THE TENSORPRODUCT OF LITTLE CUBES

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Introduction

Operads were introduced by Boardman and Vogt in 1968 to study the algebraic structure of iterated loop spaces (they called them *categories of operators in standard form*) [BV68]. Their results were refined in [BV73] and independently by May in [May72]. They proved that any n-fold loop space is homotopy equivalent to a grouplike C_n-space and vice versa, where C_n is the operad of little n-cubes.

For $n \geq 2$ the iterated loop space $\Omega^n X$ has a homotopy-commutative multiplication, satisfying an increasing number of coherence conditions, which are codified by actions of the operad C_n. This lead to the definition of E_n-spaces, as spaces on which an operad D, homotopy equivalent to C_n, operates.

Since an $(n + k)$-fold loop space can be regarded as a k-fold loop space in the category of n-fold loop spaces, one might think that an E_{n+k}-space is an E_k-space in the category of E_n-spaces. This type of structure, i.e. a D-space in the category of C-spaces, where C and D are operads, is codified by the *tensor product* $C \otimes D$ of operads (see section 2 below). Therefore the naive assumption arises, that the tensor product of an E_n-operad with an E_k-operad is homotopy equivalent to C_{n+k}, and hence an E_{n+k}-operad.

In general this is not true. The operad $\mathcal{M}$ of associative monoids is an E_1-operad, i.e. its grouplike algebras are precisely the one-fold loop spaces. But the tensor product with itself is the operad of commutative monoids, which is an E_∞-operad.

A better version of the naive approach is the following

Conjecture. *The tensor product of a cofibrant E_n-operad with an E_k-operad is an E_{n+k}-operad.*

Here the notion *cofibrant* has to be made precise. One possible choice is given in [Vog99].

A step in this direction was made by Dunn in [Dun88]. He proved that the n-fold tensor product of C_1 with itself, i.e. $C_1^{\otimes n}$ is homotopy equivalent to C_n. But unfortunately this result does not imply the equivalence of $C_n \otimes C_m$ and C_{n+m}, since the tensor product of operads does not respect homotopy equivalences.

Remark 0.1. The little cube operads C_n are not cofibrant in the sense of [Vog99].

In this paper we extend Dunn's result to our

Main Theorem. *For all $l \geq 2, n_1, \ldots, n_l \in \mathbb{N}$ and $n = n_1 + \cdots + n_l$ there exists a map $C_{n_1} \otimes \cdots \otimes C_{n_l} \to C_n$ of operads, which is a local Σ-equivalence.*

In the first three sections we recall the definition of operads in the topological setting, give a short overview over the interchange and the tensor product of operads and repeat the definition of the little cubes operads, which is extended to an operad of compact spaces in section 4. In addition we introduce a model $C_n|C_m$ for the tensor product $C_n \otimes C_m$ as a suboperad of C_{n+m}, which is based on Dunn's ideas.

The last three sections contain an analysis of this model, which leads together with some tools of Dunn to our main theorem.

Throughout this paper we work in the category $\mathfrak{Top}$ of compactly generated Hausdorff spaces in the sense of [Vog71].

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1. Topological operads and trees

Definition 1.1. A *collection* is a family $\{A(j)\}_{j \in \mathbb{N}}$ of spaces in $\mathfrak{Top}$ such that Σ_j acts on $A(j)$ from the right. For $\alpha \in A(j)$ we call j the *number of inputs* of α.

A *map of collections* $f : A \to B$ is a family $\{f_j : A(j) \to B(j)\}_{j \in \mathbb{N}}$ of equivariant maps.

The category of collections and maps between them is called $\Sigma\mathfrak{Top}$.

Definition 1.2. A *local Σ-equivalence* between two collections A and B is a map $f : A \to B$ of collections such that each $f_j : A(j) \to B(j)$ is an Σ_j-equivariant homotopy equivalence.

Definition 1.3. An *operad* A is a collection, together with a unit $\mathrm{id} \in A(1)$ and a series of *compositions* $- \circ - : A(k) \times A(j_1) \times \cdots \times A(j_k) \to A(j_1 + \cdots + j_k)$ such that

- $\alpha\sigma \circ (\beta_1, \ldots, \beta_k) = \alpha \circ (\beta_{\sigma^{-1}(1)}, \ldots, \beta_{\sigma^{-1}(k)}) \circ \bar{\sigma}$ for each $\alpha \in A(k), \beta_i \in A(j_i)$ and $\sigma \in \Sigma_k$, where $\bar{\sigma}$ permutes the blocks given by $j_1, \ldots, j_k)$ according to σ,
- $\alpha \circ (\mathrm{id}, \ldots, \mathrm{id}) = \alpha$ and $\mathrm{id} \circ \alpha = \alpha$ and
- $\alpha \circ \big(\beta_1 \circ (\gamma_1^1, \ldots, \gamma_{i_1}^1), \ldots, \beta_j \circ (\gamma_1^j, \ldots, \gamma_{i_j}^j))\big) =$
$$\big(\alpha \circ (\beta_1, \ldots, \beta_j)\big) \circ (\gamma_1^1, \ldots, \gamma_{i_1}^1, \ldots, \gamma_{i_j}^j)$$

A map $f : A \to B$ of operads is a map of the underlying collections such that $f(\mathrm{id}) = \mathrm{id}$ and
$$f\big(\alpha \circ_A (\beta_1, \ldots, \beta_k)\big) = f(\alpha) \circ_B \big(f(\beta_1), \ldots, f(\beta_k)\big),$$
where $\circ_A$ is the composition of A, and $\circ_B$ the one of B.

The category of operads and maps between them is called $oper\mathfrak{Top}$.

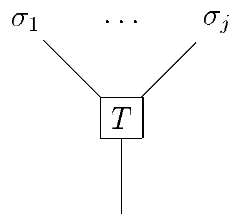
Remark 1.4. Since we require an operad to have a unit id, our notion is equivalent to the $\circ_i$-approach of Markl in [Mar96].

A very good notion for the work with operads - if not the best (free operads are constructed this way) - are trees. Since all results in the following are well-known, we just give a short description of all the terms, ideas and constructions needed. For details the reader is referred to the literature.

An edge of a graph is called *internal* if it is bounded by two vertices and *external* otherwise. External edges of directed graphs, with no vertex at their starting point are called *inputs*, and edges with no vertex at their end point are called *output*. A *tree* T is a connected, directed graph without loops, with exactly one output such that each vertex has precisely one output. The *valence* of a vertex v in a tree is the number $in(v)$ of its incoming edges. A vertex of valence 0 is called a *stump*.

Remark 1.5. The graph with no vertex and only one external edge, is a tree.

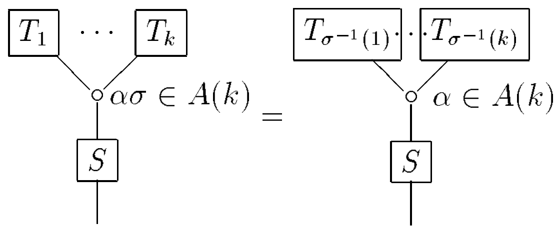
A *labeled* planar tree is a tree T together with a bijection $\sigma : in(T) \to \{1, \ldots, |in(T)|\}$ from the set of inputs of T. We represent it graphically by



where T is a tree with j inputs.

It is well-known that the labeled trees form a topological operad $\mathfrak{Tree}$ such that $\mathfrak{Tree}(j)$ is the set of trees with j inputs. The composition is given by grafting the trees along their roots and inputs.

Definition 1.6. The j-the space of the *free operad* FA of a collection A is the quotient of the space of all labeled trees with vertex labels, i.e. each vertex v of a tree is assigned a label $\alpha_v \in A(in(v))$, under the relation



The topology on $FA(j)$ is the topology of the according quotient space of

$$\coprod_{T \in \mathfrak{Tree}(j)} \left(\prod_{v \in T} A\big(in(v)\big) \right).$$

The unit of FA is the trivial tree with no vertex.

The free operads imply a functor $F : \Sigma\mathfrak{Top} \to oper\mathfrak{Top}$, which is left-adjoint to the forgetful functor $U : oper\mathfrak{Top} \to \Sigma\mathfrak{Top}$.

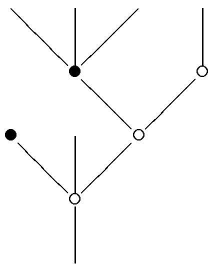
Remark 1.7. Since we need an order to define the product, we use the natural order on the vertices of a tree, given by left-traversion.

For our purposes we need a slight extension of this notion of trees.

Definition 1.8. A *bi-colored tree* (T, c) consists of a tree T and a map $c : ver(T) \to \{0, 1\}$ from the set of vertices of T. The number $c(v)$ is called the *color* of the vertex v. An internal edge is called *monochrome*, if its vertices have the same color.

Graphically we represent bi-colored trees by trees whose vertices are white ($c(v) = 0$) or black ($c(v) = 1$).

