Lattices with nonintegral character[1]

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[1] "Partially supported by the Emmy Noether Institute for Mathematics and the Minverva Foundation"

1 Introduction

Hyperbolic lattices in dimension three, i.e. discrete cofinite subgroups of $SL(2, \mathbf{C})$ show a preference for having integrally valued character functions, see, e.g. [2], [3]. The first (and only) publicly known lattice with nonintegral character seems to be the one presented by Vinberg at the very end of his fundamental paper [4] where it plays the rôle of an example for reflection groups. We in this paper first present a very geometric version of this example and then discuss a series of lattices which contains, most probably, infinitly many with no integer valued character. The main difference when compared to Vinberg's case is that the lattices exhibited here are cocompact. They appear as the result of Dehn surgery along the figure eight knot with parameter $(\pm 4n, n)$; all other Dehn surgery results are, as soon as they are hyperbolic, integrally valued on their character. We do not know of any geometric significance of this exceptional behaviour, yet. The construction hints where to look for more peculiarities of this type.

2 Vinberg's example

This is a lattice in threedimensional hyperbolic space generated by reflections. Let P be the solid in $\mathbf{H}^3$ described combinatorially as a prism with two opposite triangular and three planar quadrangular faces as shown in figure 1.

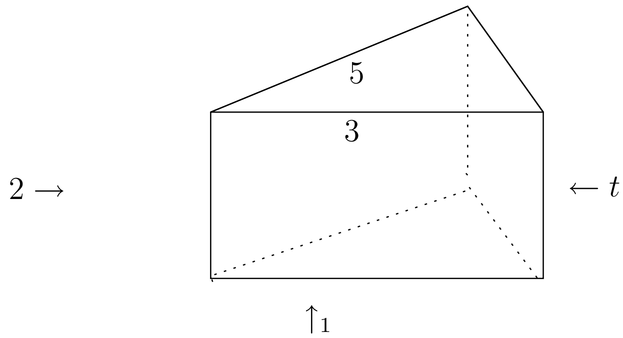

Figure 1

We number the faces the following way:
1: the bottom triangle,
2: the left hidden quadrangle,
3: the front quadrangle,
4: the right hidden quadrangle,

5: the top triangel.

This labelling is also used as a labelling of the reflections performed on the faces. Compositions of reflections should be read from right to left, so 21 indicates the orientation preserving isometry which is created by first reflecting in face number 1 and then in face number 2. We use symbols as

$1 \cap 2 = 2 \cap 1$ for the edge of P where 1 and 2 intersect,

$1 \cap 2 \cap 3$ in any ordering for the vertex where 1, 2, and 3 intersect, and

$\angle 12 = \angle 21$ for the spatial angle enclosed by the faces 1 and 2.

Vinberg shows in [4] that the combinatorial object P may be given the following geometrical realization in $\mathbf{H}^3$:

(1) angular conditions:

$\angle 12 = \angle 13 = \angle 34 = \angle 25 = \angle 35 = \angle 45 = \pi/2$,

$\angle 14 = \pi/3$,

$\angle 23 = \angle 24 = \pi/6$;

(2) vertex locations:

All vertices which are visible in figure 1 are inside hyperbolic space; the hidden vertex $1 \cap 2 \cap 4$ is at infinity.

THEOREM *(Vinberg [4]) The group Γ generated by reflections on the faces of P is a cofinite but not cocompact lattice in hyperbolic space $\mathbf{H}^3$. It is not arithmetic.*

The discreteness and cofiniteness of Γ comes from the very general discussion in [4] of reflection groups where Γ plays the rôle of an example. The nonarithmeticity comes from the observation that in order to describe Γ as a matrix group in $O(3,1)$ matrices with no longer integral traces are needed: the denominators of the traces pick up powers of the prime 2 with exponent unbounded. We, using Poincaré's model of hyperbolic geometry in dimension 3 reprove nonarithmeticity and, as a complement to [4], determine the trace field of Γ.

