Poincaré Inequality for Weighted First Order Sobolev Spaces on Loop Spaces

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Dedicated to the memory of the second named author's father,
Otto Peter Röckner

Abstract

Let E be the loop space over a compact connected Riemannian manifold with a torsion skew symmetric connection. Let L_D be the Ornstein-Uhlenback operator on a nonempty connected component D of the loop space E, and $V : D \to \mathbb{R}$ be the restriction on D of the potential in the logarithmic Sobolev inequality found by L. Gross on the loop group, S.Aida, and F.Z. Gong and Z.M. Ma, on the loop space respectively. We prove that the Schrödinger operator $-L_V := -L_D + V$ always has a spectral gap at the bottom $\lambda_0(V)$ of its spectrum, and so has its ground state transformed operator $\phi^{-1}(-L_V - \lambda_0(V))\phi$, where ϕ is the unique ground state of $-L_V$. In particular, our result proves L. Gross's conjecture about the existence of a spectral gap for the ground state transform of the Schrödinger operator studied by him on the loop group. In addition, in all the above cases we identify the domain of the Dirichlet forms associated with the ground state transforms as weighted first order Sobolev spaces with weight given by ϕ^2, thus establishing a Poincaré inequality for them. All these results are consequences from some new results in this paper on Dirichlet forms characterizing certain classes with spectral gaps and from results by S. Aida and M. Hino.

1 Introduction

A challenging open problem is to develop a De Rham-Hodge theory for loop spaces. To this end, the first step is to prove a spectral gap for some natural operators on loop spaces. The most natural one is the Ornstein-Uhlenbeck operator $\nabla_{0,\mu}^*\nabla_0$ (O-U operator for short) on the space of square integrable functions with respect to the pinned Wiener measure μ (i.e. the Brownian bridge measure) (c.f. [Gr94]), where ∇_0 is the Malliavin gradient, and $\nabla_{0,\mu}^*$ is its dual with respect to μ. A. Eberle [Eb00], however, recently proved the following important result: there is a class of simply connected compact Riemannian manifolds such that the O-U operators on the corresponding loop spaces fail to have spectral gaps, or in other words, the corresponding first order Sobolev spaces $W^{1,2}(\mu)$ in $L^2(\mu)$ fail to satisfy the Poincaré inequality. So, in order to establish a De Rham-Hodge theory on loop spaces one has to change the measure and correspondingly the O-U operators. In this paper, following an idea of L. Gross [Gr93], we shall show that in this respect ground state transforms of certain Schrödinger operators, i.e. operators of type " O-U operators plus a V" for properly chosen potentials V, are the appropriate substitutes for the O-U operator. The domain of the corresponding Dirichlet forms are on a heuristic level easily shown to be weighted Sobolev spaces $W^{1,2}(\phi^2\mu)$, where the weight is given by the ground state of the respective Schrödinger operator. The spectral gap of the transformed operator is then equivalent to a Poincaré inequality for $W^{1,2}(\phi^2\mu)$. The potentials V that serve our pupose have already been studied in the literature and serveral important results on the corresponding Schrödinger operators have been obtained.

In [Gr91] L.Gross established a log-Sobolev inequality for the O-U operator plus a certain potential on the loop group (see also [Ge91]). In [Gr93] he proved the uniqueness of the ground state for this type of Schrödinger operator on each connected component of the loop group. He pointed out in Remark 10.8 of [Gr93] that the ground state measure $\mu_\phi := \phi^2\mu$ is a natural measure on (each connected component of) the loop group, since it is determined by the given inner product of the Lie algebra of the underlying Lie group up to a constant factor. Hence, on each connected component of the loop group, one can replace the pinned Wiener measure by this new equivalent measure μ_ϕ and consider the ground state transformed operator $\nabla^*_{0,\mu_\phi}\nabla_0$ instead of the O-U operator. Moreover, L. Gross conjectured in Remark 10.8 of [Gr93] that this operator has a spectral gap. We should mention here that by a well known result of O. Rothaus in [Ro81] and B. Simon in [Si76] (or see [DSt89]) the log-Sobolev inequality for the generator of a Markovian semigroup implies the existence of a spectral gap of this generator. However, in contrast to the Markovian case, for a general Schrödinger operator its log-Sobolev inequality does not necessarily imply the existence of a spectral gap at the bottom of its spectrum.

In [Ai96], [Ai98a], and [Ai99], S. Aida extended L.Gross's results in [Gr91] and [Gr93] to loop spaces over compact connected Riemannian manifolds respectively. Gong and Ma [GM98] also extended L.Gross's result in [Gr91] to the loop space over a compact connected Riemannian manifold with the Levi-Civita connection. The potential they added depends only on the Ricci curvature and the Hessian of the heat kernel of the underlying manifold, and admits an explicit expression. In this paper we, in fact, generalize the result in [GM98] to the case of a general TSS (torsion skew symmetric) connection.

Along another direction, B. Driver and T. Lohrenz [DL96] established a log-Sobolev inequality without an added potential (see also [Fa99]), but replacing the pinned Wiener measure by the heat kernel measure of Brownian motion on the loop group constructed by P. Malliavin in [Ma90]. Since this case is Markovian, the associated operator has a spectral gap. Moreover, using a result of H. Airault and P. Malliavin in [AM92], B. Driver and V.K.Srimurthy [DS98] proved that the heat kernel measure is absolutely continuous with respect to the pinned Wiener measure on the loop group.

The aim of this paper is to prove general results in a suitable framework which imply that first, the ground state transforms of the Schrödinger operators for all potentials mentioned above have spectral gaps, and that second, the domains of the corresponding Dirichlet forms are indeed the mentioned weighted Sobolev spaces. For this among other things we use some beautiful results by Hino [Hi98] and S. Aida [Ai98b, 99]. Let us indicate the organization of this paper and describe on the way our results more precisely.

In Section 2 we describe our main results in the general framework of symmetric Dirichlet forms.

In Section 2.1 we use Duhamel's formula to analytically prove that a crucial property, i.e. the so called uniformly positivity improving (UPI) property, is invariant under zero order perturbations. Then, combining this result with [Wu00, (3.6)] and a beautiful result by Hino in [Hi98, Theorem 3.6], we give a sufficient and necessary condition for the existence of a spectral gap at the bottom of the spectrum, for a whole class of operators including Schrödinger operators of the type above.

In Section 2.2 we prove the mentioned characterization of the domain of the corresponding ground state transformed Dirichlet form as a weighted Sobolev space, provided the original Dirichlet form is conservative, quasi-regular, and has a square field operator. The proof is purely analytic.

In Section 3 we apply the results in Section 2 to the loop space.

In Section 3.1 on each connected component of the loop space over a compact connected Riemannian manifold with a TSS connection, slightly modifying [Ai98b] and [Ai99] we prove that the Ornstein-Uhlenbeck semigroup generated by the O-U operator is uniformly positivity improving. Hence by Section 2.1, all corresponding Schrödinger operators generate semigroups which on each connected component of the loop space have the UPI property.

In Section 3.2 we prove L. Gross's conjecture mentioned above. Moreover, according to a result in [GW99] we prove an F-Sobolev inequality and a super-Poincaré inequality for the ground state transform of this Schrödinger operator. Furthermore, we show that the above characterization of the domain of the ground state transformed Dirichlet form as a weighted Sobolev space holds in this case.

In Section 3.3, we first extend the log-Sobolev inequality for the Schrödinger operator on the loop space over a compact connected Riemannian manifold with the Levi-Civita connection, proved by Gong and Ma in [GM98], to the loop space over a compact connected Riemannian manifold with a TSS connection, and then extend all results from Section 3.2 to the Schrödinger operator with the potential in [GM98]. We emphasis that, similar results also hold for the Schrödinger operator given in [Ai96]. We note that the smooth cylinder functions on the loop space are all in the domain of the ground state transform of the Schrödinger operator (cf. Remark 3.2). However, to prove this result, one has to extend the expression of the O-U operator on the loop space in [ES96] to our general case, and then one can prove the crucial fact, i.e. the exponential integrability of the function $\varepsilon|Lf|^2$ for the O-U operator L for any smooth cylinder function f on the loop space, and some constant $\varepsilon := \varepsilon(f) > 0$. The details on this are contained in [Go00].

Note that, some of the results in this paper also hold for nonsymmetric operators. These are contained in [GWu00].

2 The main results in a general framework

2.1 Spectral gap for Schrödinger operators

Let $(\mathcal{O}, \mathcal{F}, \nu)$ be a probability space such that $(\mathcal{O}, \mathcal{F})$ is a Lusin space in the sense of [DM88] and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form (cf. [MR92, Chap.I, Def.4.5] and also [Fu80], [BH91], [FuOT94]) on $L^2(\nu) := L^2(\mathcal{O}, \mathcal{F}, \nu)$. We denote the strongly continuous semigroup on $L^2(\nu)$ associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ by $(P_t)_{t\geq 0}$.

Let $V : \mathcal{O} \to \mathbb{R}$ be $\mathcal{F}$-measurable and set as usual, $V^+ := max(V, 0)$, $V^- := -min(V, 0)$. Consider the following condition:

(H1): *$V^+ \in L^1(\nu)$ and there exist $a \in (0, 1)$, $b \in (0, \infty)$ such that for all $f \in \mathcal{D}(\mathcal{E}) \cap L^2(V^+ \cdot \nu)$*

$$\int f^2 V^- d\nu \leq a \left[\mathcal{E}(f, f) + \int f^2 V^+ d\nu \right] + b\|f\|^2_{L^2(\nu)}.$$

Proposition 2.1. *Suppose that* **(H1)** *holds.*

(i). *If*

$$\mathcal{E}_V(f,g) := \mathcal{E}(f,g) + \int fgV d\nu,$$

$f,g \in \mathcal{D}(\mathcal{E}_V) := \mathcal{D}(\mathcal{E}) \cap L^2(V^+ \cdot \nu)$. *Then for some* $\alpha > 0$, $(\mathcal{E}_V + \alpha(\cdot,\cdot)_{L^2(\nu)}, \mathcal{D}(\mathcal{E}_V))$ *is nonnegative definite, and for any such* α *a symmectric closed form (in the sense of [MR92, Chap.I, Def. 2.3]).*

(ii). *If* $(P_t^V)_{t\geq 0}$ *denotes the strongly continuous semigroup on* $L^2(\nu)$ *associated to* $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ *(defined under* (i)*), then for all* $f,g \in L^\infty(\nu)$, $t > 0$,

$$\int \left(P_t f - P_t^V f\right) g d\nu = \int_0^t \int (P_s f)(P_{t-s}^V g) V d\nu ds.$$

The proof of Proposition 2.1(i) is completely standard, hence omitted. Part (ii) is also essentially well-known. It follows e.g. by consecutively applying the following result from [BRZ00], first with $(\mathcal{E}^{(1)}, \mathcal{D}(\mathcal{E}^{(1)})) := (\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $(\mathcal{E}^{(2)}, \mathcal{D}(\mathcal{E}^{(2)})) := (\mathcal{E}_{V^+}, \mathcal{D}(\mathcal{E}_{V^+}))$, and then with $(\mathcal{E}^{(1)}, \mathcal{D}(\mathcal{E}^{(1)})) := (\mathcal{E}_{V^+}, \mathcal{D}(\mathcal{E}_{V^+}))$ and $(\mathcal{E}^{(2)}, \mathcal{D}(\mathcal{E}^{(2)})) := (\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$.

Proposition 2.2. *(Duhamel formula for sectorial forms, see [BRZ00, Prop. 2.2])*

Let $(\mathcal{E}^{(i)}, \mathcal{D}(\mathcal{E}^{(i)}))$ *be sectorial forms on* $L^2(\nu)$ *with corresponding semigroups* $(T_t^{(i)})_{t\geq 0}$, $i = 1, 2$. *Suppose that for some* $\alpha, c \in (0,\infty)$

$$\mathcal{E}_\alpha^{(1)}(u,u) \leq c\mathcal{E}_\alpha^{(2)}(u,u)$$

for all $u \in \mathcal{D}(\mathcal{E}^{(1)}) \cap \mathcal{D}(\mathcal{E}^{(2)})$. *Then for all* $t > 0$ *and all* $f,g \in \mathcal{D}(\mathcal{E}^{(1)}) \cap \mathcal{D}(\mathcal{E}^{(2)})$ *such that* $T_t^{(1)} f \in \mathcal{D}(\mathcal{E}^{(2)})$, $T_t^{(2)} g \in \mathcal{D}(\mathcal{E}^{(1)})$, $\forall t > 0$,

$$\begin{aligned}
&\int \left(T_t^{(1)} f - T_t^{(2)} f\right) g d\nu \\
&= \int_0^t \left[\mathcal{E}^{(2)}(T_s^{(1)} f, T_{t-s}^{(2)*} g) - \mathcal{E}^{(1)}(T_s^{(1)} f, T_{t-s}^{(2)*} g)\right] ds,
\end{aligned}$$

where $T_t^{(2)*}$ *denotes the adjoint of* $T_t^{(2)}$ *on* $L^2(\nu)$, $t > 0$.