Example 1.9.



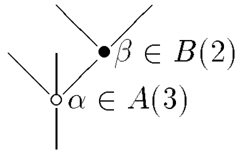
The sets $\mathfrak{BiTree}(j)$ of labeled bi-colored trees form an operad $\mathfrak{BiTree}$. As in the monochrome case the composition is given by the grafting of trees.

Bi-colored trees are very useful in the description of the direct sum $A \sqcup B$ of operads. Let T be a bi-colored tree with j inputs and $(A, B)_T$ the space

$$(A, B)_T = \prod_{\substack{v \in ver(T) \\ c(v) = 0}} A(in(v)) \times \prod_{\substack{v \in ver(T) \\ c(v) = 1}} B(in(v)).$$

Then the free operad $F(A \sqcup_\Sigma B)$, generated by the coproduct $A \sqcup_\Sigma B$ of the underlying collections, is given by the spaces $\coprod_{T \in \mathfrak{BiTree}(j)} (A, B)_T$ modulo the relations of definition 1.6. The composition is induced by the grafting of trees. The identity (or unit) is the trivial tree with no vertex.

Example 1.10.



Lemma 1.11. *$A \sqcup B(j)$ is the quotient of $F(A \sqcup_\Sigma B)(j)$, by the relations*

1. *Monochrome edges may be shrunk and their vertices composed,*
2. *the identities of A and B are identified with the trivial tree and*
3. *The relation of definition 1.6*

2. Interchange

The concept of interchange of operad structures and the tensor product of operads is well-known. Boardman and Vogt used it in [BV73] to describe homomorphisms between theories and algebras over theories, and May's notion of a pairing of two operads is closely related to the interchange of the two structures.

Definition 2.1. Let A, B and C be operads and $f : A \to C$ and $g : B \to C$ two maps of operads. We say f and g *interchange*, if the diagram

$$\begin{tikzcd}
A(j) \times B(k) \arrow[r, "\mathrm{id} \times \Delta"] \arrow[d, "\simeq"'] & A(j) \times B(k)^j \arrow[r, "f_j \times g_k^j"] & C(j) \times C(k)^j \arrow[dd, "- \circ -"] \\
B(k) \times A(j) \arrow[d, "\mathrm{id} \times \Delta"'] & & \\
B(k) \times A(j)^k \arrow[r, "g_k \times f_j^k"'] & C(k) \times C(j)^k \arrow[r, "- \circ -"'] & C(jk)
\end{tikzcd}$$

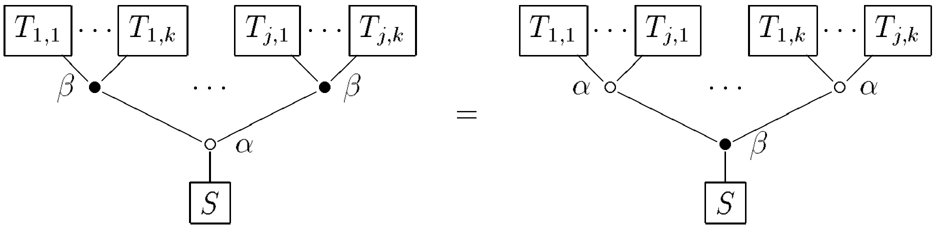
commutes for all $j, k \in \mathbb{N}$. Here Δ always means the appropriate diagonal.

If we apply this definition to algebras over A and B, i.e. if we choose $C = End_X$, then the structures of A and B on X interchange if and only if the diagrams

$$\begin{tikzcd}
(X^k)^j \arrow[rr, "\beta^j"] \arrow[d, "\simeq"'] & & X^j \arrow[d, "\alpha"] \\
(X^j)^k \arrow[r, "\alpha^k"'] & X^k \arrow[r, "\beta"'] & X
\end{tikzcd}$$

commute for all $\alpha \in A(j)$ and $\beta \in B(k)$.

The tensor product $A \otimes B$ of two operads A and B is an operad, which codifies the interchange of operad maps (cmp. [BV73]). This means that there exist two maps $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ such that the operad maps $f : A \to C$ and $g : B \to C$ interchange if and only if there exists a map $h : A \otimes B \to C$ such that $f = h \circ i_A$ and $g = h \circ i_B$. Its j-th space $A \otimes B(j)$ is the quotient of $A \sqcup B(j)$ under the additional *shuffle-relation*



As Dunn noted in [Dun88] the tensor product $A \otimes B$ is universal for pairings of operads in the sense of [May80].

3. The little cubes

For convenience we will use the following notations. The n-dimensional interval $[a_1, b_1] \times \cdots \times [a_n, b_n]$ of $\mathbb{R}^n$ will be denoted with $[a, b]$. For $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ we will write $a < b$ if $a_j < b_j$ for each j. In the same fashion we will write $a \leq b$. we denote the vector $(a_1 b_1, \ldots, a_n b_n)$ with ab.

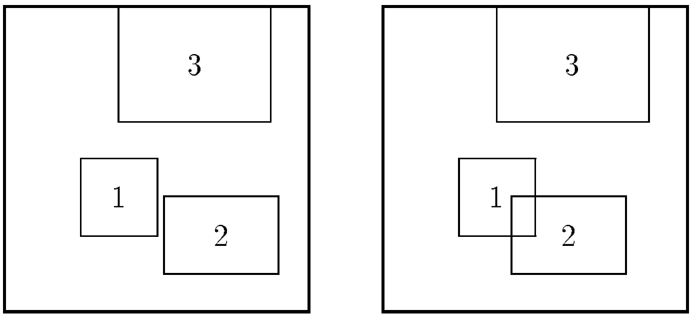


Figure 1. The left example is an element of $C_2(3)$ the right is not

Definition 3.1. Let $C_n(j), j \geq 1$, be given as the set of ordered j-tupels of n-dimensional intervals $[a^i, b^i]$ in $I^n = [0,1]^n$ with disjoint and non-empty interiors, i.e. with $a^i < b^i$. The space $C_n(0)$ consists only of the empty tupel $()$.

The composition $\alpha \circ (\beta_1, \ldots, \beta_k)$ of $\alpha = \big([a^1, b^1], \ldots, [a^k, b^k]\big) \in C_n(k)$ with $\beta_i = ([c^{1,i}, d^{1,i}], \ldots, [c^{j_i,i}, d^{j_i,i}]) \in C_n(j_i)$ for $1 \leq i \leq k$ is given by replacing the i-th interval $[a_i, b_i]$ of α with the following j-tupel

$$\big([a^i + (b^i - a^i)c^{1,i}, a^i + (b^i - a^i)d^{1,i}], \ldots$$
$$\ldots, [a^i + (b^i - a^i)c^{j_i,i}, a^i + (b^i - a^i)d^{j_i,i}]\big)$$

(recall that the $a^i, b^i, c^{i,j}$ and $d^{i,j}$ are vectors). This operation corresponds to the replacement of the i-th interval of α with a scaled-down copy of β_i.

Definition 3.2. A little cube $c \in C_n(j)$ is called *decomposable*, if ...

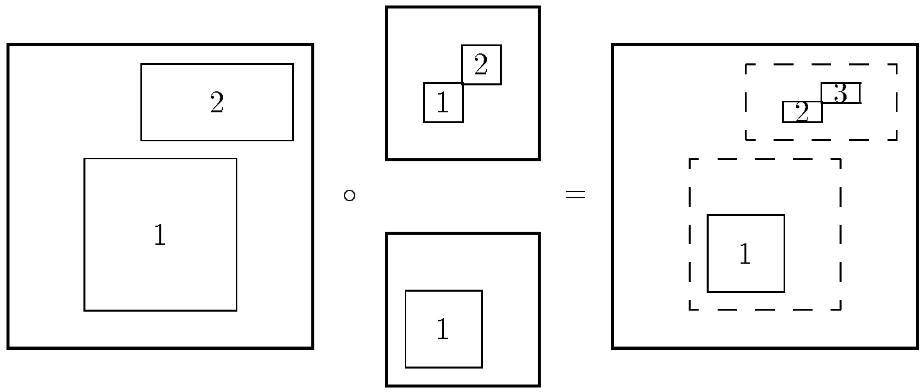


FIGURE 2. Example of a composition in C_2

1. $\ldots$ $j \in \{0, 1, 2\}$ or
2. $\ldots$ there exist a $d \in C_n(2)$ and decomposable c_1, c_2 with $c_k \in C_n(j_k)$ for $j_k > 0, k = 1, 2$ such that $c = \mu(d; c_1, c_2)$.

It is easy to see that the decomposable cubes of C_n form a suboperad D_n. Furthermore $D_1 = C_1$ and $D_n(j) = C_n(j)$ for $j \leq 3$.