Let Γ^+ the index 2 subgroup of Γ consisting of all orientation preserving isometries of Γ. The elements $\sigma_1 = 21, \sigma_2 = 25, \tau_1 = 23, \tau_2 = 24$ of Γ are contained in Γ^+, already. They allow the following presentation of Γ^+:

Generators: $\sigma_1, \sigma_2, \tau_1, \tau_2$,

Relators:

(1) $\sigma_1^2 = \sigma_2^2 = (\sigma_1\tau_1)^2 = (\sigma_2\tau_1)^2 = (\sigma_2\tau_2)^2 = (\tau_2^{-1}\tau_1)^2 =$ identity,

(2) $(\sigma_1\tau_2)^3 =$ identity,

(3) $\tau_1^6 = \tau_2^6 =$ identity.

In order to not overload the discussion with additional notation we interpret the letters $\sigma_1, \sigma_2, \tau_1, \tau_2$ as elements of $SL(2, \mathbf{C})$ instead of $PSL(2, \mathbf{C}) \cong \mathrm{Iso}^+(\mathbf{H}^3)$; consequently Γ^+ is now a subgroup of $SL(2, \mathbf{C})$. In rows (1) and (2) we then have to interprete the identity as $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and we may and shall do so in row (3), as well.

Let now
$$s \ : \ \Gamma^{+} \ \longrightarrow \ \mathbf{C}$$
be the character of the lattice Γ^{+} as a subgroup of $SL(2,\mathbf{C})$. Then lines (1), (2) and (3) translate to

$(1') : s(\sigma_1) = s(\sigma_2) = s(\sigma_1\tau_1) = s(\sigma_2\tau_1) = s(\sigma_2\tau_2) = s(\tau_2^{-1}\tau_1) = 0,$

$(2') : s(\sigma_1\tau_2) = 1,$

$(3') : s(\tau_1) = s(\tau_2) = \sqrt{3}.$

Here the value 1 for $s(\sigma_1\tau_2)$ comes from $(\sigma_1\tau_2)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; the choice of the value $\sqrt{3}$ for $s(\tau_1)$ and $s(\tau_2)$ instead of $-\sqrt{3}$ is compatible with lines (1') and (2').

We mention
$$s(\tau_2\tau_1) \ = \ 3$$
which comes from
$$s(\tau_2\tau_1) + s(\tau_2^{-1}\tau_1) = s(\tau_2\tau_1) + 0 = s(\tau_2)s(\tau_1) = 3.$$

Secondly
$$s(\sigma_1\tau_2^3) = (\sqrt{3}^2 - 1)s(\sigma_1\tau_2) - \sqrt{3}s(\sigma_1) = 2 = s(\tau_2\sigma_1\tau_2^2) = s(\tau_2^2\sigma_1\tau_2).$$

So the elements $\sigma_1\tau_2^3, \tau_2\sigma_1\tau_2^2$, and $\tau_2^2\sigma_1\tau_2$ are parabolic elements with fixed point $1 \cap 2 \cap 4$ from figure 1 which is a cusp. It is also easy to see that these three elements generate the torsion free part of the stabilizer of this cusp.

Some general character formalism (see [2]) allows to calculate $s^2(\sigma_1\tau_1\tau_2)$ and $s^2(\sigma_2\tau_1\tau_2)$ and then also $s(\sigma_1\sigma_2)$:
$$4\Big(s(\sigma_1\tau_1\tau_2) - s(\sigma_1\tau_2\tau_1)\Big)^2 + \det\begin{pmatrix} -4 & 0 & 2 \\ 0 & -1 & 3 \\ 2 & 3 & -1 \end{pmatrix} = 0$$
yields
$$\Big(2s(\sigma_1\tau_1\tau_2) - \sqrt{3}\Big)^2 \ = \ -9,$$
so
$$s^2(\sigma_1\tau_1\tau_2) \ = \ 3\Big(\frac{1+\sqrt{-3}}{2}\Big)^2.$$
Similarly
$$4\Big(s(\sigma_2\tau_1\tau_2) - s(\sigma_2\tau_2\tau_1)\Big)^2 + \det\begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} = 0$$

results to

$$s^2(\sigma_2\tau_1\tau_2) \;=\; -2.$$

Furthermore

$$4\Big(s(\sigma_1\tau_1\tau_2) - s(\sigma_1\tau_2\tau_1)\Big)\Big(s(\sigma_2\tau_1\tau_2) - s(\sigma_2\tau_2\tau_1)\Big)$$

$$+\det\begin{pmatrix} 2s(\sigma_1\sigma_2) & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} = 0$$

leads to

$$s^2(\sigma_1\sigma_2) \;=\; 9/2$$

which is not integral any more. Some more computations on these lines result in explicit values of the character s, e.g. on $\sigma_1\sigma_2\tau_1$ and $\sigma_1\sigma_2\tau_2$. We collect everything in the following

