ON THE PROP CORRESPONDING TO BIALGEBRAS

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1 Introduction

It is well-known that there exists a PROP whose category of models in the tensor category of vector spaces is equivalent to the category of bialgebras (= associative and coassociative bialgebras). In [10] there is a description of this PROP in terms of generators and relations. Here we give a more explicit construction of the same object. Our construction uses the Quillen's Q-construction for double categories given in [4].

The paper is organized as follows: In Section 2 we recall the definition of PROP and show how to obtain commutative algebras as models over $\mathcal{F}$. Here $\mathcal{F}$ is the PROP of finite sets. In the next section we construct the PROP of noncommutative sets denoted by $\mathcal{F}(as)$. The models of $\mathcal{F}(as)$ are associative algebras. In Section 4 we generalize the notion of Mackey functor for double categories and in Section 5 we describe our hero $\mathcal{QF}(as)$, which is the PROP, whose models are bialgebras. By definition of PROP the category $\mathcal{QF}(\mathrm{as})$ encodes the natural transformations $H^{\otimes n} \to H^{\otimes m}$ and relations between them. Here H runs over all bialgebras. As a sample we give the following application. For any bialgebra H, any natural number $n \in \mathbb{N}$ and any permutation $\sigma \in \mathfrak{S}_n$, we let

$$\Psi^{(n,\sigma)} : H \to H$$

be the composition $\mu^n \circ \sigma_* \circ \Delta^n : H \to H$, where $\Delta^n : H \to H^{\otimes n}$ is the $(n-1)$-th iteration of the comultiplication $\Delta : H \to H \otimes H$, $\sigma_* : H^{\otimes n} \to H^{\otimes n}$ is induced by the permutation σ, that is

$$\sigma_*(x_1 \otimes \cdots \otimes x_n) = x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n}$$

and $\mu^n : H^{\otimes n} \to H$ is the $(n-1)$-th iteration of the multiplication $\mu : H \otimes H \to H$. Moreover let $\Phi : \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ be the map constructed in Proposition 5.3. Then it is a consequence of our discussion in Section 5, that for any permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$ one has the equality

$$\Psi^{(n,\sigma)} \circ \Psi^{(m,\tau)} = \Psi^{(nm,\Phi(\sigma,\tau))}.$$

Let us note that if σ is the identity, then $\Psi^{(n,id)}$ is nothing but the Adams operation [8] and hence our formula gives the rule for the composition of Adams operations.

2 Preliminaries on PROP's

Recall that a *symmetric monoidal category* is a category $\mathbf{S}$ with a unit $0 \in \mathbf{S}$ and a bifunctor

$$\Box : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$$

together with natural isomorphisms

$$a_{X,Y,Z} : X\Box(Y\Box Z) \to (X\Box Y)\Box Z,$$

$$l_X : X\Box 0 \to X, r_X : 0\Box X \to X, c_{X,Y} : X\Box Y \to Y\Box X$$

satisfying some coherent conditions (see [5]). If in addition $a_{X,Y,Z}$, l_X, r_X are identity morphism then, $\mathbf{S}$ is called a *permutative category.* If $\mathbf{S}$ and $\mathbf{S_1}$ are symmetric monoidal categories, then a functor $M : \mathbf{S} \longrightarrow \mathbf{S_1}$ is a *monoidal functor* if there exist isomorphisms

$$u_{X,Y} : M(X)\Box M(Y) \longrightarrow M(X\Box Y)$$

satisfying the usual associativity and unit coherence conditions (see [5]). A monoidal functor is called *strict* if $u_{X,Y}$ is identity for all $X, Y \in \mathbf{S}$. According to [5] a PROP is a permutative category $(\mathbf{A}, \Box)$, with the following property: $\mathbf{A}$ has a set of objects equal to the set of natural numbers and on objects the bifunctor $\Box$ is given by $m\Box n = m + n$. A *model* of $\mathbf{A}$ is a strict monoidal functor from $\mathbf{A}$ to the tensor category $Vect$ of vector spaces over a field k.

Examples. 1) Let $\mathcal{F}$ be the category of finite sets. For any $n \geq 0$, we let $\underline{n}$ be the set $\{1, ..., n\}$. Hence $\underline{0}$ is the empty set. We assume that the objects of $\mathcal{F}$ are the sets $\underline{n}$, $n \geq 0$. The disjoint union makes the category $\mathcal{F}$ a PROP. It is well-known that *the category* *of models of $\mathcal{F}$ is equivalent to the category of commutative and associative algebras with unit.* Indeed, if A is such an algebra, then the functor $\mathcal{L}_*(A) : \mathcal{F} \to Vect$ is a model. Here the functor $\mathcal{L}_*(A)$ is given by

$$\mathcal{L}_*(A)(\underline{n}) = A^{\otimes n}.$$

For any map $f : \underline{n} \to \underline{m}$, the action of f on $\mathcal{L}_*(A)$ is given by

