}_1^\infty$ which satisfies the condition (5.1.1). Hence $\Lambda(\Gamma) \neq \emptyset$. Assume first that Γ is of divergence type. Fix $t > 0$. Let

$$\mu_{\Gamma,s,A,p,t} = \frac{1}{g_{s,p}(\Gamma, B, B)} \sum_{\gamma \in \Gamma_t} e^{-s\, d_p(A,\gamma B)} \Delta_{\gamma B}, \quad s > \delta_p(\Gamma).$$

Then for any sequence $s_m \searrow \delta_p(\Gamma)$ $\mu_{\Gamma,s_m,p,t} \to 0$. Corollary 5.1.3 and Theorem 5.1.2 yield that for any $\mu \in \mathcal{M}_{\Gamma,A,p}$ supp $\mu \subset \Lambda(\Gamma)$. Similar arguments apply if Γ is of p-convergence type. $\Box$

Lemma 5.2.7 *Let Γ be a lattice in $\mathbf{Sp}(n,\mathbb{R})$. Then Γ is strongly regular.*

Proof. Fix $t > 0$. Let $v_{n,p,t}$ be the volume growth of $B(A,r) \cap \mathbf{Sp}(n,\mathbb{R})_t$. Observe that $B(A,r) \cap \mathbf{Sp}(n,\mathbb{R})_t$ has the decomposition (5.2.7) with

$$\{y = e^x : \quad x \in \Theta_{n,p}(r), \quad x_n \leq t\}.$$

Use the arguments of the proof of Proposition 5.2.2 to deduce that $v_{n,p,t} < v_{n,p}$. As Γ is a lattice the volume estimates yield

$$\delta(\Gamma_t) = v_{n,p,t} < v_{n,p} = \delta_p(\Gamma).$$

$\Box$

Note that our results for lattices are analogous to the results of [Alb]. As in [Alb, §4], recent results of Benoist [Be2] imply the existence of many discrete Zariski dense subgroups Γ of $\mathbf{Sp}(n,\mathbb{R})$ which are p-regular for any $p \in [1,\infty]$. Let $H(\Gamma) \subset \mathbb{R}^{2n}$ be the set of rays spanned by all limit directions of the sequences

$$\frac{\log \sigma(\gamma_k)}{||\log \sigma(\gamma_k)||_2}, \quad \gamma_k \in \Gamma, \quad k = 1, \ldots, \quad \lim_{k\to\infty} ||\log \sigma(\gamma_k)||_2 = \infty.$$

As $\log \sigma(\gamma) = -\log \sigma(\gamma)$ for any $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ we deduce the $-H(\Gamma) = H(\Gamma)$. It is shown in [Be2] that if Γ is a Zariski dense subgroup in $\mathbf{Sp}(n,\mathbb{R})$ then $H(\Gamma)$ is a closed convex cone in $\mathbb{R}^{2n}$. Clearly, this cone can be identified with a subcone of $\mathbb{R}^n_+$ (the cone of all nonnegative vectors in $\mathbb{R}^n$). Benoist shows that for any closed convex cone $\mathcal{K} \subset \mathbb{R}^n_+$ there exists a Zariski dense subgroup $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ such that $PH(\Gamma) = \mathcal{K}$, where $P : \mathbb{R}^{2n} \to \mathbb{R}^n$ is the projection given by $P(x_1, \ldots, x_{2n}) = (x_1, \ldots, x_n)$.

Definition 5.2.8 *A discrete subgroup $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ is called generic if*
(a) *Γ is Zariski dense in $\mathbf{Sp}(n,\mathbb{R})$;*
(b) *Any nonzero vector $x \in H(\Gamma)$ has nonzero coordinates.*

Proposition 5.2.9 *Let Γ be a generic subgroup of $\mathbf{Sp}(n,\mathbb{R})$. The Γ is p-regular for any $p \in [1,\infty]$.*

Proof. As Γ is generic, we easily deduce that the set Γ_t is a finite set for any $t \geq 0$. The arguments of the proof of Lemma 5.2.6 yield that supp $\mu \subset \Lambda(\Gamma)$ for any $\mu \in \mathcal{M}_{\Gamma,A,p}$. $\Box$

