Constructions of Hereditary Abelian Categories with Serre Duality using Ray Quivers and Ordered Sets

Claus Michael Ringel

Let k be a field. The categories to be considered are additive k-categories (always with finite dimensional Hom and Ext-spaces). In the case of k being algebraically closed, Reiten and Van den Bergh [RV] recently have determined all hereditary abelian k-categories $\mathcal{C}$ which are noetherian and satisfy Serre duality. Of particular interest seem to be those categories $\mathcal{C}$ which in addition are connected and have at least one indecomposable projective object (the case (c) of [RV]). These categories $\mathcal{C}$ have been constructed by Reiten and Van den Bergh in two different ways, our aim is to present a third way. In contrast to Reiten and Van den Bergh, the construction to be presented here does not invoke derived categories, instead we will use a slight modification (or extension) of the notion of a quiver and its representation, namely that of a "ray quiver". Whereas the usual quivers serve to describe the spaces $\mathrm{Ext}^1(S,T)$ where S,T are simple objects, the ray quivers are a first attempt to take care of some additional Ext^1-spaces, namely $\mathrm{Ext}^1(S,U)$, where S is still simple, wheras U is a suitable uniform object without socle. The ray quiver construction of the categories $\mathcal{C}$ considered by Reiten and Van den Bergh provides additional information on the structure of these categories.

Ray categories can also be used in order to construct the categories of type (d) in [RV]. Here, we start with the category of all representations of two suitable ray quivers and obtain the required category as a full subcategory: what we do is a typical localization procedure.

It has been asked by Reiten and Van den Bergh [RV,Re] whether any hereditary abelian k-category with Serre duality may be derived equivalent to a noetherian one. The last section is devoted to construct many counter examples by considering representations of ordered sets. All the categories constructed there will be directed.

Acknowledgement. The author is indebted to Henning Krause and Otto Kerner for questions and remarks which stimulated these investigations.

Part I. Ray quivers.

1. Definition.

A linearly oriented quiver of type A_∞ with a source is usually called a *ray,* one with a sink a *coray.* We will say that a linearly oriented quiver of type A_∞^∞ is a *biray.* Thus, the upper part of the following picture shows to the left a ray, to the right a coray, and the lower part is a biray:

$$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \xrightarrow{\alpha_3} \cdots \qquad\qquad \cdots \xrightarrow{\gamma_3} \circ \xrightarrow{\gamma_2} \circ \xrightarrow{\gamma_1} \circ$$

$$\cdots \xrightarrow{\beta_{-1}} \circ \xrightarrow{\beta_0} \circ \xrightarrow{\beta_1} \circ \xrightarrow{\beta_2} \cdots$$

All the quivers we are dealing with will be assumed to be locally finite and directed (locally finite means that any vertex has only finitely many neighbors, and directed means that there are no cyclic paths). Given such a quiver, we will be interested in the subquivers which are rays (thus in infinite paths with a starting point), or better in ray classes: two rays will be said to be *equivalent* provided they differ by only finitely many vertices and arrows, the equivalence classes are called *ray classes.* We will use labels for the ray classes: given a ray with arrows α_i as above, we may use the symbol $\alpha_\bullet$ as its label; in a graphical presentation, we will use instead of the symbol $\alpha_\bullet$ a small square box $\square$ and put it besides the dots needed to mark a ray; thus it will look like $\circ \to \circ \to \cdots \square$; we call these squares *ray class boxes.* A *ray quiver* is given by the following data $Q = (Q_0, Q_1, Q_\square, Q_s)$, here (Q_0, Q_1) is a (usually non-connected) directed quiver, the set $Q_\square$ serves to label all the ray classes in (Q_0, Q_1), and the elements of Q_s are pairs $(x, \alpha_\bullet)$ where $x \in Q_0$ and $\alpha_\bullet \in Q_\square$; these elements of Q_s are called *special* arrows and are drawn in the form $x \to \alpha_\bullet$ or just $\circ \to \square$. Examples of ray quivers will be seen below.

Throughout the paper we will fix a field k, all vector spaces will be k-spaces. In order to define representations of ray quivers, we note the following: given a ray $\alpha_\bullet$ in any quiver and a representation V of this quiver, we can form the direct limit $V(\alpha_\bullet)$ along the ray, such direct limits will now be used. Note that the direct limit only depends on the ray class. A *representation* of the ray quiver $Q = (Q_0, Q_1, Q_\square, Q_s)$ is of the form $V = (V(x), V(\gamma))$, where $x \in Q_0$ and $\gamma \in Q_1 \cup Q_s$. As usual, $V(x)$ is a k-space, for every vertex $x \in Q_0$, and $V(\gamma)\colon V(x) \to V(y)$ is a k-linear map, for every arrow $\gamma\colon x \to y$ in Q_1. Given a special arrow $\gamma\colon x \to \alpha_\bullet$, then $V(\gamma)$ is required to be a k-linear map

$$V(x) \to \varinjlim V(\alpha_\bullet).$$

Note that the special arrows do not occur in any ray (by definition, but also since our ray class boxes are really sinks). Thus, in order to specify a representation of Q, we first may deal with the k-spaces $V(x)$ for $x \in Q_0$, and with all the maps

$V(\gamma)$, for $\gamma \in Q_1$, then we determine the direct limits for the rays and finally we consider the special arrows. Given two representations V, V' of the ray quiver Q, a map $f\colon V \to V'$ is given by linear maps $f(x)$ for all $x \in Q_0$, with the usual commutative diagrams for the arrows in Q_1, and with corresponding commutative diagrams for the special arrows $\gamma\colon x \to \alpha_\bullet$ in Q_s (here we use that f induces for any ray $\alpha_\bullet$ a direct limit map $\lim_\to V(\alpha_\bullet) \longrightarrow \lim_\to V'(\alpha_\bullet)$).

A representation V of the ray quiver Q is said to be *finitely generated* provided there is a finite subset $I \subseteq Q_0$ and for every $x \in I$ a finite dimensional subspace $U(x) \subseteq V(x)$ with the following property: for any vertex $y \in Q_0$ the space $V(y)$ is generated by the subspaces $w(U(x))$ where w runs through all paths in (Q_0, Q_1) starting from any vertex $x \in I$ and ending in y (note that in this definition, the special arrows are not considered at all). A representation V is finitely generated if and only if for any family V_i of subobjects of V with $V = \sum_{i \in I} V_i$, there is a finite subset $I' \subseteq I$ with $V = \sum_{i \in I'} V_i$. The category of finitely generated representations of the ray quiver Q will be denoted by $\operatorname{mod}(Q, k)$.

Remark. We write $\operatorname{mod}(Q, k)$ and not, as one could expect, the usual $\operatorname{mod} kQ$, since this would suggest that there exists something like a "path algebra" kQ, but this is not the case. There is another (and related) warning: Given a subset $I \subseteq Q_0$ and for every $x \in I$ a subspace $U(x) \subseteq V(x)$, in general one **cannot** define the "submodule generated by these subspaces". Consider for example the following ray quiver

$$\begin{tikzcd} & & & & \circ^{y} \arrow[d, "\gamma"] \\ \underset{x_0}{\circ} \arrow[r, "\alpha_1"] & \underset{x_1}{\circ} \arrow[r, "\alpha_2"] & \underset{x_2}{\circ} \arrow[r, "\alpha_3"] & \cdots & \underset{\alpha_\bullet}{\square} \end{tikzcd}$$

and its representation V with $V(x) = k$ for all vertices x and $V(\alpha)$ being the identity map for all arrows α (including the special arrow γ). The subspace $U(y) = k$ of $V(y)$ will not generate a subrepresentation of V.

Of course, this example illustrates very well the problem of dealing with intersections of chains of subobjects $V = V^{(0)} \supset V^{(1)} \supset V^{(2)} \supset \cdots$. Here, take for $V^{(i)}$ the submodule generated by $V(x_i)$ and $V(y)$, this submodule exists, but the only subobject contained in all $V^{(i)}$ is the zero object.

For the further considerations, it will be sufficient to invoke the following quite obvious principle for constructing at least some subrepresentations:

Lemma 1. *Let I be a subset of Q_0 with the following two properties: First, if $\alpha\colon x \to y$ is an arrow and $x \in I$, then $y \in I$. Second, every ray class contains a ray with all the vertices in I. Then, given a representation V of Q, the choice $V_I(x) = V(x)$ for $x \in I$ and $V_I(x) = 0$ for $x \notin I$, yields a subrepresentation V_I of V.*

Given the category $\mathcal{C} = \operatorname{mod}(Q,k)$, we want to recover information concerning the ray quiver Q. First of all, for every vertex $x \in Q_0$, we may define a corresponding simple representation S_x with $S_x(x) = k$ and $S_x(y) = 0$ for $x \neq y \in Q_0$ (and such that all the maps $S_x(\alpha)$ are zero maps, but there is no other choice, since there are no loops). Of course, for $x \neq y$, the representations S_x and S_y are not isomorphic. Also, we obtain in this way all the simple representations (up to isomorphism). Namely, if $V \neq 0$ is a representation, choose some vertex x with $V(x) \neq 0$. Now, let I be the set of vertices y of Q with no path from y to x and let I' be the set of vertices y of Q with no path of length at least 1 from y to x (thus I is obtained from I' by deleting x). Then according to Lemma 1, V has submodules $V_I \subset V_{I'}$ and $(V_{I'}/V_I)(x) = V(x)$, whereas $(V_{I'}/V_I)(y) = 0$ for all $y \neq x$. As a consequence, $V_{I'}/V_I$ is a non-zero direct sum of copies of S_x. In particular, if V is simple and $V(x) \neq 0$, then, up to isomorphism, $V = S_x$.

We have seen that we can recover the set Q_0 from $\mathcal{C} = \operatorname{mod}(Q,k)$. Of course, the k-dimension of $\operatorname{Ext}^1(S_x, S_y)$ yields the number of arrows $x \to y$; this shows that (Q_0, Q_1) is just the ordinary quiver of $\mathcal{C}$ (with vertices corresponding to the isomorphism classes of the simple objects, and such that the arrows yield the Ext^1-groups for the simple objects). So what about the boxes and the special arrows? We restrict the discussion to the case when the rays are *isolated:* by this we mean that any two non-equivalent rays have at most finitely many vertices in common. In this case, the ray classes correspond bijectively to equivalence classes of uniform objects without socle: recall that an object in an abelian category is said to be *uniform* provided the intersection of any two non-zero subobjects is non-zero again; we call two uniform objects U_1, U_2 *equivalent* provided there is a uniform object U_3 with monomorphisms $U_3 \to U_1$ and $U_3 \to U_2$. Given any uniform object U and a simple object S_x, we may look for the dimension of $\lim\limits_{\leftarrow i} \operatorname{Ext}^1(S_x, U_i)$, where U_i runs through the non-zero subrepresentations of U. In this way, we recover the number of special arrows from x to the ray class box corresponding to U.

2. The categories (c) considered by Reiten and Van den Bergh.

The aim of this section is to present a new way for constructing all the noetherian hereditary abelian k-category with Serre duality which contain non-zero projective objects. Recall that Reiten and Van den Bergh have classified, for k algebraically closed, all the noetherian hereditary abelian k-category with Serre duality, and the categories which contain non-zero projective objects said to be of type (c).