$$f_*(a_1 \otimes \cdots \otimes a_n) := b_1 \otimes \cdots \otimes b_m,$$

where

$$b_j = \prod_{f(i)=j} a_i, \ j = 1, \cdots, n.$$

Conversely, assume T is a model of $\mathcal{F}$. We let A be the value of T on $\underline{1}$. The unique map $\underline{2} \to \underline{1}$ yields a homomorphism

$$\mu : A \otimes A \cong T(\underline{2}) \to T(\underline{1}) = A.$$

On the other hand the unique map $\underline{0} \to \underline{1}$ yields a homomorphism $\eta : k = T(\underline{0}) \to T(\underline{1}) = A$. The pair (μ, η) defines on A a structure of commutative and associative algebra with unit. One can use the fact that T is strict monoidal to prove that $T \cong \mathcal{L}_*(A)$.

2) Let us note that the opposite of a PROP is still a PROP with the same $\Box$. Hence the disjoint union yields also a structure of PROP on $\mathcal{F}^{op}$. *The category of models of $\mathcal{F}^{op}$ is equivalent to the category of cocommutative and coassociative coalgebras with counit.* For any such coalgebra C we let $\mathcal{L}^*(C) : \mathcal{F}^{op} \to Vect$ be the corresponding model. On objects we still have $\mathcal{L}^*(C)(\underline{n}) = C^{\otimes n}$.

3) We let Ω be the subcategory of $\mathcal{F}$, which has the same objects as $\mathcal{F}$, but morphisms are surjections. Clearly Ω is a subPROP of $\mathcal{F}$ and the models of Ω are (nonunital) commutative algebras.

4) We let **Mon** be the category of finitely generated free monoids, which is a PROP with respect to coproduct. Similarly the category **Abmon** of finitely generated free abelian monoids, the category **Ab** of finitely generated free abelian groups and the category **Gr** of finitely generated free groups are PROP's with respect to coproducts. For the category of models of these PROP's see Theorem 5.2 and Remark 1 at the end of the paper.

In the next section we give a noncommutative generalization of Examples 1)-3).

3 Noncommutative sets

We introduce the PROP $\mathcal{F}(\mathrm{as})$ whose models are associative algebras with unit. Objects of $\mathcal{F}(\mathrm{as})$ are finite sets. So $Ob(\mathcal{F}) = Ob(\mathcal{F}(\mathrm{as}))$. A morphism from $\underline{n}$ to $\underline{m}$ is a map $f : \underline{n} \to \underline{m}$ together with a total ordering on $f^{-1}(j)$ for all $j \in \underline{m}$. By abuse of notation we will denote morphisms in $\mathcal{F}(\mathrm{as})$ by f, g etc. Moreover sometimes we write $\mid f \mid$ for the underlying map of $f \in \mathcal{F}(\mathrm{as})$. We will also say that f is a noncommutative lifting of a map $\mid f \mid$. In order to define the composition in $\mathcal{F}(\mathrm{as})$ we recall the definition of ordered union of ordered sets. Assume Λ is a totally ordered set and for each $\lambda \in \Lambda$ a totally ordered set X_λ is given. Then $X = \coprod X_\lambda$ is the disjoint union of the sets X_λ which is ordered as follows. If $x \in X_\lambda$ and $y \in X_\mu$, then $x \leq y$ in X iff $\lambda < \mu$ or $\lambda = \mu$ and $x \leq y$ in X_λ.

If $f \in \mathrm{Hom}_{\mathcal{F}(\mathrm{as})}(\underline{n}, \underline{m})$ and $g \in \mathrm{Hom}_{\mathcal{F}(\mathrm{as})}(\underline{m}, \underline{k})$, then the composite gf is $\mid g \mid\mid f \mid$ as a map, while the total ordering in $(gf)^{-1}(i)$, $i \in \underline{k}$ is given by the identification

$$(gf)^{-1}(i) = \coprod_{j \in g^{-1}(i)} f^{-1}(j).$$

Clearly one has the forgetful functor $\mathcal{F}(\mathrm{as}) \to \mathcal{F}$. A morphism f in $\mathcal{F}(\mathrm{as})$ is called a *surjection* if the map $\mid f \mid$ is a surjection. An *elementary surjection* is a surjection $f : \underline{n} \to \underline{m}$ for which $n - m \leq 1$.

Remark. The category $\mathcal{F}(\mathrm{as})$ is isomorphic to the category ΔS considered in [7], [4].

Since any injective map has the unique noncommutative lifting, we see that the disjoint union, which defines the symmetric monoidal category structure in $\mathcal{F}$ has the unique lifting in $\mathcal{F}(\mathrm{as})$. Hence $\mathcal{F}(\mathrm{as})$ is a PROP.