A more geometrical description of decomposability is given by the insertion of a hyper plane. $c \in C_n(j)$ is decomposable, if and only if there exists an $1 \leq i \leq n$ and a hyper plane L of codimension 1, parallel to the i-axis, which hits no interior of the component cubes of C, such that each of the two parts is decomposable and contains at least one component cube (cmp. [Dun88]). We call such a hyper plane *separating.*

Proposition 3.3. *(cmp.* [Dun88, Prop. 2.3.]*) The inclusion $D_n \to C_n$ is a local Σ-equivalence.*

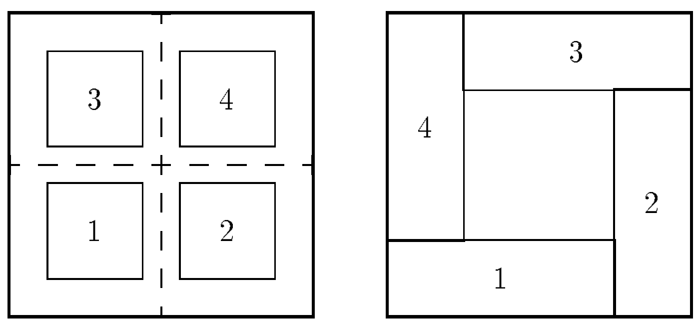


FIGURE 3. The left cube is decomposable (the dashed lines are separating hyper planes), the right is not.

Now let $H \subset C_n$ and $V \subset C_m$ be two suboperads. Each of them can be embedded into C_{n+m} as a suboperad. For H we use the inclusion

$$\big([a_1, b_1], \ldots, [a_k, b_k]\big) \mapsto \big([(a_1, 0), (b_1, 1), \ldots, (a_k, 0), (b_k, 1)]\big),$$

where $(a_i, 0)$ is the $(n+m)$-tupel $(a_i^1, \dots, a_i^n, 0, \dots, 0)$ and $(b_i, 1)$ is the tupel $(b_i^1, \dots, b_i^n, 1, \dots, 1)$. Similar we have an inclusion of V into C_{n+m} with

$$\big([c_1, d_1], \dots, [c_l, b_l]\big) \mapsto \big([(0, c_1), (1, d_1)], \dots, [(0, c_l), (1, d_l)]\big).$$

Graphically the two inclusions i_H and i_V are described by figure 4.

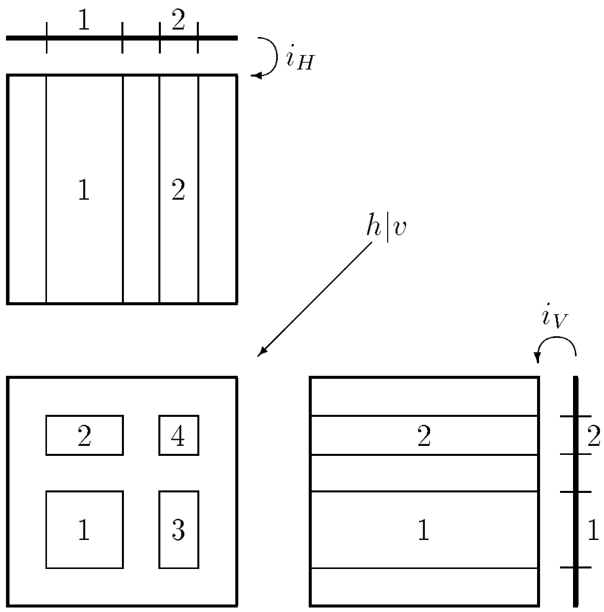


FIGURE 4. The inclusions i_H and i_V and the cube $h|v = i_H(h) \circ (i_V(v), i_V(v))$.

These two operad morphisms induce two maps $H(j) \times V(k) \to C_{n+m}(jk)$ of collections for each pair j, k of natural numbers, given by

$$(h, v) \mapsto i_H(h) \circ \underbrace{(i_V(v), \dots, i_V(v))}_{k-\text{times}}$$

and

$$(h, v) \mapsto i_V(v) \circ \underbrace{(i_H(h), \dots, i_H(h))}_{l-\text{times}}.$$

The image of the first map is called $h|v$.

It is easy to check that the first morphism is given by

$$\Big(\big([a_1, b_1], \dots, [a_k, b_k]\big), \big([c_1, d_1], \dots, [c_l, k_l]\big)\Big) \mapsto$$
$$\big([(a_1, c_1), (b_1, d_1)], \dots, [(a_1, c_l), (b_1, d_l)], [(a_2, c_1), (b_2, d_1)], \dots\big)$$

and the second by

$$\Big(\big([a_1,b_1],\dots,[a_k,b_k]\big),\big([c_1,d_1],\dots,[c_l,k_l]\big)\Big)\mapsto$$
$$\big([(a_1,c_1),(b_1,d_1)],\dots,[(a_k,c_1),(b_k,d_1)],[(a_1,c_2),(b_1,d_2)],\dots\big).$$

If we order the tupels (a_i,c_j) and (b_i,d_j) lexicographically by their indeces, the we see that the two images coincide up to a transposition. Comparing this with the interchange condition shows that i_H and i_V interchange. This leads to the existence of a morphism $H\otimes V\to C_{n+m}$. Let $H|V\subset C_{n+m}$ be the image of this morphism and $\varphi: H\otimes V\to H|V$ the induced map of morphisms.

Since this construction is based on the addition of ¨trivial¨ coordinates, it is easy to see that the suboperads $(H|M)|V$ and $H|(M|V)$ of C_{n+l+m} with $H\subset C_n$, $M\subset C_l$ and $V\subset C_m$ are equal.

4. The closed cubes

For the proofs of the main theorem we need an extension $\bar{C}_n$ of the little n-cubes such that each $\bar{C}_n(j)$ is a compact subset of $\mathbb{R}^{2nj}$. We start with an alternative description of $C_n(j)$. Let $\alpha=([a^1,b^1],\dots,[a^j,b^j])$ be an element of $C_n(j)$. The property that all intervals $[a^i,b^i]$ have non-empty interiors can be described by the inequalities $a^i<b^i$. The disjointness of the interiors of different cubes is more difficult.

Let $C_n^{(i,k)}(j)$ be the space of tupels $([a^1,b^1],\dots,[a^j,b^j])\in I^{2nj}$ with non-empty interior for $1\le i<k\le j$ such that $[a^i,b^i]$ and $[a^k,b^k]$ have disjoint interiors. Obviously we have

$$C_n(j)=\bigcap_{1\le i<k\le j} C_n^{(i,k)}(j).$$

The cube $[a^i,b^i]$ defines $2n$ parts of I^n, which are of the form

$$\Big([(0,\dots,0),(1,\dots,a^i_l,\dots,1)]\Big)\text{ or }\Big([(0,\dots,b^i_l,0),(1,\dots,1)]\Big),$$

whose union is the complement of the interior of $[a^i,b^i]$ (recall that $a^i=(a^i_1,\dots,a^i_n)$ and $b^i=(b^i_1,\dots b^i_n)$). Let $A_l^{(i,k)}(j)$ be the subspace of $C_n^{(i,k)}(j)$ such that $[a^k,b^k]$ lies in the l-th part of the first form and let $B_l^{(i,k)}(j)$ be the subspace of $C_n^{(i,k)}(j)$ such that $[a^k,b^k]$ lies in the l-th part of the second form.

$[a^k,b^k]$ and $[a^i,b^i]$ have disjoint interiors, if and only if $[a^k,b^k]$ lies in one of these parts. Hence $C_n^{(i,k)}(j)$ is the union of the $2n$ subspaces $A_l^{(i,k)}(j)$ and $B_l^{(i,k)}(j)$ of $\mathbb{R}^{2nj}$,

$$C_n(j)=\bigcap_{1\le i<k\le j}\left(\bigcup_{1\le l\le n} A_l^{(i,k)}(j)\cup B_l^{(i,k)}(j)\right).$$

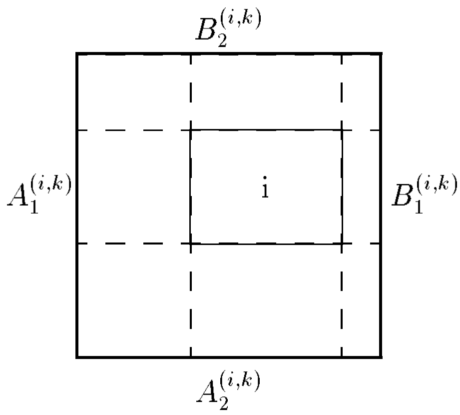


FIGURE 5. The $A_l^{(i,k)}$s and $B_l^{(i,k)}$s

Now we use this (quite complicated) description to obtain a closed (and hence compact) subset $\bar{C}_n(j)$ of I^{2nj}, which contains $C_n(j)$. We define

$$\bar{C}_n(j) = \bigcap_{1\leq i<k\leq j} \left(\bigcup_{\leq l \leq n} \bar{A}_l^{(i,k)}(j) \cup \bar{B}_l^{(i,k)}(j) \right),$$

where $\bar{A}_l^{(i,k)}(j)$ is the set of all tupels $([a^1,b^1],\ldots,[a^j,b^j])$ in I^{2nj} such that $a^i \leq b^i$, i.e. the interiors are allowed to be empty, and $[a^k,b^k]$ lies in the l-th part of I^{2nj}, generated by $[a^i,b^i]$. $\bar{B}_l^{(i,k)}(j)$ is defined accordingly. Since these properties can be described by the inequalities $a^i_m \leq b^i_m$ for $1 \leq i \leq j$ and $1 \leq m \leq n$, and either $b^k \leq (1,\ldots,a^i_l,\ldots,1)$ or $(0,\ldots,b^i_l,\ldots,0) \leq a^k$, these two spaces are closed in I^{2nj}.