THEOREM. *The lattice Γ^+ has trace field equal to $\mathbf{Q}(\sqrt{-3})$, the field of cube roots of unity. Its character values (squared) are unbounded at the nonarchimedian valuation at the prime 2 and integral at all other nonarchimedian places. It is cofinite with exactly one cusp.*

We end this paragraph by drawing a fundamental domain for Γ^+ in hyperbolic space. The cusp we locate at infinity. It is then easy to see that the matrix solution with character values (1'), (2'), (3') is, up to conjugacy by the stabilizer of the cusp:

$$\sigma_1 = \begin{pmatrix} -i & \frac{3}{\sqrt{2}}i \\ 0 & i \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\tau_1 = \begin{pmatrix} \frac{\sqrt{3}-3i}{2} & i\sqrt{2} \\ -i\sqrt{2} & \frac{\sqrt{3}+3i}{2} \end{pmatrix}, \ \tau_2 = \begin{pmatrix} \frac{\sqrt{3}+i}{2} & 0 \\ 0 & \frac{\sqrt{3}-i}{2} \end{pmatrix}.$$

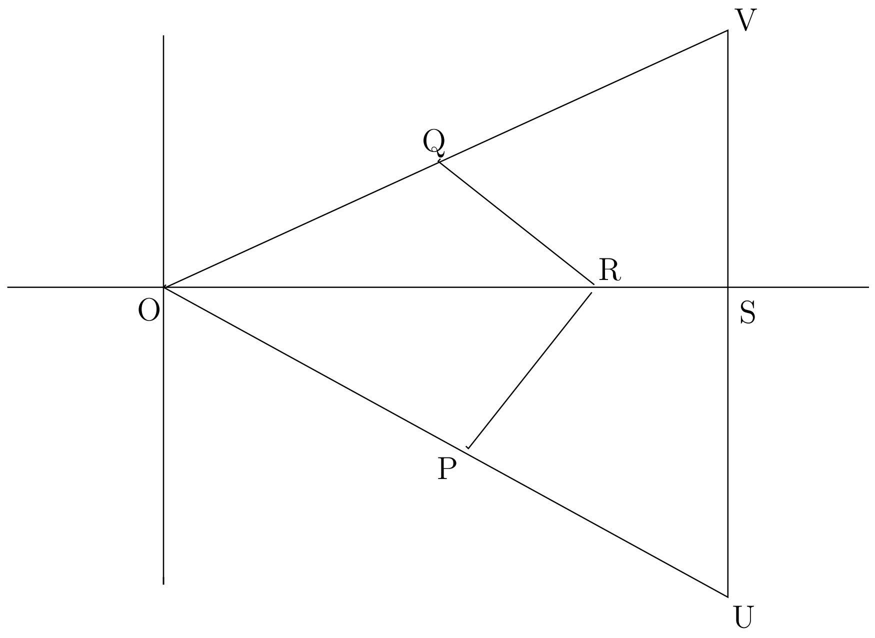

Figure 2

Figure 2 shows a Ford fundamental domain of Γ^+, viewed from the cusp at infinity. It is a triangular prism with three faces lying on the vertical Euclidian halfplains containing $\{\infty, O, V\}, \{\infty, O, U\}$, and$\{\infty$.U, V$\}$, respectively. The floor is composed by pieces of three isometric spheres:

$\{O, P, R, Q\}$ lies on the isometric sphere of σ_2,

$\{P, U, S, R\}$ lies on the isometric sphere of τ_1, and

$\{R, S, V, Q\}$ lies on the isometric sphere of τ_1^{-1}.

