Exponential Integrability for The Images of The Ornstein-Uhlenbeck Operators Acting on Cylinder Functions on Loop Spaces

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Abstract

Let E be the loop space over a compact connected Riemannian manifold with a torsion skew symmetric (TSS) connection. Let L be the Ornstein-Uhlenback operator on the loop space E, and f be a cylinder function on E. We first extend the expression of Lf, proved by O.Enchev and D.W. Stroock for the Levi-Civita connection, to a general TSS connection, and then prove that $f \in \mathcal{D}(L)$ and $\varepsilon|Lf|^2$ is exponential integrable for some constant $\varepsilon := \varepsilon(f) > 0$.

1 Introduction

Let M be an n-dimensional connected compact Riemannian manifold with a torsion skew symmetric (TSS for short) connection ∇ (for the definition see [Dr94]), and let E be defined by

$$E = \{w \in C([0,1]; M) : w(0) = x_0, w(1) = y_0\}$$

for fixed $x_0, y_0 \in M$. E is the so called loop space over M when $x_0 = y_0$.

A function f on E is called as a smooth cylinder function if there exist a function $F \in C^\infty(M^m)$ and a partition $0 < t_1 < \cdots < t_m < 1$ of $[0,1]$ such that $f(w) = F(w(t_1), \cdots, w(t_m))$ for any $w \in E$. We denote the set of all smooth cylinder functions on E by $\mathcal{F}C^\infty(E)$.

Pinned Wiener measure (i.e. Brownian bridge measure) μ on E is the unique Borel probability measure on E such that, the coordinate process (γ_t) on E is the Brownian bridge process. Let $(\mathcal{F}_t)_{0\le t\le 1}$ be the corresponding μ-completed natural filtration corresponding to it. Moreover, for a given orthonormal frame u_0 at $x_0 \in M$ there exists a unique stochastic horizontal lift (U_t) of (γ_t) determined by the TSS connection ∇ satisfying $U_0 = u_0$ (see [Dr94]). For convenience, we consider an orthonormal frame U at $x \in M$ as an isomorphism from $\mathbb{R}^n$ to T_xM. If we denote the bundle of orthonomal frames over M by $\mathcal{O}(M)$, then (U_t) is an $\mathcal{O}(M)$-valued process. We identify $T_{x_0}M$ and $\mathbb{R}^n$ via u_0 and set

$$H_0 := \{h \in C([0,1]; \mathbb{R}^n) : \|h\|_{H_0}^2 = \int_0^1 |\dot{h}(t)|^2 dt < \infty, h(0) = h(1) = 0\}.$$

Then we can define a closed densely defined operator ∇_0 from $L^2(E, \mu)$ to $L^2(E \to H_0; \mu)$ with $\mathcal{F}C^\infty(E)$ as its core, which is considered as a natural gradient operator on E with domain $\mathcal{D}(\nabla_0)$(see [Dr94]). In particular, for $f \in \mathcal{F}C^\infty(E)$ with $f(w) = F(w(t_1), \cdots, w(t_m))$ we have

$$(\nabla_0 f(\gamma))(t) = \sum_{i=1}^m (min(t_i, t) - t_i t) \nabla_{U_{t_i}}^{(i)} F(\gamma_{t_1}, \cdots, \gamma_{t_m}) \tag{1.1}$$

where $\nabla^{(i)} F$ denotes the gradient of F with respect to the i-th variable, $\nabla_U^{(i)} F$ denotes the unique element in $\mathbb{R}^n$ such that $\langle a, \nabla_U^{(i)} F \rangle_{\mathbb{R}^n} = \nabla_{Ua}^{(i)} F$ for any $a \in \mathbb{R}^n$ and $U \in \mathcal{O}(M)$. It follows from (1.1) that $\|\nabla_0 f\|_{H_0} \in L^\infty(\mu)$ $(\forall f \in \mathcal{F}C^\infty(E))$.

Let the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ in $L^2(\mu)$ be defined as follows: $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\nabla_0)$,

$$\mathcal{E}(f, g) := \int_E \langle \nabla_0 f, \nabla_0 g \rangle_{H_0} d\mu \tag{1.2}$$

for any $f, g \in \mathcal{D}(\mathcal{E})$, and denote the generator of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ by L. L is the so called O-U operator on loop space E. Obviously, $\mathcal{F}C^\infty(E) \subset \mathcal{D}(\mathcal{E})$ is the core of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

In this paper we prove that for $f \in \mathcal{F}C^\infty(E)$ we have $f \in \mathcal{D}(L)$ and

$$\int_E exp\{\varepsilon |Lf|^2\} d\mu < \infty \tag{1.3}$$

for some constant $\varepsilon := \varepsilon(f) > 0$. To this end, we extend the expression of Lf, proved by O. Enchev and D.W. Stroock in [ES96] for ∇ being the Levi-Civita connection on M, to the case of a general TSS connection ∇ on M. The result (1.3) is needed in Remark 3.2 of [GRW00] to show that f is in the domain of the so called ground state transform of a Schrödinger operator on E.

The organization of this paper is as follows:

In Section 2 we introduce some notations and describe the main result in more detail.

In Section 3 we prove the main result of this paper.

2 Notations and the main result

In this section we introduce some notations and describe the main result of this paper.