Remark 4.1. $\bar{C}_n(2)$ does not consist of all little n-cubes with arbitrary interior. For example the configuration in figure 6 is not an element in $\bar{C}_2(2)$, since each of the intervals does not lie in one of the four parts defined by the other.

In $C_2(3)$ and $C_2(4)$ the same configuration can appear, since then we can split one or two of the intervals at their intersection.

In fact a tupel $\alpha = ([a^1,b^1],\ldots,[a^j,b^j])$ of j intervals in I^n is an element of $A_l^{(i,k)}(j)$ if and only if the inequality $b^k_l \leq a^i_l$ holds. And it is an element of $B_l^{(i,k)}$ if and only if $b^i_l \leq a^k_l$. Hence α is an element of $C_n(j)$ if and only if there exists an $1 \leq l \leq n$ for each pair $1 \leq i < k \leq j$ such that either $b^k_l \leq a^i_l$ or $b^i_l \leq a^k_l$.

Now Let $\alpha = ([a^1,b^1],\ldots,[a^j,b^j])$ be an element of $\bar{C}_n(j)$ and $\gamma_i = ([c^{1,i},d^{1,i}],\ldots,[c^{k,i},d^{k,i}]), 1 \leq i \leq j$, elements of $\bar{C}_n(k_i)$. As in $\mathcal{C}_n$, we can define $\alpha \circ (\gamma_1,\ldots,\gamma_k)$. It is not very hard to see that this is an element in $\bar{C}_n(k_1+\cdots+k_j)$. Therefore the $\bar{C}_n(j)$ form an operad $\bar{C}_n$,

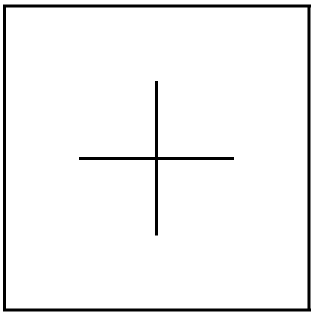


FIGURE 6. A non-example of a closed cube

which contains C_n as a suboperad. We call $\bar{C}_n$ the *operad of closed n-cubes.*

As for C_n and C_m, we obtain a suboperad $\bar{C}_n|\bar{C}_m$ of $\bar{C}_{n+m}$, which contains $C_n|C_m$ as a suboperad. Again $\bar{C}_n|\bar{C}_m$ is given as the image of a morphism $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_{n+m}$.

Definition 4.2. Let

$$\alpha = \big([a^1, b^1], \dots, [a^j, b^j]\big) \text{ and } \beta = \big([c^1, d^1], \dots, [c^k, d^k]\big)$$

be two elements of $\bar{C}_n$. α is called a *frame* of β if there exists a surjective map $\varphi : \mathbf{k} \to \mathbf{j}$ such that

$$[c^i, d^i] \subset [a^{\varphi(i)}, b^{\varphi(i)}]$$

for all $i \in \mathbf{k}$. The map φ is called a *framing* of β into α.

Definition 4.3. Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(l)$ be two frames of $\beta \in \bar{C}_n(k)$. If α' if a frame of α, then α is called *tighter* than α'.

Lemma 4.4. *Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(j')$ be two frames of $\beta \in \bar{C}_n(k)$. Then there exists a frame $\alpha \cap \alpha'$ of β, which is tighter than α and α'.*

Proof. Let α be of the form $(\dots[a^i, b^i]\dots)$ and α' of the form $(\dots[\bar{a}^i, \bar{b}^i]\dots)$ and β of the form $(\dots[c^i, d^i]\dots)$. Furthermore let φ_α and $\varphi_{\alpha'}$ be two framings of β into α and α'.

The intervals of $\alpha \cap \alpha'$ are all intervals of the form

$$[a^{\varphi_\alpha(i)}, b^{\varphi_\alpha(i)}] \cap [\bar{a}^{\varphi_{\alpha'}(i)}, \bar{b}^{\varphi_{\alpha'}(i)}]$$

for each $1 \le i \le k$. Then each interval $[c^i, b^i]$ of β is contained in the i-th intersection. The intervals of $\alpha \cap \alpha'$ can be ordered arbitrarily. In addition the map $(\varphi_\alpha, \varphi_{\alpha'}) : \mathbf{k} \to \mathbf{j} \times \mathbf{j}'$ implies a surjective map from $\mathbf{k}$ into its image. This map is a framing of β into $\alpha \cap \alpha'$ (the latter one has as many inputs as the image has elements). The maps $\mathbf{k} \to \mathbf{j} \times \mathbf{j}' \to \mathbf{j}$ and $\mathbf{k} \to \mathbf{j} \times \mathbf{j}' \to \mathbf{j}'$ induce framings of $\alpha \cap \alpha'$ into α and α'. $\square$

Lemma 4.5. *Let $\alpha \in \bar{C}_n(j)$ and $\alpha' \in \bar{C}_n(j')$ be two frames of β such that α is tighter than α' and vice versa. Then α and α' coincide up to a permutation, i.e. there exists a permutation τ such that $\alpha\tau = \alpha'$.*

Proof. Let $\varphi : \mathbf{j} \to \mathbf{j}'$ be a framing of α into α' and $\psi : \mathbf{j}' \to \mathbf{j}$ a framing of α' into α. Since both maps are surjective, their compositions are. This again implies that $\varphi\psi$ and $\psi\varphi$ are bijective and that $j = j'$.

Let $\sigma \in \Sigma_j$ be the map $\psi\varphi$. We know $\sigma^{j!} = \mathrm{id}$ and furthermore

$$[c^i, d^i] \subset [a^{\varphi(i)}, b^{\varphi(i)}] \subset [c^{\sigma(i)}, d^{\sigma(i)}] \subset \cdots \subset [c^{\sigma^{j!}(i)}, d^{\sigma^{j!}(i)}] = [c^i, d^i],$$

where $[c^i, d^i]$ is the i-th interval of α and $[a^i, b^i]$ the one of α'. The i-th interval of α is precisely the $\varphi(i)$-th interval of α'. Since φ is bijective the statement follows. $\square$

Obviously we have

Proposition 4.6. *If $\beta \in \bar{C}_n(j)$ is of the form $\alpha \circ (\gamma_1, \ldots, \gamma_k)$ with $\alpha \in \bar{C}_n(k)$ such that each γ_i has at least one input, then α is a frame of β.*

Lemma 4.7. *Let $\alpha \in \bar{C}_n(1)$ be a frame of $\beta \in \bar{C}_n(j)$. Then there exists a $\beta' \in \bar{C}_n(j)$ such that $\beta = \alpha \circ \beta'$.*

Proof. Let $[a, b]$ be the only interval of α and let $[c^i, d^i], 1 \le i \le j$, be the i-th interval of β. We define

$$\bar{c}^i_l = \begin{cases} \frac{c^i_l - a_l}{b_l - a_l} & \text{if } b_l \neq a_l \\ \frac{i-1}{j} & \text{if } b_l = a_l \end{cases} \text{ and } \bar{d}^i_l = \begin{cases} \frac{d^i_l - a_l}{b_l - a_l} & \text{if } b_l \neq a_l \\ \frac{i}{j} & \text{if } b_l = a_l, \end{cases}$$

for $1 \le i \le j$ and $1 \le l \le n$. Now let β' be given by $([\bar{c}^1, \bar{d}^1], \ldots, [\bar{c}^j, \bar{d}^j])$. We have to check, that this sequence of intervals is a complete cube.