We claim that *the category of models of* $\mathcal{F}(\mathrm{as})$ *is equivalent to the category of associative algebras with unit.* The only point here is the following. Let us denote by $\prod_{i \in I}^{<} x_i$ the product of the elements $x_i \in A$ where I is a finite totally ordered set and the ordering in the product follows to the ordering I. Here A is an associative algebra. Now we have a model $\mathcal{X}_*(A) : \mathcal{F}(\mathrm{as}) \to Vect$. Here the functor $\mathcal{X}_*(A)$ is given by the same rule as $\mathcal{L}_*(A)$ in the previous section, but to take $\prod^{<}$ in the definition of b_j. For example, if $f : \underline{4} \to \underline{3}$ is given by $f(1) = f(2) = f(4) = 3$, $f(3) = 1$ and the total ordering in $f^{-1}(3)$ is $2 < 4 < 1$ then $f_* : A^{\otimes 4} \to A^{\otimes 3}$ is nothing but $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_3 \otimes 1 \otimes a_2 a_4 a_1$.

We let $\Omega(as)$ be the subcategory of $\mathcal{F}(as)$, which has the same objects as $\mathcal{F}(as)$, but morphisms are surjections. Clearly $\Omega(as)$ is subPROP of $\mathcal{F}(as)$ and the models of $\Omega(as)$ are (nonunital) associative algebras.

Quite similarly, for any coassociative coalgebra C with counit one has a model $\mathcal{X}^*(C) : \mathcal{F}(\mathrm{as})^{op} \to Vect$ with $\mathcal{X}^*(C)(\underline{n}) = C^{\otimes n}$ and *the category of models of* $\mathcal{F}(\mathrm{as})^{op}$ *is equivalent to the category of coassociative coalgebras with counit.*

In order to put bialgebras in the picture we need the language of Mackey functors.

4 On double categories and Mackey functors

Let us recall that a *double category* consists of objects, a set of horizontal morphisms, a set of vertical morphisms and a set of bimorphisms satisfying natural conditions (see [4]). If $\mathbf{D}$ is a double category, we let $\mathbf{D}^h$ (resp. $\mathbf{D}^v$) be the category of objects and horizontal (resp. vertical) morphisms of $\mathbf{D}$.

A *Janus functor* M from a double category $\mathbf{D}$ to $Vect$ is the following data

i) a covariant functor $M_* : \mathbf{D}^h \to Vect$

ii) a contravariant functor $M_* : (\mathbf{D}^v)^{op} \to Vect$

such that for each object $S \in \mathbf{D}$ one has $M_*(S) = M^*(S) = M(S)$. A *Mackey functor* $M = (M_*, M^*)$ from a double category $\mathbf{D}$ to $Vect$ is a Janus functor M from a double category $\mathbf{D}$ to $Vect$ such that for each bimorphism in $\mathbf{D}$

$$\alpha = \begin{array}{ccc} U & \xrightarrow{f_1} & S \\ \downarrow \ \phi_1 & & \phi \ \downarrow \\ T & \xrightarrow{f} & V \end{array}$$

the following equality holds:

$$M^*(\phi)M_*(f) = M_*(f_1)M^*(\phi_1)$$

Examples 1) Let $\mathbf{C}$ be a category with pullbacks. Then one has a double category whose objects are the same as $\mathbf{C}$. Moreover $Mor^v = Mor^h = Mor(\mathbf{C})$, while bimorphisms are pullback diagrams in $\mathbf{C}$. In this case the notion of Mackey functors corresponds to pre-Mackey functors from [3]. By abuse of notation we will still denote this double category by $\mathbf{C}$. In what follows $\mathcal{F}$ is equipped with this double category structure.

2) Now we consider a double category, whose objects are still finite sets, but $Mor^v = Mor^h = Mor(\mathcal{F}(\text{as}))$, where $\mathcal{F}(\text{as})$ was introduced in Section 3. By definition a bimorphism is a diagram in $\mathcal{F}(\text{as})$

$$\alpha = \begin{array}{ccc} U & \xrightarrow{f_1} & S \\ \downarrow\ \phi_1 & & \phi\ \downarrow \\ T & \xrightarrow{f} & V \end{array}$$

such that the following holds:

i) the image $\mid \alpha \mid$ of α in $\mathcal{F}$ is a pullback diagram of sets,

ii) for all $x \in T$ the induced map $f_* : \phi_1^{-1}(x) \to \phi^{-1}(fx)$ is an isomorphism of ordered sets

iii) for all $y \in S$ the induced map $\phi_* : f_1^{-1}(y) \to f^{-1}(\phi_1 y)$ is an isomorphism of ordered sets.

By abuse of notation we will denote this double category by $\mathcal{F}(\text{as})$. Let us note that for a bimorhism α in $\mathcal{F}(\text{as})$ in general $\phi_1 \circ f \neq f_1 \circ \phi$.

One observes that for any arrows $f : T \to V$, $\phi : S \to V$ in $\mathcal{F}(as)$ there exists a bimorphism α which has f and ϕ as edges and it is unique up to natural isomorphism. Indeed, as a set we take U to be the pullback and then we lift set maps f_1 and ϕ_1 in the noncommutative world according to the properties ii) and iii). Clearly such lifting exists and it is unique.