Let $f \in \mathcal{F}C^\infty(E)$ be the form $f(w) = F(w(t_1), \cdots, w(t_m))$ with $F \in C^\infty(M^m)$. By one of the results in [GMR99] we know that for any $h \in H_0$ we have $\partial_h f :=$

$\langle \nabla_0 f, h\rangle_{H_0} \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned}\partial^2 f(h,u)(\gamma) &:= \langle \nabla_0(\partial_h f), u\rangle_{H_0} \\ &= \sum_{1\le i,j\le m} \{\nabla^{(i)}_{U_{t_i}u_{t_i}}\{\nabla^{(j)}_{U_{t_j}h_{t_j}} F\}\}(\gamma_{t_1},\cdots,\gamma_{t_m}) \\ &\quad + \sum_{1\le k\le m} \langle \Phi_{t_k}(u)(\gamma)\nabla^{(k)}_{U_{t_k}} F(\gamma_{t_1},\cdots,\gamma_{t_m}), h_{t_k}\rangle_{\mathbb{R}^n}\end{aligned} \tag{2.1}$$

for any $u \in H_0$ and μ-a.s. $\gamma \in E$, where

$$\Phi_t(u)(\gamma) := \int_0^t \Omega_{U_s}(u_s, \circ d\beta_s + \nabla_{U_s}\log p_{1-s}(\cdot, y_0)ds)$$

for any $t \in [0,1]$ and $u \in H_0$. In the above, $\Omega_U(a,b)c := U^{-1}R_{Ua,Ub}(x)Uc$, for any $a,b,c \in \mathbb{R}^n$ and $U \in \pi^{-1}(x) \subset \mathcal{O}(M)$ with $x \in M$, $R_{Ua,Ub}(x) : T_xM \to T_xM$ is the the curvature operator of the TSS connection ∇ on M, (β_t) is the martingale part of $(\int_0^t U_s^{-1}\circ d\gamma_s)$, and $\circ d\gamma_s$ stands for the Stratonovich differential of γ_s. In fact, (β_t) is a $\mathbb{R}^n$ -valued Brownian motion. Note that, as a mapping $\Omega_U(a,b) : \mathbb{R}^n \to \mathbb{R}^n$ it is skew symmetric, i.e.

$$\langle \Omega_U(a,b)c, d\rangle_{\mathbb{R}^n} = -\langle c, \Omega_U(a,b)d\rangle_{\mathbb{R}^n}$$

for any $a,b,c,d \in \mathbb{R}^n$, and we can consider $(U,a,b) \to \Omega_U(a,b)$ as a $o(n)$-valued mapping, where $o(n)$ denotes the Lie algebra consisting of skew symmetric $n\times n$ matrices. Hence, $t \to \Phi_t(u)(\gamma)$ is an $o(n)$-valued continuous function on $[0,1]$ for μ-a.s. $\gamma \in E$.

Let Ric be the Ricci curvature of the connection ∇, and $p_t(x,y)$ be the heat kernel of $\frac{1}{2}\Delta$ where Δ is the Levi-Civita Laplacian on M. For any $U \in \pi^{-1}(x) \subset \mathcal{O}(M)$ with $x \in M$ we define Ric_U and $\nabla^2_U \log p_{1-t}(\cdot, y_0)$, $(0 \le t < 1)$ by setting

$$Ric_U := U^{-1}Ric(x)U$$

and

$$\nabla^2_U \log p_{1-t}(\cdot, y_0) := U^{-1}\nabla^2 \log p_{1-t}(x, y_0)U,$$

where we consider $Ric(x)$ and $\nabla^2 log p_{1-t}(x,y_0)$ as maps from T_xM to itself.

Again, by one of the results in [GMR99] the following integration by parts formula holds: for any $f \in \mathcal{D}(\nabla_0)$

$$\int_E \langle \nabla_0 f, Y\rangle_{H_0} d\mu = -\int_E f div_1(Y)d\mu, \tag{2.2}$$

where for any $0 \le T \le 1$

$$div_T(Y) := -\int_0^T \langle \dot{Y}_t + \{\frac{1}{2}Ric^*_{U_t} - \nabla^2_{U_t}\log p_{1-t}(\cdot, y_0)^*\}Y_t, d\beta_t\rangle_{\mathbb{R}^n}, \tag{2.3}$$

and we denote the transposition of a matrix A by A^*.

On the one hand, observed that by (1.1) $\nabla_0 f$ is not only a H_0-valued random variable but also a W_0^*-valued random variable, where W_0^* denotes the dual space of the flat loop space $W_0 := \{w \in C([0,1];\mathbb{R}^n) : w(0) = w(1) = 0\}$. Let $\mathcal{M}(\mathbb{R}^n) := \mathcal{M}([0,1];\mathbb{R}^n)$ denote the space of totally finite, $\mathbb{R}^n$-valued Borel measure on $[0,1]$. For the simplicity of notation we use $\langle \cdot, \lambda \rangle$ to denote the integral with respect to the measure $\lambda \in \mathcal{M}(\mathbb{R}^n)$. One can establish an one-to-one mapping from $\mathcal{M}(\mathbb{R}^n)$ into H_0 as follows: $\Lambda : \lambda \in \mathcal{M}(\mathbb{R}^n) \to h^\lambda \in H_0$, and

