# On perturbations of pseudo differential operators with negative definite symbol

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## 1 Introduction

A generator of a Feller semigroup on $C_\infty(\mathbb{R}^n)$ satisfies the positive maximum principle and therefore by a result of Ph. Courrège [2] it has a representation as a pseudo differential operator

$$(1)\qquad -p(x,D)\varphi(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)} p(x,\xi)\cdot\hat{\varphi}(\xi)\, đ\xi$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$ in the domain of the generator. The operators arising in this situation are characterized by the property that the symbols $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, which determine the operator, are for any fixed $x \in \mathbb{R}^n$ continuous and negative definite as a function of $\xi$ (see [1] for the definition), and therefore has a representation by the Lévy-Khinchin formula

$$(2)\qquad p(x,\xi) = q(x,\xi) + i\langle b(x),\xi\rangle + c(x) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - e^{-i\langle y,\xi\rangle} - \frac{i\langle y,\xi\rangle}{1+|y|^2})\,\mu(x,dy).$$

Here for all fixed $x \in \mathbb{R}^n$, $q(x,\xi)$ is a nonnegative quadratic form, $b(x) \in \mathbb{R}^n$, $c(x) \geq 0$, and $\mu(x,dy)$ is a Lévy-measure, i.e. a Borel measure on $\mathbb{R}^n \setminus \{0\}$ such that

$$(3)\qquad \int_{\mathbb{R}^n\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\,\mu(x,dy) < \infty.$$

In [8] the technique of pseudo differential calculus was applied to this type of operator and conditions were given ensuring that a pseudo differential operator actually generates a Feller semigroup. (See also [9, 10] and [5, 6, 7] for other approaches to processes generated by pseudo differential operators.)

The inherent difficulty of this class of symbols lies in the fact that a continuous negative definite function in general is neither differentiable nor satisfies certain homogeneity properties. Hence the symbols are not accessible by a symbolic calculus. Therefore in [8] the symbol was first decomposed by splitting its Lévy-measures in (2) into two parts, one part $p_1(x,\xi)$

containing the portion concentrated around the origin, by (3) this usually is the major part of infinite total mass, and a remainder part $p_2(x, \xi)$ with Lévy-measure supported outside a neighbourhood of the origin.

It turns out that the first dominating part of the symbol is a differentiable function with respect to $\xi$ and we even can expect certain estimates for the derivatives. In [8] these estimates were used to define appropriate symbol classes and develop a calculus for these pseudo differential operators by modifying methods of H. Kumano-go and M. Nagase [11], [12]. In particular, useful estimates in $L^2$-spaces were obtained and a class of Feller generators was determined.

The purpose of this article is to consider the second part of the symbol containing the Lévy-measures restricted to the complement of some neighbourhood of the origin. We will show that the corresponding operator $p_2(x, D)$ can be regarded as a perturbation of the first part $p_1(x, D)$. In Corollary 3.4 we prove that $p_2(x, D)$ typically is a bounded operator on $C_b(\mathbb{R}^n)$ leaving the set $C_\infty(\mathbb{R}^n)$ of continuous functions vanishing at infinity invariant. In particular the property that an operator generates a Feller semigroup is not destroyed by the perturbation $p_2(x, D)$. Moreover we will consider the perturbation effect in an $L^2$-frame for symbols of a certain structure.

Observe that from the probabilistic point of view the Markov process corresponding to $p_2(x, D)$ is a pure jump type process consisting only of large jumps that occur at most finitely often in bounded time intervals. It is well-known how to handle this perturbation by stopping time arguments and therefore $p_2(x, D)$ defines also a perturbation in a probabilistic sense.

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## 2 Preliminaries

Fix a continuous negative definite function $a^2 : \mathbb{R}^n \to \mathbb{R}$. We will later on assume that the symbol satisfies certain upper and lower bounds in terms of this function. The operator $-a^2(D)$ and the corresponding Lévy-process with characteristic exponent $e^{-ta^2}$, $t \geq 0$, then play the same role as the Laplace operator and Brownian motion in the case of diffusion processes.