Choose $\le i < k \le j$. Following remark 4.1 we have to find $1 \le l \le n$ such that either $\bar{d}^k_l \le \bar{c}^i_l$ or $\bar{d}^i_l \le \bar{c}^k_l$. We know that there exists an l such that either $d^k_l \le c^i_l$ or $d^i_l \le c^k_l$ holds. If $a_l \neq b_l$ we are done. Otherwise we have two cases. In the first, $k \le i - 1$, we have

$$\bar{d}^k_l = \frac{k}{j} \le \frac{i-1}{j} = \bar{c}^i_l.$$

For $i + 1 \le k$ we have

$$\bar{d}^i_l = \frac{i}{j} \le \frac{k-1}{j} = \bar{c}^k_l.$$

$\square$

Corollary 4.8. *Let $\alpha \in \bar{C}_n(k)$ be a frame of $\beta \in \bar{C}_n(j)$. Then there exist $\beta'_i \in \bar{C}_n, 1 \le i \le k$ such that $\beta = \alpha \circ (\beta'_1, \ldots, \beta'_k)$.*

Proof. Let φ be a framing of β into α. Let $I_i \subset \{1, \ldots, j\}$ be the preimage of $i \in \{1, \ldots, k\}$ of under φ. Then we can kill all inputs of β, except for the inputs whose label is in I_i, by composition with stumps. We obtain $\beta_i \in \bar{C}_n$. Furthermore the i-th interval $[a^i, b^i]$ of α is a frame of β_i. By lemma 4.7 exists a β'_i such that $\beta_i = ([a^i, b^i]) \circ \beta'_i$. This implies

$$\beta = \alpha \circ (\beta'_1 \ldots, \beta'_k).$$

$\square$

Lemma 4.9. *Let $\alpha \in \bar{C}_n(k)$ and $\beta \in \bar{C}_n(j)$ and $([a,b]) \in \bar{C}_n(1)$ such that $([a,b]) \circ \alpha$ is a frame of $([a,b]) \circ \beta$. Then there exists a β' such that α is a frame of β' and such that*

$$([a,b]) \circ \beta = ([a,b]) \circ \beta'.$$

Proof. Let $[a^i, b^i]$ be the i-th interval of α and $[c^i, d^i]$ the i-th interval of β and φ a framing of $([a,b]) \circ \beta$ into $([a,b]) \circ \alpha$. We chose β' to be the tupel $([\bar{c}^1, \bar{d}^1], \dots, [\bar{c}^j, \bar{d}^j])$ with

$$\bar{c}^i_l = \begin{cases} c^i_l & \text{if } a_l \neq b_l \\ \frac{a_l^{\varphi(i)} + b_l^{\varphi(i)}}{2} & \text{if } a_l = b_l \end{cases} \text{ and } \bar{d}^i_l = \begin{cases} d^i_l & \text{if } a_l \neq b_l \\ \frac{a_l^{\varphi(i)} + b_l^{\varphi(i)}}{2} & \text{if } a_l = b_l, \end{cases}$$

for $1 \leq i \leq j$ and $1 \leq l \leq n$.

First we prove, that β is in $\bar{C}_n(j)$. Let $1 \leq i < k \leq j$. Since $\beta \in \bar{C}_n(j)$, we know that there exists a $1 \leq l \leq n$ such that either $d^i_l \leq c^k_l$ or $d^k_l \leq c^i_l$. If $a_l \neq b_l$, we are done. If $a_l = b_l$ we have $\bar{c}^i_l = \bar{d}^i_l$ and $\bar{c}^k_l = \bar{d}^k_l$. Hence one of the necessary inequalities holds.

Now let φ be the framing of $([a,b]) \circ \beta$ into $([a,b]) \circ \alpha$. Then the inequality

$$a_l + (b_l - a_l) a_l^{\varphi(i)} \leq a_l + (b_l - a_l) c^i_l \leq a_l + (b_l - a_l) d^i_l \leq a_l + (b_l - a_l) b_l^{\varphi(i)}$$

holds for each $1 \leq l \leq n$ and $1 \leq i \leq j$. If $a_l \neq b_l$ this immediately leads to

$$a_l^{\varphi(i)} \leq c^i_l \leq d^i_l \leq b_l^{\varphi(i)}.$$

If $b_l = a_l$ we have

$$a_l^{\varphi(i)} \leq \bar{c}^i_l = \bar{d}^i_l \leq b_l^{\varphi(i)}.$$

Thus α is a frame of β' and φ is a framing of β' into α.

The fact that $([a,b]) \circ \beta$ is equal to $([a,b]) \circ \beta'$ is easy to see. □

Together with corollary 4.8 this leads to

Corollary 4.10. *Let α and β be two elements of $\bar{C}_n$ such that $([a,b]) \circ \alpha$ is a frame of β. Then there exists a β' such that α is a frame of β' and $\beta = ([a,b]) \circ \beta$.*

5. Reduced representations

Obviously the map $\bar{C}_n \sqcup \bar{C}_m(j) \to \bar{C}_n \otimes \bar{C}_m(j)$ is a surjection for each $j \in \mathbb{N}$ and the map $\bar{C}_n \otimes \bar{C}_m(j) \to \bar{C}_n | \bar{C}_m(j)$ is surjective by definition. Therefore every element of $\bar{C}_n | \bar{C}_m(j)$ and every element of $\bar{C}_n \otimes \bar{C}_m(j)$ can be represented by an element of $(\bar{C}_n, \bar{C}_m)_T$ with T a labeled, bi-colored tree with j inputs.

Remark 5.1. In the following we denote an element of $F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ and the trees underlying its representations with the same name. It should be clear from the context whether the vertex labels are of importance.

Definition 5.2. A labeled, bi-colored tree with $j > 1$ inputs is *reduced*, if

- it contains no monochrome edge,
- it contains no vertex of valence 0 and
- it contains no sequence of valence 1, i.e. there is no subtree with more than two vertices which all have valence 1.

A tree with 0 inputs is reduced if and only if it is a stump, and a tree with one input is reduced if it contains at most two vertices of valence 1 and different colors.

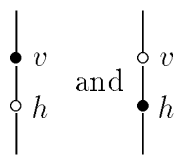
There are only finitely many reduced trees with j inputs. The maximal number of vertices a reduced tree with j inputs can have, is given by the number of vertices of a binary tree with j inputs, plus the number of all edges (split an edge by one vertex of valence 1), i.e. $(j-1)+(2j-1)$.

Lemma 5.3. *For each $c \in \bar{C}_n|\bar{C}_m(j)$ exists a reduced tree T with j inputs and an representation $T_c \in (\bar{C}_n, \bar{C}_m)_T$ of c.*

Proof. For $j = 0$ the statement is trivial, since $\bar{C}_{n+m}(0)$ consists only of one point. For a given representation $S_c \in (\bar{C}_n, \bar{C}_m)_S$ of $c \in \bar{C}_n|\bar{C}_m(j)$ for $j \geq 1$, we construct a reduced representation T_c. If S contains monochrome edges, we can shrink them them by composing the labels at their vertices. Hence we can exchange S_c by a representation which contains no monochrome edge.

Now assume that S_c contains no monochrome edge. Since the images of vertices of valence 0 of both colors coincide in $\bar{C}_{n+m}(0)$, their colors can be changed without affecting the image of the tree. Hence all outgoing edges of a vertex of valence 0 can be assumed to be monochrome. Therefore we can shrink them by composing their vertices. This kills one input of the root of the according edge and the stump.

Now we assume that S_c contains no monochrome edge and no stump. It is easy to see that the two trees



represent the same element in $\bar{C}_{n+m}(1)$. Therefore we can change the order in a sequence of valence 1 arbitrarily. Thus we can sort them by color and then shrink the obtained monochrome edges. Hence we can assume that each sequence of valence 1 consists only of two vertices of different colors. For $j = 1$ we are done now. For $j > 1$, this sequence is connected to another vertex of arbitrary color (either at the input or at the output). If the connecting edge is not monochrome, we exchange the two vertices of valence 1 and obtain at least one monochrome edge, which again can be shrunk. This last step kills (at least) one of the two vertices of valence 1. $\square$

Corollary 5.4. *$\bar{C}_n|\bar{C}_m(j)$ is the union of finitely many compact subspaces and hence compact.*

Proof. For each reduced tree T with j inputs, the space $(\bar{C}_n, \bar{C}_m)_T$ is compact, because it is a product of compact spaces. Therefore its image K_T in $\bar{C}_n|\bar{C}_m(j)$ is compact. Since each element is represented by a reduced tree, $\bar{C}_n|\bar{C}_m(j)$ is the union of the finitely many K_T. $\square$

In the same way we get

Corollary 5.5. *$\bar{C}_n \otimes \bar{C}_m(j)$ is compact.*

Since there exists a continuous morphism $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$, two elements of $F(\bar{C}_n \sqcup_\Sigma \bar{C}_M)$, which represent the same element in $\bar{C}_n \otimes \bar{C}_m$, represent the same element in $\bar{C}_n|\bar{C}_m$. For the prove of the converse situation, we construct ¨minimal¨ representations.

6. Minimal representations

Definition 6.1. A $\circ$-*representation* of $\alpha \in \bar{C}_n|\bar{C}_m(j)$, $j > 0$, is a representation of α in $(\bar{C}_n, \bar{C}_m)_T$ such that the root vertex of T has the color $\circ$ (or 0). Similarly a $\bullet$-*representation* is a representation of α, whose root has the color $\bullet$ (or 1). If there exists a $\circ$-representation of α, with $h \in \bar{C}_n$ as root, then h is called a $\circ$-*root* of α.