3) We can also consider the double category $\mathcal{F}(\text{as})_1$ whose objects are still finite sets, vertical arrows are set maps, while horizontal ones are morphisms from $\mathcal{F}(\text{as})$. The bimorphisms are diagrams similar to the diagrams in Example 2) but such that ϕ and ϕ_1 are set maps, while f and f_1 are morphisms from $\mathcal{F}(\text{as})$. Furthermore

the conditions i) and iii) from the previous example hold. We need also a double category $\mathcal{F}(\mathrm{as})_2$ which is defined similarly, but now vertical arrows are morphisms from $\mathcal{F}(\mathrm{as})$ and horizontal ones are set maps.

We have a following diagram of double categories, where arrows are forgetful functors

$$\begin{tikzcd} & \mathcal{F}(\mathrm{as})_1 \arrow[dr] & \\ \mathcal{F}(\mathrm{as}) \arrow[ur] \arrow[dr] & & \mathcal{F}. \\ & \mathcal{F}(\mathrm{as})_2 \arrow[ur] & \end{tikzcd} \tag{4.0}$$

Let $\mathbf{D}$ be one of the double categories considered in (4.0). A bimorphism α is called *elementary* if both f and ϕ are elementary surjections. The following Lemma for $\mathbf{D} = \mathcal{F}$ was proved in [1]. The proof in other cases is quite similar and hence we omit it.

Lemma 4.1 *Let $\mathbf{D}$ be one of the double categories considered in* (4.0). *Then a Janus functor M is a Mackey functor iff the following two conditions hold*

i) for any injection $g : A \to B$ one has $M^*(g)M_*(g) = id_A$

ii) for any elementary bimorphism α one has

$$M^*(\phi)M_*(f) = M_*(f_1)M^*(\phi_1).$$

Theorem 4.2 *Let V be a vector space, which is equipped simultaneously with the structure of associative algebra with unit and coassociative coalgebra with counit. Then V is a bialgebra iff*

$$\mathcal{X}(V) = (\mathcal{X}_*(V), \mathcal{X}^*(V)) : \mathcal{F}(\mathrm{as}) \to Vect$$

is a Mackey functor.

Proof. One observes that the condition 1) of the previous lemma always holds. On the other hand the diagram

$$\alpha = \begin{array}{ccc} \underline{4} & \xrightarrow{p} & \underline{2} \\ \downarrow q & & f \downarrow \\ \underline{2} & \xrightarrow{f} & \underline{1} \end{array}$$

is a bimorphism. Here $f^{-1}(1) = \{1 < 2\}$, $p^{-1}(1) = \{1 < 2\}$, $p^{-1}(2) = \{3 < 4\}$, $q^{-1}(1) = \{1 < 3\}$ and $q^{-1}(2) = \{2 < 4\}$. Clearly $f_* : V^{\otimes 2} \to V$ is the multiplication μ on V and $f^* : V \to V^{\otimes 2}$ is the comultiplication Δ on V, while $p_* = (\mu \otimes \mu) \circ \tau_{2,3}$ and $q^* = \tau_{2,3} \circ \Delta \otimes \Delta$, where $\tau_{2,3} : V^{\otimes 4} \to V^{\otimes 4}$ permutes the second and the third coordinates. Hence V is a bialgebra iff the condition ii) of the previous lemma holds for α. Since both $\mathcal{X}_*(V)$ and $\mathcal{X}^*(V)$ send disjoint union to tensor product the result follows from Lemma 4.1.

Addendum. For a cocommutative bialgebra C the Mackey functor $\mathcal{X}(C)$ factors through the double category $\mathcal{F}(\mathrm{as})_1$, for a commutative bialgebra A the Mackey functor $\mathcal{X}(A)$ factors through $\mathcal{F}(\mathrm{as})_2$ and in the case of commutative and cocommutative bialgebra H one has the Mackey functor $\mathcal{L}(H) : \mathcal{F} \to Vect$.

5 The construction of $\mathcal{QF}(\mathrm{as})$

Let $\mathbf{D}$ be one of the double categories considered in Examples 1)-3). Clearly categories $\mathbf{D}^v$ and $\mathbf{D}^h$ have the same class of isomorphisms, which we call *isomorphisms of* $\mathbf{D}$. We let $\mathcal{Q}\mathbf{D}$ be the category whose objects are finite sets, while the morphisms from T to S are equivalence classes of diagrams:

$$\begin{array}{ccc} U & \xrightarrow{f} & S \\ \downarrow \phi & & \\ T & & \end{array}$$

Here $f \in \mathbf{D}^h$ is a horizontal morphism and $\phi \in \mathbf{D}^v$ is a vertical morphism. For simplicity such data will be denoted by $T \xleftarrow{\phi} U \xrightarrow{f}$