Theorem 5.2.10 *Let Γ be a discrete Zariski dense subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then $\delta_p(\Gamma) > 0$ for any $p \in [1,\infty]$.*

Proof. The results of Tits [Tit] (see also the results on Schottky groups in [Be2]) imply that Γ contains a free subgroup Γ' on $k \geq 2$ generators such that Γ' is Zariski dense in $\mathbf{Sp}(n,\mathbb{R})$. Fix $p \in [1,\infty]$. Clearly, $\delta_p(\Gamma') \leq \delta_p(\Gamma)$. We use the results in [Fr4] to show that $\delta_p(\Gamma') > 0$. From here until the end of the proof we refer by numbers to the displayed formulas, Theorems and Corollaries in [Fr4]. Let $\gamma_1, ..., \gamma_k$ be a minimal set of generators of Γ'. Associate with these generators a subshift $\mathcal{S}$ of finite type on $2k$ letters $1, ..., 2k$. Here the letter $i \in [1,k]$ corresponds to the generator γ_i and the letter $j \in [k+1, 2k] \cap \mathbb{Z}$ corresponds the generator $\gamma_j := \gamma_{j-k}^{-1}$. $\mathcal{S}$ is the set of reduced infinite words

$$w = \gamma_{i_1}\gamma_{i_2}\ldots, \quad i_j \in [1,2k] \cap \mathbb{Z}, \quad j = 1, ..., \quad |i_j - i_{j+1}| \neq k, \quad j = 1, ... \tag{5.2.8}$$

$\mathcal{S}$ is a compact topological space respect to product topology. Let $\tau : \mathcal{S} \to \mathcal{S}$ be the shift map given by $\tau(w) = \gamma_{i_2}\gamma_{i_3}\ldots$. Let $w_m = \gamma_{1_1}\ldots\gamma_{i_m}$ be a reduced word of length m. Then $C(w_m) \subset \mathcal{S}$ is the set of all infinite words in $\mathcal{S}$ which start with w_m. Define the function $\phi_m : \mathcal{S} \to \mathbb{R}_+$ by assuming that ϕ_m is constant on each $C(w_m)$ and its value is equal to $d_p(I, w_m)$ which is denoted by $\phi_m(w_m)$. As $\mathbf{Sp}(n,\mathbb{R})$ acts as a subgroup of isometries with respect to the metric $d_p(\cdot,\cdot)$ on $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$, we deduce that the family $\{\phi_m\}_1^\infty$ satisfies the conditions (0.1). Since Γ' is discrete the condition (0.2) holds. Hence the sequence $\{\phi_m\}_1^\infty$ defines a metric $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ given by (0.3). Let $\delta(\phi)$ be the Hausdorff dimension of $\mathcal{S}$ with respect to d. Observe next that $\kappa(\phi)$ defined in (1.12) of is equal to $\delta_p(\Gamma')$. Theorem 1.14 yields $\delta_p(\Gamma') \geq \delta(\phi)$. Let $\mathcal{E}$ be the set of ergodic measures on $\mathcal{S}$ with respect to τ. For $\nu \in \mathcal{E}$ one can define the ν-Hausdorff dimension of $\mathcal{E}$ denoted by $\delta(\nu,\phi)$. By the definition $\delta(\nu,\phi) \leq \delta(\phi)$. Theorem 2.4 and Corollary 2.6 yield

$$\delta(\nu,\phi) = \frac{h(\nu)}{\alpha(\nu)} \geq \frac{h(\nu)}{\alpha_1(\nu)}.$$

Here $h(\nu)$ is the entropy of ν and

$$\alpha_1(\nu) = \sum_{j=1}^{2k} d_p(I,\gamma_j)\nu(C(\gamma_j)) \leq \max_{1 \leq j \leq 2k} d_p(I,\gamma_j).$$