The coordinates of these points are:

$O = (0, 1)$,

$P = (\frac{\sqrt{3}-i}{2\sqrt{3}}), (\frac{\sqrt{2}}{\sqrt{3}})$,

$Q = (\frac{\sqrt{3}+i}{2\sqrt{3}}), (\frac{\sqrt{2}}{\sqrt{3}})$,

$U = (\frac{3-i\sqrt{3}}{2\sqrt{2}}, 1)$,

$V = (\frac{3+i\sqrt{3}}{2\sqrt{2}}, 1)$,

$S = (\frac{3}{2\sqrt{2}}, \frac{1}{2})$,

$R = (\frac{2}{3}, \frac{\sqrt{5}}{\sqrt{3}})$.

The above generators identify the faces of the prism in the following way:

$$\sigma_1 \; : \; \begin{array}{c} \infty \\ U \\ S \\ V \end{array} \longrightarrow \begin{array}{c} \infty \\ V \\ S \\ U \end{array}, \quad \sigma_2 \; : \; \begin{array}{c} O \\ P \\ R \\ Q \end{array} \longrightarrow \begin{array}{c} O \\ Q \\ R \\ P \end{array},$$

$$\tau_1 \ : \ \begin{matrix} P \\ U \\ S \\ R \end{matrix} \ \longrightarrow \ \begin{matrix} Q \\ V \\ S \\ R \end{matrix}, \ \tau_2 \ : \ \begin{matrix} \infty \\ O \\ P \\ U \end{matrix} \ \longrightarrow \ \begin{matrix} \infty \\ O \\ Q \\ V \end{matrix}.$$

The orbifold $\Gamma^+ \setminus \mathbf{H}^3$ obviously allows an orientation reversing isometry, realized, e.g. by the reflection of the fundamental domain in figure 2 through the hyperbolic plane above the real axis.

3 Figure eight knot Dehn surgery

We recall the presentation of the fundamental group $\pi_1(S^3 \setminus 4_1)$ of the figure eight knot complement in the 3-sphere in terms of an HNN-extension:

$$\pi_1(S^3 \setminus 4_1) \cong \langle \xi, \eta, \mu \mid \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = \eta\xi\eta \rangle. \tag{4}$$

This group has a faithful lattice representation in $PSL(2.\mathbf{C})$ which is the group $Iso^+(\mathbf{H}^3)$ of of orientation preserving isometries of hyperbolic space. As it is (two-)torsion free one can lift this representation to a representation in $SL(2,\mathbf{C})$, and it is easy to see that it is legitimate to interprete the above presentation as the presentation of a lattice Γ in $SL(2,\mathbf{C})$. The character variety of $\Gamma \cong \pi_1(S^3 \setminus 4_1)$, i.e. the space of deformations of the character of Γ in $SL(2,\mathbf{C})$ is the affine algebraic curve

$$t^2 \ = \ \frac{x^2 + x - 1}{x - 1} \tag{5}$$

with the point $x = 1$ removed, see [2]. This means the following: Let $s : \Gamma \longrightarrow \mathbf{C}$ be any character on Γ which results from a deformation of the lattice character of Γ. Then s is determined by its values $s(\xi) = x$ and $s(\mu) = t$, and these values are related by equation (5).

In this paragraph we replace x by $x = q + 1$ which has the effect that formulae and notation become much more transparent respectively simplified. Equation (5) now reads

$$t^2 = \frac{q^2 + 3q + 1}{q} = q + q^{-1} + 3. \tag{5'}$$

It reveals the fact that the elliptic curve (5) allows two holomorphic involutions: $t \longrightarrow -t$ and $q \longrightarrow q^{-1}$. We may rewrite (5') as

$$q^2 - (t^2 - 3)q + 1 = 0. \tag{5''}$$

So if t is an algebraic integer so q is; it then is even a unit.

We collect some information about s, see [2]:

$$s(\xi) = x = q + 1,\ s(\eta) = y = \frac{x}{x-1} = 1 + q^{-1},\ s(\eta\xi) = x = q + 1,$$

$$s(\mu) = s(\mu^{-1}\xi) = s(\mu^{-1}\eta) = s(\mu^{-1}\eta\xi) = t.$$

Notice that the character s restricted to the rank 2 free subgroup of $\pi_1(S^3 \setminus 4_1)$ generated by ξ and η is determined by $s(\xi) = x$ and $s(\eta) = y$ which are related by $xy = x + y$, eqivalently $(x-1)(y-1) = 1$.