In order to simplify the notation we work in the following with the square root

$$\lambda(\xi) = (1 + a^2(\xi))^{1/2}. \tag{4}$$

We define the anisotropic Sobolev spaces

$$H^{s,\lambda}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_{s,\lambda} < \infty\},\ s \geq 0,$$

where

$$\|u\|_{s,\lambda} = \left(\int_{\mathbb{R}^n} \lambda(\xi)^{2s} |\hat{u}|^2 \, d\xi\right)^{1/2}.$$

In order to get a type of Sobolev embedding for $H^{s,\lambda}(\mathbb{R}^n)$ we sometimes assume a minimal growth behaviour of $a^2(\xi)$ or $\lambda(\xi)$ at infinity: there is a $r > 0$ such that

$$a^2(\xi) \geq c\,|\xi|^r \qquad \text{for } |\xi| \text{ large.} \tag{5}$$

As in [8] let us introduce the symbol class $\mathrm{S}_{\varrho}^{m,\lambda}$, $m \in \mathbb{R}$, of all symbols $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leq c_{\alpha\beta} \lambda^{m-\varrho(|\alpha|)}(\xi) \qquad \text{for all } \alpha, \beta \in \mathbb{N}_0^n, \tag{6}$$

where $\varrho(k) = k \wedge 2$, $k \in \mathbb{N}_0$.

Throughout this paper we consider pseudo differential operators with symbols

$$p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

such that for all $x \in \mathbb{R}^n$

$$\xi \mapsto p(x,\xi)$$

is a continuous negative definite function. We call such symbols negative definite. In particular we restrict to the case of real-valued symbols. In terms of the Lévy-Khinchin representation (2) this means that the linear term $i\langle b(x), \xi\rangle$ drops out and the Lévy-measures $\mu(x,dy)$ are symmetric. In this case the Lévy-Khinchin representation (2) simplifies to

$$p(x,\xi) = q(x,\xi) + c(x) + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos\langle y, \xi\rangle)\, \mu(x,dy). \tag{7}$$

Moreover we will always assume that $p$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies the estimate

$$p(x,\xi) \leq c(1 + |\xi|^2), \qquad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n. \tag{8}$$

Observe that (8) automatically holds true for fixed $x \in \mathbb{R}^n$. Sometimes we will even assume the stronger condition that the symbol is bounded from above by the fixed continuous negative definite function $a^2$:

$$p(x,\xi) \leq c(1 + a^2(\xi)) = c\lambda^2(\xi). \tag{9}$$

It is easy to see that a pseudo differential operator with (real-valued) negative definite symbol has a equivalent representation as a Lévy-type operator

$$\begin{aligned} -p(x,D)\varphi(x) &= \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} + c(x)\varphi(x) \\ &\quad + \int_{\mathbb{R}^n \setminus \{0\}} \left( \varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1 + |y|^2} \right) \mu(x,dy), \quad \varphi \in C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{10}$$

where $(a_{ij}(x))$ is the coefficient matrix of the quadratic form $q(x,\xi)$ in (7).

We recall the decomposition of the Lévy-measures in [8] as mentioned in the introduction. Let $\theta \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \theta \leq 1$, be even with $\theta(x) = 1$ in a neighbourhood of the origin. Define the symmetric Lévy kernels

$$\mu_1(x,dy) = \theta(y)\mu(x,dy), \qquad \mu_2(x,dy) = (1 - \theta(y))\mu(x,dy)$$

and define the negative definite symbols

$$\begin{aligned} p_1(x,\xi) &= q(x,\xi) + c(x) + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos\langle y, \xi\rangle)\, \mu_1(x,dy), \\ p_2(x,\xi) &= \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos\langle y, \xi\rangle)\, \mu_2(x,dy), \end{aligned} \tag{11}$$

i.e. $p(x,\xi) = p_1(x,\xi) + p_2(x,\xi)$.
Since the support of the Lévy-measures $\mu_1(x,dy)$ is bounded, it turns out (see [8], Prop.2.1) that $p_1(x,\xi)$ is smooth with respect to $\xi$. This can also be seen immediately from the formula (see [6], Proof of Lemma 3.6)

$$p_1(x,\xi) = c(x) + \int_{\mathbb{R}^n} p(x,\eta)\left(\hat{\theta}(\xi-\eta) - \hat{\theta}(\eta)\right) đ\eta. \tag{12}$$

Moreover (12) shows that $p_1$ and $p_2$ are continuous functions on $\mathbb{R}^n \times \mathbb{R}^n$