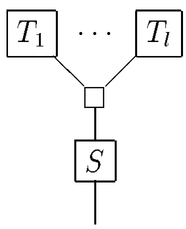
Definition 6.2. A $\circ$-*frame* of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ is an element $h \in \bar{C}_n$ such that $h|\mathrm{id}$ is a frame of α. Similarly a $\bullet$-*frame* of α is an element v of $\bar{C}_m$ such that $\mathrm{id}|v$ is a frame of α.

The following lemma is a consequence of the proof of lemma 4.4.

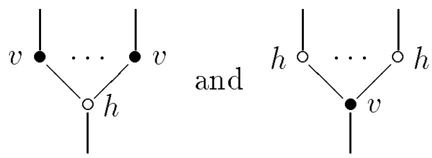
Lemma 6.3. *If h and h' are $\circ$-frames of α, then the intersection $h|id \cap h'|id$ is given by a $\circ$-frame $h \cap h'$.*

Lemma 6.4. *For $\alpha \in \bar{C}_n|\bar{C}_m(j)$ with $j > 1$, exists a $\circ$-root $h \in \bar{C}_n(k)$ or a $\bullet$-root $v \in \bar{C}_m(k)$ with $k > 1$.*

Proof. If α has more than one input, then there exists a reduced representation (either $\circ$ or $\bullet$), which has at least one vertex of a valence higher than 1. In general it is of the form



where $\square$ is either $\circ$ or $\bullet$. Since we know that the trees

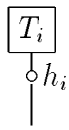


represent the same element in $\bar{C}_n|\bar{C}_m$, we can push the lowest vertex of valence > 1 down to the root. $\square$

Lemma 6.5. *$h \in \bar{C}_n(k)$ is a $\circ$-frame of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ if and only if it is a $\circ$-root.*

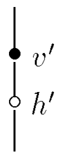
Proof. By proposition 4.6 each $\circ$-root of α is a $\circ$-frame.

Now let $h \in \bar{C}_n(k)$ be a $\circ$-frame of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ and φ a framing of α into $h|\mathrm{id}$. Assume that the statement is true for $k = 1$. Then for $k > 1$ we can kill all inputs of α, which do not lie in the i-th input of $h|id$, i.e. all $l \in \mathbf{j}$ with $\varphi(j) \neq i$. We obtain $\alpha_i \in \bar{C}_n|\bar{C}_m$, which is framed by $h_i := ([a^i, b^i])$, where $[a^i, b^i]$ is the i-th interval of h. Therefore we can find a $\circ$-representation of α_i with the root h_i, i.e. it is of the form



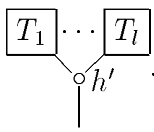
If β_i is the cube represented by T_i, we obtain $(h|\mathrm{id}) \circ (\beta_1, \dots, \beta_k)$. Therefore h is a $\circ$-root of α.

We still have to prove the theorem for $k = 1$. For $j = 1$ the statement is quite obvious, since each reduced $\circ$-representation is of the form



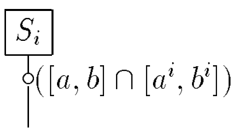
If h is a $\circ$-frame of α, it is obviously a frame of h', and by lemma 4.7 exists a $h'' \in \bar{C}_n$ with $h' = h \circ h''$.

For $j > 1$ and $k = 1$ we have to use the fact that there exists at least one representation ($\circ$ or $\bullet$), whose root has a valence greater than 1 (cmp. lemma 6.4). Let it be of the form



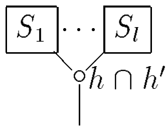
If h is a frame of h', then we are done, since by corollary 4.8 exists an $h'' \in \bar{C}_n(l)$ such that $h' = h \circ h''$.

If h is not a frame of h', we consider $\alpha_i := ([a^i, b^i])|\mathrm{id} \circ \beta_i$ for each $1 \leq i \leq l$, where $[a^i, b^i]$ is the i-th interval of h' and β_i is the cube represented by T_i. Since α_i can be obtained from α by the composition with stumps at all inputs, which do not belong to T_i, it is an element of $\bar{C}_n|\bar{C}_m(j_i)$, where $1 \leq j_i < j$. By induction there exists a $\circ$-representation of α_i, of the form



because $([a, b] \cap [a^i, b^i])$ is a $\circ$-frame of α_i.

The $\circ$-frame $h \cap h'$ of α consist of the intervals $[a, b] \cap [a^i, b^i]$, and hence α is represented by



Since h is a frame of $h \cap h'$, we can find a $h'' \in \bar{C}_n(l)$ with $h \cap h' = h \circ h''$.

If the root of valence greater than 1 has the color $\bullet$, we can proof the statement by similar means. $\square$

Lemma 6.6. *Let $\alpha \in \bar{C}_n|\bar{C}_m(j), h \in \bar{C}_n(k), H' \in \bar{C}_n(k')$ and $[a, b] \subset I^n$ an interval such that*

- *h is a $\circ$-frame of α,*
- *$([a, b]) \circ h$ is a frame of H' and*
- *H' is a $\circ$-frame of $([a, b]) \circ \alpha$.*

Then there exists $h' \in \bar{C}_n(k')$ such that

- *$H' = ([a, b]) \circ h'$,*
- *h is a frame of h'.*
- *h' is a $\circ$-frame of α and*

Proof. Let $\varphi : \mathbf{j} \to \mathbf{k}$ be a framing of α into $h|\mathrm{id}$ and hence from $([a, b])|\mathrm{id} \circ \alpha$ into $(([a, b]) \circ h)|\mathrm{id}$ and $\Phi' : \mathbf{j} \to \mathbf{k}'$ a framing from $([a, b])|\mathrm{id} \circ \alpha$ into $H'|\mathrm{id}$ and $\psi : \mathbf{k}' \to \mathbf{k}$ a framing of H' into $([a, b]) \circ h$. We can assume that $\psi \circ \Phi' = \varphi$.

We define $h' \in \bar{C}_n(k')$ to be the tupel $(\dots [\bar{c}^i, \bar{d}^i] \dots)$ with

$$\bar{c}^i_l = \begin{cases} \frac{c^i_l - a_l}{b_l - a_l} & \text{if } a_l \neq b_l \\ min\big(a^s_l : s \in {\Phi'}^{-1}(i)\big) & \text{if } a_l = b_l \end{cases}$$

and

$$\bar{d}^i_l = \begin{cases} \frac{d^i_l - a_l}{b_l - a_l} & \text{if } a_l \neq b_l \\ max\big(b^s_l : s \in {\Phi'}^{-1}(i)\big) & \text{if } a_l = b_l, \end{cases}$$

for $1 \leq l \leq n$, where $[a^i, b^i]$ is the i-th interval of α and $[c^i, d^i]$ the i-th interval of H'.

First we prove that h' is an element of $\bar{C}_n(k')$. Let $1 \leq i < h \leq k'$. If $[\bar{a}^i, \bar{b}^i]$ is the i-th interval of h, then there exists an l with $1 \leq l \leq n$ such that either $\bar{a}^{\psi(i)}_l \geq \bar{b}^{\psi(h)}_l$ or $\bar{a}^{\psi(h)}_l \geq \bar{b}^{\psi(i)}_l$. Since ψ is a framing of H' into $([a, b]) \circ h$, we know

$$a_l + (b_l - a_l)\bar{a}^{\psi(i)}_l \leq c^i_l \leq d^i_l \leq a_l + (b_l - a_l)\bar{b}^{\psi(i)}_l.$$

If $b_l \neq a_l$, this immediately leads to

$$\bar{a}^{\psi(i)}_l \leq \bar{c}^i_l \leq \bar{d}^i_l \leq \bar{b}^{\psi(i)}_l.$$

The same inequality follows for h instead of i. Together they imply that either $\bar{c}^i_l \geq \bar{d}^h_l$ or $\bar{c}^h_l \geq \bar{d}^i_l$ holds.

If $a_l = b_l$, we use the fact that $[a^s, b^s] \subset [\bar{a}^i, \bar{b}^i]$ and hence

$$\bar{a}_l^i \le a_l^s \le b_l^s \le \bar{b}_l^i$$

for all $s \in \Phi^{-1}(i)$. Since $\Phi = \psi\Phi'$ we also have ${\Phi'}^{-1}(i) \subset \Phi^{-1}(\psi(i))$, and therefore

$$\bar{c}_l^i = min\big(a_l^s : s \in {\Phi'}^{-1}(i)\big) \ge min\big(a_l^s : s \in \Phi^{-1}(\psi(i))\big) \ge \bar{a}_l^{\psi(i)}$$

and

$$\bar{d}_l^i = max\big(b_l^s : s \in {\Phi'}^{-1}(i)\big) \le max\big(b_l^s : s \in \Phi^{-1}(\psi(i))\big) \le \bar{b}_l^{\psi(i)}.$$

This implies that either $\bar{c}_l^i \ge \bar{a}_l^{\psi(i)} \ge \bar{b}_l^{\psi(h)} \ge \bar{b}_l^h$ or $\bar{c}_l^h \ge \bar{a}_l^{\psi(h)} \ge \bar{b}_l^{\psi(i)} \ge \bar{d}_l^i$. From these observations also follows, that ψ is a framing of h' into h.