We set $\lambda = \eta^{-1}\xi^{-1}\eta\xi$ and compute

$$\begin{aligned} s(\lambda) =& x^2 + y^2 + z^2 - xyz - 2 \\ =& \frac{1}{q^2}(q^4 + q^3 - 2q^2 + q + 1) = \Big(q + \frac{1}{q}\Big)^2 + \Big(q + \frac{1}{q}\Big) - 4 = t^4 - 5t^2 + 2. \end{aligned} \tag{6}$$

We also need $s(\mu^{-1}\lambda)$ and $s(\mu\lambda)$ in terms of character coordinates:

$$\begin{aligned} s(\mu^{-1}\lambda) &= s\Big((\xi\mu^{-1})(\eta^{-1}\xi^{-1})\eta\Big) \\ &= -s\Big((\eta^{-1}\xi^{-1})(\xi\mu^{-1}\eta)\Big) - s(\xi\mu^{-1})s(\eta^{-1}\xi^{-1})s(\eta) \\ &+ s(\xi\mu^{-1})s\Big((\eta^{-1}\xi^{-1})\eta\Big) + s(\eta^{-1}\xi^{-1})s\Big((\xi\mu^{-1})\eta\Big) + s(\eta)s(\xi\mu^{-1}\eta^{-1}\xi^{-1}). \end{aligned}$$

We use

$$s(\mu^{-1}\eta^{-1}) = -s(\mu^{-1}\eta) + s(\mu)s(\eta) = -t + t\Big(1 + \frac{1}{q}\Big) = \frac{t}{q}$$

and get

$$s(\mu^{-1}\lambda) = t\frac{q^3 - q^2 + 1}{q^2}. \tag{7}$$

From this follows

$$s(\mu\lambda) = t\frac{q^3 - q + 1}{q}. \tag{8}$$

We observe that $s(\mu^{-1}\lambda)$ and $s(\mu\lambda)$ are related to each other via the automorphism $q \longrightarrow q^{-1}$ of the curve (5').

To perform Dehn surgery at the knot 4_1 means to add a relator of the form $\lambda^m = \varepsilon\mu^n$ to the presentation (4) of $\Gamma < SL(2, \mathbf{C})$; here m and n are integers not simultaneously 0 and ε is central. In terms of character values we have, in this situation

$$s(\lambda^m) = \pm s(\mu^n). \tag{9}$$

Remember that $s(\gamma^k)$ is a monic polynomial with integer coefficients in $s(\gamma)$ of degree $|k|$, for $\gamma \in \Gamma$ and $k \in \mathbf{Z}$. We have seen in (6) that $s(\lambda)$ is a degree 4 polynomial in terms of $s(\mu) = t$, so if $|n| \neq 4|m|$ equation (7) defines $s(\mu) = t$ as an algebraic integer. In this case we see from (5") that q is an algebraic integer, even a unit, and so all values $s(\gamma), \gamma \in \Gamma$ are algebraically integral. We shall deal with the case $|n| = 4|m|$ separately but first collect all information in the following

Theorem. *Let m, n be integers such that the orbifold created by Dehn surgery $\lambda^m = \varepsilon\mu^n$ along the figure eight knot in S^3 is hyperbolic, and let $\Gamma_{m,n} < SL(2, \mathbf{C})$ be the corresponding lattice. Then the character s on $\Gamma_{m,n}$ is algebraically integer valued if*

(1)

$$|n| \neq 4|m|$$

or

(2)

$$n = \pm 4m \text{ and m not a power of a prime number.}$$

In case $m = \pm p^k, n = \pm 4p^k$ with p a prime number and $k \geq 1$ the situation is the following: Define $c_m = 2 + 2\cos\frac{\pi}{m} = \left(2\cos\frac{\pi}{2m}\right)^2$, and $f_m(z)$ to be the degree 4 polynomial

$$f_m(z) = z^4 + 3z^3 + z^2 + c_m.$$

Then the character s on $\Gamma_{m,n}$ is algebraically integer valued if and only if f_m splits over the field $\mathbf{Q}(c_m)$ into two factors one of which has the form $z^2 + az + b$ with b a unit in that field and discriminant $a^2 - 4b$ negative.