Note that, Proposition 2.2 in [BRZ00] is stated in a slightly less general form. But as a brief look at its proof shows, it easily generalizes to the statement above.

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of bounded operators on $L^2(\nu)$ which is positivity *preserving* (i.e., $f \geq 0 \Rightarrow T_t f \geq 0$, $\forall t > 0, f \in L^2(\nu)$). Consider the following condition (cf. e.g. [Hi98] and the references therein):

(H2): $(T_t)_{t\geq 0}$ *is uniformly positivity improving (abbreviated: UPI), i.e., for all* $\varepsilon > 0$ *there exists* $t > 0$ *such that*

$$\chi_{T_t}(\varepsilon) := \inf\left\{\int 1_A T_t 1_B d\nu : A, B \in \mathcal{F}, \nu(A), \nu(B) \geq \varepsilon\right\} > 0.$$

Both $(P_t)_{t\geq 0}$ and $(P_t^V)_{t\geq 0}$ introduced above are positivity preserving (cf. [MR92, Chap.I, Prop. 4.2 and Theorem 4.4] and [MR95, (1.3) in Prop. 1.3(i)]).

Corollary 2.3. *Suppose* **(H1)** *holds. Then* **(H2)** *holds for* $(P_t^V)_{t\geq 0}$ *provided it holds for* $(P_t)_{t\geq 0}$.

For the proof of Corollary 2.3 we need the following:

Proposition 2.4. *Let $V = V^+$ and let $t > 0$, and define finite positive measures μ_t and μ_t^V on $(\mathcal{O} \times \mathcal{O}, \mathcal{F} \otimes \mathcal{F})$ by*

$$\mu_t(G) := \int P_t(1_G(\cdot, y))(y)\nu(dy),$$

$$\mu_t^V(G) := \int P_t^V(1_G(\cdot, y))(y)\nu(dy),$$

for any $G \in \mathcal{F} \otimes \mathcal{F}$, where we write $z = (x, y) \in \mathcal{O} \times \mathcal{O}$. Then μ_t and μ_t^V are equivalent.

Proof. Since $(\mathcal{O}, \mathcal{F})$ is Lusin, by [DM88, Chap.IX.11] we know that both $(P_t)_{t \geq 0}$ and $(P_t^V)_{t \geq 0}$ have regular versions given by sub-probability kernels, so both μ_t and μ_t^V are well-defined.

Let $t > 0$ be fixed. By the symmetry of P_r and P_r^V, Proposition 2.1(ii) and a monotone class argument we have that for all $G \in \mathcal{F} \otimes \mathcal{F}$, $r > 0$,

$$\mu_r(G) = \mu_r^V(G) + \int_0^r \int P_s \left(P_{r-s}^V(1_G(\cdot, y))V \right)(y)\nu(dy)ds,$$

where the integral with respect to ν is well-defined by the same reason as mentioned above. Its measurability with respect to s then follows by a monotone class argument. Consequently,

$$\mu_t^V \leq \mu_t. \tag{2.1}$$

For $n \in \mathbb{N} \cup \{0\}$, $t > 0$, define

$$V_n := inf(V, \frac{n}{2t}), \quad P_r^{(n)} := P_r^{V-V_n}, \quad \text{and } \mu_r^{(n)} := \mu_r^{V-V_n}.$$

Then by the same arguments as above for $G \in \mathcal{F} \otimes \mathcal{F}$ and $n \in \mathbb{N}$, $r > 0$,

$$\mu_r^{(n)}(G) = \mu_r^{(n-1)}(G) + \int_0^r \int P_s^{(n)} \left(P_{r-s}^{(n-1)}(1_G(\cdot, y))(V_n - V_{n-1}) \right)(y)\nu(dy)ds.$$

Consequently,

$$\mu_r^{(n)} \geq \mu_r^{(n-1)}, \quad P_r^{(n)} \geq P_r^{(n-1)}, \quad \forall n \in \mathbb{N}, \quad r > 0, \tag{2.2}$$

and thus for $G \in \mathcal{F} \otimes \mathcal{F}$

$$\mu_t^{(n)}(G) \leq \mu_t^{(n-1)}(G) + \|V_n - V_{n-1}\|_{L^\infty(\nu)} \int_0^t \int P_s^{(n)} \left(P_{t-s}^{(n)}(1_G(\cdot, y)) \right)(y)\nu(dy)ds.$$

Since by a monotone class argument

$$\int P_s^{(n)} \left(P_{t-s}^{(n)}(1_G(\cdot, y)) \right)(y)\nu(dy) = \int P_t^{(n)}(1_G(\cdot, y))(y)\nu(dy),$$

and since $\|V_n - V_{n-1}\|_{L^\infty(\nu)} \leq \frac{1}{2t}$, it follows that

$$\mu_t^{(n)} \leq 2\mu_t^{(n-1)}, \forall n \in \mathbb{N}. \tag{2.3}$$

By (2.2) and (2.3) $\mu_t^{(n)}$ and $\mu_t^{(n-1)}$ are equivalent for any $n \in \mathbb{N}$. Since $\mu_t^{(0)} = \mu_t^V$, by (2.1), (2.2), and (2.3) the assertion follows if we can show that for the finite measure

$$\nu_t := lim_{n\to\infty} \mu_t^{(n)}$$

we have

$$\nu_t = \mu_t.$$

For this it suffices to show that for all $A, B \in \mathcal{F}$

$$\nu_t(A \times B) = \mu_t(A \times B). \tag{2.4}$$

But the left hand side of (2.4) equals $lim_{n\to\infty} \int 1_A P_t^{(n)} 1_B d\nu$, while the right hand side equals $\int 1_A P_t 1_B d\nu$. So, it is enough to show that

$$lim_{n\to\infty} e^{-t} P_t^{(n)} f = e^{-t} P_t f, \quad \forall f \in L^2(\nu), \tag{2.5}$$

where the limit is taken with respect to $\| \cdot \|_{L^2(\nu)}$. But by [Mo94, Corollary 2.6.1] this is equivalent to proving the following two claims (cf. [Mo94, Def. 2.1.1]), i.e. to proving *Mosco-convergence* of

$$(\mathcal{E}_1^{(n)}, \mathcal{D}(\mathcal{E}^{(n)})) := (\mathcal{E}_{V-V_n,1}, \mathcal{D}(\mathcal{E}_{V-V_n}))$$

to $(\mathcal{E}_1, \mathcal{D}(\mathcal{E}))$ (where $\mathcal{E}_{V-V_n,1} := \mathcal{E}_{V-V_n} + (\cdot,\cdot)_{L^2(\nu)}$).

Claim 1: Let $f_n, f \in L^2(\nu)$, $n \in \mathbb{N}$, so that $f_n \to f$ weakly in $L^2(\nu)$ as $n \to \infty$. Then

$$\mathcal{E}_1(f,f) \le liminf_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n), \tag{2.6}$$

where here and in the following *Claim 2* we set $\mathcal{E}_1(f,f) := \infty$ and $\mathcal{E}_1^{(n)}(f,f) := \infty$ if $f \notin \mathcal{D}(\mathcal{E})$ resp. if $f \notin \mathcal{D}(\mathcal{E}^{(n)})$.

Claim 2: Let $f \in L^2(\nu)$. Then there exist $f_n \in L^2(\nu)$, $n \in \mathbb{N}$, such that $lim_{n\to\infty} \|f - f_n\|_{L^2(\nu)} = 0$ and

$$limsup_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n) \le \mathcal{E}_1(f,f). \tag{2.7}$$

So, it remains to prove *Claim 1* and *Claim 2*:

(*Proof of Claim 1*): We may assume that the right hand side of (2.6) is finite, and, selecting a subsequence if necessary, that

$$liminf_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n) = lim_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n).$$

Since $\mathcal{E}_1(f_n, f_n) \le \mathcal{E}_1^{(n)}(f_n, f_n)$, it follows by the Banach-Alaoglu Theorem that, selecting a subsequence if necessary, $f \in \mathcal{D}(\mathcal{E})$ and $f_n \to f$ weakly in the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ as $n \to \infty$. Consequently,

$$\mathcal{E}_1(f,f) \le liminf_{n\to\infty} \mathcal{E}_1(f_n, f_n) \le liminf_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n).$$

(*Proof of Claim 2*): We may assume that $\mathcal{E}_1(f,f) < \infty$, hence that $f \in \mathcal{D}(\mathcal{E})$. Define $f_n := sup(inf(f,n), -n)$, $n \in \mathbb{N}$. Then $f_n \to f$ in $L^2(\nu)$ as $n \to \infty$ and $\mathcal{E}_1(f_n, f_n) \le \mathcal{E}_1(f,f) (\forall n \in \mathbb{N})$, consequently,

$$limsup_{n\to\infty} \mathcal{E}_1^{(n)}(f_n, f_n) \le \mathcal{E}_1(f,f) + limsup_{n\to\infty} \int f^2 (V - V_n) d\nu = \mathcal{E}_1(f,f),$$

since $V_n \uparrow V$ and $V \in L^1(\nu)$.

Thus, the proof of the Proposition is completed. $\square$

Proof. (of Corollary 2.3): Suppose the assertion holds if $V = V^+$. Then by Propostion 2.1(ii) applied to $(P_t^{V^+})_{t\geq 0}$ instead of $(P_t)_{t\geq 0}$ we see that for all $t \geq 0$

$$\int 1_B P_t^{V+} 1_A d\nu \leq \int 1_B P_t^V 1_A d\nu$$

for all $A, B \in \mathcal{F}$. So, **(H2)** then also holds for $(P_t^V)_{t\geq 0}$. So, we may assume that $V = V^+$.

Let $\varepsilon > 0$ and let $t > 0$ be such that $\chi_{P_t}(\varepsilon) > 0$. Define μ_t and μ_t^V as in Proposition 2.4. Let $\rho_t \in L^1(\mu_t^V)$ such that

$$\mu_t = \rho_t \mu_t^V.$$

Let $n \in \mathbb{N}$ such that $\int_{[\rho_t > n]} \rho_t d\mu_t^V < \frac{\chi_{P_t}(\varepsilon)}{2}$. Then for all $A, B \in \mathcal{F}$ such that $\nu(A), \nu(B) \geq \varepsilon$ we have that

$$\begin{aligned}
\chi_{P_t}(\varepsilon) \leq \int 1_B P_t 1_A d\nu &= \mu_t(A \times B) \\
&= \int 1_{A\times B} \rho_t d\mu_t^V \\
&\leq \int_{[\rho_t > n]} \rho_t d\mu_t^V + \int_{[\rho_t \leq n]} 1_{A\times B} \rho_t d\mu_t^V \\
&\leq \frac{\chi_{P_t}(\varepsilon)}{2} + n \int 1_B P_t^V 1_A d\nu,
\end{aligned}$$

and the assertion follows. □

Remark 2.1. We emphasize that above we only work with a measurable space $(\mathcal{O}, \mathcal{F})$, which is Lusin, no particular topology is required. If $(\mathcal{O}, \mathcal{F})$ comes from a topological space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is quasi-regular (in the sense of [MR92, Chap.IV, Def. 3.1]) with respect to this topology, then Corollary 2.3 can be proved more easily using the Feynman-Kac formula and by exactly the same arguments as in the proof of [Hi98, Prop. 4.5].