Construction. The following ray quiver will be the essential ingredient in our further investigation, we call it a *ray-coray.*

$$(*) \qquad \overset{a_0}{\circ} \xrightarrow{\alpha_1} \overset{a_1}{\circ} \xrightarrow{\alpha_2} \overset{a_2}{\circ} \xrightarrow{\alpha_3} \cdots \quad \square \xleftarrow{\gamma_1} \overset{c_1}{\circ} \xleftarrow{\gamma_2} \overset{c_2}{\circ} \xleftarrow{\gamma_3} \cdots$$

Let Δ be a connected quiver without paths of infinite length (of course, as all our quivers, also directed and locally finite). Let $\mathbf{m} : \Delta_0 \to \mathbb{N}_0$ be a function. To every vertex x attach $\mathbf{m}(x)$ (otherwise disjoint) ray-corays $(*)$, always identifying the source a_0 with x. The ray quiver obtained in this way will be denoted by $\Delta^{\mathbf{m}}$.

Theorem A. *Let Δ be a connected quiver without paths of infinite length and $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$ a function. The category $\mathcal{C} = \operatorname{mod}(\Delta^{\mathbf{m}}, k)$ is a connected noetherian hereditary abelian k-category with Serre duality, with non-zero projective representations. If k is algebraically closed, then any category $\mathcal{C}$ with these properties is obtained in this way.*

The proof will be given in the next two sections. Before we start with the proof, let us discuss whether it is possible to recover the given quiver Δ from $\Delta^{\mathbf{m}}$. This is often possible, the only exception is the following: Assume that x is a sink of Δ, that precisely one arrow $x' \to x$ ends in x, and that $\mathbf{m}(x) = 1$. In this case, we can delete the vertex x as well as the arrow $x' \to x$ from Δ and increase $\mathbf{m}(x')$ by 1, and we will obtain the same result. In order to exclude this ambiguity, we can assume that the function $\mathbf{m}$ is *reduced* in the following sense: If x is a sink of Δ and precisely one arrow ends in x, then $\mathbf{m}(x) \neq 1$. Using this additional assumption, the quiver Δ is uniquely determined by the ray quiver $\Delta^{\mathbf{m}}$: *Given a reduced function $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$, then a vertex x of $\Delta^{\mathbf{m}}$ does not belong to Δ if and only if the following property is satisfied: precisely one arrow starts at x and if we remove this arrow, then we obtain the disjoint union of a ray quiver with at least two vertices and of a coray or a copy of $(*)$.* Since Δ is a convex subquiver of $\Delta^{\mathbf{m}}$, it is sufficient to know its vertices in order to recover it from $\Delta^{\mathbf{m}}$. On the other hand, it is sufficient to work with reduced functions: *For every function $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$ there is a ray quiver Δ' and a reduced function $\mathbf{m}'\colon \Delta'_0 \to \mathbb{N}_0$ such that $\Delta^{\mathbf{m}} = (\Delta')^{\mathbf{m}'}$.*

Let Δ be a ray quiver and $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$ a reduced function, then the ray subquivers of $\Delta^{\mathbf{m}}$ of the form $(*)$ used in the construction will be called the *normalized* ray-corays. These are the ray-corays with $a_0 \in \Delta_0$, but $a_1 \notin \Delta_0$. Since we assume that $\mathbf{m}$ is reduced, a ray subquiver of $\Delta^{\mathbf{m}}$ of the form $(*)$ is normalized if and only if the following conditions are satisfied: (1) If a_0 is endpoint of precisely one arrow, then it is starting point of at least two arrows. (2) The only arrows starting or ending in the vertices a_i and c_i with $i \geq 1$, or ending in the ray class box $\alpha_\bullet$ are the given arrows α_j and γ_j.

3. The category $\mathcal{C}$ is a noetherian hereditary abelian category.

Let Δ be a quiver without infinite paths and $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$ a function, and let us assume that $\mathbf{m}$ is reduced. Let $\mathcal{C} = \operatorname{mod}(\Delta^{\mathbf{m}}, k)$.

Recall that the ray quiver $\Delta^{\mathbf{m}}$ was obtained from Δ by attaching the ray-coray quiver $(*)$. We will need similarly defined quivers obtained from Δ by attaching only the ray part (this quiver will be denote by $\Delta^{\mathbf{mr}}$) or by attaching suitably oriented quivers of type A_{2i+1} (this quiver will be denoted by $\Delta^{\mathbf{m},i}$). Thus, $\Delta^{\mathbf{mr}}$ is the ray quiver obtained from Δ by attaching to every vertex x precisely $\mathbf{m}(x)$ rays:

$$\overset{a_0}{\circ} \xrightarrow{\alpha_1} \overset{a_1}{\circ} \xrightarrow{\alpha_2} \overset{a_2}{\circ} \xrightarrow{\alpha_3} \cdots \ \square$$

always identifying the source a_0 with x (we could delete the boxes and consider $\Delta^{\mathbf{mr}}$ just as an ordinary quiver). Let $\mathcal{C}' = \operatorname{mod}(\Delta^{\mathbf{mr}}, k)$, of course we may consider this (in a canonical way) as a full subcategory of $\mathcal{C}$.

Remark. When dealing with the representations of $\Delta^{\mathbf{mr}}$, the ray class boxes do not play any role; the corresponding quiver (obtained by deleting the ray class boxes) is the one considered by Reiten and Van den Bergh [RV] as the starting quiver for constructing the category $\mathcal{C}$.

Lemma 2. *Each representation V of $\Delta^{\mathbf{m}}$ has a largest subrepresentation V' which belongs to $\mathcal{C}'$ and V/V' is finite dimensional.*

Proof: The first assertion follows directly from Lemma 1. In order to see that V/V' is of finite length, note that $(V/V')(x)$ is non-zero only for vertices $x = c_i$ lying on the coray-part of a ray-coray. Thus V/V' lies on a disjoint union of corays, thus its support is a disjoint union of finitely many quivers of type A_n.

Lemma 3. *If x is a vertex of $\Delta^{\mathbf{mr}}$, then the simple representation S_x has a projective cover and an injective envelope in $\mathcal{C}$ and both lie already in $\mathcal{C}'$.*

Proof. Let x belong to $\Delta^{\mathbf{mr}}$. Clearly, S_x has a projective cover and also an injective envelope in $\mathcal{C}'$. Let P_x be the projective cover of S_x in $\mathcal{C}'$. Let V, W be representations of $\Delta^{\mathbf{m}}$ and let $f\colon V \to W$ be a surjective map. Let I be the set of vertices of $\Delta^{\mathbf{mr}}$. Any map $P_x \to W$ factors through W_I, and if $f\colon V \to W$ is surjective, then also the induced map $f_I\colon V_I \to W_I$ is surjective. Since P_x is projective in $\mathcal{C}'$, the map $P_x \to W_I$ can be lifted to a map $P_x \to V_I$.

On the other hand, let I_x be the injective envelope of S_x in $\mathcal{C}'$. Note that I_x is of finite length. We want to show that I_x remains injective in $\mathcal{C}$. Given a representation V of Q, let V' be its maximal subrepresentation in $\mathcal{C}'$. Then $\operatorname{Ext}^1(V, I_x) = 0$. Note that $\operatorname{Ext}^1(S, I_x) = 0$ for all simple representations, since I_x is of finite length and $\Delta^{\mathbf{mr}}$ is a connected component of the ordinary quiver (Q_0, Q_1). Since V/V' is of finite length, it follows that $\operatorname{Ext}^1(V/V', I_x) = 0$. Altogether we see that $\operatorname{Ext}^1(V, I_x) = 0$.

Note that the indecomposable projective objects P_x with x a vertex of $\Delta^{\mathbf{mr}}$ are noetherian. Since all the objects in $\mathcal{C}'$ have a projective cover, we see immediately that the category $\mathcal{C}'$ is a noetherian hereditary abelian category. (Of course, this is well-known and has been used also in [RV].) For us, the following consequence of the previous results is of interest:

Corollary 1. *If V is in $\mathcal{C}$, then $V(\alpha)$ is bijective for almost all arrows α.*

Proof: Let V' be the largest subrepresentation of V which belongs to $\mathcal{C}'$. It is sufficient to show that both maps $V'(\alpha)$ and $(V/V')(\alpha)$ are bijective for almost all arrows α. Since V/V' is of finite length, we only have to consider V'. Now, the objects in $\mathcal{C}'$ have projective covers, let $p\colon P \to V'$ be a projective cover, and P' the kernel of p. Note that both P, P' are finite direct sums of modules of the form P_x with x a vertex in $\Delta^{\mathbf{mr}}$, thus we only have to consider the case of $V = P_x$. Since P_x is projective, all the maps $P_x(\alpha)$ have to be injective. Let us consider an arrow $\alpha\colon y \to z$ such that $P_x(\alpha)$ is not surjective. First of all, this happens in case $z = x$, but there are only finitely many arrows ending in x. Second, let us assume that $z \neq x$. In this case, there has to exist another arrow which also ends in z, say $\beta\colon y' \to z$, and such that there is a path from x to y'. Since there are at least two arrows which end in z, we see that z has to belong to Δ (all the additional vertices in the ray-corays are endpoint of precisely one arrow). But inside Δ there are only finitely many paths starting in x (since Δ is locally finite and has no infinite path), and therefore there are only finitely many possibilities for α.

We denote by $\mathcal{C}_i$ the full subcategory of all finitely generated representations V of $\Delta^{\mathbf{m}}$ such that for any normalized ray-coray $(*)$, the map $V(\alpha_s)$ is an isomorphism and $V(c_s) = 0$ for all $s > i$. It is eays to see that the category $\mathcal{C}_i$ is equivalent to the category of representations of the quiver $\Delta^{\mathbf{m},i}$ obtained from Δ by attaching to every vertex x precisely $\mathbf{m}(x)$ copies of the following quiver, always identifying the source a_0 with x:

$$(**) \qquad \overset{a_0}{\circ} \xrightarrow[\alpha_1]{} \overset{a_1}{\circ} \cdots \overset{a_{i-1}}{\circ} \xrightarrow[\alpha_i]{} \overset{a_i}{\circ} \xleftarrow[\gamma_1]{} \overset{c_1}{\circ} \cdots \circ \xleftarrow[\gamma_i]{} \overset{c_i}{\circ}$$

To be more precise, define functors

$$F_i\colon \Delta^{\mathbf{m},i} \to \mathcal{C}$$

as follows: Given a representation V of $\Delta^{\mathbf{m},i}$, let $(F_i(V))(x) = V(x)$ for all vertices $x \in \Delta_0$ and $(F_i(V))(\beta) = V(\beta)$ for every arrow $\beta \in \Delta_1$. Also, given a normalized ray-coray $(*)$, let $(F_i)(V))(x) = V(x)$ for $x = a_s$ with $s \leq i$, and for $x = c_s$ with $s \leq i$; let $(F_i)(V))(a_s) = V(a_i)$ for $s > i$ and $(F_i)(V))(c_s) = 0$ for $s > i$; similarly, let $(F_i)(V))(\beta) = V(\beta)$ for $\beta = \alpha_s$ and $\beta = \gamma_s$ with $s \leq i$, finally, let $(F_i)(V))(\alpha_s)$ be the identity map for $s > i$ (and, of course, $(F_i)(V))(\gamma_s)$ the zero map for $s > i$). Clearly, *the functor F_i is a full exact embedding and its image is dense in $\mathcal{C}_i$.*

The quiver $\Delta^{\mathbf{m},i}$ has no paths of infinite length, thus $\operatorname{mod}(\Delta^{\mathbf{m},i}, k)$ is a hereditary abelian k-category.

Corollary 2.

$$\mathcal{C} = \bigcup_i \mathcal{C}_i.$$

Proof: This is an immediate consequence of Corollary 1 and Lemma 2.

Corollary 3. *The category $\mathcal{C}$ is a noetherian hereditary abelian k-category.*

Proof: Since all the categories $\mathcal{C}_i$ are hereditary abelian k-categories, the same is true for the union: all the conditions to be checked involve only finitely many objects. That $\mathcal{C}$ is noetherian follows from Lemma 2: Given any object V, there is a unique maximal subobject V' which belongs to $\mathcal{C}'$ and V/V' has finite length. But, as we know, the category $\mathcal{C}'$ is noetherian, thus V' is noetherian. Since V/V' is of finite length, it follows that also V is noetherian.