What is left to prove is a detailed study of the situation $n = \pm 4m$. We assume m positive which does not mean any loss; requiring hyperbolicity means $m \geq 2$. First $n = -4m$. We have, from $\lambda^m = \varepsilon\mu^{-4m}$: $(\mu^4\lambda)^m = \varepsilon = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as λ and μ commute. So $s(\mu^4\lambda) = -2\cos\frac{\pi}{m}$. Here the negative sign is mandatory for odd m: The group $\Gamma_{m,-4m}$ regarded as a group of isometries of hyperbolic space has no 2-torsion and so may be lifted to a matrix group; omitting the negative sign would create $(\mu^4\lambda)^m = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. For even m the negative sign is expected as, for $m \longrightarrow \infty$ this value should converge to -2 which is the character value on λ in the complete hyperbolic case where λ and μ are parabolic. We derive from equations (6) and (7):

$$s(\mu^{-4}\lambda) = (t^3 - 2t)s(\mu^{-1}\lambda) - (t^2 - 1)s(\lambda)$$

$$= t^2(t^2 - 2)\frac{q^3 - q^2 + 1}{q^2} - (t^2 - 1)(t^4 - 5t^2 + 2).$$

We express using (5') t^2 in terms of q and get

$$s(\mu^{-4}\lambda) = \frac{1}{q^4}(2q^4 + q^2 + 3q + 1).$$

So

$$s(\mu^4\lambda) = -s(\mu^{-4}\lambda) + s(\mu^4)s(\lambda) = q^4 + 3q^3 + q^2 + 2.$$

So the algebraic equation for q is

$$q^4 + 3q^3 + q^2 + c_m = 0 \text{ if} (\mu^4\lambda)^{\mathrm{m}} = \text{identity},$$
$$c_m q^4 + q^2 + 3q + 1 = 0 \text{ if} (\mu^{-4}\lambda)^{\mathrm{m}} = \text{identity}.$$

Remember that both q and q^{-1} enter into character values. So if a lattice character is integer valued it is necessary (and sufficient) that q be an algebraic unit which is not real (as otherwise the trace field would be real).

Proposition. *The number $c_m = 2 + 2\cos\frac{\pi}{m} = (2\cos\frac{\pi}{2m})^2$ is, for $m \geq 2$ an algebraic integer. It is a unit if and only if m is not a prime power; if $m = p^k$ with p a prime, $k \geq 1$ then it is a prime in the ring of integers in $\mathbf{Q}(c_m)$ with degree 1. Its norm as an element of this field is p.*

We sketch the proof which is presumably in the literature: Let $g_m(z)$ be the minimal polynomial of the primitive $(2m)^{th}$ root of unity $e^{\frac{\pi i}{m}}$:

$$g_m(z) = \prod_{0<d|2m} \left(z^d - 1\right)^{\mu(\frac{2m}{d})}$$

with μ the Möbius function. Its degree is $\varphi(2m)$ which is an even natural number; φ is of course Euler's function. Its constant term is

$$g_m(0) = \Big(-1\Big)^{\sum_{0<d|2m}\mu(\frac{2m}{d})} = 1.$$

The rational function $g_m(z)/z^{\varphi(2m)/2}$ is invariant under change from z to z^{-1}. This means that it may be written as a monic polynomial $\tilde{g}_m$ with integer coefficients in terms of $z+2+z^{-1}$:

$$\frac{1}{z^{\varphi(2m)/2}} g_m(z) = \tilde{g}_m(z+2+z^{-1}).$$

$g_m(z)$ is, of course, the minimal polynomial of c_m. We have, in order to compute the norm of c_m to compute its constant coefficient:

$$\tilde{g}_m(0) = \frac{1}{z^{\varphi(2m)/2}} g_m(z)|_{z=-1} = (-1)^{\varphi(2m)/2} g_m(-1).$$