Let $(L, \mathcal{D}(L))$ denote the generator of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and, provided **(H1)** holds, $(L_V, \mathcal{D}(L_V))$ that of $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ (cf. [MR92, Chap.I, Sect. 2]). In accordance with situations studied in mathematical physics we shall call $-L_V$ *(corresponding) Schrödinger operator*. Let $\sigma(L_V)$ denote its spectrum and set

$$\lambda_0(V) := -sup\sigma(L_V).$$

As a consequence of Corollary 2.3, [Wu00, (3.6)], and a beautiful result by M. Hino in [Hi98, Theorem 3.6] we obtain the following

Theorem 2.5. *Suppose* **(H1)** *and that* **(H2)** *holds for* $(P_t)_{t\geq 0}$. *Then the following are equivalent:*

(i) *There exists* $t > 0$ *such that*

$$limsup_{K\to\infty} sup_{\|f\|_{L^2(\nu)}\leq 1} \int \left(e^{t\lambda_0(V)} P_t^V f\right)^2 1_{[|P_t^V f|\geq K]} d\nu < 1.$$

(ii) $\lambda_0(V)$ *is an eigenvalue of* $-L_V$ *with corresponding eigenspace spanned by a* ν*-a.e. strictly positive eigenfunction* ϕ *(" ground state") and*

$$\lambda_1(V) := inf\{\lambda - \lambda_0(V) : \lambda > \lambda_0(V), \lambda \in \sigma(-L_V)\} > 0.$$

(i.e. the Schrödinger operator $(-L_V, \mathcal{D}(L_V))$ *has a spectral gap at the bottom of its spectrum).*

Proof. (i) $\Rightarrow$ (ii): Suppose (i). We first prove that $\lambda_0(V)$ is an eigenvalue of $-L_V$ and that there exists a corresponding eigenfunction ϕ such that $\phi \geq 0$, ν-a.e.. By the spectral theorem it is suffices to show that for

$$S := e^{t\lambda_0(V)} P_t^V$$

(where $t > 0$ is as in (i)) there exists $\phi \in L^2(\nu)$, $\phi \geq 0$, ν-a.e., $\|\phi\|_{L^2(\nu)} > 0$, such that

$$S\phi = \phi.$$

Let $\{\lambda_k\}_{k\in\mathbb{N}} \subset (1,\infty)$ be a decreasing sequence such that $lim_{k\to\infty}\lambda_k = 1$. Then by the fact (3.6) in [Wu00] we know that there exists a nonzero and nonnegative function $f \in L^2(\nu)$ such that

$$lim_{k\to\infty} C_k = \infty, \tag{2.8}$$

where $C_k := \|G_{\lambda_k} f\|_{L^2(\nu)}$, and G_λ denotes the resolvent operator of S. Set

$$\phi_k := \frac{G_{\lambda_k} f}{C_k},$$

for any $k \in \mathbb{N}$. Obviously, $\|\phi_k\|_{L^2(\nu)} = 1$, $\phi_k \geq 0$, ν-a.e., and

$$S\phi_k = \lambda_k \phi_k - \frac{f}{C_k},$$

for any $k \in \mathbb{N}$. Hence, there exists a subsequence of $\{\phi_k\}_{k\in\mathbb{N}}$ which is weakly convergent in $L^2(\nu)$ to a function $\phi \in L^2(\nu)$. Assume for simplicity that $w - lim_{k\to\infty}\phi_k = \phi$. It follows that $S\phi = \phi$ and $\phi \geq 0$, ν-a.e.. We want to prove that ϕ is not identically zero.

Suppose that $\phi \equiv 0$. Then we get

$$lim_{k\to\infty} \int \phi_k d\nu = lim_{k\to\infty} \langle 1, \phi_k \rangle_{L^2(\nu)} = 0. \tag{2.9}$$

But for any $L > 0$ and $k \in \mathbb{N}$

$$\begin{aligned}
1 &= \int \phi_k^2 d\nu \\
&= \frac{1}{\lambda_k} \int \phi_k |S\phi_k + \frac{f}{C_k}| d\nu \\
&\leq \frac{1}{\lambda_k} \int \phi_k |S\phi_k| d\nu + \frac{1}{\lambda_k C_k} \int \phi_k f d\nu \\
&\leq \frac{1}{\lambda_k} \int \phi_k |S\phi_k| 1_{[|S\phi_k| \geq L]} d\nu + \frac{L}{\lambda_k} \int \phi_k d\nu + \frac{\|f\|_{L^2(\nu)}}{\lambda_k C_k} \\
&\leq \frac{1}{\lambda_k} sup_{\|g\|_{L^2(\nu)} \leq 1} \int |Sg|^2 1_{[|Sg| \geq L]} d\nu + \frac{L}{\lambda_k} \int \phi_k d\nu + \frac{\|f\|_{L^2(\nu)}}{\lambda_k C_k}.
\end{aligned} \tag{2.10}$$

Taking $limsup_{L\to\infty} lim_{k\to\infty}$ in both sides of (2.10), by (i),(2.8), (2.9), and the fact that $lim_{k\to\infty}\lambda_k = 1$ we get a contradiction. Hence, ϕ is not identically zero.

Let $\phi \in L^2(\nu)$, $\phi \geq 0$, ν-a.e., $\|\phi\|_{L^2(\nu)} > 0$, such that $S\phi = \phi$, then (as above) $\forall s > 0$

$$e^{s\lambda_0(V)} P_s^V \phi = \phi,$$

and a standard argument using **(H2)** (cf. [Hi98, Proposition 3.3(iii)]) shows that $\phi > 0$, ν-a.e., and that $dim(1 - S) = 1$. It remains to prove the spectral gap. But, since by Corollary 2.3 $(e^{t\lambda_0(V)} P_t^V)_{t\geq 0}$ satisfies **(H2)** and because of Remark 2.2(i) below, this is now an immediate consequence of [Hi98, Theorem3.6(ii)] which implies that, if $\phi > 0$, ν-a.e., is as above such that $\|\phi\|_{L^2(\nu)} = 1$, then there exist $M, \delta > 0$ such that for all $f \in L^2(\nu)$ and $t > 0$

$$\|e^{t\lambda_0(V)} P_t^V f - < f, \phi >_{L^2(\nu)} \phi\|_{L^2(\nu)} \leq M e^{-\delta t} \|f\|_{L^2(\nu)}, \tag{2.11}$$

which by the spectral theorem implies the last part of assertion (ii).

(ii) $\Rightarrow$ (i): Clearly, (ii) implies (2.11). Therefore, (i) follows by Remark 2.2(i) below and the other half of [Hi98, Theorem 3.6(ii)]. □

Remark 2.2. (i) Since for all $K > 0$, $N \in \nabla$, $g \in L^2(\nu)$

$$(|g| - K)^+ \leq |g| 1_{[|g| \geq K]} \leq (|g| - \frac{K}{N})^+ + \frac{K}{N} 1_{[|g| \geq K]} \leq (|g| - \frac{K}{N})^+ + \frac{|g|}{N},$$

assertion (i) in Theorem 2.5 is indeed equivalent to Property (I) in [Hi98, Theorem 3.6(ii)].

(ii) The condition in Theorem 2.5(i) is e.g. obviously fulfilled if

$$\{e^{t\lambda_0(V)} P_t^V f : f \in L^2(\nu), \|f\|_{L^2(\nu)} \leq 1\}$$

is uniformly ν-square integrable. See [GW99] for a characterization of this property in terms of functional inequalities of $\mathcal{E}_V - \lambda_0(V)(\cdot,\cdot)_{L^2(\nu)}$. Obviously, the above uniformly ν-square integrability holds if for some $p \in (2, \infty)$ we have that $P_t^V : L^2(\nu) \to L^p(\nu)$ is continuous. This, in turn, is the case if $V \geq 0$ and $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ satisfies a (defective) logarithmic Sobolev inequality (cf. [Gr91, Theorem 5.1(ii)], [Gr93a], and p.242 in [DSt89]). We emphasize that in the latter case, the existence and uniqueness of a ground state ϕ with $\phi \geq 0$, ν-a.e., was already proved in [Gr72].

2.2 The corresponding ground state transform and the characterization of its domaim

Assume **(H1)** and that **(H2)** holds for $(P_t)_{t\geq 0}$ and that one (hence both) of the equivalent conditions (i) or (ii) in Theorem 2.5 is fulfilled. Then we can define the *ground state transform* $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ of $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ by

$$\begin{aligned} &\mathcal{E}_\phi(f, g) := \mathcal{E}_V(f\phi, g\phi) - \lambda_0(V)(f, g)_{L^2(\phi^2\nu)}, \\ &f, g \in \mathcal{D}(\mathcal{E}_\phi) := \{f \in L^2(\phi^2\nu) : f\phi \in \mathcal{D}(\mathcal{E}_V)\}. \end{aligned} \tag{2.12}$$

It is easily to check that $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ is a symmetric Dirichlet form on $L^2(\phi^2\nu)$ (cf. [MR95, Theorem 3.5]). The corresponding generator $(L_\phi, \mathcal{D}(L_\phi))$, semigroup $(P_t^\phi)_{t\geq 0}$ respectively are given by

$$L_\phi f = \frac{L_V(f\phi)}{\phi} + \lambda_0(V) f$$

for $f \in \mathcal{D}(L_\phi) := \{f \in L^2(\phi^2\nu) : f\phi \in \mathcal{D}(L_V)\}$, and

$$P_t^\phi f = \frac{e^{t\lambda_0(V)} P_t^V(f\phi)}{\phi}, f \in L^2(\phi^2\nu)$$

(cf. [MR95, Remark 3.2(iii)]).

Since $f \to f\phi$ is a unitary isomorphism from $L^2(\phi^2\nu)$ to $L^2(\nu)$, the spectral properties of $(L_\phi, \mathcal{D}(L_\phi))$ are uniquely determined by those of $(L_V, \mathcal{D}(L_V))$. In particular, $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ has a spectral gap of size $\lambda_1(V)$ above 0 (which in turn is of course an eigenvalue with eigenspace spanned by the constant function 1). However, since not much is known about ϕ, we can hardly handle $\mathcal{D}(\mathcal{E}_\phi)$ at all. The aim of this subsection is to give handable explicit description of $\mathcal{D}(\mathcal{E}_\phi)$. For this we need the following additional assumption:

(H3) *There exists a topology on $\mathcal{O}$ such that its Borel σ-algebra is equal to $\mathcal{F}$ and is generated by the corresponding continuous functions. Furthermore, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is quasi-regular in the sense of [MR92, Chap.IV, Def. 3.1] with respect to this topology.*

For the convenience of the reader we recall the definition of quasi-regularity from [MR92]. First we recall that a sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of $\mathcal{O}$ is called an $\mathcal{E}$-*nest* if

$$\mathcal{D}_0(\mathcal{E}, (F_k)_{k\in\mathbb{N}}) := \{f \in \mathcal{D}(\mathcal{E}) : f = 0 \quad \nu - a.e. \text{ on } \ \mathcal{O} \setminus F_k \text{ for some } \ k \in \mathbb{N}\}$$

is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm $\mathcal{E}_1(\cdot,\cdot)^{\frac{1}{2}} := (\mathcal{E}(\cdot,\cdot) + \|\cdot\|^2_{L^2(\nu)})^{\frac{1}{2}}$. Furthermore, a set $N \subset \mathcal{O}$ is called $\mathcal{E}$-*exceptional* if it is contained in the complement of some $\mathcal{E}$-nest, and a function $f : \mathcal{O} \to R$ is called $\mathcal{E}$-*quasi-continuous*, if $f|_{F_k}$ is continuous for all $k \in \mathbb{N}$ and some $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$.

Definition 2.1. A (symmetric) Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\nu)$ is called *quasi-regular* if:

(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\in\mathbb{N}}$ consisting of compact subsets.

(ii) There exists an $\mathcal{E}_1^{\frac{1}{2}}$-dense subset of $\mathcal{D}(\mathcal{E})$ whose elements have $\mathcal{E}$-quasi-continuous ν-versions.

(iii) There exist $u_n \in \mathcal{D}(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous ν-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional subset $N \subset \mathcal{O}$ such that $\{\tilde{u}_n : n \in \mathbb{N}\}$ separates the points of $\mathcal{O} \setminus N$.

We also recall that by [MR92, Chap.IV, Prop. 3.3(ii)] every $f \in \mathcal{D}(\mathcal{E})$ has a ν-version $\tilde{f}$ which is $\mathcal{E}$-quasi-continuous, provided **(H3)** holds.

The following can now be proved in exactly the same way as Theorem 3.7 in [Wu00a].

Proposition 2.6. *Consider the situation of Theorem 2.5 and assume that its part* (i) *or equivalently its part* (ii) *holds as well as* **(H3)**. *Suppose furthermore that*

(i) $V \in L^p(\nu)$ *for some* $p \in (2, \infty)$*;*

(ii) $M := log\left(sup_{0\le t\le 1} max(\|P_t^V\|_p, \|P_t^V\|_{\frac{2p}{p-2}})\right) < \infty$ *(which in in our case automatically holds if* $V^- \in L^\infty(\nu)$*).*

Then for $\lambda \in (M, \infty)$, $f \in L^{\frac{2p}{p-2}}(\nu)$, $f \ge 0$,

$$G_\lambda^V f := \int_0^\infty e^{-t\lambda} P_t^V f dt \in \mathcal{D}(L).$$

and for any $\mathcal{E}$-quasi-continuous ν-version $\widetilde{G_\lambda^V f}$ of $G_\lambda^V f$, the subset $\{\widetilde{G_\lambda^V f} = 0\}$ is $\mathcal{E}$-exceptional provided $\|f\|_{L^2(\nu)} > 0$. In particular, for any $\mathcal{E}$-quasi-continuous ν-version $\tilde{\phi}$ of the ground state ϕ, we have $\{\tilde{\phi} = 0\}$ is $\mathcal{E}$-exceptional.

Now we introduce our last hypothesis:

(H4) *$1 \in \mathcal{D}(\mathcal{E})$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ has a square field operator, i.e., there exists a positive definite symmetric bilinear mapping $\Gamma : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(\nu)$ such that for all $f, g, h \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\nu)$*

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h),$$

and

$$\mathcal{E}(f, g) = \int \Gamma(f, g) d\nu, \forall f, g \in \mathcal{D}(\mathcal{E}).$$

As usual we set $\Gamma(f) := \Gamma(f, f)$. It is easy to check that **(H4)** implies that $\Gamma(1) = 0$, hence $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *conservative*, i.e., $\mathcal{E}(1, 1) = 0$ or equivalently $P_t 1 = 1 (\forall t \geq 0)$. Furthermore, **(H4)** implies that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local in the sense of [BH91, Def. I.5.1.2] (cf. [Sch93, Prop.2.3]). In particular, by [BH91, Theorem I.7.1.1] for all $h \in \mathcal{D}(\mathcal{E})$

$$\Gamma(h) = 0 \quad \nu - a.e. \quad on \quad \{h = 0\}. \tag{2.13}$$

This in turn implies that for all $L > 0$

$$\Gamma\left(min(h, L)\right) \leq 1_{[h \leq L]} \Gamma(h)$$

and

$$\Gamma\left(max(h, L)\right) \leq 1_{[h \geq L]} \Gamma(h),$$

which we shall use below without further notice.