4. Serre duality for $\mathcal{C}$.

Recall that the simple objects in $\mathcal{C} = \operatorname{mod}(\Delta^{\mathbf{m}}, k)$ are the one-dimensional representations and that they are of the form S_x, with x a vertex of $\Delta^{\mathbf{m}}$.

Proposition 1. *The following assertions are equivalent for a vertex x of $\Delta^{\mathbf{m}}$*

(i) *S_x has a projective cover.*
(ii) *S_x has an injective envelope.*
(iii) *x is a vertex of $\Delta^{\mathbf{mr}}$.*

Proof. According to Lemma 3, we know that (iii) implies both (i) and (ii). Conversely, let us show that for a vertex x of $\Delta^{\mathbf{m}}$ which does not belong to $\Delta^{\mathbf{mr}}$, the simple representation S_x has neither a projective cover nor an injective envelope.

The vertex x is of the form $x = c_t$ for some t and some ray-coray $(*)$. First, assume that S_x has a projective cover P_x in $\mathcal{C}$. There is an i such that P_x belongs to $\mathcal{C}_i$. Of course, P_x is the projective cover of S_x also in $\mathcal{C}_i$. But then we know precisely the structure of P_x, namely P_x is a representation of Q such that $P_x(a_j) = k$ for $j \geq i$ and $P_x(c_j) = k$ for all $j \leq t$, whereas $P_x(y) = 0$ otherwise. However, such a representation is not local, it has two different simple factor modules, namely S_x and also S_{a_i}. This is impossible.

Next, assume that S_x has an injective envelope I_x in $\mathcal{C}$. Then I_x is an indecomposable representation with socle S_x. It is easy to see that any representation with socle S_x is of finite length and uniserial, say with top S_{c_i} for some vertex c_i belonging to the given coray. But then $\operatorname{Ext}^1(S_{c_{i+1}}, I_x) \neq 0$, impossible.

We know in this way that there is a bijection ν between the indecomposable projective modules P and the indecomposable injective modules $I = \nu(P)$ such that $P/\operatorname{rad} P$ is isomorphic to $\operatorname{soc} I$. Thus in order to know that the category $\mathcal{C}$ has Serre duality, it remains to be shown that $\mathcal{C}$ has almost split sequences (see [RV]).

We can assume that $\mathbf{m} \neq 0$, since it is well-known that for any locally finite quiver Q without infinite paths, the category $\operatorname{mod} kQ$ has almost split sequences.

Recall that the categories $\mathcal{C}_i$ with $i \in \mathbb{N}_0$ are abelian categories, and we will use their properties. First of all, note that the category $\mathcal{C}_i$ has almost split sequences, since $\Delta^{\mathbf{m},i}$ is a locally finite quiver without infinite paths. Let us denote the Auslander-Reiten translation in $\mathcal{C}_i$ by τ_i and the inverse by τ_i^-.

Lemma 4. *Let $0 \le i < j$ be natural numbers. Assume V belongs to $\mathcal{C}_i$. Then both $\tau_j V$ and $\tau_j^- V$ belong to $\mathcal{C}_{i+1}$.*

Proof: We work in the category $\operatorname{mod} k\Delta^{\mathbf{m},j}$ and denote the Auslander-Reiten translation in this category by τ and its inverse by τ^-. Let V be a representation of $\Delta^{\mathbf{m},j}$. Note that $F_j(V)$ belongs to $\mathcal{C}_i$ if and only if $V(\alpha_s)$ is an isomorphism for all $s > i$ and $V(c_s) = 0$ for all $s > i$. Thus, we have to show: If $V(\alpha_s)$ is an isomorphism for all $s > i$ and $V(c_s) = 0$ for all $s > i$, then for both representations $W = \tau V$ and $W = \tau^- V$, the maps $W(\alpha_s)$ are isomorphisms for all $s > i+1$ and $W(c_s) = 0$ for all $s > i+1$.

We know that for every indecomposable representation W of $\Delta^{\mathbf{m},j}$, the maps $W(\alpha_s)$ are injective or surjective, thus in order to show that $W(\alpha_s)$ is bijective, we only have to check that $\dim W(a_{s-1}) = \dim W(a_s)$. This shows that it is sufficient to look at the dimension vector of W. Now the dimension vectors of τV and $\tau^- V$ are obtained from the dimension vector of V by applying the corresponding Coxeter transformations Φ and Φ^{-1}. Here are the calculations for such an arm: the upper line shows the labels x of the vertices, below are the dimensions $\dim(\tau V)(x)$, $\dim V(x)$, and $\dim(\tau^- V)(x)$.

$$
\begin{array}{lcccccccccccccccccccccc}
 & \cdots a_{i-1} & \to & a_i & \to & a_{i+1} & \to \cdots & a_j & \leftarrow & c_1 & \leftarrow \cdots & c_{i-1} & \leftarrow & c_i & \leftarrow & c_{i+1} & \leftarrow & c_{i+2} & \leftarrow \cdots \\
\tau V & \cdots u' & \to & u & \to & v_1 & = \cdots & v_1 & \leftarrow & v_2 & \leftarrow \cdots & v_i & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow \cdots \\
V & \cdots v_1' & \to & v_0 & = & v_0 & = \cdots & v_0 & \leftarrow & v_1 & \leftarrow \cdots & v_{i-1} & \leftarrow & v_i & \leftarrow & 0 & \leftarrow & 0 & \leftarrow \cdots \\
\tau^- V & \cdots w & = & w & = & w & = \cdots & w & \leftarrow & v_0 & \leftarrow \cdots & v_{i-2} & \leftarrow & v_{i-1} & \leftarrow & v_i & \leftarrow & 0 & \leftarrow \cdots
\end{array}
$$

with $u = v_1 + v_1' - v_0$ and suitable u', w (they depend on the dimensions $\dim V(y)$ with $y \in \Delta$).

Corollary 4. *Assume that V is indecomposable and belongs to $\mathcal{C}_{i-1}$ for some $i > 0$. If V is not projective in $\mathcal{C}$, then V is not projective in $\mathcal{C}_i$ and the almost split sequence in $\mathcal{C}_i$ ending in V is an almost split sequence in $\mathcal{C}$. If V is not injective in $\mathcal{C}$, then V is not injective in $\mathcal{C}_i$ and the almost split sequence in $\mathcal{C}_i$ starting in V is an almost split sequence in $\mathcal{C}$.*

Proof: Let V be an object of $\mathcal{C}$ which is indecomposable and not projective, and assume V belongs to $\mathcal{C}_{i-1}$ for some $i > 0$.

First, let us assume that V is projective in $\mathcal{C}_i$, say $V = F_i(P_x)$, where P_x is the indecomposable projective representation corresponding to the vertex x of $\Delta^{\mathbf{m},i}$. It follows that x cannot be one of the vertices a_i or $c_1, \ldots, c_i$ on an arm $(**)$, since V belongs to $\mathcal{C}_{i-1}$ and this means in particular that $V(a_{i-1})$ and

$V(a_i)$ have the same dimension, whereas $F_i(P_x)(a_{i-1}) = 0$ and $F_i(P_x)(a_i) = k$, for these vertices x. But for the remaining vertices x, one easily sees that $F_i(P_x)$ is projective in $\mathcal{C}$, in contrast to our assumption.

Since V is not projective in $\mathcal{C}_i$, there exists an almost split sequence $0 \to \tau_i V \to W \to V \to 0$ in $\mathcal{C}_i$. We want to see that this is an almost split sequence in $\mathcal{C}$. Since $\mathcal{C} = \bigcup_j \mathcal{C}_j$, it is sufficient to show that the sequence is almost split in any $\mathcal{C}_j$ with $j \geq i$. Let $j \geq i$. Since V belongs to $\mathcal{C}_{i-1}$, the previous lemma asserts that $\tau_j V$ belongs to $\mathcal{C}_i$. Since $\mathcal{C}_i$ is closed under extensions, the almost split sequence in $\mathcal{C}_j$ ending in V lies inside $\mathcal{C}_i$ and thus coincides with the sequence $0 \to \tau_i V \to W \to V \to 0$. This completes the proof of the first assertion.

The proof of the second assertion is similar. We consider an object V of $\mathcal{C}$ which is indecomposable and not injective, and we assume that V belongs to $\mathcal{C}_{i-1}$ for some $i > 0$. We claim that V cannot be injective in $\mathcal{C}_i$. Namely, assume that $V = F_i(I_x)$, where I_x is the indecomposable injective representation of $\Delta^{\mathbf{m},i}$ corresponding to the vertex x. Again, it follows that x cannot be one of the vertices a_i or $c_1, \ldots, c_i$ on an arm $(**)$, since $V(c_i) = 0$, whereas $F_i(P_x)(c_i) = k$ for $x \in \{a_i, c_1, \ldots, c_i\}$. But for the remaining vertices x, one easily sees that $F_i(I_x)$ is injective in $\mathcal{C}$, in contrast to our assumption.

Now take an almost split sequence $0 \to V \to W' \to \tau^- V \to 0$ in $\mathcal{C}_i$ and, as in the first part of the proof, we show that this sequence is almost split in any $\mathcal{C}_j$ with $j \geq i$. Of course, here we use again the previous lemma.

Proof of Theorem A. We have shown that the category $\mathcal{C}$ has all the required properties. Conversely, assume that k is algebraically closed. Let $\mathcal{D}$ be a connected noetherian hereditary abelian k-category with Serre duality and with non-zero projectives. We may consider the full subcategory $\mathcal{D}'$ of all objects in $\mathcal{D}$ generated by projective objects. According to [RV], this category $\mathcal{D}'$ is of the form $\operatorname{mod}(\Delta^{\mathbf{mr}}, k)$ for some connected locally-finite directed quiver Δ without infinite paths and a function $\mathbf{m}\colon \Delta_0 \to \mathbb{N}_0$. Since both categories $\mathcal{D}$ and $\operatorname{mod}(\Delta^{\mathbf{m}}, k)$ have Serre duality, any equivalence $\operatorname{mod}(\Delta^{\mathbf{mr}}, k) \simeq \mathcal{D}'$ lifts to an equivalence $\operatorname{mod}(\Delta^{\mathbf{m}}, k) \simeq \mathcal{D}$, again referring to [RV].

Remark. We may use these considerations in order to provide an effective recipe for calculating τV and $\tau^- V$ for any indecomposable representation V of $\Delta^{\mathbf{m}}$. Namely, choose $i > 0$ such that V belongs to $\mathcal{C}_{i-1}$, thus V may be considered as a representation of the quiver $\Delta^{\mathbf{m},i}$. Since $\Delta^{\mathbf{m},i}$ is a locally finite quiver without infinite paths, the usual procedures for determining τV and $\tau^- V$ work: we either may use the Auslander-Reiten constructions $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively, or else we may work with the Bernstein-Gelfand-Ponomarev reflection functors, taking into account that the corresponding Coxeter functors yield $D\operatorname{Tr}$ and $\operatorname{Tr} D$ up to the use of signs, see [Ga].

For $i \in \mathbb{N}_0$, let $\mathcal{C}_{\mathbf{r},i}$ be the full subcategory of all representations V of $\Delta^{\mathbf{m}}$ such that $V(c_s) = 0$ for all the coray vertices labeled c_s with $s > i$. Note that

$\mathcal{C}_{\mathbf{r},0} = \mathcal{C}'$. In addition, we denote by $\mathcal{C}'' = \mathcal{C}_{\mathbf{r},-1}$ the full subcategory of all finite dimensional representations in $\mathcal{C}_{\mathbf{r},0} = \mathcal{C}'$.