S. Two diagrams $T \overset{\phi}{\hookleftarrow} U \xrightarrow{f} S$ and $T \overset{\phi_1}{\hookleftarrow} U_1 \xrightarrow{f_1} S$ are equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} T & \overset{\phi}{\hookleftarrow} & U & \xrightarrow{f} & S \\ \| & & h\downarrow & & \| \\ T & \overset{\phi_1}{\hookleftarrow} & U_1 & \xrightarrow{f_1} & S \end{array}$$

such that h is an isomorphism. The composition of $T \overset{\phi}{\hookleftarrow} U \xrightarrow{f} S$ and $S \overset{\psi}{\hookleftarrow} V \xrightarrow{g} R$ in $\mathcal{Q}\mathbf{D}$ is by definition $T \overset{\psi_1\phi}{\hookleftarrow} W \xrightarrow{gf_1} R$, where

$$\begin{array}{ccccc} W & & \xrightarrow{f_1} & & V \\ \downarrow & \psi_1 & & \psi & \downarrow \\ U & & \xrightarrow{f} & & S. \end{array}$$

is a bimorphism in $\mathbf{D}$. One easily checks that $\mathcal{Q}\mathbf{D}$ is a category and for any object S the diagram $S \overset{1_S}{\hookleftarrow} S \xrightarrow{1_S} S$ is an identity morphism in $\mathcal{Q}\mathbf{D}$.

Clearly the disjoint union yields a structure of PROP on $\mathcal{Q}\mathbf{D}$ and $\underline{0}$ is not only a unit object with respect to this monoidal structure, but also a zero object.

For a horizontal morphism $f : S \to T$ in $\mathbf{D}$ we let $i_*(f) : S \to T$ be the following morphism in $\mathcal{Q}\mathbf{D}$:

$$S \overset{1_S}{\hookleftarrow} S \xrightarrow{f} T.$$

Similarly, for a vertical morphism $\phi : S \to T$ we let $i^*(f) : T \to S$ be the following morphism in $\mathcal{Q}\mathbf{D}$:

$$T \overset{f}{\hookleftarrow} S \xrightarrow{1_S} S.$$

In this way one obtains the morphisms of PROP's: $i_* : \mathbf{D} \to \mathcal{Q}\mathbf{D}$ and $i^* : \mathbf{D}^{op} \to \mathcal{Q}\mathbf{D}$.

Remark. The construction of $\mathcal{Q}\mathbf{D}$ is a particular case of the generalized Quillen's Q-constuction [11] considered by Fiedorowicz and Loday in [4]. The following lemma is a variant of a result of [6].

Lemma 5.1 *The category of Mackey functors from* $\mathbf{D}$ *to* $Vect$ *is equivalent to the category of functors* $M : \mathcal{Q}\mathbf{D} \to Vect$.

Proof. Let $M : \mathcal{Q}\mathbf{D} \to Vect$ be a functor. For any arrow $f : S \to T$ we put $M_*(f) := M(i_*(f))$ and $M^*(f) := M(i^*(f))$. In this way we get a Mackey functor on $\mathbf{D}$. Conversely, if M is a Mackey functor on $\mathbf{D}$, then we put

$$M(S \overset{g}{\hookleftarrow} V \xrightarrow{f} T) = M_*(f)M^*(g).$$

One easily shows that in this way we get a covariant functor $\mathcal{Q}\mathbf{D}$ to $Vect$ and the proof is finished.

By applying the Q-construction to the diagram (4.0) one obtains the following (noncommutative) diagram of PROP's:

$$\begin{array}{ccccc} & & \mathcal{Q}(\mathcal{F}(\mathrm{as})_1) & & \\ & \nearrow & & \searrow & \\ \mathcal{Q}(\mathcal{F}(\mathrm{as})) & & & & \mathcal{Q}(\mathcal{F}) \\ & \searrow & & \nearrow & \\ & & \mathcal{Q}(\mathcal{F}(\mathrm{as})_2) & & \end{array}$$

The following theorem gives the identification of the terms involved in the diagram, except for $\mathcal{Q}((\mathcal{F}(\mathrm{as})))$.

Theorem 5.2 *i) The category of models of* $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ *is equivalent to the category of bialgebras.*

ii) The category of models of $\mathcal{Q}(\mathcal{F}(\mathrm{as})_1)$ *is equivalent to the category of cocommutative bialgebras and* $\mathcal{Q}(\mathcal{F}(\mathrm{as})_1)$ *is isomorphic to the PROP* $\mathbf{Mon}^{op}$.

iii) The category of models of $\mathcal{Q}(\mathcal{F}(\mathrm{as})_1)$ *is equivalent to the category of commutative bialgebras and* $\mathcal{Q}(\mathcal{F}(\mathrm{as})_2)$ *is isomorphic to the PROP* $\mathbf{Mon}$.

iv) The category of models of $\mathcal{Q}(\mathcal{F})$ *is equivalent to the category of cocommutative and commutative bialgebras and* $\mathcal{Q}(\mathcal{F})$ *is isomorphic to the PROP* $\mathbf{Abmon}$.