$$h_t^\lambda := \Lambda(\lambda)_t := \int_{[0,1]} (min(t,s) - ts)\lambda(ds)$$

for any $t \in [0,1]$. It follows from above that $\langle h, h_\lambda \rangle_{H_0} = \langle h, \lambda \rangle$ for any $h \in H_0$. Following the idea of [ES96] we define $\nabla_0^{W_0^*} f : E \to \mathcal{M}(\mathbb{R}^n)$ by the following equation: for any $t \in [0,1]$

$$\nabla_0 f(\gamma)_t = \int_{[0,1]} (min(t,s) - ts)\nabla_0^{W_0^*} f(\gamma)(ds),$$

i.e. $\nabla_0^{W_0^*} f = \Lambda^{-1}\nabla_0 f$. It follows from (1.1) that

$$\nabla_0^{W_0^*} f(\gamma) = \sum_{1 \le i \le m} \nabla_{U_{t_i}}^{(i)} F(\gamma_{t_1}, \cdots, \gamma_{t_m})\delta_{t_i}, \tag{2.4}$$

where δ_t denotes the Draic measure on $[0,1]$ at the point $t \in [0,1]$. Obviously, $\nabla_0^{W_0^*} f(\gamma)\left([\frac{1+t_m}{2}, 1]\right) = 0 (\forall \mu - a.s. \gamma \in E)$.

On the other hand, observed that there are two terms in (2.1). The first term concerns with the second order derivative of F, and the second term conerns the first order derivative of F associated with an action of an $o(n)$-valed random variable which is independent on F. Also following the idea of [ES96] we define $\nabla_{0;H_0}^2 f : E \to H_0 \otimes H_0$ by

$$\begin{aligned} \left\langle \nabla_{0;H_0}^2 f(\gamma), h \otimes u \right\rangle_{H_0 \otimes H_0} &:= Q(h,u)(\gamma) \\ &:= \sum_{1 \le i,j \le m} \nabla_{U_{t_i} u_{t_i}}^{(i)} \{\nabla_{U_{t_j} h_{t_j}}^{(j)} F\}(\gamma_{t_1}, \cdots, \gamma_{t_m}) \end{aligned}$$

for any $h, u \in H_0$. Since there exists a constant $C := C(F) \in (0,\infty)$ such that for μ-a.s. $\gamma \in E$

$$|Q(h,u)(\gamma)| \le C\|h\|_{H_0}\|u\|_{H_0},$$

the above definition is well-defined. We can also define $Tr_{H_0}\nabla_{0;H_0}^2 f : E \to \mathbb{R}$ by

$$Tr_{H_0}\nabla_{0;H_0}^2 f(\gamma) = \sum_\alpha \left\langle \nabla_{0;H_0}^2 f(\gamma), h_\alpha \otimes h_\alpha \right\rangle_{H_0 \otimes H_0},$$

where $\{h_\alpha\}$ is any ONB of H_0. According to the above definition we obtain that for μ-a.s. $\gamma \in E$

$$\nabla_{0;H_0}^2 f(\gamma)_{t,s} = \sum_{1 \le i,j \le m} (min(t,t_i) - tt_i)(min(s,t_j) - st_j)\nabla_{U_{t_i}}^{(i)} \{\nabla_{U_{t_j}}^{(j)} F\}(\gamma_{t_1}, \cdots, \gamma_{t_m}),$$

where $\nabla^{(i)}_{U_1}\{\nabla^{(j)}_{U_2}F\} \in \mathbb{R}^n \otimes \mathbb{R}^n$ denotes the unique element in $\mathbb{R}^n \otimes \mathbb{R}^n$ such that

$$\left\langle a \otimes b, \nabla^{(i)}_{U_1}\{\nabla^{(j)}_{U_2}F\}\right\rangle_{\mathbb{R}^n\otimes\mathbb{R}^n} := \nabla^{(i)}_{U_1 a}\{\nabla^{(j)}_{U_2 b}F\}$$

for any $a, b \in \mathbb{R}^n$ and $U_1, U_2 \in \mathcal{O}(M)$. Moreover, for μ-a.s. $\gamma \in E$ we have

$$Tr_{H_0}\nabla^2_{0;H_0} f(\gamma) = \sum_{1\le i,j\le m} (min(t_i,t_j) - t_i t_j)\nabla^{(i)}_{U_{t_i}}\{\nabla^{(j)}_{U_{t_j}}F\}(\gamma_{t_1},\cdots,\gamma_{t_m}). \tag{2.5}$$

By (2.5) we know that $Tr_{H_0}\nabla^2_{0;H_0} f \in L^\infty(\mu)$. By (2.1) we have for μ-a.s. $\gamma \in E$

$$-\langle \Phi(u)h, \nabla^{W_0^*}_0 f(\gamma)\rangle = \sum_{1\le k\le m} \langle \Phi_{t_k}(u)\nabla^{(k)}_{U_{t_k}}F(\gamma_{t_1},\cdots,\gamma_{t_m}), h_{t_k}\rangle_{\mathbb{R}^n},$$

and hence

$$\partial^2 f(h,u)(\gamma) = \left\langle \nabla^2_{0;H_0} f(\gamma), h\otimes u\right\rangle_{H_0\otimes H_0} - \langle \Phi(u)(\gamma)h, \nabla^{W_0^*}_0 f(\gamma)\rangle. \tag{2.6}$$

The following is the main result of this paper:

Theorem 2.1. *Let us define* $\mathcal{U} : [0,1]\times E \to C([0,1];\mathbb{R}^n)$ *by*

$$\begin{aligned}\mathcal{U}(T)_t &:= \beta_t - t\beta_1 - \mathcal{R}_t + \hat{\mathcal{R}}(T)_t \\ &+ \int_0^t s(1-t)\{Ric_{U_s}\nabla_{U_s}\log p_{1-s}(\cdot,y_0) - \frac{1}{2}\nabla_{U_s}\kappa\}ds\end{aligned} \tag{2.7}$$

for any $t, T \in [0,1]$, *where*

$$\kappa_U := \sum_{1\le k\le n} \nabla_{Ue_k} Ric_U e_k, (\forall U \in \mathcal{O}(M)),$$

$\{e_k : 1 \le k \le n\}$ *is an ONB of* $\mathbb{R}^n$,

$$\mathcal{R}_t := \int_0^t s(1-t)Ric_{U_s}d\beta_s,$$

and

$$\hat{\mathcal{R}}(T)_t := \int_0^T (min(t,s) - ts)\{\frac{1}{2}Ric_{U_s} - \nabla^2_{U_s}\log p_{1-s}(\cdot,y_0)\}d\beta_s,$$

for any $t, T \in [0,1]$. *Then for any* $f \in \mathcal{F}C^\infty(E)$ *we have:*

(i)*:* $f \in \mathcal{D}(L)$, *and*

$$Lf = Tr_{H_0}\nabla^2_{0;H_0} f - \langle \mathcal{U}(1), \nabla^{W^*}_0 f\rangle. \tag{2.8}$$

(ii)*: There exists a constant* $\varepsilon := \varepsilon(f) > 0$ *such that*

$$\int_E \exp\{\varepsilon|Lf|^2\}d\mu < \infty. \tag{2.9}$$

Moreover,

$$max_{t\in[0,1]}|\mathcal{U}(1)_t| \in \bigcap_{1<p<\infty} L^p(\mu). \tag{2.10}$$

(iii)*: If we set: for any* $0 \le t \le 1$

$$b(t) := \beta_t + \int_0^t \nabla_{U_s} \log p_{1-s}(\cdot, y_0)ds,$$

then for any $t \in [0,1]$

$$\begin{aligned}\mathcal{U}(1)_t =& b(t) - tb(1) + \frac{1}{2}\int_0^1 (min(t,s) - st)Ric_{U_s}db(s) - \mathcal{R}_t \\ &+ \int_0^t s(1-t)\{Ric\nabla_{U_s}\log p_{1-s}(\cdot, y_0) - \frac{1}{2}\kappa_{U_s}\}ds.\end{aligned} \tag{2.11}$$

Hence, for the compact Lie group $M = G$ *with the right Cartan connnection, we know that* $Ric = 0$ *and*

$$\mathcal{U}(1)_t = b(t) - tb(1), \tag{2.12}$$

for any $t \in [0,1]$. *In particular, we can write the L. Gross's Schrödinger operator on* $\mathcal{F}C^\infty(E)$ *as*

$$-Tr_{H_0}\nabla^2_{0;H_0} + \langle b(\cdot) - \cdot b(1), \nabla_0^{W_0^*}\rangle + \alpha|b(1)|^2, \tag{2.13}$$

for any $\alpha > 0$.

(iv)*: Let* D *be a nonempty connected component of* E. *By the facts that* $1_D \in \mathcal{D}(\nabla_0)$ *and* $\nabla_0 1_D = 0$ *we know that* $1_D \in \mathcal{D}(L)$ *and* $L1_D = 0$. *Hence,* $f1_D \in \mathcal{D}(L)$, $L(f1_D) = 1_D Lf$, $f|_D \in \mathcal{D}(L_D)$, $L_D(f|_D) = (Lf)|_D$, *and*

$$\int_D \exp\{\varepsilon|L_D(f|_D)|^2\}d\mu_D < \infty \tag{2.14}$$

for some $\varepsilon := \varepsilon(f) > 0$.

3 The proof of Theorem 2.1

In this section we prove Theorem 2.1.

For convenience, we recall the following estimates which are proved in [Dr94], [Sh91], and [St96] respectively:

$$\begin{aligned}& max\left\{|\nabla \log p_{1-s}(\cdot, y_0)|^2, \|\text{Hess } logp_{1-s}(\cdot, y_0)\|_{H.S.}\right\} \\ &\le c(\frac{1}{1-s} + \frac{d(\cdot, y_0)^2}{(1-s)^2}), \quad s \in [0,1)\end{aligned} \tag{3.1}$$

and

$$E_\mu[d(\gamma_s, y_0)^{2p}] \le c(p)(1-s)^p, \quad s \in [0,1), \quad p \in [1,\infty), \tag{3.2}$$

where $d(\cdot,\cdot)$ is the Riemannian distance on M.

Proof. (of Theorem 2.1) We formulate the proof in two claims.

Claim 1: (2.8)-(2.9) and (iii) hold.

Proof of Claim 1: We prove *Claim 1* by six steps. The proof of (2.8) with (2.7) is a slight modification of that of Theorem 3.13 in [ES96].