Define $L^0(\nu)$ to be the set of all ν-classes of $\mathcal{F}$-measurable functions on $\mathcal{O}$, and we define the set $\mathcal{D}_{loc}(\mathcal{E})$ by: $f \in \mathcal{D}_{loc}(\mathcal{E})$ if and only if $f \in L^0(\nu)$ and there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$ such that $f = f_k$ ν-a.e. on F_k for some $f_k \in \mathcal{D}(\mathcal{E})$, $(\forall k \in \mathbb{N})$. For $f, g \in \mathcal{D}_{loc}(\mathcal{E})$ with corresponding $\mathcal{E}$-nests $(F_k^f)_{k \in \mathbb{N}}$, $(F_k^g)_{k \in \mathbb{N}}$, $f_k, g_k \in \mathcal{D}(\mathcal{E})$, $k \in \mathbb{N}$, define

$$\Gamma(f, g) := \Gamma(f_k, g_k), \quad \nu - a.e., \text{on } F_k^f \cap F_k^g, \quad \forall k \in \mathbb{N}.$$

Since $(F_k^f \cap F_k^g)_{k \in \mathbb{N}}$ is again an $\mathcal{E}$-nest and because of (2.13) $\Gamma(f, g)$ is well-defined and independent of the specially chosen F_k^f, F_k^g, f_k, g_k, $k \in \mathbb{N}$, above. Obviously, $\Gamma : \mathcal{D}_{loc}(\mathcal{E}) \times \mathcal{D}_{loc}(\mathcal{E}) \to L^0(\nu)$ inherits all properties of $\Gamma : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(\nu)$. In particular, (2.13) holds and $\Gamma : \mathcal{D}_{loc}(\mathcal{E}) \times \mathcal{D}_{loc}(\mathcal{E}) \to L^0(\nu)$ is bilinear, symmetric, and positive definite.

Now we can prove the main result of this subsection, i.e., the characterization of $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi)$ mentioned above.

Theorem 2.7. *Consider the situation of Proposition 2.6, but instead of Conditions* (i) *and* (ii) *there, just assume that $\{\tilde{\phi} = 0\}$ is $\mathcal{E}$-exceptional. Suppose that* **(H4)** *holds. Then:*

(i) *$\mathcal{D}_0(\mathcal{E}_\phi) := \{f \in \mathcal{D}(\mathcal{E}) : \int (f^2 + \Gamma(f))\phi^2 d\nu < \infty\}$ is contained in $\mathcal{D}(\mathcal{E}_\phi)$ and*

$$\mathcal{E}_\phi(f, g) = \int \Gamma(f, g)\phi^2 d\nu, \quad \forall f, g \in \mathcal{D}_0(\mathcal{E}_\phi). \tag{2.14}$$

Moreover, $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ is the closure of $(\mathcal{E}_\phi, \mathcal{D}_0(\mathcal{E}_\phi))$ on $L^2(\phi^2\nu)$, hence $\Gamma : \mathcal{D}_0(\mathcal{E}_\phi) \times \mathcal{D}_0(\mathcal{E}_\phi) \to L^1(\phi^2\nu)$ extends to a mapping $\bar{\Gamma} : \mathcal{D}(\mathcal{E}_\phi) \times \mathcal{D}(\mathcal{E}_\phi) \to L^1(\phi^2\nu)$ such that

$$\mathcal{E}_\phi(f, g) = \int \bar{\Gamma}(f, g)\phi^2 d\nu, \quad \forall f, g \in \mathcal{D}(\mathcal{E}_\phi).$$

(ii) *Define*

$$\mathcal{D}_1(\mathcal{E}_\phi) := \{f \in \mathcal{D}_{loc}(\mathcal{E}) : \int (f^2 + \Gamma(f))\phi^2 d\nu < \infty\}.$$

Then $\mathcal{D}(\mathcal{E}_\phi) = \mathcal{D}_1(\mathcal{E}_\phi)$ *and* $\Gamma(f,g) = \bar{\Gamma}(f,g)\ \forall f,g \in \mathcal{D}(\mathcal{E}_\phi)$, *in particular,*

$$\mathcal{E}_\phi(f,g) = \int \Gamma(f,g)\phi^2 d\nu, \forall f,g \in \mathcal{D}(\mathcal{E}_\phi).$$

Remark 2.3. By Theorem 2.7 $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ is characterized as a (generalized) weighted Sobolev space of first order. Furthermore, by Theorem 2.5 the Poincaré inequality holds for this Sobolev space, i.e.,

$$\int \left(f - \int f d\nu\right)^2 d\nu \leq const. \int \Gamma(f)\phi^2 d\nu, \quad \forall f \in \mathcal{D}(\mathcal{E}_\phi).$$

Proof. (of Theorem 2.7): The proof is performed in several steps, formulated in three claims.

Claim 1: $\mathcal{D}_0(\mathcal{E}_\phi) \subset \mathcal{D}(\mathcal{E}_\phi)$ and (2.14) holds.

(*Proof of Claim 1*): Let $f \in \mathcal{D}_0(\mathcal{E}_\phi)$. Then $f\phi \in L^2(\nu)$.

Suppose first that $f \in L^\infty(\nu)$. Then for $\phi_n := inf(\phi, n)$, $n \in \mathbb{N}$, it follows that $f\phi_n \in \mathcal{D}(\mathcal{E})$ and

$$\Gamma(f\phi_n) \leq 2\left(f^2\Gamma(\phi_n) + \phi_n^2\Gamma(f)\right).$$

But since $\phi_n^2\Gamma(f) \leq \phi^2\Gamma(f) \in L^1(\nu)$ by the definition of $\mathcal{D}_0(\mathcal{E}_\phi)$, it follows by [MR92, Chap.I, Prop. 4.17], that $f\phi \in \mathcal{D}(\mathcal{E})$ and $\Gamma(f\phi) = f^2\Gamma(\phi) + \phi^2\Gamma(f) + 2f\phi\Gamma(f,\phi)$. Hence, clearly, $f\phi \in \mathcal{D}(\mathcal{E}_V)$, i.e., $f \in \mathcal{D}(\mathcal{E}_\phi)$. Furthermore, since

$$L_V\phi = -\lambda_0(V)\phi \quad and \quad \Gamma(f\phi) = \Gamma(f^2\phi, \phi) + \phi^2\Gamma(f),$$

we have

$$\begin{aligned}
\mathcal{E}_\phi(f,f) &= \mathcal{E}(f\phi, f\phi) + \int (V - \lambda_0(V))\phi f^2 \phi d\nu \\
&= \mathcal{E}_V(\phi, f^2\phi) - \lambda_0(V)\int \phi f^2\phi d\nu + \int \Gamma(f)\phi^2 d\nu \\
&= \int \Gamma(f)\phi^2 d\nu.
\end{aligned}$$

If $f \in \mathcal{D}_0(\mathcal{E}_\phi)$, consider

$$f_n := sup(inf(f,n), -n).$$

Then clearly, $f_n \in \mathcal{D}_0(\mathcal{E}_\phi) \cap L^\infty(\nu)$, hence $f_n \in \mathcal{D}(\mathcal{E}_\phi)$. Since $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ is a Dirichlet form, it follows by [MR92, Chap.I, Prop.4.17], that $f \in \mathcal{D}(\mathcal{E}_\phi)$. Since

$$\begin{aligned}
|\Gamma(f)^{\frac{1}{2}} - \Gamma(f_n)^{\frac{1}{2}}|^2 &\leq \Gamma(f - f_n) \\
&\leq 1_{[|f| \geq n]}\Gamma(f - f_n) \\
&\leq 2 \cdot 1_{[|f| \geq n]}\Gamma(f),
\end{aligned}$$

$\forall n \in \mathbb{N}$, we also obtain that

$$\mathcal{E}_\phi(f,f) = \int \Gamma(f)\phi^2 d\nu.$$

Now (2.14) follows by polarisation.

Claim 2: $\mathcal{D}_1(\mathcal{E}_\phi) \subset \overline{\mathcal{D}_0(\mathcal{E}_\phi)}$:= closure of $\mathcal{D}_0(\mathcal{E}_\phi)$ in $\mathcal{D}(\mathcal{E}_\phi)$ with respect to the norm $\mathcal{E}_{\phi,1}^{\frac{1}{2}} := \left(\mathcal{E}_\phi + (\cdot,\cdot)_{L^2(\phi^2\nu)}\right)^{\frac{1}{2}}$. Furthermore, $\Gamma(f) = \bar{\Gamma}(f)$, $\forall f \in \mathcal{D}_1(\mathcal{E}_\phi)$.

(*Proof of Claim 2*): We prove this *Claim* by three steps.

Step 1. Let $f \in D_1(\mathcal{E}_\phi)$ and for $n \in \mathbb{N}$ define

$$f_n := sup(inf(f,n),-n)(\in \mathcal{D}_1(\mathcal{E}_\phi)).$$

Then $f_n \to f$ in $L^2(\phi^2\nu)$ as $n \to \infty$ and by (2.13)

$$\Gamma(f-f_n) \leq 1_{[|f|\geq n]}\Gamma(f-f_n) \leq 2\cdot 1_{[|f|\geq n]}\Gamma(f).$$

Hence

$$lim_{n\to\infty}\int \left[\Gamma(f-f_n)+(f-f_n)^2\right]\phi^2 d\nu = 0.$$

Step 2. Let $f \in \mathcal{D}_1(\mathcal{E}_\phi)\cap L^\infty(\nu)$. For $L \geq 1$ set

$$h_L := \frac{inf(\phi,L)}{L}, \text{ and } \ f_L := (1-h_L)f \quad (\in L^\infty(\nu)).$$

Then $f_L = 0$ on $[\phi \geq L]$, and (2.13) implies that $f_L \in \mathcal{D}_1(\mathcal{E}_\phi)$. Let us show that

$$f - f_L \to 0 \ \text{ in } \ L^2(\phi^2\nu) \ \text{ and } \ \Gamma(f-f_L)\to 0 \text{ in } \ L^1(\phi^2\nu) \ \text{ as } \ L\to\infty. \tag{2.15}$$

Since $f - f_L = h_L f$ converges to zero $\nu - a.e.$ (hence $\phi^2\nu - a.e.$), and $0 \leq h_L \leq 1$ we have $f - f_L \to 0$ in $L^2(\phi^2\nu)$ as $L \to \infty$, by Lebesgue's dominated convergence theorem. For the second convergence in (2.15), note that by the Cauchy-Schwarz inequality,

$$\Gamma(f-f_L)\phi^2 = \Gamma(fh_L)\phi^2 \leq 2h_L^2\Gamma(f)\phi^2 + 2f^2\Gamma(h_L)\phi^2.$$

The first term on the r.h.s. tends to zero $\nu - a.e.$ and is bounded by $2\Gamma(f)\phi^2 \in L^1(\nu)$, so it tends to zero in $L^1(\nu)$. For the second term on the r.h.s. above, letting $C := \|f\|^2_{L^\infty(\nu)}$ and noting that $\Gamma(inf(\phi,L)) = 0$, $\nu - a.e.$, on $[\phi \geq L]$, we have

$$f^2\Gamma(h_L)\phi^2 \leq C\phi^2\frac{\Gamma(inf(\phi,L))}{L^2} \leq C\Gamma(\phi)1_{[\phi\leq L]}\frac{\phi^2}{L^2} \leq C\Gamma(\phi)\left(inf(1,\frac{\phi^2}{L^2})\right).$$

Since the last term tends to zero $\nu - a.e.$ and is bounded by $C\Gamma(\phi) \in L^1(\nu)$, it converges to zero in $L^1(\nu)$. So, we have proved (2.15).