Corollary 5. *Let $i \in \mathbb{N}_0$. Then $\tau(\mathcal{C}_{\mathbf{r},i}) \subseteq \mathcal{C}_{\mathbf{r},i-1}$. If V belongs to $\mathcal{C}_{\mathbf{r},i+1} \backslash \mathcal{C}_{\mathbf{r},i}$, then τV belongs to $\mathcal{C}_{\mathbf{r},i} \setminus \mathcal{C}_{\mathbf{r},i-1}$.*

For $i = 0$, this means that $\tau(\mathcal{C}') \subseteq \mathcal{C}''$. In particular, $\tau(\mathcal{C}'') \subseteq \mathcal{C}''$.

Proof: This is a direct consequence of our calculations and we may reformulate the last sentence also in the following way:

Corollary 6. *Let M be indecomposable in $\mathcal{C}$ and $i \geq 0$. Then the following assertions are equivalent:*
(i) *M belongs to $\mathcal{C}_{\mathbf{r},i}$, but not to $\mathcal{C}_{\mathbf{r},i-1}$.*
(ii) *$\tau^i M$ is infinite dimensional and $\tau^{i+1} M$ is finite dimensional and belongs to $\mathcal{C}'$.*
(iii) *For any $j \in \mathbb{N}_0$, the representation $\tau^{-j} M$ belongs to $\mathcal{C}_{\mathbf{r},i+j}$, but not to $\mathcal{C}_{\mathbf{r},i+j-1}$.*

Here another consequence of our calculation of τ and τ^-:

Corollary 7. *The functor τ maps injective maps to injective maps. The functor τ^- maps surjective maps to surjective maps.*

Proof: Given any map $f\colon V \to V'$ in $\mathcal{C}$, we may suppose that V, V' both belong to $\mathcal{C}_{i-1}$ for some $i > 0$, then we can construct $\tau(f)$ and $\tau^-(f)$ inside $\mathcal{C}_i$. Instead of looking at $\mathcal{C}_i$ we may work in the module category $\operatorname{mod} k\Delta^{\mathbf{m},i}$ and there we know that τ preserves injectivity and that τ^- preserves surjectivity.

5. The components of the Auslander-Reiten quiver.

Let us recall the following observation which is valid in any hereditary abelian category with almost split sequences: Let X, Y be indecomposable and let $X \to Y$ be an irreducible map. If Y is projective, then X is projective. As a consequence, we also see that X projective implies that either Y or τY is projective. Also dually, if X is injective, then Y is injective. And if Y is injective, then X or $\tau^- X$ is injective.

The preprojective component. Any arrow $\gamma\colon x \to y$ in the quiver $\Delta^{\mathbf{mr}}$ gives rise to an irreducible monomorphism $P_y \to P_x$, this shows that all the indecomposable projective objects lie in one component, which we denote by $\mathcal{P}$. Using the observation just mentioned, we see that for any indecomposable object Z in $\mathcal{P}$, there exists some $t \in \mathbb{N}_0$ with $\tau^t Z$ projective. For any vertex x lying on a ray, the representation P_x is infinite dimensional, and Corollary 6 implies that the τ-orbit containing P_x is infinite. If we assume that $\mathbf{m} \neq 0$, then we obtain in this way τ-orbits in $\mathcal{P}$ which are infinite. But using the injectivity part of the observation, we see that then all the τ-orbits have to be infinite. Also for $\mathbf{m} = 0$, usually all the τ-orbits in $\mathcal{P}$ are infinite, the only exceptions are the cases where Δ is of type A_n, D_n, E_6, E_7, or E_8.

The objects in the preprojective component $\mathcal{P}$ generate $\mathcal{C}$. Proof: Given any indecomposable object M in $\mathcal{C}$, there is some $t \in \mathbb{N}_0$ such that $\tau^t M$ is non-zero and belongs to $\mathcal{C}'$. Any object in $\mathcal{C}'$ is generated by the projective objects, thus there is a surjective map $f\colon P \to \tau^t M$ with P projective. Now apply τ^{-t} and note that τ^{-t} maps surjective maps to surjective map, according to Corollary 7.

The preinjective component. Any arrow $\gamma\colon x \to y$ in the quiver $\Delta^{\mathbf{mr}}$ gives rise to an irreducible monomorphism $I_y \to I_x$, this shows that also all the indecomposable injective objects lie in one component, which we denote by $\mathcal{Q}$. For any indecomposable object Z in $\mathcal{Q}$, there exists some $t \in \mathbb{N}_0$ with $\tau^{-t} Z$ injective. Again, all the τ-orbits are infinite except in case $\mathbf{m} = 0$ and Δ is of type A_n, D_n, E_6, E_7, or E_8. Of course, in the latter case, $\mathcal{Q} = \mathcal{P}$ is the only component.

The regular components. Let us consider now the shape of the regular components. The following extends the main result of [R1].

Proposition 2. *Any regular component of the Auslander-Reiten quiver is of the form $\mathbb{Z}A_\infty$.*

Proof. If we deal with a component which contains only representations of finite length, then we can use the arguments as presented in [R1]. Thus, let us assume that we deal with a component which is not contained in $\mathcal{C}''$. Let X be indecomposable and $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ an almost split sequence. and assume that Z does not belong to $\mathcal{C}''$. Thus Z belongs to $\mathcal{C}_{\mathbf{r},j}$, but not to $\mathcal{C}_{\mathbf{r},j-1}$, for some j, and $X = \tau Z$ belongs to $\mathcal{C}_{\mathbf{r},j-1}$. Decompose $Y = \bigoplus_{i=1}^t Y_i$ with indecomposable direct summands Y_i, and write $f = (f_i)_i$, where $f_i\colon X \to Y_i$.

(1) Not all maps f_i are surjective. For, otherwise X^t maps onto Y, thus also onto Z (using the map g); but Z does not belong to $\mathcal{C}_{\mathbf{r},j-1}$, whereas $X = \tau Z$ belongs to $\mathcal{C}_{\mathbf{r},j-1}$. Impossible.

(2) If $t \geq 2$, then the map $(f_1, f_2) : X \to Y_1 \oplus Y_2$ is injective. For the proof, assume (f_1, f_2) is not injective. Then it has to be surjective, since it is an irreducible map. Applying τ^-, we also obtain a surjective map $(\tau^- f_1, \tau^- f_2)\colon Z \to \tau^- Y_1 \oplus \tau^- Y_2$. Using the exact sequence $0 \to X \to Y_1 \oplus Y_2 \oplus Y' \to Z$ and the almost split sequences $0 \to Y_i \to Z \oplus C_i \to \tau^- Y_i$ for $i = 1, 2$, we see that for any vertex x of $\Delta^{\mathbf{m}}$, we have

$$\begin{aligned} \dim X(x) + \dim Z(x) &\geq \dim Y_1(x) + \dim Y_2(x) + \dim \tau^- Y_1(x) + \dim \tau^- Y_2(x) \\ &\geq 2 \dim Z(x), \end{aligned}$$

thus $\dim X(x) \geq \dim Z(x)$. But this contradicts again the fact that Z does not belong to $\mathcal{C}_{\mathbf{r},j-1}$, whereas $X = \tau Z$ belongs to $\mathcal{C}_{\mathbf{r},j-1}$.

(3) Assume that we can decompose $Y = Y' \oplus Y''$ and write f accordingly as $f = (f', f'')$ with $f'\colon X \to Y'$ and $f''\colon X \to Y''$. Then it is impossible that both

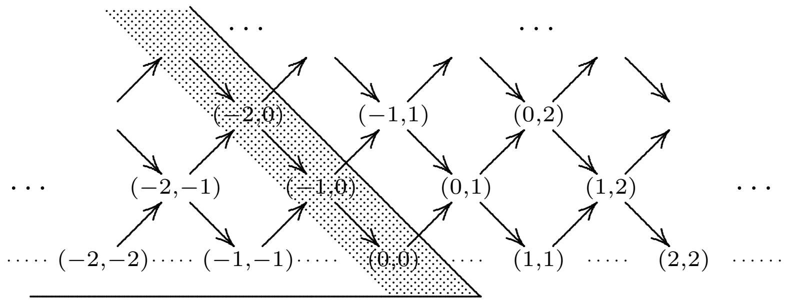
maps f', f'' are injective. Proof: We write also $g = (g', g'')$ with $g'\colon Y' \to Z$, and $g''\colon Y'' \to Z$. Let us assume that both f', f'' are injective. Then also g', g'' are injective maps, thus we have a proper monomorphism $\tau Z = X \to Z$. Applying τ^t with $t \geq 1$, we obtain correspondingly proper monomorphisms $\tau^{t+1}Z \to \tau^t Z$. But for some $t \geq 1$, the module $\tau^t Z$ is finite dimensional, and then the infinite chain of proper submodules

$$\tau^t Z \supset \tau^{t+1} Z \supset \tau^{t+2} Z \supset \cdots$$

yields a contradiction.

(4) Altogether we see that t has to be 1 or 2, and in case $t = 2$, one of the maps $f_i\colon X \to Y_i$ with $i = 1, 2$ is injective, the other is surjective. For, we know from (1) that at least one of the maps f_i, say f_t, is injective. If $t = 2$, then we use (3) in order to see that f_1 cannot be injective, thus it has to be surjective. If $t \geq 3$, then consider the property (3) for the decomposition $Y' = \bigoplus_{i=1}^{t-1} Y_i$ and $Y'' = Y_t$. According to (2), we know that $f'\colon X \to Y'$ is injective. Again, we get a contradiction. Thus the case $t \geq 3$ cannot happen.

Coordinization of regular components. Let us stress a quite interesting phenomenon for the components which contain at least one infinite-dimensional representation, namely that they have well-defined coordinates: Thus, let M be an infinite-dimensional representation and consider the component $\mathcal{R}_M$ which contains M. Of course, at least one of the quasi-simple objects in $\mathcal{R}_M$ has to be infinite-dimensional too, say N (recall that N is said to be quasi-simple provided the almost split sequence ending in N has an indecomposable middle term), and according to Corollary 5 we may assume that τN is finite-dimensional. We call such a representation *central.* We label the indecomposables in the $\mathcal{R}_M$ inductively by pairs $i \leq j$ of integers as follows: Let $X_{ii} = \tau^{-i}N$ and if X_{ij} is already defined, let $X_{i-1,j}$ be the indecomposable representation with a surjective irreducible map $X_{i-1,j} \to X_{ij}$. It is clear that all the objects X_{ij} with $j < 0$ are of finite length, whereas those of the form X_{i0} are infinite-dimensional. Of course, it follows that all the objects X_{ij} with $i \leq 0 \leq j$ are infinite-dimensional. Here are the labels of such a regular component:



The dark coray marks the infinite dimensional indecomposables M' with $\tau M'$ being finite dimensional; the encircled co-cone is the part of the component which belongs to $\mathcal{C}'$. The representation with label $(0,0)$ is the central one. We can reformulate Corollary 6 as follows: Let $j \in \mathbb{N}_0$. An indecomposable representation M' belongs to $\mathcal{C}_{\mathbf{r},j}$, but not to $\mathcal{C}_{\mathbf{r},j-1}$ if and only if M' is labeled by a pair of the form (i,j).