Step 1. Let $\{e_k\}_{1\le k\le n}$ be an ONB of $\mathbb{R}^n$. For any m $\mathbb{N}$ we set: for any $t\in[0,1]$

$$h_m(t) := \frac{\sqrt{2}sin(m\pi t)}{m\pi},$$

and for any $m,k\in\mathbb{N}$, $1\le k\le n$, and $t\in[0,1]$

$$h_\alpha(t) := h_{(m,k)}(t) := \frac{\sqrt{2}sin(m\pi t)}{m\pi}e_k,$$

where $\alpha := (m,k)$. Then

$$\{h_\alpha : \alpha=(m,k), 1\le m<\infty, 1\le k\le n\}\subset W_0^*$$

is an ONB of H_0, and for any $t,s\in[0,1]$

$$\sum_m h_m(t)h_m(s) = min(t,s) - ts, \tag{3.3}$$

where the convergence is absolute and uniform on $[0,1]^2$.

For any $\alpha=(m,k)$ and $f\in\mathcal{D}(\mathcal{E})$ we set $\partial_\alpha f := \partial_{h_\alpha} f$ and $\partial_\alpha^2 f := \partial^2 f(h_\alpha, h_\alpha)$.

Now, for the fixed $\mathcal{F}_T$-measurable $f,g\in\mathcal{F}C^\infty(E)$ with $0<T<1$ we have

$$-\mathcal{E}(f,g) = -\sum_\alpha\int_E \partial_\alpha f\partial_\alpha g d\mu. \tag{3.4}$$

Since $\partial_\alpha f\in\mathcal{D}(\mathcal{E}) = \mathcal{D}(\nabla_0)$ and $g\partial_\alpha^2 f = \partial_\alpha\{g\partial_\alpha f\} - \partial_\alpha f\partial_\alpha g$ for any α, by (2.2) with (2.3) and the fact $f,g\in\mathcal{F}_T$ we get: for any α

$$-\int_E \partial_\alpha f\partial_\alpha g d\mu = \int_E g div_T(h_\alpha)\partial_\alpha f d\mu + \int_E g\partial_\alpha^2 f d\mu. \tag{3.5}$$

Step 2: Using (2.6) and $f,g\in\mathcal{F}_T$ we get: for any α and μ-a.s. $\gamma\in E$

$$\begin{aligned}\partial_\alpha^2 f(\gamma) &= \langle\nabla^2_{0;H_0} f(\gamma), h_\alpha\otimes h_\alpha\rangle\\ &\quad -\int_{[0,T]}\langle\Phi_t(h_\alpha)h_\alpha(t), \nabla_0^{W_0^*} f(\gamma)(dt)\rangle.\end{aligned}$$

Hence, by the fact that for any α and μ-a.s. $\gamma\in E$

$$|g(\gamma)\langle\nabla^2_{0;H_0} f(\gamma), h_\alpha\otimes h_\alpha\rangle| \le C < \infty$$

we obtain that

$$\begin{aligned}\sum_\alpha\int_E g\partial_\alpha^2 f d\mu &= \int_E g Tr_{H_0}\nabla^2_{0;H_0} f d\mu\\ &\quad -\sum_\alpha\int_E g\langle\Phi(h_\alpha)h_\alpha, \nabla_0^{W_0^*} f\rangle d\mu.\end{aligned} \tag{3.6}$$

Step 3: Note that for any $a \in \mathbb{R}^n$ and $U \in \mathcal{O}(M)$

$$Ric_U a = \sum_{1 \le k \le n} \Omega_U(a, e_k) e_k.$$

Hence, it follows from (3.1) and (3.2) that there exists a $E' \in \mathcal{F}_1$ with $\mu(E') = 1$ such that for any $t \in [0,1]$ and $\gamma \in E'$

$$\begin{aligned} S(\gamma)_t : &= \{\int_0^t s(1-t)\{\frac{1}{2}\kappa_{U_s} - Ric_{U_s}\nabla_{U_s}\log p_{1-s}(\cdot, y_0)\}ds\}(\gamma) \\ &= \sum_{m<\infty}\{h_m(t)\int_0^t h_m(s)\{\frac{1}{2}\kappa_{U_s} - Ric_{U_s}\nabla_{U_s}\log p_{1-s}(\cdot, y_0)\}ds\}(\gamma), \end{aligned}$$

$t \in [0,1] \to S(\gamma)_t$ is continuous, and

$$max_{t\in[0,1]}|S_t| \in \bigcap_{1<p<\infty} L^p(\mu). \tag{3.7}$$

Since for any $N \in \mathbb{N}$ and $t_1, t_2 \in [0,1]$ we have

$$\int_E |Y_N(t_1) - Y_N(t_2)|^{2p} d\mu \le C_p |t_1 - t_2|^p$$

for any $p \ge 1$, where

$$Y_N(t) := \sum_{m \le N} h_m(t) \int_0^t h_m(s) Ric_{U_s} d\beta_s,$$

for any $N \in \mathbb{N}$ and $t \in [0,1]$. Hence, by (3.3), the Burkholder-Davis-Gundy inequality, and the Kolmogorov's continuity criterion, we know that there also exists a $E' \in \mathcal{F}_1$ with $\mu(E') = 1$ such that for any $t \in [0,1]$ and $\gamma \in E'$

$$\begin{aligned} &\sum_{m\in\mathbb{N}} h_m(t)\{\int_0^t h_m(s) Ric_{U_s} d\beta_s\}(\gamma) \\ &= \mathcal{R}(\gamma)_t := \{\int_0^t s(1-t) Ric_{U_s} d\beta_s\}(\gamma), \end{aligned}$$