If $p_1$ satisfies (9) then (6) automatically holds for $m = 2$ and $|\beta| = 0$. Assuming the same behaviour for the derivatives $\partial_x^\beta p_1(x,\xi)$ we see that $p_1 \in \mathrm{S}_\varrho^{2,\lambda}$ is reasonable assumption. We recall some results of [8]

**Theorem 2.1.** *Let $p_1 \in S_\varrho^{2,\lambda}$ and assume that for some $\delta \geq 0$ and $\tau$ sufficiently large*

$$p_1(x,\xi) + \tau \geq \delta\,\lambda^2(\xi). \tag{13}$$

*Then for $\kappa$ sufficiently large the bilinear form $((p_1(x,D)+\kappa)u,v)_{L^2}$, $u,v \in C_0^\infty(\mathbb{R}^n)$ has a continuous extension to a coercive form on $H^{1,\lambda}(\mathbb{R}^n)$.*
*Moreover $p_1(x,D)$ is closable in $L^2(\mathbb{R}^n)$, the closure is given by the continuous extension*

$$p_1(x,D) : H^{2,\lambda}(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

*the estimates*

$$c_1 \|u\|_{2,\lambda} \leq \|p(x,D)u\|_{L^2} + \|u\|_{L^2} \leq c_2 \|u\|_{2,\lambda}, \qquad c_1, c_2 > 0, \tag{14}$$

*hold and for $\kappa$ sufficiently large the equation*

$$(p_1(x,D) + \kappa)u = f$$

*has a unique solution $u \in H^{2,\lambda}(\mathbb{R}^n)$ for every $f \in L^2(\mathbb{R}^n)$. In particular $-(p_1(x,D)+\kappa)$with domain $H^{2,\lambda}(\mathbb{R}^n)$ is the generator of a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$.*
*If in addition $p_1$ is a negative definite symbol and (5) holds, then $-p_1(x,D)$ has an extension that generates a Feller semigroup, i.e. a strongly continuous, positivity preserving contraction semigroup on $C_\infty(\mathbb{R}^n)$.*

# 3 Generators of Feller semigroups

The central question of this article is, whether the original operator $p(x,D)$ which we get from $p_1(x,D)$ by the perturbation $p_2(x,D)$ has the same properties as $p_1(x,D)$. For the readers' convenience we recall a standard perturbation result for generators of semigroups (see for example [3]).

**Theorem 3.1.** *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|)$ and $(B, D(B))$ a linear dissipative operator in $X$ such that $D(A) \subset D(B)$ and*

$$\|Bu\| \leq \alpha \|Au\| + \beta \|u\|, \qquad u \in D(A),$$

*for some $0 \leq \alpha < 1$ and $\beta \geq 0$. Then $(A+B, D(A))$ generates a strongly continuous contraction semigroup.*

The theorem in particular applies to bounded perturbations. Note that in the case of Feller semigroups the generator $A$ satisfies the positive maximum principle, i.e for all $u \in D(A)$ such that $u(x_0) = \sup u(x) \geq 0$ we have $Au(x_0) \leq 0$. Conversely the positive maximum principle implies the dissipativity of an operator on $C_\infty(\mathbb{R}^n)$ and if it generates a strongly continuous contraction semigroup, this is positivity preserving, hence Feller. So if $A$ is the generator of a Feller semigroup then by the above the operator $A+B$ is a generator of a Feller semigroup for every bounded operator $B$ on $C_\infty(\mathbb{R}^n)$ that satisfies the positive maximum principle.

In order to see that $p_2(x, D)$ defines a perturbation of this type first note

**Proposition 3.2.** *Let $p_2$ be as in (11). Then (8) implies that the Lévy-measure $\mu_2(x, dy)$, $x \in \mathbb{R}^n$, have uniformly bounded mass.*