The simple representations outside of $\mathcal{C}'$. The only indecomposable representations of finite length outside of $\mathcal{C}'$ are those living on one of the corays attached to Δ and these are serial objects. In order to point out the position of these representations, let us label the ray-coray quivers $(*)$ attached to Δ at the vertex $x \in \Delta_0$ by the pairs (x,i) with $1 \le i \le \mathbf{m}(i)$. The vertices of such a ray-coray will be labeled as follows:

$$\begin{array}{ccccccccccccc} x & & (x,i,a_1) & & (x,i,a_2) & & & & & (x,i,c_1) & & (x,i,c_2) & \\ \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \cdots & \square & \longleftarrow & \circ & \longleftarrow & \circ & \longleftarrow \ \cdots \end{array}$$

Consider the projective representation $P_{(x,i,a_1)}$. It is uniform and has a relative injective envelope $I'_{(x,i)}$ inside $\mathcal{C}'$ (note that in $\mathcal{C}'$ there are two kinds of indecomposable injectives, namely the injective envelopes I_x of the simple objects S_x with x a vertex of $\Delta^{\mathbf{mr}}$, and the objects of the form $I'_{(x,i)}$ and one obtains in this way sufficiently many injective objects). The representation $I'_{(x,i)}$ is no longer injective when considered as an object of $\mathcal{C}$, we have

$$\tau^{-t} I_{(x,i)} = S_{(x,i,c_t)} \quad \text{for all} \quad t \in \mathbb{N}_1.$$

Thus, if $I_{(x,i)}$ is regular, then $I_{(x,i)}$ is central in its component and carries the label $(0,0)$, whereas $S_{(x,i,c_t)}$ carries the label (t,t). As the first example in the next section shows, $I_{(x,i)}$ may be preprojective, then also all the simple representations $S_{(x,i,c_t)}$ are preprojective.

Proposition 3. *If $\Delta^{\mathbf{mr}}$ is not of type A_∞, then all the simple objects of the form $S_{(x,i,a_t)}$ or $S_{(x,i,c_t)}$ with $t \ge 1$ are regular. If $\Delta^{\mathbf{mr}}$ is of type A_∞, then the objects $S_{(x,i,a_t)}$ are preinjective, those of the form $S_{(x,i,c_t)}$ are preprojective.*

Proof. For the second assertion, we refer to the next section, where the full Auslander-Reiten quiver will be displayed for this case. Also, the cases where $\Delta^{\mathbf{mr}}$ is of type A_∞^∞ will be treated in the next section. Thus, we assume that $\Delta^{\mathbf{mr}}$ is not of type A_∞ or A_∞^∞. Let $t \ge 1$. First of all, we note that the object $S_{(x,i,a_t)}$ cannot be preprojective, since $\tau^j(S_{(x,i,a_t)} = S_{(x,i,a_{t+j})}$ for any $j \ge 0$. Second, the object $S_{(x,i,c_t)}$ cannot be preinjective, since $\mathcal{Q} \subset \mathcal{C}''$. Since $\Delta^{\mathbf{mr}}$ is of not of type A_∞ or A_∞^∞, thus there is some braching vertex x in $\Delta^{\mathbf{mr}}$. It is easy to construct an indecomposable representation V of $\Delta^{\mathbf{m}}$ with subobject $S_{(x,i,a_t)}$ and factor object $S_{(x,i,c_t)}$. Namely, start with an indecomposable representation V' of $\Delta^{\mathbf{mr}}$ with $\dim V'(x) = \dim V'(x,i,a_t) = 2$ and $\dim V'(x,i,a_j) = 1$ for all $j > t$ and extend it by the simple objects $S_{(x,i,c_s)}$ with $1 \le s \le t$ in order to obtain V. The non-zero maps $S_{(x,i,a_t)} \to V \to S_{(x,i,c_t)}$ show that $S_{(x,i,a_t)}$ cannot be preinjective (since $S_{(x,i,c_t)}$ is not preinjective) and that $S_{(x,i,c_t)}$ cannot be preprojective (since $S_{(x,i,a_t)}$ is not preprojective.)

6. Bounded representation type.

We say that a ray quiver Q (or also the category $\operatorname{mod}(Q,k)$) is of *bounded type* provided there is a bound $b \in \mathbb{N}$ such that for any finitely generated indecomposable representation V of Q, we have $\dim_k V(x) \leq b$ for all vertices x. If we want to specify the bound b, we say that Q is *b-bounded.*

Consider the following quivers Δ and functions $\mathbf{m}$; always, the non-oriented edges may be oriented arbitrarily (but in the case A_∞ we have to use infinitely many arrows in either direction) and we list only the values $\mathbf{m}(x)$ (below the vertex x) provided these values are non-zero:

A_n $(n \geq 2)$ $$\begin{tikzcd}[row sep=tiny] \circ \arrow[r] & \circ \arrow[r, dash] & \circ & \cdots & \circ \arrow[r, dash] & \circ \\ 1 \end{tikzcd}$$

A_1 $$\begin{tikzcd}[row sep=tiny] \circ \\ 1 \end{tikzcd}$$

A_n $(n \geq 3)$ $$\begin{tikzcd}[row sep=tiny] \circ \arrow[r] & \circ \arrow[r, dash] & \circ & \cdots & \circ \arrow[r, dash] & \circ & \circ \arrow[l] \\ 1 & & & & & & 1 \end{tikzcd}$$

A_1 $$\begin{tikzcd}[row sep=tiny] \circ \\ 2 \end{tikzcd}$$

A_∞ $$\begin{tikzcd}[row sep=tiny] \circ \arrow[r] & \circ \arrow[r, dash] & \circ & \cdots & \circ \arrow[r, dash] & \circ & \cdots \\ 1 \end{tikzcd}$$

D_n $(n \geq 4)$ $$\begin{tikzcd}[row sep=tiny] & & & & & & \circ \\ \circ \arrow[r] & \circ \arrow[r, dash] & \circ & \cdots & \circ \arrow[r, dash] & \circ \arrow[ur, dash] \arrow[dr, dash] \\ 1 & & & & & & \circ \end{tikzcd}$$

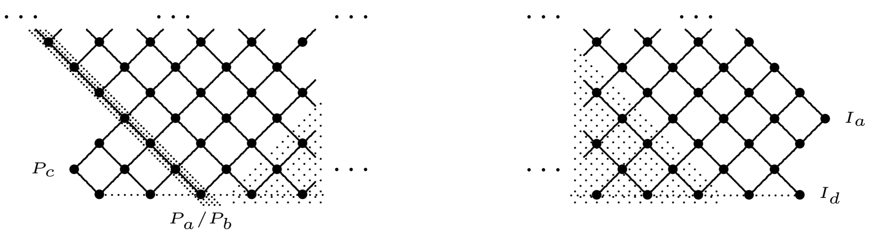
D_3 $$\begin{tikzcd}[row sep=tiny] & \circ \\ \circ \arrow[ur, dash] \arrow[dr, dash] \\ 1 & \circ \end{tikzcd}$$

We are going to present the corresponding Auslander-Reiten quivers.

(1) First, consider the A_4 quiver:

$$\begin{tikzcd}[row sep=tiny] a & b & c & d \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] \\ 1 \end{tikzcd}$$

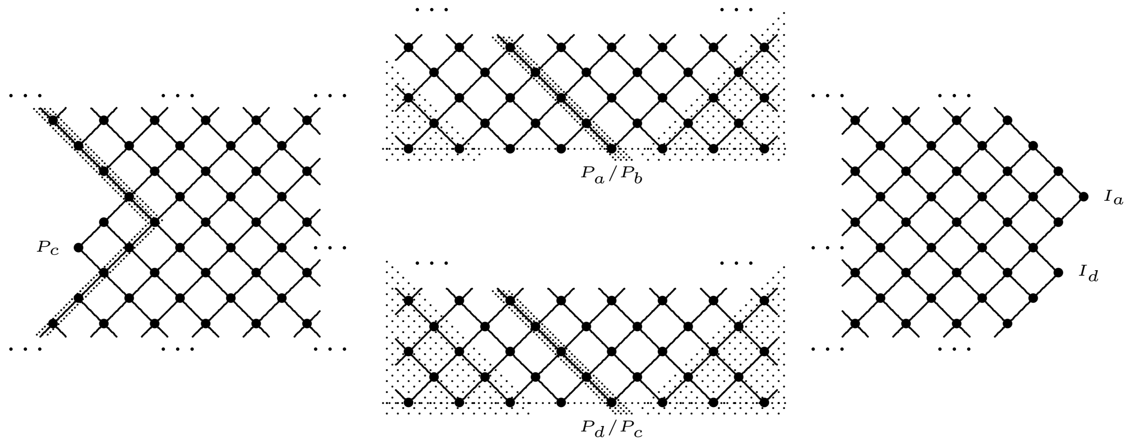
Here is the Auslander-Reiten quiver:



(2) The next case, again we start with an A_4 quiver:

$$\begin{tikzcd}[row sep=tiny] a & b & c & d \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] \\ 1 & & & 1 \end{tikzcd}$$

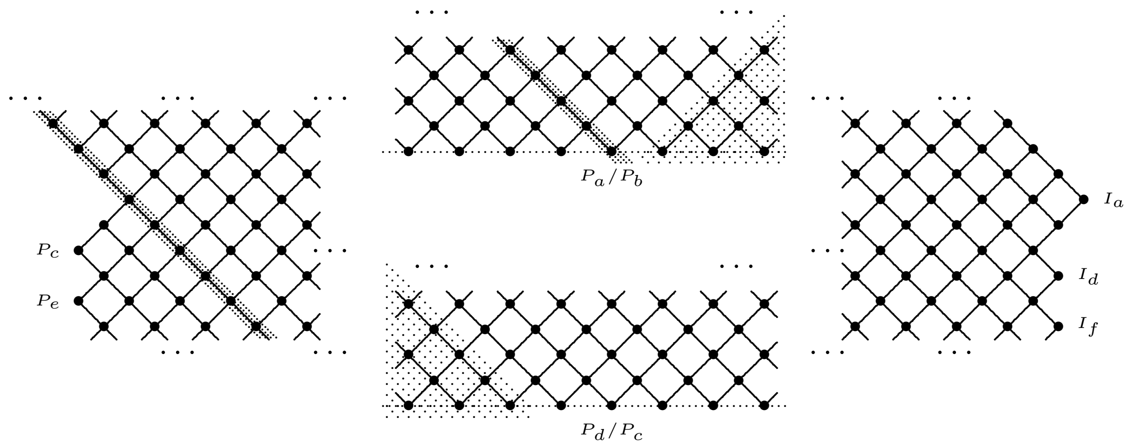
Here is the Auslander-Reiten quiver:



(3) The third case; we start with an A_∞ quiver:

$$\begin{tikzcd}[row sep=tiny] a & b & c & d & e & f \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] & \cdots \\ 1 \end{tikzcd}$$

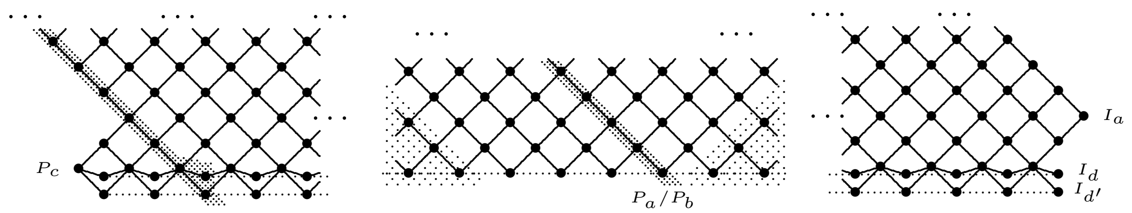
Here is the Auslander-Reiten quiver:

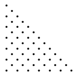


(4) The last case which we will consider is the following D_5-case:

$$\begin{tikzcd}[row sep=small] & & & \circ\ d \arrow[dl] \\ a\ \circ \arrow[r] & b\ \circ \arrow[r] & c\ \circ & \\ 1 & & & \circ\ d' \arrow[ul] \end{tikzcd}$$

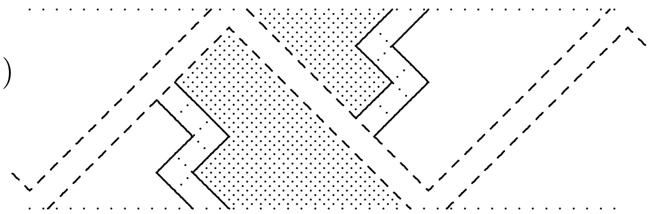
Here is the Auslander-Reiten quiver:



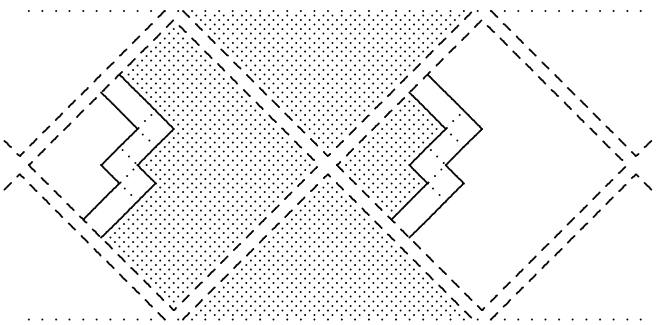
Always, the dark shaded area shows the infinite dimensional representations M with τM finite dimensional. In addition, we have shaded the cones  which contain the simple representations corresponding to the vertices on the corays and the co-cones  which contain the simple representations corresponding to the vertices on the rays.