Note that for any nontrivial $\gamma \in \Gamma'$ $d_p(I,\gamma) > 1$. Otherwise $\gamma \in \mathbf{K}_n$ and $< \gamma >$ is a discrete, hence a finite subgroup of $\mathbf{K}_n$. This contradicts the freeness of Γ'. Let ν_P be the equidistributed measure given by

$$\nu_P(C(w_m)) = \frac{1}{2k(2k-1)^{m-1}}, \quad m = 1, ...$$

Then Corollary 2.10 yields

$$\delta_p(\Gamma') \geq \delta(\phi) \geq \delta(\nu_P, \phi) \geq \frac{\log(2k-1)}{\alpha_1(\nu_P)},$$

$$\alpha_1(\nu_P) = \frac{1}{2k} \sum_{j=1}^{2k} d_p(I, \gamma_j),$$

and the theorem follows. □

In [Fr4] we show that for a Kleinian Schottky group Γ we have equalities

$$\kappa(\phi) = \delta(\phi) = \sup_{\nu \in \mathcal{E}} \frac{h(\nu)}{\alpha(\nu)}.$$

It is an interesting problem if the above equalities hold for a generic subgroup $\Gamma < \mathbf{Sp}(n, \mathbb{R})$. Theorem 5.2.10 can be considered as a generalization of the result of Beardon [B1] that the Hausdorff dimension of a nonelementary Kleinian group is positive (see [Fr4, (4.1)]).

Corollary 5.2.11 *Let Γ be a generic subgroup of $\mathbf{Sp}(n, \mathbb{R})$. Then Γ is strongly p-regular for any $p \in [1, \infty]$.*

Proof. As Γ_t is a finite set $\delta_p(\Gamma_t) = 0$. Theorem 5.2.10 implies that $\delta_p(\Gamma) > 0$. □

In what follows we restrict our attention to $p = 1$. Recall that 1-Busemann compactification of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ gives the compactification of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ as a bounded domain. In view of Corollary 4.3.9 we identify the 1-Busemann boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ with the Shilov boundary of $\mathbf{SH}_n$. Let Γ be a discrete subgroup of $\mathbf{Sp}(n, \mathbb{R})$. By abuse of notation we view $\Lambda(\Gamma)$ as a closed subset of 1-Busemann boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. Use the definition of the Busemann functions and the standard arguments for Patterson-Sullivan measure as in [Nic] and [Alb] to obtain:

Theorem 5.2.12 *Let Γ be a discrete 1-regular subgroup of $\mathbf{Sp}(n, \mathbb{R})$. Choose a family of Patterson-Sullivan measures $\mu_X \in \mathcal{M}_{\Gamma, X, 1}$ dependending on a parameter $X \in \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. Then*
(a) $\gamma^* \mu_X = \mu_{\gamma^{-1}(X)}$ *for* $\gamma \in \Gamma$;
(b) $\frac{d\mu_X}{d\mu_{X_0}}(\xi) = -\delta_1(\Gamma) b_1(\xi, X_0, X)$ *for any* $\xi \in \Lambda(\Gamma)$.

Note that Proposition 4.3.8 gives a simple explicit formula for $b_1(\xi, X_0, X)$. It is worth to mention that if we want to consider the p versions of Theorem 5.2.12 for $p \in (1, \infty]$ we should consider the p-Busemann compactification of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ instead of 1-Busemann compactification.

5.3 Modified Patterson-Sullivan measures

In this subsection we assume that

$$\Lambda(\Gamma) \neq \emptyset. \tag{5.3.1}$$

We suggest here another definition of the PS measures $\tilde{M}_{\Gamma,A,p}$ so that supp $\mu \subset \Lambda(\Gamma)$ for any $\mu \in \tilde{M}_{\Gamma,A,p}$. For any subgroup $G \leq \mathbf{Sp}(n,\mathbb{R})$ let