Proof. Theorem 4.2 together with Lemma 5.1 shows that any bialgebra V gives rise to the model $\mathcal{X}(V)$ of $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$. Conversely assume M is a model of $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ and let $V = M(\underline{1})$. Then $M \circ i_*$ and $M \circ i^*$ are models of $\mathcal{F}(\mathrm{as})$ and $\mathcal{F}(\mathrm{as})^{op}$. Thus M carries natural structures of associative algebra and coassociative coalgebra. Since $M = (M \circ i_*, M \circ i^*)$ is a Mackey functor on $\mathcal{F}(\mathrm{as})$, it follows from Theorem 4.2 that V is indeed a bialgebra. To prove the remaining parts of the theorem, let us observe that $(\mathcal{Q}(\mathcal{F}(\mathrm{as})_2))^{op} \cong \mathcal{Q}(\mathcal{F}(\mathrm{as})_1)$, where equivalence is identity on objects and sends $T \overset{\phi}{\hookleftarrow} U \xrightarrow{f} S$ to $S \overset{f}{\hookleftarrow} U \xrightarrow{\phi} T$. We now show that $\mathcal{Q}(\mathcal{F}(\mathrm{as})_2) \cong \mathbf{Mon}$. The main observation here is the fact that if $f : X \to S_1 \coprod S_2$ is a morphism in $\mathcal{F}(\mathrm{as})$ then $f = f_1 \coprod f_2$ in the category $\mathcal{F}(\mathrm{as})$, where f_i as a map is the restriction of f on $f^{-1}(S_i), i = 1, 2$. Since $f_i^{-1}(y) = f^{-1}(y)$ for all $y \in f^{-1}(S_i)$ we can take the same total ordering in $f_i^{-1}(y)$ to turn f_i into a morphism in $\mathcal{F}(\mathrm{as})$. A conclusion of this observation is the fact that disjoint union defines not only a symmetric monoidal category structure but it is the coproduct in $\mathcal{Q}(\mathcal{F}(\mathrm{as})_2)$. Clearly $\underline{n}$ is an n-fold coproduct of $\underline{1}$. On the other hand, we may assume that the objects of **Mon** are natural numbers, while the set of morphisms from k to n is the same as $\mathrm{Hom}_{monoids}(F_k, F_n)$, where F_n is the free monoid on n generators. This set can be identified with the set of k-tuples of words on n variables $x_1, \cdots, x_n$. Since $\mathcal{Q}(\mathcal{F}(\mathrm{as})_2)$ and **Mon** are categories with finite coproducts and any object in both categories is a coproduct of some copies of $\underline{1}$, we need only to identify the set of morphisms originating from $\underline{1}$. A morphism $\underline{1} \to \underline{n}$ in $\mathcal{Q}(\mathcal{F}(\mathrm{as})_2)$ is a diagram $\underline{1} \overset{\phi}{\hookleftarrow} U \xrightarrow{f} \underline{n}$, where ϕ is a map of noncommutative sets. We can associate to this morphism a word w of length m on n variables $x_1, \cdots, x_n$. Here $m = Card(U)$ and the i-th place of w is $x_{f(y_i)}$, where $U = \{y_1 < \cdots < y_m\}$. In this way one sees immediately that this correspondence defines the equivalence of categories $\mathcal{Q}(\mathcal{F}(as)_2) \cong \mathbf{Mon}$. We refer the reader to [1] for the fact that $\mathcal{Q}(\mathcal{F})$ is equivalent to **Abmon**. Argument in this case is even simpler than the previous one and can be sketched as follows. Since the PROP $\mathcal{Q}(\mathcal{F})$ is isomorphic to its opposite disjoint union yields not only the coproduct in $\mathcal{Q}(\mathcal{F})$ but also the product. Next, morphisms $\underline{1} \to \underline{1}$ in $\mathcal{Q}(\mathcal{F})$ are diagrams of maps $\underline{1} \leftarrow U \to \underline{1}$, whose equivalence class is completely determined by the cardinality of U. This gives identification of morphisms from $\underline{1} \to \underline{1}$ with natural numbers and the proof is done.

Thus the above diagram of PROP's is equivalent to the diagram

$$\begin{tikzcd}
 & \mathbf{Mon}^{op} \arrow[dr] & \\
\mathcal{Q}(\mathcal{F}(\mathrm{as})) \arrow[ur] \arrow[dr] & & \mathbf{Abmon} \\
 & \mathbf{Mon} \arrow[ur] &
\end{tikzcd}$$

Here $\mathbf{Mon} \to \mathbf{Abmon}$ is given by abelization functor. Let us note that $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ and **Abmon** are self dual PROP's, and the arrows are surjection on morphisms. If one looks at endomorphisms of $\underline{1}$ we see that the endomorphism monoid $End_{\mathbf{C}}(\underline{1})$ for $\mathbf{C} = \mathbf{Mon}^{op}, \mathbf{Mon}, \mathbf{Abmon}$ is isomorphic to the multiplicative monoid of natural numbers. This corresponds to the fact that the operations $\Psi^{(n,\sigma)}$ from the introduction for commutative or cocommutative bialgebras are independent of σ and $\Psi^n \circ \Psi^m = \Psi^{nm}$ [8].