If we want to visualize the four categories of bounded type, it is not sufficient to deal only with the Auslander-Reiten components, but we also have to consider the maps between objects which belong to different components, (and also maps f between objects in one component which cannot be written as a linear combination of compositions of irreducible maps — but, as we will see, this does not happen). Here is an attempt to present a global picture of these categories (inside the corresponding derived categories):

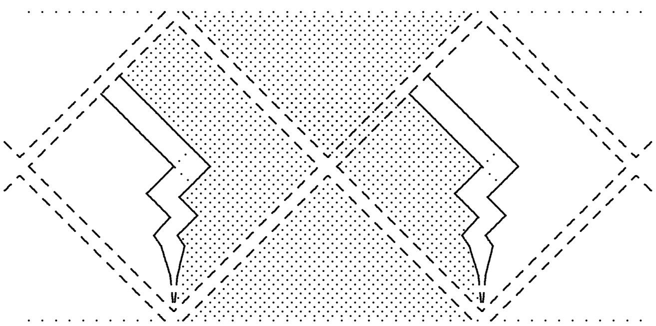
(1)



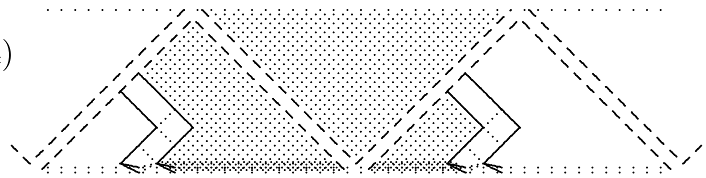
(2)



(3)



(4)



Of course, a better feeling for these categories may be obtained by using the quilt approach of sewing together the components, as described in [R2].

Recall that an additive category is said to be *directed* proved all objects are finite direct sums of indecomposables and given any sequence $M_0, M_1, \dots, M_n$ of indecomposable objects and any sequence of non-zero maps $f_i \colon M_{i-1} \to M_i$, for $1 \le i \le n$, then $M_0 = M_n$ implies that all the maps f_i are isomorphisms.

Proposition 4. *The category $\mathcal{C}$ is of bounded type if and only if $\mathcal{C}$ is directed.*

Proof: If the quiver $\Delta^{\mathbf{m},i}$ contains a Euclidean subquiver, then the category $\operatorname{mod}(\Delta^{\mathbf{m}}, k)$ is neither of bounded type nor directed. Thus we only have to discuss the cases where no Euclidean subquiver is involved. In case $\mathbf{m} = 0$, these are the finite quivers of type A_n, D_n, E_6, E_7, E_8 as well as the infinite quivers of type $A_\infty, A_\infty^\infty, D_\infty$. In all these cases, the category $\operatorname{mod} k\Delta$ is known to be both of bounded type as well as directed. Now assume that $\mathbf{m} \neq 0$. Then clearly we are in one of the four cases exhibited above. We know that $\mathcal{C} = \bigcup \mathcal{C}_i$ with $\mathcal{C}_i$ being equivalent to $\operatorname{mod} k\Delta^{\mathbf{m},i}$. Now in the cases (1), (2), (3), these quivers $\Delta^{\mathbf{m},i}$ are of type A_n, in the case (4), they are of type D_n, thus always $\mathcal{C}_i$ is directed. As a consequence, also $\mathcal{C}$ is directed.

Remarks. *If $\mathcal{C}$ is of bounded type, then it is of 6-bounded type. If $\mathcal{C}$ is of bounded type and has infinitely many simple objects, then $\mathcal{C}$ is even of 2-bounded type. If $\mathbf{m} \neq 0$ and $\mathcal{C}$ is not of bounded type, then $\mathcal{C}$ is wild.*

7. The categories (d) considered by Reiten and Van den Bergh.

The aim of this section is to use ray quivers in order to present also a new construction of the two categories listed by Reiten and Van den Bergh as case (d).

First, consider the following ray quiver Q:

$$\begin{tikzcd}[row sep=small]
\cdots \arrow[r, "\beta_{-1}"] & \circ \arrow[r, "\beta_0"] & \circ \arrow[r, "\beta_1"] & \cdots & \square & \\
 & & & & & \circ \arrow[ul, "\gamma"'] \arrow[dl, "\gamma'"] \\
\cdots \arrow[r, "\beta'_{-1}"] & \circ \arrow[r, "\beta'_0"] & \circ \arrow[r, "\beta'_1"] & \cdots & \square &
\end{tikzcd}$$

We denote by $\mathcal{D}$ the full subcategory of $\operatorname{mod}(Q, k)$ given by all representations V such that both maps $V(\gamma)$ and $V(\gamma')$ are isomorphisms. Clearly, $\mathcal{D}$ is a hereditary abelian category, and it is noetherian. Of course, $\mathcal{D}$ has neither non-zero projective objects nor non-zero injective objects.

Similar to previous considerations, we write $\mathcal{D}$ as a union of an ascending chain of subcategories $\mathcal{D}_i$, for $i \in \mathbb{N}$. Here, $\mathcal{D}_i$ is the full subcategory of all representations V in $\mathcal{D}$ with $V(\beta_j)$ and $V(\beta'_j)$ being isomorphisms, for all $j > i$. Of course, $\mathcal{D}_i$ has as full and dense subcategory the category of all finitely generated representations V of Q such that all the maps $V(\gamma), V(\gamma'), V(\beta_j), V(\beta'_j)$ with $j > i$ are even identity maps, and this subcategory may be identified with the category of finitely generated representations of the following quiver $Q^{(i)}$

$$\begin{tikzcd}
\cdots \arrow[r, "\beta_{i-2}"] & \circ \arrow[r, "\beta_{i-1}"] & \circ \arrow[dr, "\beta_i"] & \\
& & & \circ \\
\cdots \arrow[r, "\beta'_{i-2}"'] & \circ \arrow[r, "\beta'_{i-1}"'] & \circ \arrow[ur, "\beta'_i"'] &
\end{tikzcd}$$

This is a quiver of type A_∞^∞, and we know very well all its indecomposable representations as well as the maps. In particular, the category $\operatorname{mod} kQ^{(i)}$ is directed.

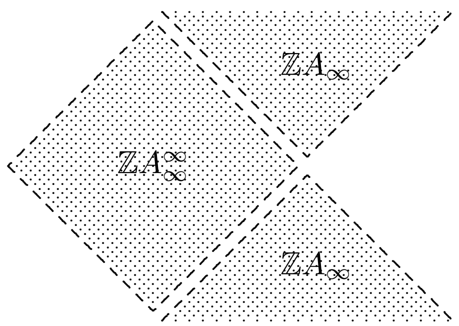
As a consequence, we obtain the full classification of the indecomposable objects as well as the maps in $\mathcal{D}$. Thus, we see that for V indecomposable in $\mathcal{D}$, all the vector spaces $V(x)$ are one-dimensional or zero, that $\mathcal{D}$ is directed and that it is a category with Serre duality.

To be more precise, let us consider the indecomposable objects of $\mathcal{D}$ in detail. The objects of finite length have their support either on the upper biray (formed by the arrows β_i) or on the lower biray (formed by the arrows β'_i). We obtain in this way two Auslander-Reiten components of the form $\mathbb{Z}A_\infty$. In addition, given a pair (z, z') of integers, there is a unique indecomposable object $M_{z,z'}$ in $\mathcal{D}$ such that $\beta_i \neq 0$ iff $i \geq z$ and $\beta'_i \neq 0$ iff $i \geq z'$. The inclusion maps $M_{z,z'} \to M_{z-1,z'}$ and $M_{z,z'} \to M_{z,z'-1}$ all are irreducible, and we obtain in this way an Auslander-Reiten component of the form $\mathbb{Z}A_\infty^\infty$.

$$\begin{tikzcd}
& \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & M_{0,1} \arrow[ur] \arrow[dr] & & M_{-1,0} \arrow[ur] \arrow[dr] & & \cdots \\
& M_{1,1} \arrow[ur] \arrow[dr] & & M_{0,0} \arrow[ur] \arrow[dr] & & M_{-1,-1} \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & M_{1,0} \arrow[ur] \arrow[dr] & & M_{0,-1} \arrow[ur] \arrow[dr] & & \cdots \\
& \cdots \arrow[ur] & & \cdots \arrow[ur] & & \cdots \arrow[ur] &
\end{tikzcd}$$

Remark: Note that the known components of the form $\mathbb{Z}A_\infty^\infty$ which occur in the Auslander-Reiten quiver say of a special biserial algebra, all contain a unique module of smallest length, the so called Geiß module [Gei,Ri]. The component constructed here is quite different: all the maps are injective.

Altogether, the Auslander-Reiten quiver of $\mathcal{D}$ looks as follows:



This picture looks familiar to us: Clearly, the derived category of $\mathcal{D}$ coincides with the derived categories of the ray quivers of type (2) discussed in section 6.

Second, let us consider the following ray quiver Q':

$$\begin{tikzcd}[row sep=small]
 & & & & \circ\ c \arrow[dl, "\gamma"'] \\
\cdots \arrow[r, "\beta_{-1}"] & \circ \arrow[r, "\beta_0"] & \circ \arrow[r, "\beta_1"] & \cdots\ \overset{\beta_\bullet}{\square} & \\
 & & & & \circ\ d \arrow[ul, "\delta"]
\end{tikzcd}$$

(when necessary, we will label the vertices different from c, d by the integers, with $\beta_i \colon i \to i{+}1$). Let us denote by $\mathcal{D}'$ the full subcategory of $\operatorname{mod}(Q', k)$ given by all representations V such that the map

$$[V(\gamma) \quad V(\delta)] \colon V(c) \oplus V(d) \to V(\beta_\bullet)$$

is an isomorphism. Clearly, $\mathcal{D}'$ is a hereditary abelian category, and has neither non-zero projective objects nor non-zero injective objects.