Step 3. (cf. [RZ94,Proof of Theorem 3.1]) Let $g := f_L$, $L > 1$, fixed, as defined in *Step 2.* Since $g \in \mathcal{D}_{loc}(\mathcal{E})\cap L^\infty(\nu)$, there exist an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ and $u_k \in \mathcal{D}(\mathcal{E})\cap L^\infty(\nu)$, $k \in \mathbb{N}$, such that $g = u_k$ $\nu - a.e.$ on F_k ($\forall k \in \mathbb{N}$). For $k \in \mathbb{N}$, let $e_k := h_{\mathcal{O}\setminus F_k}$ be the 1-reduced function on $\mathcal{O}\setminus F_k$ of the constant function $h \equiv 1$ (cf. [MR92, Chap.III, Prop.1.5]). Then by [MR92, Chap.III, Prop. 2.12],

$$lim_{k\to\infty}\mathcal{E}_1(e_k,e_k) = 0, \tag{2.16}$$

where $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(\nu)}$. Fix $k \in \mathbb{N}$ and define

$$g_k := (1-e_k)g.$$

Then, since $0 \leq e_k \leq 1$ and $e_k = 1$ ν-a.e. on $\mathcal{O}\setminus F_k$,

$$g_k = (1-e_k)u_k \in \mathcal{D}(\mathcal{E})\cap L^\infty(\nu),$$

and by (2.13)

$$\Gamma(g_k) \le 2\left[(1-e_k)^2\Gamma(g) + g^2 1_{[\phi\le L]}\Gamma(e_k)\right] \in L^1(\phi^2\nu),$$

so $g_k \in \mathcal{D}_0(\mathcal{E}_\phi)$. Furthermore, since $g - g_k = e_k g$ and thus

$$\Gamma(g-g_k) \le 2\left[e_k^2\Gamma(g) + g^2 1_{[\phi\le L]}\Gamma(e_k)\right],$$

it follows by (2.16) that

$$lim_{k\to\infty}\int\left[\Gamma(g-g_k)+(g-g_k)^2\right]\phi^2 d\nu = 0.$$

Steps 1-3, and *Claim 1* imply that for every $f \in D_1(\mathcal{E}_\phi)$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_0(\mathcal{E}_\phi)$ which converges to f in $L^2(\phi^2\nu)$ and which is a Cauchy sequence with respect to $\mathcal{E}_{\phi,1}^{\frac{1}{2}}$. Therefore, $f \in \overline{\mathcal{D}_0(\mathcal{E}_\phi)}$ and by (2.14)

$$\bar{\Gamma}(f) = lim_{n\to\infty}\Gamma(f_n), \text{ in } L^2(\phi^2\nu).$$

Furthermore, we have shown in *Steps 1-3* that $\Gamma(f-f_n)\to 0$ in $L^2(\phi^2\nu)$ as $n\to\infty$, hence

$$limsup_{n\to\infty}|\Gamma(f)^{\frac{1}{2}} - \Gamma(f_n)^{\frac{1}{2}}| \le lim_{n\to\infty}\Gamma(f-f_n)^{\frac{1}{2}} = 0$$

in $L^2(\phi^2\nu)$. Therefore,

$$\bar{\Gamma}(f) = \Gamma(f)$$

and *Claim 2* is completely proved.

Claim 3: $\mathcal{D}(\mathcal{E}_\phi) \subset \mathcal{D}_1(\mathcal{E}_\phi)$.

(*Proof of Claim 3*): Let $f \in \mathcal{D}(\mathcal{E}_\phi)$. Then $f\phi \in \mathcal{D}(\mathcal{E})$. Let $\widetilde{f\phi}$ be one of its $\mathcal{E}$-quasi-continuous ν-versions and $(E_k)_{k\in\mathbb{N}}$ an $\mathcal{E}$-nest of compact sets so that $\widetilde{f\phi}|_{E_k}$ and $\tilde{\phi}|_{E_k}$ are continuous for all $k \in \mathbb{N}$ and $\tilde{\phi}(x) > 0$ for all $x \in \cup_{k\in\mathbb{N}}E_k$. Fix $k \in \mathbb{N}$ and let $\delta_k > 0$ so that $\tilde{\phi} > \delta_k$ on E_k. Set

$$M_k := sup\{|\widetilde{f\phi}(x)| : x \in E_k\},$$

and

$$\hat{f}_k := \frac{sup(inf(f\phi, M_k), -M_k)}{sup(\phi,\delta_k)}.$$

Then $\hat{f}_k \in \mathcal{D}(\mathcal{E})$ and $\hat{f}_k = f$ ν-a.e. on E_k. So, $f \in \mathcal{D}_{loc}(\mathcal{E})$.

Set for $n \in \mathbb{N}$, $f_n := sup(inf(f,n),-n)$ $(\in \mathcal{D}(\mathcal{E}_\phi))$. Then by (2.13) for all $n \in \mathbb{N}$

$$|\Gamma(f)^{\frac{1}{2}} - \Gamma(f_n)^{\frac{1}{2}}|^2 \le \Gamma(f-f_n) \le 2\cdot 1_{[|f|\ge n]}\Gamma(f),$$

so $lim_{n\to\infty}\Gamma(f_n) = \Gamma(f)$, ν-a.e.. Furthermore, for all $n \in \mathbb{N}$

$$\begin{aligned}\Gamma(f_n\phi) &= \left(\phi\Gamma(f_n)^{\frac{1}{2}} - |f_n|\Gamma(\phi)^{\frac{1}{2}}\right)^2 \\ &\quad + 2\phi|f_n|\Gamma(f_n)^{\frac{1}{2}}\Gamma(\phi)^{\frac{1}{2}} + 2f_n\phi\Gamma(f_n,\phi) \\ &\ge \left(\phi\Gamma(f_n)^{\frac{1}{2}} - |f_n|\Gamma(\phi)^{\frac{1}{2}}\right)^2,\end{aligned}$$

hence

$$\phi\Gamma(f_n)^{\frac{1}{2}} \le \Gamma(f_n\phi)^{\frac{1}{2}} + |f_n|\Gamma(\phi)^{\frac{1}{2}} \in L^2(\nu).$$

So, $f_n\phi \in \mathcal{D}_1(\mathcal{E}_\phi)$ and thus by *Claims 1, 2* and Fatou's Lemma

$$\int \Gamma(f)\phi^2 d\nu \le liminf_{n\to\infty} \int \bar{\Gamma}(f_n)\phi^2 d\nu = liminf_{n\to\infty}\mathcal{E}_\phi(f_n, f_n) < \infty.$$

Consequently, $f \in \mathcal{D}_1(\mathcal{E}_\phi)$.

Claims 1-3 prove both assertion (i) and assertion (ii) of the Theorem. □

Remark 2.4. (i) Let $\mathcal{D}_{loc}^{BH}(\mathcal{E})$ be the local Dirichlet space in the sense of [BH91, Chap.I, Def. 7.1.3], i.e., $f \in \mathcal{D}_{loc}^{BH}(\mathcal{E})$ if and only if $f \in L^0(\nu)$ and there exist $(\mathcal{O}_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\cup_{n\in\mathbb{N}}\mathcal{O}_n = \mathcal{O}$, $f = f_n$ ν-a.e. on $\mathcal{O}_n$ for some $f_n \in \mathcal{D}(\mathcal{E})$ $(\forall n \in \mathbb{N})$. Let $\Gamma : \mathcal{D}_{loc}^{BH}(\mathcal{E}) \times \mathcal{D}_{loc}^{BH}(\mathcal{E}) \to L^0(\nu)$ be defined by [BH91, Chap.I, Prop.7.1.4] and polarisation, and $\hat{\mathcal{D}}_{loc}(\mathcal{E})$ be a *subalgebra* of $\mathcal{D}_{loc}^{BH}(\mathcal{E})$ such that

(a). $\mathcal{D}_{loc}(\mathcal{E}) \subset \hat{\mathcal{D}}_{loc}(\mathcal{E})$.

(b). $f \in \hat{\mathcal{D}}_{loc}(\mathcal{E})$, $\int \big(f^2 + \Gamma(f)\big)\, d\nu < \infty \Rightarrow f \in \mathcal{D}(\mathcal{E})$.

Then, we can also prove that

$$(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi)) = (\hat{\mathcal{E}}_\phi, \hat{\mathcal{D}}_1(\mathcal{E}_\phi)) = (\mathcal{E}_\phi, \mathcal{D}_1(\mathcal{E}_\phi)),$$

where we define

$$\hat{\mathcal{D}}_1(\mathcal{E}_\phi) := \{f \in \hat{\mathcal{D}}_{loc}(\mathcal{E}) : \int \big(f^2 + \Gamma(f)\big)\, \phi^2 d\nu < \infty\},$$

and

$$\hat{\mathcal{E}}_\phi(f, g) := \int \Gamma(f, g)\phi^2 d\nu, \quad f, g \in \hat{\mathcal{D}}_1(\mathcal{E}_\phi).$$

In fact, since $\hat{\mathcal{D}}_{loc}(\mathcal{E})$ is a subalgebra of $\mathcal{D}_{loc}^{BH}(\mathcal{E})$, and condition (a) above implies $\mathcal{D}(\mathcal{E}) \subset \hat{\mathcal{D}}_{loc}(\mathcal{E})$, so *Steps 1, 2* in the proof of *Claim 2* are true for $f \in \hat{\mathcal{D}}_1(\mathcal{E}_\phi)$. Let $(F_k)_{k\in\mathbb{N}}$ be a fixed $\mathcal{E}$-nest, and define g_k as in *Step 3* in the proof of *Claim 2*, then $g_k \in \hat{\mathcal{D}}_{loc}(\mathcal{E})$ and $\int \big(g_k^2 + \Gamma(g_k)\big)\, \phi^2 d\nu < \infty$. It follows from condition (b) above that $g_k \in \mathcal{D}(\mathcal{E})$, and all the remain in the proof of *Claim 2* are true for $\hat{\mathcal{D}}_1(\mathcal{E}_\phi)$ replacing $\mathcal{D}_1(\mathcal{E}_\phi)$. Hence,

$$\hat{\mathcal{D}}_1(\mathcal{E}_\phi) \subset \overline{\mathcal{D}_0(\mathcal{E}_\phi)},$$

and combining this with condition (a), *Claim 1*, and *Claim 3* we have proved the above assertion.

There are some $\hat{\mathcal{D}}_{loc}(\mathcal{E})$ satisfying the conditions (a) and (b), for example, we can choose $\hat{\mathcal{D}}_{loc}(\mathcal{E})$ as follows:

$f \in \hat{\mathcal{D}}_{loc}(\mathcal{E})$ if $f \in L^0(\nu)$ and there exist $(\mathcal{O}_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\mathcal{O}_n \uparrow \mathcal{O}(n \to \infty)$, $\overline{\mathcal{D}_0(\mathcal{E}, (\mathcal{O}_n)_{n\in\mathbb{N}})}^{\mathcal{E}_1^{\frac{1}{2}}} = \mathcal{D}(\mathcal{E})$, and $f = f_n$ ν-a.e. on $\mathcal{O}_n$ for some $f_n \in \mathcal{D}(\mathcal{E})$ $(\forall n \in \mathbb{N})$, where

$$\mathcal{D}_0(\mathcal{E}, (\mathcal{O}_n)_{n\in\mathbb{N}}) := \{f \in \mathcal{D}(\mathcal{E}) : f = 0 \quad \nu - a.e. \text{ on } \ \mathcal{O} \setminus \mathcal{O}_n \text{ for some } \ n \in \mathbb{N}\}.$$

(ii) By Theorem 2.7 the form

$$(f, g) \to \int \Gamma(f, g)\phi^2 d\nu$$

with domain $\mathcal{D}_0(\mathcal{E}_\phi)$ and $\mathcal{D}_1(\mathcal{E}_\phi)$ is closable, closed respectively on $L^2(\phi^2\nu)$. Thus, we have given new analytic proofs for corresponding results in [Eb96], [Fi97] respectively, both proved there by probabilistic methods. In [Eb96], however, the case where not necessarily $\tilde{\phi} > 0$ $\mathcal{E}$-q.e., not covered by our result, was also considered.

Finally, we prove a representation of the generator $(L_\phi, \mathcal{D}(L_\phi))$ of $(\mathcal{E}_\phi, \mathcal{D}(\mathcal{E}_\phi))$ for certain functions in $\mathcal{D}(L_\phi)$ in terms of the generator L of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and ϕ.