The following proposition describes the endomorphism monoid $End_{\mathbf{C}}(\underline{1})$ for $\mathbf{C} = Q(\mathcal{F}(as))$.

Let $n \in \mathbb{N}$ be a natural number and let $\sigma \in \mathfrak{S}_n$ be a permutation. Here $\mathfrak{S}_n$ is the group of permutations on n letters. We let $[\sigma]$ be the morphism $\underline{n} \to \underline{1}$ in $\mathcal{F}(\mathrm{as})$ corresponding to the ordering $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$. For example $[id_n]$, or simply $[id]$ denotes the morphism $\underline{n} \to \underline{1}$ in $\mathcal{F}(\mathrm{as})$ corresponding to the ordering $1 < 2 < \cdots < n$. Moreover we let $(n, \sigma) : \underline{1} \to \underline{1}$ be the morphism in $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ corresponding to the diagram $\underline{1} \overset{[\sigma]}{\hookleftarrow} \underline{n} \xrightarrow{[id]} 1$.

Proposition 5.3 *The monoid of endomorphisms of* $\underline{1} \in \mathcal{Q}(\mathcal{F}(\mathrm{as}))$ *is isomorphic to the monoid of pairs* (n, σ)*, where* $\sigma \in \mathfrak{S}_n$ *and* $n \in \mathbb{N}$*, with the following multiplication*

$$(n, \sigma) \circ (m, \tau) = (nm, \Phi(\sigma, \tau)).$$

Here

$$\Phi : \mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$$

is a map, which is defined by

$$\Phi(\sigma,\tau)(x) = \tau(q+1) + m(\sigma(q)-1), \quad 1 \le x \le nm,$$

where $x = pn + q,\ 1 \le q \le n$ *and* $0 \le p \le m-1$.

Proof. A morphism $\underline{1} \to \underline{1}$ in $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ is a diagram $\underline{1} \overset{\phi}{\hookleftarrow} U \xrightarrow{f} \underline{1}$, where ϕ and f are morphisms of noncommutative sets. Hence U has two total orderings corresponding to ϕ and f. We will identify U to $\underline{n}$, via ordering corresponding to f. Here n is the cardinality of U. We denote the first (resp. the second, $\cdots$) element in the ordering corresponding to ϕ by $\sigma(1)$ (resp. $\sigma(2), \cdots$). In this way we get a permutation $\sigma \in \mathfrak{S}_n$. Thus any morphism $\underline{1} \to \underline{1}$ in $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ is of the form (n,σ). In order to identify the composition law it is enough to note the following two facts:

i) The diagram

$$\begin{array}{ccc} \underline{nm} & \xrightarrow{f} & \underline{n} \\ g \downarrow & & \downarrow [\sigma] \\ \underline{m} & \xrightarrow{[id]} & \underline{1} \end{array}$$

is a bimorphism in $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$. Here f and g are given by

$$f^{-1}(j) = \{1+(j-1)m < 2+(j-1)m < \cdots < (m-1)+(j-1)m < jm\},$$

$$g^{-1}(i) = \{i+(\sigma(1)-1)m < i+(\sigma(2)-1)m < \cdots < i+(\sigma(n)-1)m\},$$

for $i \in \underline{m}$ and $j \in \underline{n}$.

ii) One has $[\Phi(\sigma,\tau)] = [\tau] \circ g$ and $[id_{\underline{n}}] \circ f = [id_{\underline{nm}}]$.

Remarks 1) It is well known that the PROP corresponding to cocommutative Hopf algebras is $\mathbf{Gr}^{op}$ (see next remark), the PROP corresponding to commutative Hopf algebras is $\mathbf{Gr}$, while the PROP corresponding to commutative and cocommutative Hopf algebras is $\mathbf{Ab}$. Of course the category of Hopf algebras are also models over some PROP, which can be easily described via generators and relations [10]. An explicit description of this particular PROP will be the subject of the forthcoming paper.

2) Any cocommutative Hopf algebra A gives rise to the functor $\mathcal{X}(A) : \mathbf{Gr}^{op} \to Vect$ which takes $< n >$ to $A^{\otimes n}$. Here $< n >$ is a free group on $x_1, \cdots, x_n$ and $\mathcal{X}(A)$ is defined as follows. Since $\otimes$ is a product in the category **Coalg** of cocommutative coalgebras, A is a group object in this category. On the other hand any group object in any category $\mathbf{A}$ with finite products gives rise to the model in $\mathbf{A}$ of the algebraic theory of groups in the sense of Lawvere [12]. But the algebraic theory of groups is nothing but $\mathbf{Gr}^{op}$ and hence we have the functor $\mathcal{X}(A) : \mathbf{Gr}^{op} \to \mathbf{Coalg}$, which assigns $A^{\otimes n}$ to $< n >$. Moreover it assigns μ to the morphism $< 1 > \to < 2 >$ given by $x_1 \mapsto x_1 x_2$. Similarly $\mathcal{X}(A)$ assigns Δ to the homomorphism $< 2 > \to < 1 >$ given by $x_1, x_2 \mapsto x_1$. Of course it assigns the antipode $S : A \to A$ to $x_1 \mapsto x_1^{-1}$. Having these facts in mind one easily describes the action of $\mathcal{X}(A)$ on more complicate morphisms. For example one checks that $\mathcal{X}(A)$ assigns