Again, we write $\mathcal{D}'$ as a union of an ascending chain of subcategories $\mathcal{D}'_i$, for $i \in \mathbb{N}$, with $\mathcal{D}'_i$ being given by all representations V in $\mathcal{D}'$ with $V(\beta_j)$ an isomorphism, for all $j > i$. Of course, up to isomorphism, the objects in $\mathcal{D}'_i$ are those finitely generated representations V of Q' such that all the maps $V(\beta_j)$ with $j > i$ are identity maps, and such that the maps $V(\gamma)$ and $V(\delta)$ are the inclusion maps of the direct summands in a direct sum decomposition $V(\beta_\bullet) = V(c) \oplus V(d)$. The category $\mathcal{D}'_i$ may be identified with the category of finitely generated representations of the following quiver

$$\begin{tikzcd}[row sep=small]
 & & & & \circ\ c \arrow[dl, "\gamma"'] \\
\cdots \arrow[r, "\beta_{i-2}"] & \circ \arrow[r, "\beta_{i-1}"] & \circ \arrow[r, "\beta_i"] & \overset{b}{\circ} & \\
 & & & & \circ\ d \arrow[ul, "\delta"]
\end{tikzcd}$$

with the corresponding direct sum condition: that the maps $V(\gamma)$ and $V(\delta)$ are the inclusion maps of the direct sum decomposition $V(b) = V(c) \oplus V(d)$.

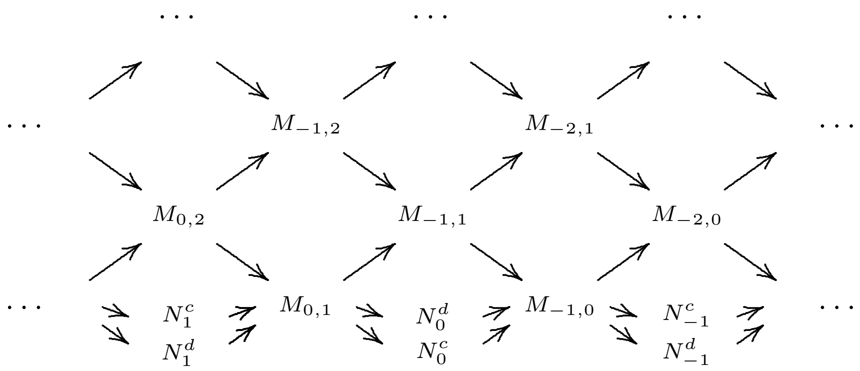
Of course, this quiver is of type D_∞, thus again we know very well all the indecomposables and all the maps. And again, we deal with a directed category.

As a consequence, we obtain the full classification of the indecomposable objects as well as the maps in $\mathcal{D}'$. We see that for V indecomposable in $\mathcal{D}'$, all the vector spaces $V(x)$ are at most two-dimensional, that $\mathcal{D}'$ is directed and that it is a category with Serre duality. The objects of finite length have their support on the biray (formed by the arrows β_i), and we obtain in this way one Auslander-Reiten component of the form $\mathbb{Z}A_\infty$. In addition, there is one component of the form $\mathbb{Z}D_\infty$ formed by those indecomposable representations V with $V(\beta_\bullet) \neq 0$. There are two kinds: $V(\beta_\bullet)$ may be one-dimensional or two-dimensional. The first ones we may denote by N_z^c or N_z^d with $z \in \mathbb{Z}$: these are those indecomposables V with $V(i) = k$ iff $i \geq z$ and with $N_z^c(c) = k$, and $N_z^d(d) = k$, respectively.

Those with $V(\beta_\bullet)$ two-dimensional may be denoted by $M_{z,z'}$, where $z<z'$; here $M_{z,z'}(c)=M_{z,z'}(d)=k$ and, for $i\in\mathbb{Z}$,

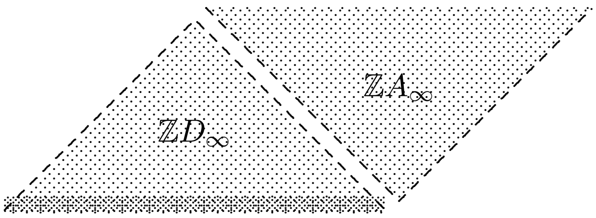
$$M_{z,z'}(i)=\begin{cases}0 & \\ k & \text{in case} \\ k^2 & \end{cases}\begin{array}{l} i<z,\\ z\le i<z',\\ z'\le i.\end{array}$$

Here is part of the $\mathbb{Z}D_\infty$-component:



Also in this component, nearly all the maps are monomorphisms. The only exceptions are the maps of the form $M_{z,z+1}\to N_z^c$ and $M_{z,z+1}\to N_z^d$.

The Auslander-Reiten quiver of $\mathcal{D}'$ looks as follows:



Again, this picture is familiar to us: Clearly, the derived category of $\mathcal{D}'$ coincides with the derived categories of the ray quivers of type (3) discussed in section 6.

Part II. Examples using representations of ordered sets.

Let I be a (totally) ordered set. Let $k[I]$ be the incidence algebra of I; it is the k-algebra with basis the pairs (i,j) where $i,j\in I$ and $i\le j$, with multiplication $(i,j)(i',j')=(i,j')$ provided $j=i'$ and equal to zero, otherwise. In case I is an infinite set, then the algebra $k[I]$ does not have an identity element, but it always has sufficiently many idempotents, namely the pairs (i,i) with $i\in I$. The aim of this section is to show the following result.

Theorem B. *Let T be an ordered set, let $I = T\vec{\times}\mathbb{Z}$ be the product $T\times\mathbb{Z}$ with the lexicographical ordering. Then the category $\mathcal{D}(I)$ of all $k[I]$-modules which are finitely generated and finitely cogenerated is a hereditary abelian k-category with Serre duality and does not have non-zero projective objects. This category $\mathcal{D}(I)$ is derived equivalent to a noetherian hereditary abelian category only in case T is of cardinality at most 2.*

1. The serial representations of an ordered set.

Let $T \neq \emptyset$ be an ordered set. Let us construct the serial $k[T]$-modules (recall that a module is said to be *serial* provided its submodules are ordered). As usual, we call a subset $J \subseteq T$ an ideal provided for any $j \in J$ any element $i \in T$ with $i < j$ belongs to J. If J is an ideal of T, we may consider the vector space $M(J) = kJ$ with basis J as an $k[T]$-module with scalar multiplication $(i,i') * j = i$ in case $i' = j$ and zero otherwise (here, (i,i') is considered as a basis element of $k[T]$, thus we assume that $i \le i'$, whereas the letter j as well as the letter i on the right side of the quality sign are basis elements of M). Of course, if $I \subseteq J$ are ideals of T, then $M(I)$ is a submodule of $M(J)$. We denote $M(J/I) = M(J)/M(I)$. Note that we obtain a bijection between the submodules of $M(J/I)$ and the ideals I' with $I \subseteq I' \subseteq J$; in particular, all these modules $M(J/I)$ are serial, and any serial modules is obtained (up to isomorphism) in this way. We denote by $\mathcal{S}(T)$ the full subcategory of $\operatorname{Mod} k[T]$ given by all finite direct sums of serial modules.

Let us consider some maps and some extensions in the category $\mathcal{S}(T)$. If $I \subseteq I' \subseteq J$ are ideals of T, there is the canonical projection $M(J/I) \to M(J/I')$, and such a map will be denoted by π; if $I \subseteq J \subseteq J'$, there is the canonical inclusion $M(J/I) \to M(J'/I)$. it will be denoted by ι. In general, we have

$$\operatorname{Hom}(M(J/I), M(J'/I')) = \begin{cases} k & \text{in case} \quad I \subseteq I' \subset J \subseteq J', \\ 0 & \text{otherwise} \quad . \end{cases}$$

In case $I \subseteq I' \subset J \subseteq J'$, a non-zero map $M(J/I) \to M(J'/I')$ is given by the composition map in the following commutating diagram:

$$\begin{tikzcd} & M(J'/I) \arrow[dr, "\pi"] & \\ M(J/I) \arrow[ur, "\iota"] \arrow[dr, "\pi"'] & & M(J'/I') \\ & M(J/I') \arrow[ur, "\iota"'] & \end{tikzcd}$$

Of course, the commutative diagram yields also the following exact sequence

$$(*) \qquad 0 \to M(J/I) \xrightarrow{\left[\begin{smallmatrix}\iota\\ \pi\end{smallmatrix}\right]} M(J'/I) \oplus M(J/I') \xrightarrow{[\pi\ -\iota]} M(J'/I') \to 0$$

A similar exact sequence is obtained in case we deal with $I \subseteq I' = J \subseteq J'$:

$$0 \to M(J/I) \xrightarrow{\iota} M(J'/I) \xrightarrow{\pi} M(J'/I') \to 0.$$

Actually, we don't have to distinguish the two cases, the exact sequence $(*)$ exists for all quadruples I, I', J, J' with $I \subseteq I' \subseteq J \subseteq J'$, and reduces to the second one in case $I' = J$ so that $M(J'/I) = 0$. In general, the sequence $(*)$ is non-split provided $I \subset I'$ and $J \subset J'$.

The description of the Hom-sets of indecomposable objects has the following consequence: *For any ordered set T, the category $\mathcal{S}(T)$ is directed.*

We will use certain full subcategories $\mathcal{G}$ of $\mathcal{S}(T)$ which are constructed as follows: Let $\mathcal{I} = (I_0, I_1, \dots, I_m)$ be a chain of ideals of T, thus $I_0 \subset I_1 \subset \cdots \subset I_m$. Denote by $\mathcal{G} = \mathcal{G}(\mathcal{I})$ the full subcategory of all $k[T]$-modules which have a finite filtration with factors of the form $N_r = M(I_r/I_{r-1})$, for $1 \le r \le m$. This subcategory $\mathcal{G}$ will be said to be the *grid* subcategory given by $\mathcal{I}$. Note that $\mathcal{G}$ is a full exact abelian subcategory and the objects N_r are simple objects in $\mathcal{G}$. Since $\dim_k \operatorname{Ext}^1(N_r, N_s) = 1$ for $s = r-1$ and 0 otherwise, we see that $\mathcal{G}$ is equivalent to the category of finitely generated B-modules, where B is a factor algebra of the path algebra kQ of the linearly oriented quiver Q of type A_m. The module $M(I_m/I_0)$ belongs to $\mathcal{G}$; it is indecomposable and has a filtration with all the factors $N_1, \dots, N_m$. This shows that actually $B = kQ$.

Let $M_1, \dots, M_n$ be serial $k[T]$-modules. Let $M_r = M(J_r/J'_r)$, for $1 \le r \le n$ and ideals $J'_r \subseteq J_r$ of T. Let $\{J_r, J'_r \mid 1 \le r \le n\} = \{I_0, \dots, I_m\} = \mathcal{I}$ with $I_0 \subset I_1 \subset \cdots \subset I_m$. Of course, $m \le 2n-1$. Let $\mathcal{G} = \mathcal{G}(\mathcal{I})$ be the grid subcategory given by $\mathcal{I}$, we call it the *grid subcategory generated by* $M_1, \dots, M_n$ and write $\mathcal{G} = \mathcal{G}(M_1, \dots, M_n)$.

These considerations may be interpreted as follows: *The category $\mathcal{S}(T)$ is a filtered union of grid subcategories, and any of these subcategories is equivalent to the category of* $\operatorname{mod} kQ$ *where Q is the linearly oriented quiver of type A_m for some finite m.* We will use this observation quite often.

Remark: As we have seen, the grid category $\mathcal{G}$ generated by the serial $k[T]$-modules $M_1, \dots, M_n$ is a full exact abelian subcategory of $\mathcal{S}(T)$ containing the given modules $M_1, \dots, M_n$, but it is not necessarily the smallest such subcategory. For example, if we start with ideals $J_0 \subset J_1 \subset J_2 \subset J_3$ and take $M_1 = M(J_3/J_0)$ and $M_2 = M(J_2/J_1)$, then the category of all direct sums of copies of M_1 and M_2 is a full exact abelian subcategory and contains both M_1 and M_2, but this is a proper subcategory of $\mathcal{G}(M_1, M_2)$.