$$G^t := G \backslash \mathbf{Sp}(n,\mathbb{R})_t.$$

The assumption (5.3.1) yields that Γ^t is an infinite set for any $t \geq 0$. We now consider families $\mathcal{M}_{\Gamma^t,A,p}$. Clearly, supp $\mu \subset \mathrm{BCl}(\Gamma^t(B))$ for any $\mu \in \mathcal{M}_{\Gamma^t,A,p}$. Let $\tilde{\mathcal{M}}_{\Gamma,A,p}$ be the set of weak limits of measures in $\mathcal{M}_{\Gamma^t,A,p}$ as $t \to \infty$. Note that each $\mu \in \mathcal{M}_{\Gamma^t,B,p}$ is a probability measure on $\mathrm{BCl}(\mathbf{SH}_n)$. Hence $\tilde{\mathcal{M}}_{\Gamma,B,p}$ is a set of probability measures which is supported on $\Lambda(\Gamma)$. Thus $\tilde{\mathcal{M}}_{\Gamma,A,p}$ is a set of positive finite measures which is supported on $\Lambda(\Gamma)$.

Proposition 5.3.1 *Assume that Γ is strongly p-regular. If Γ is of p-divergence type then for each $t \geq 0$ $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$. In particular $\tilde{\mathcal{M}}_{\Gamma,A,p} = \mathcal{M}_{\Gamma,A,p}$. Assume that Γ is of p-convergence then for each $t \geq 0$ it is possible to choose $\mathcal{M}_{\Gamma^t,A,p}$ to be equal to $\mathcal{M}_{\Gamma,A,p}$. For these choices $\tilde{\mathcal{M}}_{\Gamma,A,p} = \mathcal{M}_{\Gamma,A,p}$.*

Proof. Assume first that Γ is of p-divergence type. Then Γ^t is of p-divergence type. The arguments of the proof of Lemma 5.2.6 imply the equality $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$. Assume that Γ is of convergence type. Fix the function $h_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that the series $g^*_{s,p}(\Gamma, A, B)$ given by (5.2.4) diverges for $s = \delta_p(\Gamma)$. For $t \geq 0$ choose $h_{p,t} = h_p$. Then the series $g^*_{s,p}(\Gamma^t, A, B)$ diverges for $s = \delta_p(\Gamma)$ and converges for $s > \delta_p(\Gamma)$. For this choice of $h_{p,t}$ $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$. $\square$

Let

$$\overline{\delta}_p(\Gamma) = \limsup_{t\to\infty} \delta_p(\Gamma^t),$$
$$\underline{\delta}_p(\Gamma) = \liminf_{t\to\infty} \delta_p(\Gamma^t),$$

We conjecture that for any Γ satisfying (5.3.1) each $\mu_X \in \tilde{\mathcal{M}}_{\Gamma,X,p}$ is a β density:

$$\frac{d\mu_X}{d\mu_{X_0}}(\xi) = e^{-\beta b_1(\xi,X_0,X)}, \quad \xi \in \Lambda(\Gamma), \quad \beta \in [\underline{\delta}_1(\Gamma), \overline{\delta}_1(\Gamma)].$$

Theorem 5.3.2 *Let Γ be a discrete Zariski dense subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then $\underline{\delta}_p(\Gamma) > 0$ for any $p \in [1,\infty]$.*

Proof. We use the notations and the results of the proof of Theorem 5.2.10. Consider the free Zariski dense group Γ' on $k > 1$ generators, the assiciated subshift of finite type $\mathcal{S}$

on $2k$ letters and an ergodic measure $\nu \in \mathcal{E}$. With each infinite reducible word $w \in \mathcal{S}$ of the form (5.2.8) we associate the matrix cocycle

$$\mathcal{A}(w, m) = w_m = \gamma_{i_1} \ldots \gamma_{i_m} \in \mathbf{Sp}(n, \mathbb{R}).$$