From ( ) we get

$$g_m(-1) = \prod_{\substack{d|2m \\ d \text{ odd}}} \Big(-2\Big)^{\mu(\frac{2m}{d})} \prod_{\substack{d|2m \\ d \text{ even}}} \lim_{z\to 1} \Big(\frac{z^d-1}{z-1}\Big)^{\mu(\frac{2m}{d})} \lim_{z\to 1} \prod_{\substack{d|2m \\ d \text{ even}}} \Big(z-1\Big)^{\mu(\frac{2m}{d})}.$$

The first and the third product are easily seen to be 1, and the second is

$$\prod_{\substack{d|2m \\ d \text{ even}}} d^{\mu(\frac{2m}{d})} = 2^{\sum_{d|m}\mu(\frac{m}{d})} \prod_{d|m} d^{\mu(\frac{m}{d})} = \left\{ \begin{array}{ll} p & \mathrm{m = p^k} \\ 1 & \text{otherwise} \end{array} \right. .$$

This is the statement of the proposition.

The polynomial $f_m(z) = z^4 + 3z^3 + z^2 + c_m$ is easily seen to have 2 real and one pair of complex conjugate roots. Let m be a prime power, so c_m not a unit. If first f_m is irreducible over the field $\mathbf{Q}(c_m)$ then no root is an algebraic unit. If f_m splits over this field into a cubic and a linear factor one sees at once that the cubic factor cannot have a unit as its constant term (and, of cours, the zero of the linear factor being real cannot lead to a lattice character). So in this case no character can be integer valued. Remains the case that f_m decomposes into two quadratic factors one having two real roots and constant term a unit times c_m and the other one having constant term a unit and two complex conjugates as the roots. It is undecided to which extent this situation has to be expected.

4 Examples

For $m = 2, 3, 4,$ and 5 we have $c_m = 2, 3, 2+\sqrt{2},$ and $\frac{5+\sqrt{5}}{2}$, respectively. The polynomial

$$f_m(z) = z^4 + 3z^3 + z^2 + c_m$$

is in all cases, seen to be irreducible over $\mathbf{Q}(c_m)$. So the lattices $\Gamma_{m,\pm 4m}$, for these values of m have trace fields which are degree 4 extensions of $\mathbf{Q}(c_m)$. Their character is not integer valued; its values have denominators which are powers of the prime number c_m. If m goes to infinity the polynomial f_m becomes

$$f_\infty(z) = z^4 + 3z^3 + z^2 + 4 = (z+2)^2(z^2 - z + 1).$$

The root of the quadratic factor defines the lattice representation of the fundamental group of $S^3 \setminus 4_1$, again.

We briefly sketch a situation where the figure eight knot complement is replaced by the manifold M with fundamental group

$$\pi_1(M) \cong \langle \xi, \eta, \mu \mid \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^3\eta \rangle.$$

This is another standard hyperbolic manifold. It fibers over the circle with fibre the once punctured torus. $\langle \xi, \eta \rangle$ again is the (free) fundamental group of the fibre and μ represents the pseudo-Anosov on the level of $\langle \xi, \eta \rangle$. The character variety of $\pi_1(M)$ is a hyperelliptic curve:

$$t^2 = (z-1)^2 \frac{z^3 + z^2 - 2z - 1}{z^2 - z - 1},$$

see [2]. Here the point (z, t) on this curve serves as a coordinate for the character s on $\pi_1(M)$ determined by

$$s(\xi) = z,\ s(\eta) = z\frac{z-1}{z^2 - z - 1} = 1 + \frac{1}{z^2 - z - 1},\ s(\eta\xi) = z,$$

$$s(\mu) = t,\ s(\mu^{-1}\xi) = \frac{1}{z-1}t,\ s(\mu^{-1}\eta) = t,\ s(\mu^{-1}\eta\xi) = \frac{1}{z-1}t.$$

In this case Dehn surgery leads to lattices with nonintegral character in case $\mu^3 = \lambda^{\pm 4}$, and the algebraic equation for its character is, in analogy to the figure eight knot situation, $s(\mu^{\pm 3}\lambda^4) = -2\cos\frac{\pi}{m}$ for m sufficiently large in order to guarantee hyperbolicity. The resulting polynomial which replaces f_m has degree 16.

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