Proposition 2.8. *Consider the situation of Theorem 2.7 and let $f \in \mathcal{D}(L)$ such that $\Gamma(f) \in L^\infty(\nu)$ and $Lf \in L^2(\phi^2\nu)$. Then $f \in \mathcal{D}(L_\phi)$ and*

$$L_\phi f = Lf + 2\frac{\Gamma(\phi, f)}{\phi}.$$

Proof. Let $g \in \mathcal{D}(\mathcal{E}_\phi) \cap L^\infty(\nu)$. Then $g\phi \in \mathcal{D}(\mathcal{E})$ and by Theorem 2.7 both g and f are in $\mathcal{D}_{loc}(\mathcal{E})$. Below again for a function $h : \mathcal{O} \to \mathbb{R}$ we set

$$h_n := (h)_n := sup(inf(h, n), -n), \quad n \in \mathbb{N}.$$

Then $\phi_n, (g\phi)_n \in \mathcal{D}(\mathcal{E})$ for all $n \in \mathbb{N}$ and

$$|\Gamma(f, g\phi - (g\phi)_n)| \leq \|\Gamma(f)^{\frac{1}{2}}\|_{L^\infty(\nu)} \Gamma\left(g\phi - (g\phi)_n\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty$$

in $L^2(\nu)$ and likewise for ϕ replacing $g\phi$. Hence,

$$\begin{aligned}
&\int \Gamma(f, g)\phi^2 d\nu \\
&= \int \Gamma(f, g\phi)\phi d\nu - \int \Gamma(f, \phi) g\phi d\nu \\
&= lim_{n\to\infty} \int \Gamma(f, (g\phi)_n)\phi_n d\nu - \int \Gamma(f, \phi) g\phi d\nu \\
&= lim_{n\to\infty} \int \Gamma(f, (g\phi)_n\phi_n) d\nu - 2\int \Gamma(f, \phi) g\phi d\nu \\
&= lim_{n\to\infty} \int (-Lf)(g\phi)_n\phi_n d\nu - \int 2\frac{\Gamma(f, \phi)}{\phi} g\phi^2 d\nu \\
&= -\int \left(Lf + 2\frac{\Gamma(f, \phi)}{\phi}\right) g\phi^2 d\nu,
\end{aligned}$$

where the last step is justified by Lebesgue's domainated convergence theorem, since

$$\left|\frac{(g\phi)_n\phi_n}{\phi^2}\right| \leq |g| \in L^\infty(\nu).$$

Since $\mathcal{D}(\mathcal{E}_\phi) \cap L^\infty(\nu)$ is dense in $\mathcal{D}(\mathcal{E}_\phi)$ with respect to $\mathcal{E}_{\phi,1}^{\frac{1}{2}}$, the assertion follows by [MR92, Chap.I, Prop.2.16]. ☐

3 Applications to Schrödinger operators on loop spaces

Let M be an n-dimensional connected compact Riemannian manifold with a torsion skew symmetric (TSS for short) connection ∇ (for the definition see [Dr92]), and let E be defined by

$$E = \{w \in C([0, 1]; M) : w(0) = x_0, w(1) = y_0\}$$

for fixed $x_0, y_0 \in M$. E is the so called loop space over M when $x_0 = y_0$.

A function f on E is called as a smooth cylinder function if there exist a function $F \in C^\infty(M^m)$ and a partition $0 < t_1 < \cdots < t_m < 1$ of $[0,1]$ such that $f(w) = F(w(t_1), \cdots, w(t_m))$ for any $w \in E$. We denote the set of all smooth cylinder functions on E by $\mathcal{F}C^\infty(E)$.

Pinned Wiener measure (i.e. Brownian bridge measure) μ on E is the unique Borel probability measure on E such that, the coordinate process (γ_t) on E is the Brownian bridge process. Let $(\mathcal{F}_t)_{0\le t\le 1}$ be the corresponding μ-completed natural filtration corresponding to it. Moreover, for a given orthonormal frame u_0 at $x_0 \in M$ there exists a unique stochastic horizontal lift (U_t) of (γ_t) determined by the TSS connection ∇ satisfying $U_0 = u_0$ (see [Dr94]). For convenience, we consider an orthonormal frame U at $x \in M$ as an isomorphism from $\mathbb{R}^n$ to T_xM. If we denote the bundle of orthonomal frames over M by $\mathcal{O}(M)$, then (U_t) is an $\mathcal{O}(M)$-valued process. We identify $T_{x_0}M$ and $\mathbb{R}^n$ via u_0 and set

$$H_0 := \{h \in C([0,1];\mathbb{R}^n) : \|h\|_{H_0}^2 = \int_0^1 |\dot{h}(t)|^2 dt < \infty, h(0) = h(1) = 0\}.$$

Then we can define a closed densely defined operator ∇_0 from $L^2(E,\mu)$ to $L^2(E \to H_0;\mu)$ with $\mathcal{F}C^\infty(E)$ as its core, which is considered as a natural gradient operator on E with domain $\mathcal{D}(\nabla_0)$(see [Dr94], [DR92]). In particular, for $f \in \mathcal{F}C^\infty(E)$ with $f(w) = F(w(t_1), \cdots, w(t_m))$ we have

$$(\nabla_0 f(\gamma))(t) = \sum_{i=1}^m (min(t_i, t) - t_i t) \nabla^{(i)}_{U_{t_i}} F(\gamma_{t_1}, \cdots, \gamma_{t_m}) \tag{3.1}$$

where $\nabla^{(i)}F$ denotes the gradient of F with respect to the i-th variable, $\nabla^{(i)}_U F$ denotes the unique element in $\mathbb{R}^n$ such that $\langle a, \nabla^{(i)}_U F\rangle_{\mathbb{R}^n} = \nabla^{(i)}_{Ua} F$ for any $a \in \mathbb{R}^n$ and $U \in \mathcal{O}(M)$. It follows from (3.1) that $\|\nabla_0 f\|_{H_0} \in L^\infty(\mu)$ $(\forall f \in \mathcal{F}C^\infty(E))$.

Let D be a given nonempty connected component of E, and $\mu_D = \frac{\mu|_D}{\mu(D)}$. One can easily check that $1_D \in \mathcal{D}(\nabla_0)$ and $\nabla_0 1_D = 0$, μ-a.e., on E (see [Ai98a]). Recall that if M is simply connected then $D = E$. In the general case we define a pre-Dirichlet form (see [MR92]) $(\mathcal{E}^0_D, \mathcal{D}(\mathcal{E}^0_D))$ on $L^2(\mu_D)$ as

$$\mathcal{D}(\mathcal{E}^0_D) := \{f \in L^2(\mu_D) : 1_D f \in \mathcal{D}(\nabla_0), \|\nabla_0(1_D f)\|_{H_0} \in L^2(\mu_D)\},$$

$$\mathcal{E}^0_D(f,g) := \int_D \langle \nabla_0(1_D f), \nabla_0(1_D g)\rangle_{H_0} d\mu_D,$$

for any $f, g \in \mathcal{D}(\mathcal{E}^0_D)$. It is known that the form $(\mathcal{E}^0_D, \mathcal{D}(\mathcal{E}^0_D))$ is closable in $L^2(\mu_D)$ (see [DR92]). We denote its closure by $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$. Obviously $\mathcal{E}_D(1,1) = 0$. The generator L_D of $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ is the so called Ornstein-Uhlenbeck operator on D (O-U operator for short), and $\mathcal{D}(\mathcal{E}_D) = \{f|_D : f \in \mathcal{D}(\nabla_0)\}$. The strongly continuous semigroup $(P^D_t)_{t\ge 0}$ associated to $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ is the so called O-U semigroup. For convenience, we set $\mathcal{F}C^\infty(D) := \{f|_D : f \in \mathcal{F}C^\infty(E)\}$, and call a function in $\mathcal{F}C^\infty(D)$ a smooth cylinder function on D. $\mathcal{F}C^\infty(D)$ is a form core of $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$.

If $M = G$ is a compact connected Lie group with an $Ad(G)$ -invariant inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra, we choose the TSS connection ∇ on G as the right Cartan connection ∇^R on G. In this case, (U_t) is just the right translation (R_{γ_t}), and we set $x_0 = e$ (the unit element of G). Following L.Gross we define

$$V_G := |b(1)|^2 := \left| \int_0^1 U_s^{-1} \circ d\gamma_s \right|^2 = \left| \int_0^1 R^{-1}_{\gamma_s} \circ d\gamma_s \right|^2. \tag{3.2}$$

With this quadratic potential L. Gross in [Gr91] (or see [Ge91] and [Gr93]) proved the following defective logarithmic Sobolev inequality denoted by $LSI(C, \alpha V_G, A)$: for each $\alpha > 0$, there exist two constants $C, A \geq 0$ such that for all $f \in \mathcal{D}(\nabla_0)$,

$$\int_E f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu)}} d\mu \leq C \int_E \left(|\nabla_0 f|^2_{H_0} + \alpha V_G f^2\right) d\mu + A\|f\|^2_{L^2(\mu)}.$$

In the general case, let Ric be the Ricci curvature of M, and $p_t(x, y)$ be the heat kernel of $\frac{1}{2}\Delta$ where Δ is the Levi-Civita Laplacian on M. For any $U \in \pi^{-1}(x) \subset \mathcal{O}(M)$ with $x \in M$ we define Ric_U and $\nabla^2_U \log p_{1-t}(\cdot, y_0)$, $(0 \leq t < 1)$ by setting

$$Ric_U := U^{-1} Ric(x) U$$

and

$$\nabla^2_U \log p_{1-t}(\cdot, y_0) := U^{-1} \nabla^2 \log p_{1-t}(x, y_0) U,$$

where we consider $Ric(x)$ and $\nabla^2 log p_{1-t}(x, y_0)$ as maps from $T_x M$ to itself. We define

$$v_t := \int_t^1 \{id_{\mathbb{R}^n} - \frac{1}{2}(1-s) Ric_{U_s} + (1-s)\nabla^2_{U_s} \log p_{1-s}(\cdot, y_0)\} d\beta_s, \tag{3.3}$$

and

$$V_M := \int_0^1 \left|\frac{v_t}{1-t}\right|^2 dt, \tag{3.4}$$

where (β_t) is the martingale part of $(\int_0^t U_s^{-1} \circ d\gamma_s)$, and $\circ d\gamma_s$ stands for the Stratonovich differential of γ_s. In fact, (β_t) is an $\mathbb{R}^n$ -valued Brownian motion. When the TSS connection ∇ is the Levi-Civita connection, Gong and Ma in [GM98] have proved that, for each $\alpha > 0$ the log-Sobolev inequality $LSI(2(1+\alpha), \frac{1}{4\alpha} V_M, 0)$ holds, where V_M is given by (3.4).

The aim of this section is to show that if we take $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ as the initial Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, then all results in Section 2 hold for the closed symmetric form $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ and the corresponding Schrödinger operator $-L_V$, in case in $V := \alpha V_G|_D$ and $V := \frac{1}{4\alpha} V_M|_D$ $(\alpha > 0)$ respectively. In particular, for $V := \alpha V_G|_D$ our result proves the conjecture formulated by L. Gross in [Gr93, Remark 10.8]. To this end, we need to prove that assumptions **(H1)**-**(H4)** in Section 2, condition (i) in Theorem 2.5, and conditions (i)-(ii) in Proposition 2.6 hold.

For $(\mathcal{O}, \mathcal{F}, \nu) := (D, \mathcal{B}(D), \mu_D)$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E})) := (\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ assumption **(H4)** holds with $\Gamma(f, g) := \langle \nabla_0 f, \nabla_0 g\rangle_{H_0}$ $(\forall f, g \in \mathcal{D}(\nabla_0))$. Furthermore, using similar arguments as in the proofs of Theorem 4 and Theorem 4' in [DR92] we know that the Dirichlet form $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ is quasi-regular in the sense of Definition 2.1, i.e. assumption **(H3)** also holds for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Hence, we only need to prove that assumptions **(H1)**-**(H2)**, condition (i) in Theorem 2.5, and conditions (i) – (ii) in Proposition 2.6 hold for $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ and $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ respectively with V as specified above.

3.1 The UPI property of O-U semigroups

In this subsection we will prove the UPI property of the O-U semigroup, i.e., we will prove that assumption **(H2)** holds for $(P_t^D)_{t\geq 0}$.

Let $(\mathcal{O}, \mathcal{F}, \nu)$ be a probability space, and $(P_t)_{t\geq 0}$ be a strongly continuous symmetric positivity preserving semigroup in $L^2(\nu)$. Note that the UPI property for $(P_t)_{t\geq 0}$ is just condition (E) in [Hi98]. Another form of the UPI property, i.e. $\chi_{P_t}(\varepsilon) > 0$ for each $\varepsilon > 0$ and each $t > 0$, was introduced by S. Kusuoka in [Ku92], and was applied in [Mat98], [Ai98a], [Ai98b], [Ai99], [Hi98], [Hi00], and [RW00] etc.. In particular, it is known that if $(P_t)_{t\geq 0}$ is Markovian (i.e. $P_t 1 = 1 (\forall t \geq 0)$), the following are equivalent:

(i) $\chi_{P_t}(\varepsilon) > 0$ for each $\varepsilon > 0$ and each $t > 0$.

(ii) The *weak spectral gap property* of its associated Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, i.e., for any sequence $(f_m)_{m\geq 1}$ in $\mathcal{D}(\mathcal{E})$, if $sup_{m\geq 1}\|f_m\|_{L^2(\nu)} < \infty$, $\int f_m d\nu = 0$, $(\forall m \geq 1)$, and $lim_{m\to\infty}\mathcal{E}(f_m, f_m) = 0$, then $f_m \to 0$ in probablity-ν as $m \to \infty$.

(iii) $\forall \varepsilon > 0$, there is a $t > 0$ such that $\chi_{P_t}(\varepsilon) > 0$.

(iv) For any $r > 0$ there is $\alpha(r) > 0$ so that

$$\|f\|^2_{L^2(\nu)} \leq \alpha(r)\mathcal{E}(f, f) + r\|f\|^2_{L^\infty(\nu)}$$

for all $f \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\nu)$ with $\int_{\mathcal{O}} f d\nu = 0$.