Proof: Denoting by $\nu$ the bounded measure on $\mathbb{R}^n$ with Fourier transform $(1+|y|^2)^{-1} = \int \cos(y, \xi)\, \nu(d\xi)$. Then

$$\begin{aligned} \mu_2(x, \mathbb{R}^n \setminus \{0\}) &= \int_{\mathbb{R}^n\setminus\{0\}} (1-\theta(y))\, \mu(x, dy) \leq c \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - \frac{1}{1+|y|^2}\right) \mu(x, dy) \\ &= c \int_{\mathbb{R}^n\setminus\{0\}} \int_{\mathbb{R}^n} (1-\cos(y,\xi))\, \nu(d\xi)\mu(x, dy) \leq c \int_{\mathbb{R}^n} p(x,\xi)\, \nu(d\xi) \\ &\leq c \int_{\mathbb{R}^n} (1+|\xi|^2)\, \nu(d\xi) < \infty \end{aligned}$$

by [6], Lemma 2.2. $\square$

If we apply the representation (10) to the operator $p_2(x, D)$ then the third term in the integro-differential part becomes integrable with respect to the finite Lévy-measures $\mu_2(x, dy)$ and vanishes by the symmetry of the measures. Thus we find

$$-p_2(x, D)\varphi(x) = \int_{\mathbb{R}^n\setminus\{0\}} (\varphi(x+y) - \varphi(x))\, \mu_2(x, dy), \qquad \varphi \in C_0^\infty(\mathbb{R}^n). \tag{15}$$

Moreover for all $\psi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi(0) = 0$,

$$\int_{\mathbb{R}^n\setminus\{0\}} \psi(y)\, \mu_2(x, dy) = -\int_{\mathbb{R}^n\setminus\{0\}} \int_{\mathbb{R}^n} (1-\cos(y,\xi))\hat{\psi}(\xi)\, đ\xi\, \mu_2(x, dy) = -\int_{\mathbb{R}^n} p_2(x,\xi)\hat{\psi}(\xi) đ\xi$$

and the continuity of $p_2$ proves continuous dependence of $\mu_2(x, dy)$ on $x$ with respect to the vague topology. Thus Proposition 3.2 shows that $p_2(x, D)$ has extensions to the bounded Borel measurable functions $\mathbf{B}_b(\mathbb{R}^n)$ and the bounded continuous functions $C_b(\mathbb{R}^n)$

$$\begin{aligned} p_2(x, D) &: \mathbf{B}_b(\mathbb{R}^n) \to \mathbf{B}_b(\mathbb{R}^n) \\ &: C_b(\mathbb{R}^n) \to C_b(\mathbb{R}^n), \end{aligned}$$

which are continuous with a bound given by $2 \sup \|\mu_2(x, \cdot)\|_\infty$. In order to apply Theorem 3.1 to the perturbation $-p_2(x, D)$ in the case of Feller semigroups we have to show that $C_\infty(\mathbb{R}^n)$ is invariant under $p_2(x, D)$. In general this is not true, since the non-local character of $p_2(x, D)$ may destroy the behaviour at infinity. A reasonable condition to control the non-locality is the tightness of the Lévy-measures $\mu_2(x, dy)$. Here we consider the Lévy-measures $\mu$ as measures extended to $\mathbb{R}^n$ by $\mu(\{0\}) = 0$. We give a complete characterization in terms of the symbol.

**Theorem 3.3.** *Let $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite symbol such that $p(x,\xi) \leq c(1+|\xi|^2)$ and with Lévy-Khinchin representation*

$$p(x,\xi) = q(x,\xi) + c(x) + \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,\mu(x,dy).$$

*Then the following are equivalent:*

*(i) the family $\mu(x,dy)$, $x \in \mathbb{R}^n$ of measures on $\mathbb{R}^n$ is tight,*

*(ii) $\sup\limits_{x\in\mathbb{R}^n} (p(x,\xi) - p(x,0)) \to 0 \quad$ as $\xi \to 0$.*

*In this case $p(x,D)$ maps $C_0^\infty(\mathbb{R}^n)$ into $C_\infty\mathbb{R}^n$.*

Note that typically a condition on the symbol for small $\xi$, here the equicontinuity at $\xi = 0$, implies properties of the Lévy-measures are infinity.

Proof: Note that by the assumptions $q(x,\xi) + c(x) \leq c(1+|\xi|^2)$ therefore $c(x)$ and the coefficients of $q(x,\cdot)$ are bounded. Thus they are equicontinuous at $\xi = 0$ and we may assume that

$$p(x,\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,\mu(x,dy).$$

Assume that (ii) holds true. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\mathrm{supp}\varphi \subset B_1(0)$ and $\varphi_R(x) = \varphi(x/R)$, $R \geq 1$. Then