One can view $\mathcal{A}(w, m)$ as a random product of m matrices from the set $\{\gamma_1, ..., \gamma_{2k}\}$ with respect to the stationary measure ν. The fundamental result of Oseledets [Ose] claims that

$$\lim_{m \to \infty} \frac{\log \sigma_i(w_m)}{m} = \lambda_i(w, \nu), \quad i = 1, ..., 2n, \tag{5.3.2}$$

for almost all $w \in \mathcal{S}$ with respect ν. Since $\Gamma' < \mathbf{Sp}(n, \mathbb{R})$ we deduce that

$$\lambda_i(w, \nu) = -\lambda_{2n+1-i}(w, \nu), \quad i = 1, ..., 2n.$$

As ν is ergodic we obtain $\lambda_i(w, \nu) = \lambda_i(\nu)$, $i = 1, ..., 2n$ are ν-Lyapunov exponents of Γ'. Since $\mathbf{Sp}(n, \mathbb{R})$ is a simple group, the fundamental result of Goldsheid-Margulis [GM1], [GM2] claims that that all the ν-Lyapunov exponents are simple:

$$\lambda_1(\nu) > \ldots > \lambda_n(\nu) > \lambda_{n+1}(\nu) > \ldots > \lambda_{2n}(\nu).$$

As $\lambda_{n+1}(\nu) = -\lambda_n(\nu)$ we deduce that $\lambda_n(\nu) > 0$. That is, for a.a. w with respect to ν

$$\sigma_n(w_m) \asymp e^{m\lambda_n(\nu)}.$$

The arguments of the proofs of Theorem 2.4 and Corollary 2.6 in [Fr4] imply that

$$\delta_p((\Gamma')^t) \geq \delta(\phi) \geq \delta(\nu, \phi)$$

for any $t \geq 0$. Hence

$$\underline{\delta}_p(\Gamma') \geq \delta(\phi) \geq \delta(\nu, \phi).$$

Choose $\nu = \nu_p$ to deduce that

$$\underline{\delta}_p(\Gamma) \geq \underline{\delta}_p(\Gamma') \geq \delta(\nu_p, \phi) > 0.$$

□

5.4 Critical exponent

We now define the critical exponent for the action of Γ on the Shilov boundary $\partial_n \mathbf{SH}_n$ as it done for Kleinian groups [Bis, (1.1)]. As in Proposition 4.3.8 we identify ξ with a unit vector in $\mathbb{R}^{\binom{2n}{n}}$ in the one dimensional subspace $\wedge_n \Xi \subset \mathbb{R}^{\binom{2n}{n}}$. Note that ξ is determined up to a sign. Then

$$\mathrm{dist}(\xi, \eta) := \min(||\xi - \eta||_2, ||\xi + \eta||_2), \quad \xi, \eta \in \partial_n \mathbf{SH}_n.$$

Definition 5.4.1 *A discrete group $\Gamma < \mathbf{Sp}(n, \mathbb{R})$ is called a Siegel group, if $n > 1$, $\Lambda(\Gamma) \neq \emptyset$ and $\Lambda_d(\Gamma)$ is strictly contained in the Shilov boundary of $\mathbf{SH}_n$.*

Note that the Siegel modular group $\mathbf{Sp}(n,\mathbb{Z})$ is a lattice in $\mathbf{Sp}(n,\mathbb{R})$. It can be shown that $\Lambda(\mathbf{Sp}(n,\mathbb{Z}))$ is the Shilov boundary of $\mathbf{SH}_n$. Hence $\mathbf{Sp}(n,\mathbb{Z})$ is not a Siegel group. It is not difficult to show that if $\Gamma_i \in \mathbf{SL}(2,\mathbb{R})$ are Schottky Fuchsian groups for $i = 1, ..., n$ then $\Gamma := \Gamma_1 \odot \ldots \odot \Gamma_n < \mathbf{Sp}(n,\mathbb{R})$ is a Siegel group. The critical exponent $\epsilon(\Gamma,\xi)$ of Siegel group Γ is defined as

$$\epsilon(\Gamma,\xi) := \inf\ \{s > 0 : \quad \sum_{\gamma\in\Gamma} \mathrm{dist}(\gamma(\xi), \Lambda_d(\Gamma))^s\}, \quad \xi \in \Omega(\Gamma). \tag{5.4.1}$$

It seems that $\epsilon(\Gamma,\xi)$ does not depend on $\xi \in \Omega(\Gamma)$. The interesting question is how $\epsilon(\Gamma,\xi)$ is related to $\delta_1(\Gamma)$.

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