Here (i) $\Longrightarrow$ (ii) is due to Kusuoka [Ku92]. (ii) $\Longrightarrow$ (i) is an observation due to Mattieu [Mat98] and Aida [Ai98b]. (iii) $\Longrightarrow$ (i) is contained in [Hi00] (the inverse is trivial), and the equivalence between (ii) and (iv) is proved by F.Y. Wang and the second named author in [RW00].

Note that the O-U semigroup $(P^D_t)_{t\geq 0}$ associated with $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ is a Markovian semigroup in $L^2(\mu_D)$. Hence, to prove the UPI property of $(P^D_t)_{t\geq 0}$, we only need to prove the weak spectral gap property of $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ mentioned in (ii) above. To this end, we will use some results essentially proved by S. Aida in [Ai99]. For the reader's convenience we first recall some notions used in [Ai99].

Let $n = dim M$, and $N = (n + 1)(2n + 1)n^3$. We can choose a bundle homomorphism $\sigma : M \times \mathbb{R}^N \to TM$ such that the associated Le Jan-Watanabe connection (see [ELL97] and [ELL99]) is just the TSS connection ∇ on M, and $\sigma(x)\sigma(x)^* = id_{T_xM}$ for any $x \in M$. This fact was proved by Elworthy-Le Jan-Li in Section 2H of [ELL97] (or see [ELL99] and [Ai99]). We consider the following SDE on M:

$$dX(t, x, \omega) = \sigma(X(t, x, \omega)) \circ d\omega(t), X(0, x, \omega) = x \in M,$$

where $\omega \in W^N$, and W^N denotes N-dimensional Wiener space. We denote the Cameron-Martin space of W^N by $H := H(\mathbb{R}^N)$, and the Wiener measure on W^N by P. Then $X(t, x, \cdot) : W^N \to M$ is a nondegenerate smooth mapping in the sense of Malliavin for each $x \in M$ and each $t \in [0, 1]$, and $X(\cdot, x, \omega) : [0, 1] \to M$ is continuous for each $x \in M$ and P-a.e. $\omega \in W^N$ (see [Ai93] and [Ai99]). We also use $X(1, x, \cdot)$ to denote the quasi-continuous version of $X(1, x, \cdot)$. Set

$$S_{x_0,y_0} := \{\omega \in W^N : X(1, x_0, \omega) = y_0\}.$$

Let D_S denote the H-derivative (i.e. Malliavin derivative) along S_{x_0,y_0}, and let P_{x_0,y_0} be the probability measure on S_{x_0,y_0} obtained as the normalization of the measure

$$\delta_{y_0}(X(1, x_0, \omega))P(d\omega)$$

on S_{x_0,y_0} (this is a positive Watanabe distribution on the Wiener space, then a measure by Sugita's theorem in [Su88]). $X(\cdot, x_0, \cdot) : S_{x_0,y_0} \to E$ is an isomorphism in the sense of measure theory. Let $\mathcal{D}(S_{x_0,y_0})$ be the domain of the closure of the following pre-Dirichlet form

$$\mathcal{E}_S(f, g) = \int_{S_{x_0,y_0}} \langle D_S f, D_S g\rangle_H dP_{x_0,y_0}$$

with the pre-domain $\mathcal{F}C_b^\infty(W^N)$ on $L^2(P_{x_0,y_0})$. Define

$$X^{-1}(D) := \{\omega \in S_{x_0,y_0} : X(\cdot, x_0, \omega) \in D\}$$

for the given connected component D of E, and set $\nu_{x_0,y_0} := \frac{P_{x_0,y_0}}{P_{x_0,y_0}(X^{-1}(D))}$. Then using similar arguments as in the proofs of Lemma 2.16 and Corollary 2.17 in [Ai98a] (see Lemma 3.1 in [Ai99] for $x_0 = y_0$) we obtain

Lemma 3.1. *For any $f \in L^1(\mu_D)$ we have*

$$\int_{X^{-1}(D)} f \circ X(\cdot, x_0, \omega)\nu_{x_0,y_0}(d\omega) = \int_D f d\mu_D.$$

Moreover, for any $f \in \mathcal{D}(\nabla_0)$ we have $f \circ X(\cdot, x_0, \cdot) \in \mathcal{D}(S_{x_0,y_0})$, and there exists a positive constant C_0 independent of f such that

$$\int_{X^{-1}(D)} |D_S\{f \circ X(\cdot, x_0, \omega)\}|_H^2 \nu_{x_0,y_0}(d\omega) \le C_0 \int_D \|\nabla_0 f\|_{H_0}^2 d\mu_D.$$

Since D is a connected open subset of E, by similar arguments as in Remark 1.2 and in the proof of Lemma 3.2 in [Ai99] for $x_0 = y_0$ we get

Lemma 3.2. *There exist a non-empty H-connected (in the sense of [Ku92], or see Definition 2.8 in [Ai98a]) measurable subset U_D of W^N, a measurable mapping $\Phi : U_D \to X^{-1}(D)$, and two constants $C_1, C_2 > 0$ such that, if $g \in \mathcal{D}(S_{x_0,y_0})$, then $g|_{X^{-1}(D)} \circ \Phi \in \mathcal{D}(\mathcal{E}_{U_D})$, and*

$$\int_{U_D} g|_{X^{-1}(D)} \circ \Phi dP = \int_{X^{-1}(D)} gz d\nu_{x_0,y_0},$$

$$\int_{U_D} |D\{g|_{X^{-1}(D)} \circ \Phi\}|_H^2 dP \le C_1 \int_{X^{-1}(D)} |D_S g|^2 d\nu_{x_0,y_0},$$

where z satisfies that $0 < z \le C_2$, ν_{x_0,y_0}-a.e. on $X^{-1}(D)$, and $(\mathcal{E}_{U_D}, \mathcal{D}(\mathcal{E}_{U_D}))$ is the Dirichlet form on $L^2(U_D, P)$ defined in Section 6 of [Ku92].

Obviously, $\mathcal{E}_{U_D}(1, 1) = 0$. By Lemma 6.13 and Lemma 6.15 in [Ku92] (or see Theorem 5.3 in [Ai98b] and Theorem 3.3 in [Ai99]) the Dirichlet form $(\mathcal{E}_{U_D}, \mathcal{D}(\mathcal{E}_{U_D}))$ has the *weak spectral gap property.*

Now, we can prove the following:

Proposition 3.3. *The O-U Dirichlet form $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ has the weak spectral gap property or equivalently, the associated O-U semigroup $(P_t^D)_{t\ge0}$ has the UPI property, i.e. assumption* **(H2)** *holds for $(P_t^D)_{t\ge0}$.*

Proof. The proof is a slight modification of that of Lemma 5.1 in [Ai98b]. Set $T := X(\cdot, x_0, \cdot)\circ\Phi : U_D \to D$. By Lemma 3.1 and Lemma 3.2 we obtain that for any $f \in \mathcal{D}(\mathcal{E}_D)$ we have $f\circ T \in \mathcal{D}(\mathcal{E}_{U_D})$ and

$$\int_{U_D} f\circ T dP = \int_D f z_D d\mu_D, \tag{3.5}$$

$$\int_{U_D} |D(f\circ T)|_H^2 dP \leq C\int_D \|\nabla_0 f\|_{H_0}^2 d\mu_d, \tag{3.6}$$

where $z_D := z\circ X(\cdot, x_0, \cdot)|_{X^{-1}(D)}^{-1}$, and $C := C_0C_1$. Obviously, $0 < z_D \leq C_2$, μ_D-a.s. on D.

Now, let $(f_m)_{m\geq 1}$ be as in the definition of the weak spectral gap property (cf. (ii) above), and $F_m := f_m\circ T - \int_{U_D} f_m \circ T dP$ for any $m \geq 1$. Using (3.5), (3.6), and the fact $\mathcal{E}_{U_D}(1,1) = 0$ one can easily check that $(F_m)_{m\geq 1} \subset \mathcal{D}(\mathcal{E}_{U_D})$, $\sup_{m\geq 1}\|F_m\|_{L^2(U_D,P)} < \infty$, $\int_{U_D} F_m dP = 0$, $(\forall m \geq 1)$, and $lim_{m\to\infty}\mathcal{E}_{U_D}(F_m, F_m) = 0$. In particular, $(F_m)_{m\geq 1}$ is uniformly integrable in $L^1(U_D, P)$. Hence, by the weak spectral gap property of $(\mathcal{E}_{U_D}, \mathcal{D}(\mathcal{E}_{U_D}))$ we obtain $\lim_{m\to\infty}\int_{U_D}|F_m|dP = 0$. Using (3.5) we get

$$\lim_{m\to\infty}\int_D |f_m - c_m| z_D d\mu_D = 0, \tag{3.7}$$

where $c_m := \int_{U_D} f_m\circ T dP$ for any $m \geq 1$.

By (3.7), $|f_m - c_m|$ converges to zero in measure $z_D\mu_D$, thus also in measure μ_D (since the two measures are equivalent).

By (3.5) and the fact $0 < z_D \leq C_2$, μ_D-a.s. on D we get

$$\sup_{m\geq 1}|c_m| \leq C_2 \sup_{m\geq 1}\|f_m\|_{L^2(\mu_D)}.$$

Hence, $(|f_m - c_m|)_{m\geq 1}$ is uniformly integrable in $L^1(\mu_D)$, and

$$\lim_{m\to\infty}\int_D |f_m - c_m| d\mu_D = 0. \tag{3.8}$$

Let $c := \lim_{i\to\infty} c_{m_i}$ be any accumulation point of $(c_m)_{m\geq 1}$, then by (3.8)

$$0 = lim_{i\to\infty}\int_D f_{m_i} d\mu_D = c.$$

Therefore $f_m \to 0$ in probability-μ_D on D. □

Note that for the Levi-Civita connection on a simply connected compact Riemanian manifold M (then $D = E$), the weak spectral gap property of the Dirichlet form $(\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$ was proved by S. Aida in Theorem 5.2 in [Ai98b]. Hence, Proposition 3.3 is an extension of this result.

As a consequence of Corollary 2.3 and Proposition 3.3 we obtain that

Corollary 3.4. *Let $V : D \to \mathbb{R}$ be a measurable function satisfying assumption* **(H1)** *for $(\mathcal{O}, \mathcal{F}, \nu) := (D, \mathcal{B}(D), \mu_D)$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E})) := (\mathcal{E}_D, \mathcal{D}(\mathcal{E}_D))$. Then the semigroup $(P_t^V)_{t\geq 0}$ has the UPI property.*

3.2 Proof of Gross's conjecture

In this subsection, we will apply the results in Section 2 to prove L. Gross's conjecture given in [Gr93, Remark 10.8] mentioned above, i.e., we will prove the following result:

Theorem 3.5. *Let $M = G$ be a compact connected Lie group, $(\mathcal{E}, \mathcal{D}(\mathcal{E})) := (\mathcal{E}_D, \mathcal{D}(E_D))$, $\alpha > 0$ be a fixed constant, and $V := \alpha V_G|_D$ for V_G given by (3.2). Then the symmetric form $(\mathcal{E}_V, \mathcal{D}(\mathcal{E}_V))$ (as defined in Proposition 2.1*(i) *above) on $L^2(\mu_D)$ is a local symmetric Dirichlet form. $\lambda_0(V) := inf\sigma(-L_V)$ is an eigenvalue of $-L_V$ with corresponding eigenspace spanned by a μ_D-a.s. strictly positive eigenfunction ϕ, and any $\mathcal{E}_D$-quasi-continuous μ_D-version $\tilde{\phi}$ of ϕ is $\mathcal{E}_D$-q.e. strictly positive. Moreover, we have:*

(a) The Schrödinger operator $-L_V$ has a spectral gap at $\lambda_0(V)$, i.e.,

$$\lambda_1(V) := \inf\{\lambda - \lambda_0(V) : \lambda \in \sigma(-L_V), \lambda > \lambda_0(V)\} > 0.$$

(b) Consider the ground state transformed operator

$$L_\phi := \phi^{-1}(L_V + \lambda_0(V))\phi$$

with the domain $\mathcal{D}(L_\phi) := \{f \in L^2(\phi^2\mu_D) : f\phi \in \mathcal{D}(L_V)\}$ of L_V, i.e. the generator of the corresponding ground state transformed Dirichlet form $((\mathcal{E}_D)_\phi, \mathcal{D}((\mathcal{E}_D)_\phi)) = ((\mathcal{E}_D)_\phi, \mathcal{D}_1((\mathcal{E}_D)_\phi))$, defined in subsection 2.2, which was characterized as a weighted Sobolev space in Theorem 2.7. Then $-L_\phi$ has a spectral gap, i.e. the following Poincaré inequality holds: $\forall f \in \mathcal{D}((\mathcal{E}_D)_\phi) = \mathcal{D}_1((\mathcal{E}_D)_\phi)$

$$\lambda_1(V)\left\{\|f\|^2_{L^2(\phi^2\mu_D)} - \left(\int_D f\phi^2 d\mu_D\right)^2\right\} \le \int_D \|\nabla_0 f\|^2_{H_0}\phi^2 d\mu_D. \tag{3.9}$$

(c) There exist a function $F \in C(0,\infty)$ with $sup_{r\in(0,1)}|rF(r)| < \infty$ and $lim_{r\to\infty}F(r) = \infty$, and a positive decreasing function $\alpha \in C(0,\infty)$ such that for any $f \in \mathcal{D}((\mathcal{E}_D)_\phi) = \mathcal{D}_1((\mathcal{E}_D)_\phi)$

$$\int_D f^2 F(f^2)\phi^2 d\mu_D \le \int_D \|\nabla_0 f\|^2_{H_0}\phi^2 d\mu_D, \ \|f\|_{L^2(\phi^2\mu_D)} = 1, \tag{3.10}$$

and for any $r > 0$,

$$\|f\|^2_{L^2(\phi^2\mu_D)} \le r\int_D \|\nabla_0 f\|^2_{H_0}\phi^2 d\mu_D + \alpha(r)\|f\|^2_{L^1(\phi^2\mu_D)}. \tag{3.11}$$

Remark 3.1. The first part of Theorem 3.5 has been proved by L. Gross in [Gr93, Theorem 10.7] except for the $\mathcal{E}_D$-q.e. strict positivity of any $\mathcal{E}$-quasi-continuous μ_D-version of the ground state ϕ.