$$\begin{aligned}
\mu(x,\complement B_R(0)) &\leq \int_{\mathbb{R}^n\setminus\{0\}} (\varphi_R(0)-\varphi_R(y))\,\mu(x,dy) \\
&= \int_{\mathbb{R}^n\setminus\{0\}} \int_{\mathbb{R}^n} (1-\cos(y,\xi))\hat{\varphi_R}(\xi)\,đ\xi\,\mu(x,dy) \leq \int_{\mathbb{R}^n} p(x,\xi)\,|\hat{\varphi_R}(\xi)|\,đ\xi \\
&\leq \int_{|\xi|\leq\frac{1}{\sqrt{R}}} p(x,\xi)R^n\,|\hat{\varphi}(R\xi)|\,đ\xi + c\int_{|\xi|>\frac{1}{\sqrt{R}}} (1+|\xi|)^2 R^n (R\xi)^{-(n+3)}\,đ\xi \\
&\leq c \sup_{|\xi|\leq\frac{1}{\sqrt{R}}} p(x,\xi)\cdot\|\hat{\varphi}\|_{L^1} + cR^{-3/2} \to 0 \text{ as } R\to\infty
\end{aligned} \tag{16}$$

uniformly with respect to $x$, i.e. $\mu(x,\cdot)$ is tight.
Conversely, if (ii) is not true, then there is a sequence $\xi_n \to 0$ such that

$$\sup_{x\in\mathbb{R}^n} p(x,\xi_n) = \eta > 0$$

and we can choose $x_n \in \mathbb{R}^n$, $n \in \mathbb{N}$, such that $p(x_n,\xi_n) > \eta/2$. Then for $K \subset \mathbb{R}^n$ compact and any $x,\xi \in \mathbb{R}^n$

$$\begin{aligned}
\mu(x,\complement K) &\geq \frac{1}{2}\int_{\complement K} (1-\cos(y,\xi))\,\mu(x,dy) \\
&= \frac{1}{2}p(x,\xi) - \frac{1}{2}\int_K (1-\cos(y,\xi))\,\mu(x,dy).
\end{aligned} \tag{17}$$

Let again $\nu$ be the representation measure of the positive definite function $(1+|y|^2)^{-1} = \int_{\mathbb{R}^n} e^{-i(y,\xi)}\,\nu(d\xi)$ and $A = \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} p(x,\xi)\,\nu(d\xi) \leq c\int_{\mathbb{R}^n}(1+|\xi|^2)\,\nu(d\xi) < \infty$. Choose $a>0$ such that $a < \dfrac{\eta}{4A}$. There is a $n_0 = n_0(K) \in \mathbb{N}$ such that for $n \geq n_0$

$$1-\cos(y,\xi_n) \leq a\Big(1-\frac{1}{1+|y|^2}\Big) \qquad \text{for all } y \in K$$

and therefore for all $n \geq n_0$

$$\begin{aligned}
&\int_K (1-\cos(y,\xi_n))\,\mu(x,dy) \leq a\int_{\mathbb{R}^n\setminus\{0\}}\Big(1-\frac{1}{1+|y|^2}\Big)\,\mu(x,dy)\\
&= \; a\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n}(1-\cos(y,\xi))\,\nu(d\xi)\,\mu(x,dy) = a\int_{\mathbb{R}^n} p(x,\xi)\,\nu(d\xi) \leq a\cdot A < \frac{\eta}{4}.
\end{aligned}$$

Hence by (17) for all $n \geq n_0$

$$\mu(x_n, \complement K) \geq \frac{1}{2}p(x_n,\xi_n) - \frac{1}{2}\cdot\frac{\eta}{4} \geq \frac{\eta}{8} > 0.$$

and $\mu(x,\cdot)$, $x \in \mathbb{R}^n$, is not tight.
Finally, if (i) or (ii) is satisfied and $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $p(x,D)\varphi$ is continuous and by (10) and (16) for $x \notin \operatorname{supp}\varphi$

$$|p(x,D)\varphi(x)| = \left|\int_{\mathbb{R}^n\setminus\{0\}} \varphi(x+y)\,\mu(x,dy)\right| \leq \|\varphi\|_\infty \cdot \mu(x, \operatorname{supp}\varphi(x+\cdot)) \to 0$$

as $|x| \to \infty$, i.e. $p(x,D)\varphi \in C_\infty(\mathbb{R}^n)$. □

As $C_0^\infty(\mathbb{R}^n)$ is dense in $C_\infty(\mathbb{R}^n)$ the result implies in particular in the situation considered above