$$(\mu, \mu) \circ (\mu, id, \mu, id) \circ (S, id_{A^{\otimes 4}}) \circ \tau_{2,3} \circ (id_{A^{\otimes 3}}, \Delta, id) \circ (\Delta, \Delta, id)$$

to the morphism $< 2 > \to < 3 >$ corresponding to the pair of words $(x_1^{-1} x_2 x_1, x_1^2 x_3)$. Here $\tau_{2,3}$ permutes the second and third coordinates. Conversely any linear map $A^{\otimes n} \to A^{\otimes m}$ constructed using the structural data of a cocommutative Hopf algebra A is comming in this way. Hence to check whether a complicated diagram involving such maps commutes it is enough to look to the corresponding diagram in $\mathbf{Gr}$, which is usualy simpler to handle.

3) It is well known that the morphism $\underline{n} \to \underline{m}$ in **Abmon** can be identified with $(m \times n)$-matrices over natural numbers. Under this identification the equivalence $\mathcal{Q}(\mathcal{F}) \cong \mathbf{Abmon}$ is given by assigning the matrix whose (i, j)-component is the cardinality of $f^{-1}(j) \bigcap g^{-1}(i)$, $1 \le i \le m$, $1 \le j \le n$ to the diagram $\underline{n} \xleftarrow{f} X \xrightarrow{g} m$. It is less known that the morphisms $\underline{n} \to \underline{m}$ in **Mon** can be described via shuffles. In order to explain this connection let us start with particular case. Consider a word $x^2 y x y^3 x^2$ of bidegree $(5, 4)$. It defines a morphism $\underline{1} \to \underline{2}$ in **Mon**. One associates a $(5, 4)$-shuffle $(1, 2, 4, 8, 9, 3, 5, 6, 7)$ to this word, whose first five values are just the numbers of places where x lies. Similarly morphisms $\underline{n} \to \underline{m}$ in **Mon** are in 1-1-correspondence with collections $\{A = (a_{ij}), (\varphi_1, \cdots, \varphi_n)\}$, where A is an $(m \times n)$-matrix over natural numbers and φ_i is a $(a_{i1}, \cdots, a_{im})$-shuffle, $i = 1, \cdots, n$.

The functor **Mon** $\to$ **Abmon** corresponds to forgetting the shuffles. Now combine this observation with Proposition 5.3 to get the description of morphisms $\underline{n} \to \underline{m}$ in $\mathcal{Q}(\mathcal{F}(\mathrm{as}))$ as collections $\{A = (\alpha_{ij}), (\varphi_1, \cdots, \varphi_n)\}$, where $\alpha = (a_{ij}, \sigma_{ij})$ and a_{ij} is a natural number, while $\sigma_{ij} \in \mathfrak{S}_{a_{ij}}$ is a permutation and finally φ_i is a $(a_{i1}, \cdots, a_{im})$-shuffle.

4) In the recent preprint [2] the authors defined the action of $\mathfrak{S}_{k+1}$ on $A^{\otimes k}$ for any commutative or cocommutative Hopf algebra A. Actually they implicitly constructed the group homomorphism

$$\xi_k : \mathfrak{S}_{k+1} \to \mathfrak{G}_k,$$

where $\mathfrak{G}_k$ is the automorphism group of $< k >$. Then the action of $x \in \mathfrak{S}_{k+1}$ on $A^{\otimes k}$ is obtained by applying the functor $\mathcal{X}(A)$ to $\xi_k(x)$. The homomorphism ξ_k is given by

$$\sigma_1(x_1) = x_1^{-1},\ \sigma_1(x_2) = x_1 x_2,\ \sigma_1(x_i) = x_i,\ i \geq 2$$

$$\sigma_i(x_{i-1}) = x_{i-1}x_i,\ \sigma_i(x_i) = x_i^{-1},\ \sigma_i(x_{i+1}) = x_i x_{i+1},\ \sigma_i(x_j) = x_j,$$

for $1 < i < k,\ j \neq i-1, i, i+1$ and

$$\sigma_k(x_{k-1}) = x_{k-1}x_k,\ \sigma_k(x_k) = x_k^{-1},\ \sigma_k(x_j) = x_j \text{ if } j < n-1.$$

Here $\sigma_i \in \mathfrak{S}_{k+1}$ is the transposition $(i, i+1),\ 1 \leq i \leq k$.

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