Part (a) and Part (b) positively confirm L. Gross's conjecture formulated in [Gr93, Remark 10.8].

Part (c) is motivated by another conjecture of L. Gross given in [Gr93, Remark 10.8], i.e. whether the Dirichlet form associated to the ground state transform satisfies a log-Sobolev inequality $LSI(C, 0, 0)$. The first inequality in Part (c) is a very small step in this direction.

Proof. By Corollary 4.10 in [Gr91] we know that $\int_D \exp\{\varepsilon V\} d\mu_D < \infty$ for sufficiently small $\varepsilon > 0$, hence $V \in \cap_{1\le p<\infty} L^p(\mu_D)$. Obviously, $V \geq 0$, μ_D-a.s.. Hence, by Proposition 3.3 we know that assumptions **(H1)-(H4)** in Section 2, and conditions (i)-(ii) in Proposition 2.6 hold.

According to Theorem 4.1 in [Gr91] (or see Proposition 10.5 in [Gr93]) we know that the Schrödinger operator $-L_V$ satisfies the defective log-Sobolev inequality $LSI(C, V, A)$ for constants $C, A > 0$. Hence,using Theorem 5.1(ii) in [Gr91] we get: for any $t > 0$, $f \in L^2(\mu_D)$,

$$\|P_t^V f\|_{L^{p(t)}(\mu_D)} \leq \exp\{M(t)\}\|f\|_{L^2(\mu_D)}, \tag{3.12}$$

where $p(t) := 1 + \exp\{\frac{2t}{C}\} > 2$, and $M(t) := 2A(\frac{1}{2} - \frac{1}{p(t)})$. It follows that condition (i) in Theorem 2.5 holds.

Hence, by Theorem 2.5, Proposition 2.6, and Theorem 2.7 we obtain all the assertions except for Part (c).

Since $P_t^\phi := \phi^{-1} e^{t\lambda_0(V)} P_t^V \phi$, $t \geq 0$, is the semigroup associated to the Dirichlet form $((\mathcal{E}_D)_\phi, \mathcal{D}((\mathcal{E}_D)_\phi))$ on $L^2(\phi^2\mu_D)$, for any $t > 0$ we get

$$limsup_{K\to\infty} sup_{\|f\|_{L^2(\phi^2\mu_D)}\leq 1} \int_D |P_t^\phi f|^2 1_{[|P_t^\phi f|\geq K]} \phi^2 d\mu_D = 0,$$

and by Theorem 1.2 in [GW99] Part (c) follows. $\square$

3.3 A spectral gap for Schrödinger operators on loop spaces

In this subsection we will treat the case $V := \frac{1}{4\alpha} V_M$ and V_M given by (3.4) with (3.3). To this end, we need to extend Theorem 1.1 in [GM98] to the loop space over M with a TSS connection ∇, i.e., we need to prove the following:

Proposition 3.6. *Let V_M be given by (3.4). Then*

$$V_M \in \cap_{1\leq p<\infty} L^p(E, \mu),$$

and for any $\alpha > 0$ we have: for all $f \in \mathcal{D}(\nabla_0)$

$$\int_E f^2 \log \frac{f^2}{\|f\|^2_{L^2(\mu)}} d\mu \leq 2(1+\alpha) \int_E \|\nabla_0 f\|^2_{H_0} d\mu + \frac{1+\alpha}{2\alpha} \int_E V_M f^2 d\mu. \tag{3.13}$$

Proof. We firstly prove that $V_M \in \cap_{p\in(1,\infty)} L^p(\mu)$.

Let $\overline{Ric}$ be the Ricci curvature of the Levi-Civita connection $\bar\nabla$ on M, and $T\langle\cdot,\cdot\rangle$ be the torsion tensor of the TSS connection ∇ on M. Then we know that $\nabla = \bar\nabla + \frac{1}{2}T\langle\cdot,\cdot\rangle$, $Ric = \overline{Ric} + \hat{T}$, and $\nabla^2 F = \bar\nabla^2 F + \frac{1}{2} T\langle\cdot,\cdot\rangle F$ for any $F \in C^\infty(M)$, where $\hat T$ is a tensor determined only by T and $\bar\nabla T$. Using the above facts and (3.3) we get $v_t = \bar v_t + \hat v_t$, where

$$\bar v_t := \int_t^1 \{id_{\mathbb{R}^n} - \frac{1-s}{2}\overline{Ric}_{U_s} + (1-s)\bar\nabla^2_{U_s} \log p_{1-s}(\cdot, y_0)\} d\beta_s,$$

$$\hat v_t := \frac{1}{2}\int_t^1 (1-s)\{\Xi_{U_s}\langle \nabla_{U_s} \log p_{1-s}(\cdot), y_0), d\beta_s\rangle - \hat T_{U_s} d\beta_s\}.$$

and $\Xi_U\langle a,b\rangle := U^{-1}T\langle Ua,Ub\rangle \in \mathbb{R}^n$, $\hat{T}_U a := U^{-1}\hat{T}(Ua) \in \mathbb{R}^n$ for any $U \in \mathcal{O}(M)$ and $a, b \in \mathbb{R}^n$. Obviously, by (3.4) we know that for any $p \in (1,\infty)$ we have

$$\|V_M\|_{L^p(\mu)} \leq 2\left\|\int_0^1 |\frac{\bar{v}_t}{1-t}|^2 dt\right\|_{L^p(\mu)} + 2\left\|\int_0^1 |\frac{\hat{v}_t}{1-t}|^2 dt\right\|_{L^p(\mu)}.$$

By Theorem 1.1(i) in [GM98] we get

$$\left\|\int_0^1 |\frac{\bar{v}_t}{1-t}|^2 dt\right\|_{L^p(\mu)} < \infty.$$

Hence we only need to prove

$$\left\|\int_0^1 |\frac{\hat{v}_t}{1-t}|^2 dt\right\|_{L^p(\mu)} < \infty.$$

Set

$$C := sup\{|\Xi_U\langle a,b\rangle|^2 + |\hat{T}_U a|^2 : |a|_{\mathbb{R}^n}, |b|_{\mathbb{R}^n} \leq 1, U \in \mathcal{O}(M)\},$$

Since M is compact, $C < \infty$. By the Burkholder-Davis-Gundy inequality we get

$$\begin{aligned} E_\mu[|\hat{v}_t|^{2p}] &\leq 2^{-p} c_p C^p E_\mu\left[\left(\int_t^1 (1-s)^2\{1 + |\nabla \log p_{1-s}(\gamma_s, y_0)|^2\} ds\right)^p\right] \\ &\leq \frac{1}{2} c_p C^p \left(\frac{1}{3^p}(1-t)^{3p} + E_\mu\left[\left(\int_t^1 |(1-s)\nabla \log p_{1-s}(\gamma_s, y_0)|^2 ds\right)^p\right]\right). \end{aligned}$$

Since

$$|\nabla \log p_{1-s}(\cdot, y_0)| \leq c(\frac{1}{\sqrt{1-s}} + \frac{d(\cdot, y_0)}{1-s}), \quad s \in [0,1) \tag{3.14}$$

and

$$E_\mu[d(\gamma_s, y_0)^{2p}] \leq c(p)(1-s)^p, \quad s \in [0,1), \tag{3.15}$$

we have the following estimate (see [Dr94]): there exists a constant $C_0 > 0$ such that

$$\begin{aligned} &E_\mu\left[\left(\int_t^1 |(1-s)\nabla \log p_{1-s}(\gamma_s, y_0)|^2 ds\right)^p\right] \\ &\leq c\left((1-t)^{2p} + (1-t)^{p-1}\int_t^1 (1-s)^p ds\right) \\ &\leq C_0(1-t)^{2p}. \end{aligned}$$

Using the above estimates we obtain that there exists a constant $C_1 > 0$ such that

$$E_\mu[|\hat{v}_t|^{2p}] \leq C_1(1-t)^{2p}$$

for any $t \in [0,1)$. By this fact we can easily prove $\| \int_0^1 |\frac{\hat{v}_t}{1-t}|^2 dt\|_{L^p(\mu)} < \infty$ and $\|V_M\|_{L^p(\mu)} < \infty$ for any $p \in (1,\infty)$.

Secondary, if $Y \in \cup_{p\in(2,\infty)} L^p(\mu; H_0)$, and the process $(Y_t := Y(t))_{t\in[0,1]}$ is adapted, then by one of the results in [GMR99] the following integration by parts formula holds: for any $f \in \mathcal{D}(\nabla_0)$

$$\int_E \langle \nabla_0 f, Y\rangle_{H_0} d\mu = -\int_E f div_1(Y) d\mu, \tag{3.16}$$

where for any $0 \le T \le 1$

$$div_T(Y) := -\int_0^T \langle \dot{Y}_t + \{\frac{1}{2} Ric^*_{U_t} - \nabla^2_{U_t} \log p_{1-t}(\cdot, y_0)^*\} Y_t, d\beta_t\rangle_{\mathbb{R}^n}, \tag{3.17}$$

where we denote the transposition of a matrix A by A^*. Hence, by the same argument as in the proofs of Theorem 5.2 and Theorem 5.3 in [GM98] we can prove the log-Sobolev inequality (3.13). $\square$

Using exactly the same arguments in the proof of Theorem 3.5 we get

Theorem 3.7. *Let M be a compact connected Riemannian manifold with a TSS connection ∇. Consider the potential $V := \frac{1}{4\alpha} V_M|_D$ for V_M given by (3.4), where α is a fixed positive constant. Then, all the conclusions in Theorem 3.5 remain valid.*

Note that, if the TSS connection ∇ on M is the Levi-Civita connection, and we consider the potential term in [Ai96], then again all the conclusions in Theorem 3.5 hold.

Remark 3.2. In the situation of both Theorems 3.5 and 3.7 we have (by (3.12)) that

$$\phi \in \cap_{1\le p<\infty} L^p(\mu_D).$$

Let $f \in \mathcal{F}C^\infty(D)$. By a result in [Go00] we also have

$$L_D f \in \cap_{1\le p<\infty} L^p(\mu_D).$$

Hence (since also $\Gamma(f) = \|\nabla_0 f\|^2_{H_0} \in L^\infty(\mu_D)$, as mentioned before) Proposition 2.8 implies that $f \in \mathcal{D}(L_\phi)$ and that

$$L_\phi f = L_D f + 2\phi^{-1} \langle \nabla_0 \phi, \nabla_0 f\rangle_{H_0}.$$

Acknowlegement We thank S. Aida, B.K. Driver, A. Eberle, K.D. Elworthy, Fang Shizian, L. Gross, M. Hino, Xuemei Li, Ma Zhiming, Wang Fengyu, W. Stannat, Zhang Tusheng, Zhao Huaizhong for useful discussions. We also thank L. Gross and B.K. Driver for interesting comments. The first named author would like to thank the financial support by Max-Planck Society, SFB343, BiBoS, NNSFC(19631030), Science and Technology Ministry 973 project. He thanks the Laboratoire de Mathématiques Appliquées, Université Blaise Pascal, for inviting him to visit for one week. He also thanks Prof. K.D. Elworthy to invite and support him to take part in the conference " Stochastic analysis on path spaces and quantisation", July 2000, in Warwick University. The main results of this paper have been presented during the above conference at Warwick Universty and during the "Workshop on stochastic analysis and application" at Bielefeld University in August 14-18,2000, respectively.

Note added in proof. Since this paper was submitted for publication we received a preprint [Ai00] of S. Aida. In fact, S. Aida in [Ai00] has proved a lower bound for the spectral gap at the bottom of the spectrum of a Schrödinger operator which generates a hyperbounded semigroup, by using the function $\alpha(\cdot)$ in (iv) of Section 3.1 in this paper (for more detalis see [RW00]) and the distribution function of the ground state ϕ. Although, up to now, both these functions are unkown in the general case, useful information for the spectral gap and other proofs of some results in this paper were given in [Ai00].

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