To prove that h' is a $\circ$-frame of α with framing Φ', we have to check that for each $1 \le l \le n$ and $1 \le i \le j$, the inequality

$$\bar{c}_l^{\Phi'(i)} \le a_l^i \le b_l^i \le \bar{d}_l^{\Phi'(i)}$$

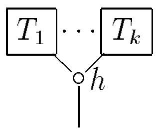
is true. If $a_l \ne b_l$ this follows from

$$c_l^{\Phi'(i)} \le a_l + (b_l - a_l)a_l^i \le a_l + (b_l - a_l)b_l^i \le d_l^{\Phi'(i)},$$

which again holds, because Φ' is a framing of $H'|\mathrm{id}$ to $([a, b]) \circ \alpha$. For $a_l = b_l$ the inequality is fulfilled since $i \in {\Phi'}^{-1}(\Phi(i))$.

It remains to check, that $H' = ([a, b]) \circ h'$. But this is an immediate consequence of the definition of h'. $\square$

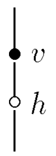
Theorem 6.7. *For each $\alpha \in \bar{C}_n|\bar{C}_m(j)$ there exists an (up to permutations) uniquely determined $\circ$-frame $h \in \bar{C}_n(k)$ of α and a $\circ$-representation of the form*



such that each $\circ$-frame h' of α is also a frame of h.

Proof. We prove the theorem via induction over the number j of inputs of α. For $j = 0$ the only reduced $\circ$-representation is the stump of color $\circ$. Hence the theorem holds trivially.

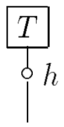
For $j = 1$ the reduced $\circ$-representation of α is uniquely determined and of the form



.

Obviously h is tighter than any other $\circ$-frame of α.

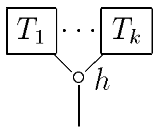
If α has more than one input we have to differentiate between two basic cases. First let all $\circ$-representations of α have a root of valence 0, i.e. every $\circ$-representation is of the form



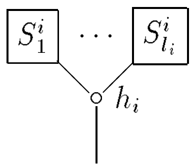
For each $1 \leq i \leq k$ we can chose an interval $h^i = ([c^i, d^i]) \in \bar{C}_n(1)$ such that $d^i_i - c^i_i$ is minimal under all $\circ$-roots. (Intervals of this kind exist, since $\bar{C}_n|\bar{C}_m(j)$ is a Hausdorff-space and since $(\bar{C}_n, \bar{C}_m)_T$ is compact for each bi-colored, labelled tree T.) Each h^i is a $\circ$-frame of α. Hence their intersection h is a $\circ$-frame and therefore a $\circ$-root of α.

If h' is another $\circ$-frame of α, then the intersection $h'' := h \cap h'$ would be a frame, which is tighter than h and h'. Hence there exists a $\circ$-representation with h'' as root. Since h is the intersection of ¨minimal¨ roots h^i, this implies $h'' = h$, because otherwise there has to exists a coordinate such that h'' is ¨smaller¨ in the i-th direction than the according h^i.

In the second case we assume that there exists a reduced $\circ$-representation of α of the form

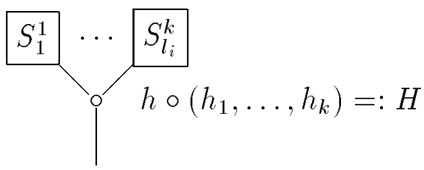


for $k > 1$. Then each T_i has k_i inputs with $1 \leq k_i < j$. By induction we can find $h_i \in \bar{C}_n(l_i)$ and $\circ$-representations of $\beta_i \in \bar{C}_n|\bar{C}_m(k_i)$, represented by T_i, of the form



such that every $\circ$-frame of β_i is wider than h_i.

Together these form a $\circ$-representation



of α.

Now let H' be another frame of α. Without restriction we can assume that H' is tighter than H (replace it with $H \cap H'$). By composition with stumps, we can kill all inputs of α, H and H', which are represented by inputs on another subtree than T_i. We obtain $\alpha_i \in \bar{C}_n|\bar{C}_m(k_i)$ and two $\circ$-frames H_i and H'_i such that H'_i is tighter than H_i. The cube α_i is the composition $([a^i, b^i])|\mathrm{id} \circ \beta_i$ and H_i the composition $([a^i, b^i]) \circ h_i$, where

$[a^i, b^i]$ is the i-th interval of h. By lemma 6.6 there exists an $h'_i \in \bar{C}_n$ such that

- $H'_i = ([a^i, b^i]) \circ h'_i$,
- h'_i is a $\circ$-frame of β_i and
- h_i is a frame of h'_i.

The second property implies that h'_i is a frame of h_i. Together with the third property and lemma 4.5 this implies that h_i and h'_i coincide up to permutation. Hence we can assume that they are equal. Therefore we have $H_i = H'_i$. This again implies that H and H' are equal up to a permutation. Hence H is a minimal $\circ$-root.

The uniqueness of H is an immediate consequence of lemma 4.5. □

Definition 6.8. We call the (up to permutation) unique root of theorem 6.7 the *minimal $\circ$-root*. We define a *minimal $\bullet$-root* analogously.

Definition 6.9. A reduced representation T of $\alpha \in \bar{C}_n|\bar{C}_m$ is called *minimal*, if every vertex is a minimal root of the element represented by the subtree with the vertex as root.

The algorithm for the construction of a minimal representation is quite clear. We choose the color of the root, construct the minimal root of this color, and then recursively construct the minimal representations of the subtrees whose root has the other color. Since the minimal roots are uniquely determined (up to permutation) we obtain the following

Proposition 6.10. *There exists an (up to permutations) uniquely determined minimal $\circ$-representation for each $\alpha \in \bar{C}_n|\bar{C}_m(j)$-*

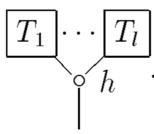
Remark 6.11. With ¨up to permutations¨ we mean one permutation for each vertex of the tree.

Theorem 6.12. *The images of an arbitrary reduced $\circ$-representation $T \in F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ of $\alpha \in \bar{C}_n|\bar{C}_m(j), j > 0$, and the minimal $\circ$-representation under the projection $p : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n \otimes \bar{C}_m$ coincide.*

Proof. First recall, that the application of a permutation to a vertex of a tree in $F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ does not change the image under p. Hence we can ignore ambiguities which occur when choosing permutations.

For $j = 1$ the reduced $\circ$-representation is minimal. Hence the statement is trivial.

If α has more than one input and its minimal $\circ$-root h_{min} has more than one input, then, as seen in the proof of theorem 6.7, h_{min} is the composition of the root h of T and the minimal $\circ$-roots h_i of the $\beta_i := pr(T_i)$, where T is of the form

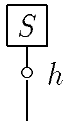


.

Since each T_i has at least one input and less than j, the statement follows by induction.

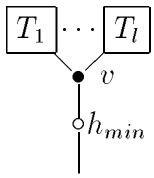
If α has more than one input and its $\circ$-root only has one input, then its $\bullet$-root has more than one input (follows from lemma 6.4). As above we can prove that each reduced $\bullet$-representation has the same image as the minimal $\bullet$-representation, if the minimal $\bullet$-root has more than one input.

T has to be of the form

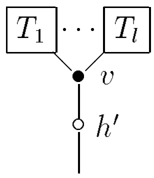


where S is a $\bullet$-representation of an element $\beta \in \bar{C}_n|\bar{C}_m(j)$. The minimal $\bullet$-root of T has more than one input (otherwise the minimal $\circ$-root of T has to have more than one). Thus $p(S) = p(S_{min})$, where S_{min} is the minimal $\circ$-representation of β.

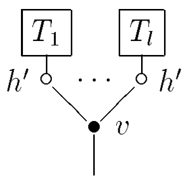
Let the minimal $\circ$-representation T_{min} of α be of the form



with $l > 1$. We know that h is a frame of the minimal $\circ$-root h_{min} of α. By corollary 4.8 we can find an $h' \in \bar{C}_n(1)$ such that $h_{min} = h \circ h'$. Thus the tree



is a representation of β. Since the interchange condition holds, the image of this tree coincides with the image of the tree T', which is of the form



which is a $\bullet$-representation of β. Hence we now that it image under p coincides with the one of S_{min} and hence with the one of S. Since p is a map of operads, it follows that the images of T and T_{min} coincide. □

The construction of a reduced from an arbitrary representation in lemma 5.3 shows that their image under p coincide. Hence we obtain the following

Corollary 6.13. *Let T and S be two $\circ$-representations of $\alpha \in \bar{C}_n|\bar{C}_m(j)$ for $j < 0$. Then $p(T) = p(S) \in \bar{C}_n \otimes \bar{C}_m(j)$.*

7. The tensor product of little cubes

Now we use the minimal representations, to construct a homeomorphism between the two operads $\bar{C}_n|\bar{C}_m$ and $\bar{C}_n \otimes \bar{C}_m$. We then show, that $C_n \otimes C_m$ is locally Σ-equivalent to $\mathcal{C}_{n+m}$. One direction of the homeomorphism, namely $\bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$, is already known. The minimal representations make it possible to construct an inverse map.