$t \in [0,1] \to \mathcal{R}(\gamma)_t$ is continuous, and

$$max_{t\in[0,1]}|\mathcal{R}_t| \in \bigcap_{1\le p<\infty} L^p(\mu). \tag{3.8}$$

By the definition of h_α, (3.3), and the above facts, we obtain that for any $t \in [0,1]$

$$\begin{aligned} &-\sum_\alpha \Phi_t(h_\alpha)h_\alpha(t) \\ &= (S\mathcal{R})_t := \mathcal{R}_t + \int_0^t s(1-t)\{\frac{1}{2}\kappa_{U_s} - Ric_{U_s}\nabla_{U_s}\log p_{1-s}(\cdot, y_0)\}ds, \end{aligned} \tag{3.9}$$

$t \in [0,1] \to (S\mathcal{R})_t$ is continuous μ-a.s. on E, and

$$max_{t\in[0,1]}|(S\mathcal{R})_t| \in \bigcap_{1<p<\infty} L^p(\mu). \tag{3.10}$$

Step 4: Recall that for any $m \in \mathbb{N}$ and $t, s \in [0,1]$

$$\int_0^1 h_m(t)(\dot{h}_m(s))d\beta_s = (-1)^m h_m(t)\beta_1 + \int_0^1 \beta_s sin(m\pi t) sin(m\pi s) ds.$$

Hence, we obtain that

$$lim_{N\to\infty} \int_0^1 \sum_{m\leq N} h_m(t)(\dot{h}_m(s))d\beta_s = \beta_t - t\beta_1$$

uniformly with respect to $t \in [0,1]$ in $L^p(\mu)$, $(\forall 1 \leq p < \infty)$, and

$$\sum_\alpha \int_E g\partial_\alpha f\{\int_0^T \langle \dot{h}_\alpha, d\beta_s\rangle_{\mathbb{R}^n}\}d\mu = \int_E g\{\int_0^T \langle \beta_t - t\beta_1, \nabla_0^{W_0^*} f(dt)\rangle_{\mathbb{R}^n}\}.$$

Note that

$$max_{s\in[0,T],x\in M}|\nabla^2 \log p_{1-s}(x, y_0)| < \infty.$$

Hence, again by using (3.3), the Burkholder-Davis-Gundy inequality, and the Kolmogorov's continuity criterion we obtain that there exists a $E' \in \mathcal{F}_1$ with $\mu(E') = 1$ such that for any $\gamma \in E'$ and $t \in [0,1]$

$$\begin{aligned}&\sum_\alpha h_\alpha(t)\{\int_0^T \langle h_\alpha(s), (Rc)_s d\beta_s\rangle_{\mathbb{R}^n}\}(\gamma)\\ &= (RL)(T,\gamma)_t := \{\int_0^T (min(t,s) - ts)(Rc)_s d\beta_s\}(\gamma),\end{aligned}$$

$t \in [0,1] \to (RL)(T,\gamma)_t$ is continuous, and

$$max_{t\in[0,1]}|(RL)(T)_t| \in \bigcap_{1\leq p<\infty} L^p(\mu),$$

where for any $s \in [0,T]$

$$(Rc)_s := \frac{1}{2} Ric_{U_s} - \nabla^2_{U_s} \log p_{1-s}(\cdot, y_0).$$

It follows from above facts that

$$\begin{aligned}&-\sum_\alpha \int_E g div_T(h_\alpha)\partial_\alpha f d\mu\\ &= \int_E g\{\int_0^T \langle \beta_t - t\beta_1 + (RL)(T)_t, \nabla_0^{W_0^*} f(dt)\rangle_{\mathbb{R}^n}\}.\end{aligned} \tag{3.11}$$

Step 5: According to (3.4)-(3.6), (3.9), (3.11) and the fact that

$$\nabla_0^{W_0^*}(\gamma)\left([\frac{T+1}{2},1]\right)=0,(\mu-a.s.\gamma\in E)$$

we know that for any $\mathcal{F}_T$-measurable functions $f,g\in\mathcal{F}C^\infty(E)$ we have

$$-\mathcal{E}(f,g)=\int_E g\{Tr_{H_0}\nabla^2_{0;H_0}f-\langle\mathcal{U}(T),\nabla_0^{W_0^*}f\rangle\}d\mu. \tag{3.12}$$

In order to prove (2.8) with (2.7) we need to prove that for the fixed $f\in\mathcal{F}C^\infty(E)$ the following equation

$$-\mathcal{E}(f,g)=\int_E g\{Tr_{H_0}\nabla^2_{0;H_0}f-\langle\mathcal{U}(1),\nabla_0^{W_0^*}f\rangle\}d\mu \tag{3.13}$$

holds for any $g\in\mathcal{F}C^\infty(E)$, and

$$Tr_{H_0}\nabla^2_{0;H_0}f-\langle\mathcal{U}(1),\nabla_0^{W_0^*}f\rangle\in L^2(\mu). \tag{3.14}$$

By (2.4) we get that for any $0<T<1$

$$\langle\mathcal{U}(T),\nabla_0^{W_0^*}f\rangle=\sum_i\langle\mathcal{U}(T)_{t_i},\nabla^{(i)}_{U_{t_i}}F\rangle_{\mathbb{R}^n}. \tag{3.15}$$

Hence, by (2.7) and (3.15) we know that (3.13) holds.