**Corollary 3.4.** *Let $p_2$ be as in (11) and assume*

(18) $$\sup_{x\in\mathbb{R}^n} p_2(x,\xi) \to 0 \;\; as \;\; \xi \to 0.$$

*Then $p_2(x,D)$ maps $C_\infty(\mathbb{R}^n)$ continuously into itself.*

Observe that (18) is determined directly by the original symbol $p(x,\xi)$, because $p_1$ in (11) always satisfies $\sup_{x\in\mathbb{R}^n}(p_1(x,\xi) - p_1(x,0)) \to 0$ as $\xi \to 0$. This can be seen for example using Theorem 3.3, since the Lévy-measures of $p_1$ are supported in a bounded set. Thus (18) is equivalent to the condition

(19) $$\sup_{x\in\mathbb{R}^n}(p(x,\xi) - p(x,0)) \to 0 \text{ as } \xi \to 0.$$

We combine the results with the perturbation argument of Theorem 3.1 and the subsequent remark. Note that (15) shows that $-p_2(x,D)$ obviously satisfies the positive maximum principle on $C_\infty(\mathbb{R}^n)$. Thus we have proven

**Theorem 3.5.** *Let $p$ be a continuous negative definite symbol that satisfies (8) and (19) with decomposition (11). Assume that $-p_1(x,D)$ extends to the generator of a Feller semigroup. Then $-p(x,D)$ has the same property.*

**Corollary 3.6.** *Let $p$ be a continuous negative definite symbol with (8) and $\lambda(\xi)$ be as in (4), (5). If $p$ satisfies (19), $p(x,\xi) \geq \delta\lambda^2(\xi)$ and the mollified symbol*

$$(x,\xi) \mapsto \int_{\mathbb{R}^n} p(x,\eta)\hat{\theta}(\xi-\eta)\, đ\eta$$

*is in $S_\varrho^{2,\lambda}$. Then $-p(x,D)$ has an extension that generates a Feller semigroup.*

Proof: By (11) and (12) we decompose $p$ as

$$p(x,\xi) = \int_{\mathbb{R}^n} p(x,\eta)\hat{\theta}(\xi-\eta)\, đ\eta - \int_{\mathbb{R}^n} p(x,\eta)\hat{\theta}(\eta)\, đ\eta + p_2(x,\xi) + c(x).$$

The assumptions yield that $x \mapsto \int_{\mathbb{R}^n} p(x,\eta)\hat{\theta}(\eta)\, đ\eta$ is in $C_b^\infty(\mathbb{R}^n)$ and $c \in C_b(\mathbb{R}^n)$. Therefore the symbol $\int_{\mathbb{R}^n} p(x,\eta)(\hat{\theta}(\xi-\eta) - \hat{\theta}(\eta))\, đ\eta$ is in $\mathrm{S}_\varrho^{2,\lambda}$ and since $p_2(x,\xi)+c(x)$ is bounded, it has a lower bound as in (13). Thus it satisfies the assumptions of Theorem 2.1 and the corresponding pseudo differential operator has an extension that generates a Feller semigroup. Moreover the symbol $p_2(x,\xi)+c(x)$ defines a bounded operator on $C_\infty(\mathbb{R}^n)$ and we conclude as above. $\square$

# 4 $L^2$ – estimates

We decompose a continuous negative definite symbol as in (11)

$$p(x,\xi) = p_1(x,\xi) + p_2(x,\xi).$$

Whereas for the part $p_1(x,\xi)$ which is smooth with respect to $\xi$ we can expect estimates in an $L^2$-framework (see Theorem 2.1 and also [8] for more detailed results), such estimates are difficult to prove for the non-smooth part $p_2(x,\xi)$ and in general it is even not true. Therefore we restrict in this section to symbols of a more specific structure. More precisely, we consider symbols with Lévy-Khinchin representation (7) such that each measure $\mu(x,\cdot)$ is absolutely continuous with respect to a given fixed Lévy-measure $\mu$. This is to say

$$\mu(x,dy) = f_x(y)\,\mu(dy). \tag{20}$$

Let $\theta$ be as above. We decompose $\mu$ in the same way as the Lévy-kernel:

$$\mu_1(dy) = \theta(y)\mu(dy), \qquad \mu_2(dy) = (1-\theta(y))\mu(dy).$$

then $\mu_1$ has bounded support and $\mu_2$ is a bounded measure. The decomposition (11) becomes

$$\begin{aligned} p_1(x,\xi) &= \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))f_x(y)\,\mu_1(dy), \\ p_2(x,\xi) &= \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))f_x(y)\,\mu_2(dy). \end{aligned} \tag{21}$$

For reasons of simplicity we omit the local terms $q(x,\xi)+c(x)$ in this section. We are again interested in results stating that $p_2(x,D)$ is a small perturbation $p_1(x,D)$, in particular that $p_2(x,D)$ is bounded in $L^2(\mathbb{R}^n)$.