Theorem 7.1. *The morphism $\varphi : \bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$ is a homeomorphism of operads.*

Proof. First we construct an inverse map $\psi_j : \bar{C}_n|\bar{C}_m(j) \to \bar{C}_n \otimes \bar{C}_m(j)$ for each $j \geq 0$. For $j = 0$ the map is trivial, since both spaces are. For $j > 0$ we choose ψ to be given by $\psi(x) := p(T)$ where $T \in F(\bar{C}_n \sqcup_\Sigma \bar{C}_m)$ is a $\circ$-representation of x, and $p : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n \otimes \bar{C}_m$ and $q : F(\bar{C}_n \sqcup_\Sigma \bar{C}_m) \to \bar{C}_n|\bar{C}_m$ are the projections. Let T and T' be two $\circ$-representations of $x \in \bar{C}_n|\bar{C}_m(j)$. By corollary 6.13 this implies $p(T) = p(T')$ and hence ψ is well-defined for each j.

Since $\varphi \circ p = q$ holds, we have

$$\psi \circ \varphi \circ p(T) = \psi \circ q(T) = p(T)$$

for each $\circ$-representation T. Every element of $\bar{C}_n \otimes \bar{C}_M$ has a $\circ$-representation and this implies $\psi \circ \varphi = \mathrm{id}$. On the other hand we have

$$\varphi \circ \psi \circ q(T) = \varphi \circ p(T) = q(T).$$

which leads to $\varphi \circ \psi = \mathrm{id}$. Hence φ and ψ are bijective maps of set operads.

It remains to prove that ψ is continuous. By corollary 5.5 $\bar{C}_n \otimes \bar{C}_m(j)$ is compact. Since φ is continuous and bijective and $\bar{C}_n|\bar{C}_m(j)$ is a Hausdorff-space, φ is a homeomorphism. □

Since $\varphi : \bar{C}_n \otimes \bar{C}_m \to \bar{C}_n|\bar{C}_m$ maps non-degenerated cubes, i.e. elements of $C_n \otimes C_m$, surjectively to non-degenerate cubes, i.e. elements of $C_n|C_m$, we obtain

Corollary 7.2. *$\varphi : C_n \otimes C_m \to C_n|C_m$ is a homeomorphism of operads.*

Up to this point we just examined tensor products of two little cubes operads. But a closer look at the results of the preceding sections reveals, that it is possible to adapt the proof to the tensor product of three or more little cubes. In the following we just give a short overview over the necessary changes.

In the combinatoric part, i.e. the construction of the minimal representations, we just have to use multi-colored trees instead of trees with only two colors, i.e. we need one color i for each factor C_{n_i} in the tensor product $C_{n_1} \otimes \cdots \otimes C_{n_l}$. In addition we have to modify the notion of reduced representations. They still are not allowed to have monochrome edges and vertices of valence 0. But they are allowed to have sequences of valence 1 of a length less than l such that each vertex of this sequence has a different color. As in the bicolored case we can find a reduced representation for each element of $\bar{C}_{n_1}|\dots|\bar{C}_{n_l} \subset \bar{C}_n$ with $n = n_1 + \dots + n_l$.

With this modifications the results of sections 5 and 6 remain valid. We just have to take the increased number of colors into account. Basically this results in more bookkeeping. But we still obtain

Corollary 7.3. *The map $\varphi : C_{n_1} \otimes \dots \otimes C_{n_l} \to C_{n_1}|\dots|C_{n_l}$ is a homeomorphism of operads for each $l \geq 2$ and each choice $n_1, \dots, n_l \in \mathbb{N}$*

In the following we use the suboperad of decomposable cubes $D_n(j) \subset C_n(j)$, to obtain our final result.

Lemma 7.4. *$D_n|D_m(j)$ is precisely the space $D_{n+m}(j)$.*

Proof. For $j = 0, 1$ the spaces $D_n(j)$ and $C_n(j)$ coincide. Hence the equality of $D_n|D_m(j)$ and $D_{n+m}(j)$ follows directly from the fact that $C_n|C_m(j) = C_{n+m}(j)$ for $j = 0, 1$.

An element α of $C_{n+m}(j), j \geq 2$, is decomposable, if and only if there exists an $i \in \{1, \dots, n+m\}$ and a hyper plane orthogonal to the i-th axis, which separates α into two non-trivial parts (i.e. parts with at least on input each). This is equivalent to the existence of $\beta \in C_{n+m}(2)$ of the form

$$\beta = \Big(\big[(0, \dots, 0), (1, \dots, \underset{i}{r}, \dots, 1)\big], \big[(0, \dots, \underset{i}{r}, \dots, 0), (1, \dots, 1)\big] \Big)$$

and $\beta_k \in D_{n+m}(j_k)$ for $k = 1, 2$ such that $1 \leq j_k < j$ and $\alpha = \beta \circ (\beta_1, \beta_2)$. Obviously β is an element of $C_n|C_m(2)$. It even is of one of the forms $h|\mathrm{id}$ or $\mathrm{id}|v$ with $h \in D_n(2) = C_n(2)$ or $v \in D_m(2) = C_m(2)$, depending whether i is less or equal to n or not. Hence we see, by induction over the number of inputs of α, that $D_{n+m}(j)$ is a subspace of $D_n|D_m(j)$ for each j.

On the other hand each element of the form $h|\mathrm{id}$ with $h \in D_n(j)$ is decomposable in $C_{n+m}(j)$ and the same holds for $\mathrm{id}|v$ for $v \in D_m(j)$. This implies that each element of $D_n|D_m(j)$ is a composition of decomposable elements in $C_{n+m}(j)$ and hence itself decomposable, what again leads to $D_n|D_m(j) \subset D_{n+m}(j)$ for each j. □

Corollary 7.5. *$D_{n_1}|\dots|D_{n_l}(j) \subset C_{n_1}|\dots|C_{n_l}(j)$ is precisely the space $D_n(j) \subset C_n(j)$ for all $j, n_1, \dots, n_l \in \mathbb{N}, l \geq 2$ and $n = n_1 + \dots + n_l$.*

Lemma 7.6. *For each $j \in \mathbb{N}, l \geq 2$ and all $n_1, \dots, n_l \in \mathbb{N}$ the space $D_{n_1}|\dots|D_{n_l}(j)$ is a Σ_j-equivariant deformation retract of $C_{n_1}|\dots|C_{n_l}(j)$.*

Proof. Since $D_n(j) = C_n(j)$ for all n and $j = 0, 1, 2$, the statement is trivial in this cases.

Now let $n = n_1 + \cdots + n_l$. Following [Dun88, Lem. 2.2.] there exists an equivariant deformation retraction $h : I \times C_n(j) \to C_n(j)$ of $C_n(j)$ to $D_n(j)$ and a map $u : C_n \to I$ such that

$$h(s, \alpha) = \alpha \circ \big(\beta_{su(\alpha)}, \ldots, \beta_{su(\alpha)}\big),$$

where $\beta_t \in C_n(1)$ is of the form

$$\beta_t = \left(\left[\frac{1}{2}t, 1 - \frac{1}{2}t \right]^n \right).$$

h maps $C_{n_1}|\ldots|C_{n_l}(j)$ into itself, because $C_{n_1}|\ldots|C_{n_l}(1) = C_n(1)$. Together with the equality of $D_{n_1}|\ldots|D_{n_l}(j)$ and $D_n(j)$, this implies the statement. □

Putting together all the collected pieces, we obtain the diagram

$$\begin{tikzcd}
C_{n_1} \otimes \cdots \otimes C_{n_l}(j) \arrow[r] \arrow[dd, "\varphi"'] & C_n(j) \\
& D_n(j) \arrow[u, hook, "\simeq"'] \arrow[d, equal] \\
C_{n_1}|\ldots|C_{n_l}(j) \arrow[uur] & D_{n_1}|\ldots|D_{n_l}(j) \arrow[l, "\simeq"']
\end{tikzcd}$$

for each $j \geq 0$. Since all maps, except for the diagonal and the map at the top, are known to be either homeomorphisms or local Σ-equivalences, we obtain

Main Theorem. *The operad-map $C_{n_1} \otimes \cdots \otimes C_{n_l} \to C_n$ is a local Σ-equivalence for all $l \geq 2, n_1, \ldots, n_l \in \mathbb{N}$ and $n = n_1 + \cdots + n_l$.*

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