Step 6: Let us define $J_s:=\nabla_{U_s}\log p_{1-s}(\cdot,y_0)$. By Ito's formula one can prove that (or see [GMR99])

$$dJ_s=\nabla^2_{U_s}\log p_{1-s}(\cdot,y_0)d\beta_s+\frac{1}{2}Ric_{U_s}J_sds.$$

It follows from (3.1), (3.2), and the above formula that (2.11) holds. According to (2.11) we get (2.12) and (2.13).

By (2.5), (2.11), (3.1), (3.2) and (3.15) we obtain (3.14). Hence, $f\in\mathcal{D}(L)$ and (2.8) with (2.7) holds. Moreover, by (3.1) and the fact that

$$\int_E\exp\{\varepsilon|\int_0^1\frac{d(\gamma_s,y_0)}{1-s}ds|^2\}d\mu<\infty$$

for some $\varepsilon>0$ (see [Dr94]) we know that there exists a constant $\varepsilon>0$ such that

$$\int_E\exp\{\varepsilon\left(\int_0^1|J_s|ds\right)^2\}d\mu<\infty. \tag{3.16}$$

According to (2.5), (2.8), (2.11), (3.16), and exponential inequality of martingale IV-(3.16) in [RY91] we obtain (2.9).

By *Steps 1-6* we get (2.8)-(2.9) and (iii). This completes the proof of *Claim 1*.

Claim 2: (2.10) holds.

Proof of Claim 2: We prove *Claim 2* by three steps.

Step 1: For any $0 < T_1 < T_2 < 1$ and $t \in [0,1]$ we have

$$\hat{\mathcal{R}}(T_2)_t - \hat{\mathcal{R}}(T_1)_t = \int_{T_1}^{T_2} (min(t,s) - ts)\{\frac{1}{2}Ric_{U_s} - \nabla^2_{U_s} \log p_{1-s}(\cdot, y_0)\} d\beta_s.$$

By using the similar trick as in the proof of Proposition 3.6 in [GRW00] we get $\hat{\mathcal{R}}(T_2)_t - \hat{\mathcal{R}}(T_1)_t = I_1(t) + I_2(t)$. where

$$I_1(t) := \int_{T_1}^{T_2} (min(t,s) - ts)\{\frac{1}{2}\overline{Ric}_{U_s} - \mathrm{Hess}^2_{U_s} \log p_{1-s}(\cdot, y_0)\} d\beta_s,$$

$$I_2(t) := \frac{1}{2}\int_{T_1}^{T_2} (min(t,s) - ts)\{\Xi_{U_s}\langle \nabla_{U_s} \log p_{1-s}(\cdot, y_0), d\beta_s\rangle - \hat{T}_{U_s} d\beta_s\},$$

$\overline{Ric}$ is the Ricci curvature of the Levi-Civita connection $\bar{\nabla}$ on M, $T\langle \cdot, \cdot\rangle$ is the torsion tensor of the TSS connection ∇ on M, $\nabla = \bar{\nabla} + \frac{1}{2}T\langle \cdot, \cdot\rangle$, $Ric = \overline{Ric} + \hat{T}$, and $\hat{T}$ is a tensor determined only by T and $\bar{\nabla}T$. Hence, it follows from the Burkholder-Davis-Gundy inequality and (3.1)-(3.2) that for any $1 < p < \infty$ there exists a constant $C_{p,1} > 0$ such that

$$\int_E |\hat{\mathcal{R}}(T_2)_t - \hat{\mathcal{R}}(T_1)_t|^{2p} d\mu \le C_{p,1}|T_2 - T_1|^p \tag{3.17}$$

for any $T_1, T_2 \in (0,1)$ and $t \in [0,1]$.

Step 2: By the Burkholder-Davis-Gundy inequality and (3.1)-(3.2), we obtain that for any $1 < p < \infty$ there exists a constant $c_{p,0} > 0$ such that for any $0 < T < 1$, $0 \le t_1 \le t_2 \le 1$

$$\int_E |\hat{\mathcal{R}}(T)_{t_2} - \hat{\mathcal{R}}(T)_{t_1}|^{2p} d\mu \le c_{p,0}\left[\int_0^T |G_s(t_1,t_2)|^{2p} ds + \{\int_0^T |\frac{G_s(t_1,t_2)}{1-s}|^{\frac{2p}{p-1}} ds\}^{p-1}\right], \tag{3.18}$$

where $G_s(t_1,t_2) := \{min(t_2,s) - t_2 s\} - \{min(t_1,s) - t_1 s\}$. But for any $0 \le s \le T < 1$ we have

$$\begin{aligned} G_s(t_1,t_2) =& 1_{[0,t_1]}(T)\int_0^T s(t_1 - t_2) ds \\ &+ 1_{(t_1,t_2]}(T)\left(\int_0^{t_1} s(t_1 - t_2) ds + \int_{t_1}^T (s(1 + t_1 - t_2) - t_1) ds\right) \\ &+ 1_{(t_2,1)}(T)\left(\int_0^{t_1} s(t_1 - t_2) ds + \int_{t_1}^{t_2} (s(1 + t_1 - t_2) - t_1) ds\right) \\ &+ 1_{(t_2,1)}(T)\int_{t_2}^T (t_2 - t_1)(1 - s) ds. \end{aligned}$$