**Proposition 4.1.** *Assume that $f_x(y) \le M$ for some $M \ge 0$. Then $p_2(x,D)$ is bounded in $L^2(\mathbb{R}^n)$:*

$$\|p_2(x,D)u\|_{L^2} \le c\,\|u\|_{L^2}\,. \tag{22}$$

Proof: It is enough to prove (22) for $u \in C_0^\infty(\mathbb{R}^n)$. Since the Lévy-kernel of $p_2$ consists of finite measures we have

$$\begin{aligned} -p_2(x,D)u(x) &= \int_{\mathbb{R}^n\setminus\{0\}} (u(x+y)-u(x))f_x(y)\,\mu_2(dy) \\ &= \int_{\mathbb{R}^n\setminus\{0\}} u(x+y)f_x(y)\,\mu_2(dy) - \left(\int_{\mathbb{R}^n\setminus\{0\}} f_x(y)\,\mu_2(dy)\right)\cdot u(x) \end{aligned}$$

and it is enough to prove continuity in $L^2(\mathbb{R}^n)$ for the first term. Let $v \in L^2(\mathbb{R}^n)$.

$$\begin{aligned} \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus\{0\}} u(x+y)\;\; f_x(y)\,\mu_2(dy)\cdot v(x)\,dx\right| &\le M\int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n} |u(x+y)|\cdot|v(x)|\;dx\,\mu_2(dy) \\ \le\quad & M\int_{\mathbb{R}^n\setminus\{0\}}\left(\int_{\mathbb{R}^n}|u(x+y)|^2\,dx\right)^{1/2}\cdot\left(\int_{\mathbb{R}^n}|v(x)|^2\,dx\right)^{1/2}\mu_2(dy) \\ =\quad & M\cdot\mu_2(\mathbb{R}^n\setminus\{0\})\cdot\|u\|_{L^2}\cdot\|v\|_{L^2}\,. \end{aligned}$$

Dividing by $\|v\|_{L^2}$ and taking the supremum over all $v$ with $\|v\|_{L^2}=1$ gives the result □

**Theorem 4.2.** *Decompose $p = p_1 + p_2$ as in (21) and assume that $p_1$ is as in Theorem 2.1. If $f_x(y) \le M$ for some $M \ge 0$ then the following estimates hold*

$$\begin{aligned} \|p(x,D)u\|_{L^2} &\le c\,\|u\|_{2,\lambda}\,, \\ \|u\|_{2,\lambda} &\le c(\|p(x,D)u\|_{L^2} + \|u\|_{L^2}), \\ |(p(x,D)u,v)_{L^2}| &\le c\,\|u\|_{1,\lambda}\cdot\|v\|_{1,\lambda}\,, \\ (p(x,D)u,v)_{L^2} &\ge c_1\,\|u\|_{1,\lambda}^2 - c_2\,\|u\|_{L^2}^2 \end{aligned}$$

*for $u, v \in C_0^\infty(\mathbb{R}^n)$. Moreover the extension of $-p(x,D)$ to $H^{2,\lambda}(\mathbb{R}^n)$ is the generator of a strongly continuous semigroup in $L^2(\mathbb{R}^n)$.*

Proof: This follows immediately from the corresponding estimates for $p_1(x,D)$, the boundedness of $p_2(x,D)$ in $L^2(\mathbb{R}^n)$, and the continuous embeddings $H^{2,\lambda}(\mathbb{R}^n) \hookrightarrow H^{1,\lambda}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. □

**Remark:** The assumption $p_1 \in \mathrm{S}_\varrho^{2,\lambda}$ of Theorem 4.2 is fulfilled if the densities $f_x$ depend smoothly on $x$ and the derivatives are bounded, i.e.

$$\left|\partial_x^\beta f_x(\cdot)\right| \le M_\beta \qquad \text{for all } \beta \in \mathbb{N}_0^n. \tag{23}$$

Then $p_1 \in \mathrm{S}_{\varrho}^{2,\lambda}$ for $\lambda = (1+a^2)^{1/2}$ and $a^2(\xi) = \int_{\mathbb{R}^n\setminus\{0\}}(1-\cos(y,\xi))\,\mu(dy)$. In fact $\left|\partial_x^\beta p_1(x,\xi)\right| \leq M_0\,\lambda^2(\xi)$ and for $|\alpha| \geq 1$

$$\left|\partial_\xi^\alpha \partial_x^\beta p_1(x,\xi)\right| \leq M_\beta \cdot \int_{\mathbb{R}^n\setminus\{0\}} \left|y^\alpha \cdot \cos^{(|\alpha|)}(y,\xi)\right| \mu_1(dy),$$

which gives for $|\alpha| = 1$ by the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\partial_\xi^\alpha \partial_x^\beta p_1(x,\xi)\right| &\leq M_\beta \left\|y^{2\alpha}\right\|_{L^2(\mu_1)} \cdot \left(\int_{\mathbb{R}^n\setminus\{0\}} \sin^2(y,\xi)\,\mu_1(dy)\right)^{1/2} \\ &\leq c_{\alpha\beta}\left(\int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,\mu_1(dy)\right)^{1/2} \leq c_{\alpha\beta}\lambda(\xi)\end{aligned}$$

and for $|\alpha| \geq 2$

$$\left|\partial_\xi^\alpha \partial_x^\beta p_1(x,\xi)\right| \leq M_\beta \int_{\mathbb{R}^n\setminus\{0\}} |y|^{|\alpha|}\ \mu_1(dy) = c_{\alpha,\beta}.$$

If moreover $f_x(y) \geq m > 0$ we also have the ellipticity bound:

$$p_1(x,\xi) \geq m \cdot \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,\mu_1(dy) \geq m\ a^2(\xi) - 2m \cdot \mu_2(\mathbb{R}^n \setminus \{0\}),$$

i.e. for $\tau$ sufficiently large

$$p_1(x,\xi) + \tau \geq\ \lambda^2(\xi). \tag{24}$$

**Example:** Consider symbols of the type

$$p(x,\xi) = \sum_{j=1}^N b_j(x) \cdot a_j^2(\xi),$$

where $N \in \mathbb{N}$, $b_j : \mathbb{R}^n \to \mathbb{R}_+$ are bounded functions, and $a_j^2 : \mathbb{R}^n \to \mathbb{R}$ are continuous negative definite functions with Lévy-Khinchin representation

$$a_j^2(\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,\mu_j(dy).$$

Define a Lévy-measure $\mu = \sum_{j=1}^N \mu_j$ that corresponds to the continuous negative definite function $a^2(\xi) = \sum_{j=1}^N a_j^2(\xi)$, and let again $\lambda(\xi) = (1+a^2(\xi))^{1/2}$
Then there are functions $h_j : \mathbb{R}^n \to \mathbb{R}_+$, $|h| \leq 1$ such that $\mu_j = h_j \cdot \mu$ and the Lévy-Khinchin representation of $p(x,\xi)$ is given by

$$p(x,\xi) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\ \sum_{j=1}^N b_j(x)h_j(y)\,\mu(dy),$$

i.e. $f_x(y) = \sum_{j=1}^N b_j(x) \cdot h_j(y)$ in the above notation.
Therefore if $b_j \in C_b^\infty(\mathbb{R}^n)$ for $j = 1, \ldots, N$ then (23) holds and $p_1 \in \mathrm{S}_{\varrho}^{2,\lambda}$.
If moreover $b_j \geq K > 0$, then

$$\begin{aligned}p_1(x,\xi) &= \sum_{j=1}^N \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\,b_j(x)\,\theta(y)\,h_j(y)\mu(dy) \geq K\int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(y,\xi))\theta(y)\mu(dy) \\ &\geq a^2(\xi) - 2K\,((1-\theta)\cdot\mu)(\mathbb{R}^n \setminus \{0\}),\end{aligned}$$

i.e.we also have the lower bound (24).

Note that this example extends results obtained in [9] and [4] to the case where $b_j$ does not vanish asymptotically at infinity.

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