Hence, there exists $c_{p,1} > 0$ such that

$$\int_0^T |G_s(t_1,t_2)|^{2p} ds \le c_{p,1}|t_1 - t_2|^{2p} \quad (\forall t_1, t_2 \in [0,1], \quad T \in (0,1)). \tag{3.19}$$

In the following we prove that for $q := \frac{2p}{p-1}$ there exists $c_{p,2} > 0$ such that

$$\int_0^T |\frac{G_s(t_1,t_2)}{1-s}|^q ds \le c_{p,2}|t_1 - t_2| \quad (\forall t_1, t_2 \in [0,1], \quad T \in (0,1)). \tag{3.20}$$

In fact, for $0 < T \le t_1$ we have $\frac{t_2-t_1}{1-T} \le 1$ and

$$\begin{aligned}
&\int_0^{t_1} |\frac{s(t_1-t_2)}{1-s}|^q ds \\
&\le |t_2 - t_1| \frac{|t_2 - t_1|^{q-1}}{(q-1)(1-T)^{q-1}} \\
&\le \frac{|t_1 - t_1|}{q-1}.
\end{aligned}$$

Similarly, $\int_0^{t_1} |\frac{s(t_1-t_2)}{1-s}|^q ds \le \frac{|t_2-t_1|}{q-1}$ for $t_1 < T \le t_2$, and

$$\int_0^{t_1} |\frac{s(t_1-t_2)}{1-s}|^q ds \le \frac{|t_2-t_1|}{q-1}$$

for $t_2 < T < 1$. Note that for $t_1 \le s \le \frac{t_1}{1+t_1-t_2} \le t_2$ we have $t_1(t_1 - t_2) \le s(1 + t_1 - t_2) - t_1 \le 0$, and

$$\begin{aligned}
&\int_{t_1}^{\frac{t_1}{1+t_1-t_2}} |\frac{s(1+t_1-t_2)-t_1}{1-s}|^q ds \\
&\le |t_2 - t_1| \frac{|t_2 - t_1|^{q-1}}{(q-1)(1-t_2)^{q-1}} \\
&\le \frac{|t_2 - t_1|}{q-1}.
\end{aligned}$$

Moreover, for $\frac{t_1}{1+t_1-t_2} \le s \le T \le t_2$ we have $0 \le s(1+t_1-t_2)-t_1 \le (t_2-t_1)(1-t_2)$, and

$$\begin{aligned}
&\int_{\frac{t_1}{1+t_1-t_2}}^{T} |\frac{s(1+t_1-t_2)-t_1}{1-s}|^q ds \\
&\le |t_2 - t_1| \frac{|t_2 - t_1|^{q-1}}{(q-1)(1-T)^{q-1}} \\
&\le \frac{|t_2 - t_1|}{q-1}.
\end{aligned}$$

Hence, we get: for $t_1 < T \le t_2$

$$\int_{t_1}^{T} |\frac{s(1+t_1-t_2)-t_1}{1-s}|^q ds \le c_{p,3}|t_2 - t_1|.$$

Similarly, for any $t_2 < T < 1$, $\int_{t_1}^{t_2} |\frac{s(1+t_1-t_2)-t_1}{1-s}|^q ds \le c_{p,4}|t_2 - t_1|$. By the above inequalities we get (3.20), and by (3.18), (3.19), and (3.20) we know that for any

$1 < p < \infty$ there exists a constant $c_p > 0$ such that for any $0 < T < 1$, $0 \le t_1 \le t_2 \le 1$

$$\int_E |\hat{\mathcal{R}}(T)_{t_2} - \hat{\mathcal{R}}(T)_{t_1}|^{2p} d\mu \le c_p |t_2 - t_1|^{p-1}. \tag{3.21}$$

Step 3: Using (3.17) and (3.21) we get: for any $1 < p < \infty$ there exists a constant $C_{p,2} > 0$ such that for any $0 < T_1, T_2 < 1$, $0 \le t_1, t_2 \le 1$

$$\int_E |\hat{\mathcal{R}}(T_2)_{t_2} - \hat{\mathcal{R}}(T_1)_{t_1}|^{2p} d\mu \le C_{p,2}\{|t_2 - t_1|^{p-1} + |T_2 - T_1|^p\}. \tag{3.22}$$

By (3.22) and the Kolmogorov's continuity criterion we obtain that there exists a mapping $((T,t),\gamma) \in [0,1]^2 \times E \to \hat{\mathcal{R}}(T,\gamma)_t \in \mathbb{R}^n$ such that $\hat{\mathcal{R}}(T,\gamma)_t$ is continuous with respect to $T, t \in [0,1]$ for μ-a.s. $\gamma \in E$, and it is an extension of $\{\hat{\mathcal{R}}(T)\}_{0 \le T < 1}$. Moreover, for any $2 < p < \infty$

$$\int_E sup_{T\in[0,1]} \|\hat{\mathcal{R}}(T)\|^p_{C([0,1];\mathbb{R}^n)} d\mu < \infty. \tag{3.23}$$

Using (3.7), (3.8), and (3.10) we get: for any $2 < p < \infty$

$$\int_E max_{t\in[0,1]} |\mathcal{U}(1)_t - \hat{\mathcal{R}}(1)_t|^p d\mu < \infty. \tag{3.24}$$

.

Claim 2 follows from (3.23) and (3.24).

Claims 1, 2 imply (i) – (iii), and (iv) follows from (ii). □

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