2. The diamond category $\mathcal{D}(T)$ of an ordered set T.

Since (i,i) with $i \in T$ is a primitive idempotent of $A = k[T]$, the module $P(i) = A(i,i)$ is projective and serial. Of course, $P(i) = M(\langle i\rangle)$, where $\langle i\rangle$ is the ideal generated by i, namely the set of all elements $j \le i$. Also, let $\langle \hat{i}\rangle$ be the ideal of all elements $j < i$. Then $S(i) = M(\langle i\rangle/\langle \hat{i}\rangle)$ is a simple module and one obtains all the simple modules in this way.

Let us denote $M(i,j) = M(\langle j\rangle/\langle \hat{i}\rangle)$. In general, the module $M(i,j)$ has a simple top isomorphic to $S(j)$ and a simple socle isomorphic to $S(i)$. The following assertion is quite obvious:

Lemma 5. *The following assertions are equivalent for an A-module M.*
(i) *M is finitely generated and finitely cogenerated.*
(ii) *M is a finite direct sum of modules of the form $M(i,j)$.*

We denote by $\mathcal{D} = \mathcal{D}(T)$ the full subcategory of $\operatorname{mod} k[T]$ given by all modules which are finitely generated and finitely cogenerated (the modules $M(i,j)$ are said to be the *diamonds* of $\operatorname{mod} k[T]$). Note that $\mathcal{D}(T)$ is a full subcategory of $\mathcal{S}(T)$.

In general, the category $\mathcal{D}(T)$ is not closed under kernels or cokernels.

Lemma 6. *The following conditions are equivalent:*
(i) *$\mathcal{D}(T)$ is closed under kernels.*
(ii) *For any non-minimal element $j \in T$, there exists $j' < j$ such that j' and j are neighbors (i.e. there is no $t \in T$ with $j' < t < j$).*
(iii) *The $k[T]$-modules $M(i,j)$ are finitely presented, for all $i < j$ in T.*

Proof: (i) $\implies$ (ii): If $i \in T$ is non-minimal, let $i < j$. The kernel of $\pi\colon M(i,j) \to S(i)$ is $M(\langle \hat{i}\rangle/\langle \hat{j}\rangle)$. The latter module belongs to $\mathcal{D}(T)$ only in case $\langle \hat{j}\rangle$ is of the form $\langle j'\rangle$, but this means that $j' < j$ is a neighbor of j.

(ii) $\implies$ (iii) is clear: $M(i,j) = P(j)/P(i-1)$.

(iii) $\implies$ (i): Let $f\colon N \to N'$ be a map in $\mathcal{D}(T)$. It follows from (iii) that the kernel of f is finitely generated. As a submodule of N, the kernel is also finitely cogenerated. Thus, the kernel of f belongs to $\mathcal{D}(T)$.

3. Ordered sets which are locally discrete.

Let I be an ordered set. Let $i < j$ in I be neighbors, then we write $j = i+1$ or $i = j-1$.

We say that I is *locally discrete* provided no element of I is an accumulation point: this means, first, that for any non-maximal element $i \in I$, the neighbor $i+1$ exists, and second, that for any non-minimal element $i \in I$, the neighbor $i-1$ exists.

The previous lemma together with its dual yields the following result.

Corollary 8. *$\mathcal{D}(I)$ is an exact abelian subcategegory of $\mathcal{S}(I)$ if and only if I is locally discrete.*

Lemma 7. *Let $I \neq \emptyset$ be a locally discrete ordered set. The following conditions are equivalent:*
(i) *I has a minimal element.*
(ii) *$\mathcal{D}(I)$ has at least one indecomposable projective object.*
(iii) *$\mathcal{D}(I)$ has projective covers.*

Proof: (ii) $\implies$ (i): Let $M(i,j)$ be indecomposable projective. Assume i is not minimal. Then there is $i' < i$ and the canonical projection $\pi\colon M(i',j) \to M(i,j)$ is an epimorphism which does not split. This contradiction shows that i has to be minimal.

(i) $\implies$ (iii): Let t be minimal in I. Then the canonical projection $\pi\colon M(t,j) \to M(i,j)$ is a projective cover for any $i \le j$.

Of course, there is also the dual assertion: *I has a maximal element, if and only if $\mathcal{D}(I)$ has at least one indecomposable injective object, if and only if $\mathcal{D}(I)$ has injective envelopes.*

Also we remark: If we start with a finite number of diamonds $M_1, \dots, M_n$, then the grid subcategory of $\mathcal{S}(I)$ generated by the modules $M_1, \dots, M_n$ is a subcategory of $\mathcal{D}(I)$. Namely, we write $M_r = M(j_r, j_r')$, for $1 \le r \le n$, with elements $j_r \le j_r'$ in I, and consider the set $\{j_r - 1, j_r' \mid 1 \le r \le n\} = \{i_0, \dots, i_m\}$ with $i_0 < i_1 < \cdots < i_m$. Then $\mathcal{G}(M_1, \dots, M_n)$ is the full subcategory of all $k[I]$-modules which have a filtration with factors $M(i_{r-1}+1, i_r)$, where $1 \le r \le m$.

Construction. Let $T \neq \emptyset$ be any ordered set and consider the product $I = T \times \mathbb{Z}$ with lexicographical ordering (we will write $I = T \overrightarrow{\times} \mathbb{Z}$), thus if $t, t' \in T$ and $z, z' \in \mathbb{Z}$, then

$$(t,z) < (t',z') \quad \iff \quad \text{either} \quad t < t' \quad \text{or} \quad t = t',\ z < z'.$$

Given $i = (t,z) \in I$ and $z' \in \mathbb{Z}$, we denote by $i + z' = (t, z + z')$. Note that the operation $+z'$ provides an automorphism of I.

We obtain in this way an ordered set I with no minimal nor maximal element, and I is locally discrete. Also conversely, if I is an ordered set with no minimal or maximal element and such that I is locally discrete, then I is obtained in this way. Namely, we can recover T from I as follows: Call two elements i, i' equivalent provided the sets $\{j \mid i < j < i'\}$ and $\{j \mid i' < j < i\}$ are both finite (one will be empty), then T may be identified with $I/\!\sim$ (see also [Sch]).

The Auslander-Reiten translation. *Let I be a locally discrete ordered set. Then $\mathcal{D}(I)$ has Auslander-Reiten sequences and the Auslander-Reiten translation is given by $\tau M(i,j) = M(i-1, j-1)$, for $i \le j$ and i not minimal.*

Remark. If $i \in I$ is minimal, and $i \le j$, then $M(i,j)$ is projective and its radical is $M(i,j-1)$ (using again the convention that $M(i,i-1)=0$). If I has no minimal element, then $i \mapsto i-1$ is an injective map $I \to I$, and $\tau\colon \mathcal{D}(I) \to \mathcal{D}(I)$ is the induced functor. If I has neither a minimal nor a maximal element, then $i \mapsto i-1$ is bijective and the induced functor $\tau\colon \mathcal{D}(I) \to \mathcal{D}(I)$ will be an equivalence!

Proof. Let us assume that $i \le j$ and that i is not minimal. Then $M(i–1,j–1)$ is defined. Of course, we may consider the exact sequence

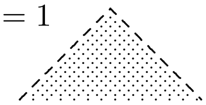
$$(*)\qquad 0 \longrightarrow M(i-1,j-1) \xrightarrow{\left[\begin{smallmatrix}\iota\\ \pi\end{smallmatrix}\right]} M(i-1,j)\oplus M(i,j-1) \xrightarrow{[\pi\ -\iota]} M(i,j) \longrightarrow 0,$$

we claim that this is an almost split sequence in $\mathcal{S}(I)$. Consider an indecomposable object N in $\mathcal{S}(I)$, and take $\mathcal{G} = \mathcal{G}(M(i,j), M(i-1,j-1), N)$. This category is equivalent to $\operatorname{mod} kQ$, where Q is a linearly ordered quiver of type A_m with $m \le 5$. The sequence $(*)$ lies in $\mathcal{G}$, and clearly is an almost split sequence in $\mathcal{G}$. Thus it has the desired lifting properties with respect to the maps $N \to M(i,j)$ and $M(i-1,j-1) \to N$.

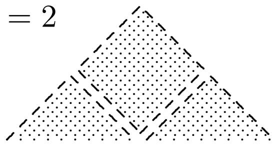
The Auslander-Reiten quiver Γ of $\mathcal{D}(T\overrightarrow{\times}\mathbb{Z})$. Note that two modules $M((t,z),(t',z'))$ belong to the same component of Γ if and only if $t=t'$, thus the components of Γ correspond bijectively to the pairs (t,t') of T with $t \le t'$. For $t \le t'$, we denote by $\Gamma_{t,t'}$ the component which contains all the modules of the form $M((t,z),(t',z'))$. There are two kinds of components: For $t=t'$, all these modules are of finite length and $\Gamma_{t,t'}$ is of the form $\mathbb{Z}A_\infty$. For $t<t'$, the modules $M((t,z),(t',z'))$ are neither artinian nor noetherian and the component $\Gamma_{t,t'}$ is of the form $\mathbb{Z}A_\infty^\infty$.

Let us sketch the structure of the Auslander-Reiten quivers in case $|T| \le 3$.

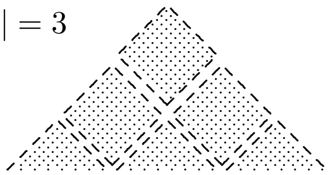
$|T|=1$



$|T|=2$



$|T|=3$



Observe that the second category (that for $|T|=2$) is one which we know already very well: its derived category has been considered both in sections 6 and 7 of part I.

Corollary 9. *$\mathcal{D}(T\overrightarrow{\times}\mathbb{Z})$ is derived equivalent to a noetherian hereditary abelian category if and only if the cardinality of T is 1 or 2.*

Proof. As we have seen, the category $\mathcal{S}(I)$, and therefore also $\mathcal{D}(I)$ is directed, for any ordered set I. Now, let $I = T\overrightarrow{\times}\mathbb{Z}$, thus $\mathcal{D}(I)$ is abelian. Since $\mathcal{D}(I)$ is directed, also the derived category $D^b(\mathcal{D}(I))$ is directed. Now assume $\mathcal{D}(I)$ is derived equivalent to some noetherian hereditary abelian category $\mathcal{H}$. With $\mathcal{D}(I)$ also $\mathcal{H}$ is a k-category and satisfies Serre duality, thus $\mathcal{H}$ is one of the categories classified by Reiten and Van den Bergh [RV]. We have to see which of the categories listed under (a),(b),(c),(d) in [RV] are directed. In case (a), consider the category of finite dimensional representations of the linearly oriented quiver of type A_∞^∞, but this category is just the category $\mathcal{D}(\mathbb{Z})$, thus we deal with the case $|T| = 1$. In case (b), no category is directed. The case (c) has been discussed here in part I, at least provided we deal with at least one ray. According to section 7, the directed categories are just those of bounded representation type. As we have seen, there are three essentially different cases. In one of these cases, we encounter Auslander-Reiten triangles with three middle terms, this is for $D^b(\mathcal{D}(I))$. In the remaining two cases $D^b(\mathcal{H})$ has 1 or 3 shift orbits of Auslander-Reiten components, and for $|T| = n$, the number of shift orbits for $D^b(\mathcal{D}(I))$ is $\binom{n}{2}$, thus $n = 1$ or $n = 2$. Of course, in case (c) without rays, we deal with the category of representations of a quiver Q without infinite paths: Again, only the cases where Q is of type A_∞ or A_∞^∞ are of interest, and the corresponding categories $D^b(\operatorname{mod} kQ)$ have 1 or 3 shift orbits of Auslander-Reiten components. Finally, the categories noted in case (d) are derived equivalent to categories already mentioned.

4. Krull-Gabriel dimension.

The construction presented in this section seems to be of interest also for another reason: we may construct in this way nice abelian categories with arbitrary finite Krull-Gabriel dimension. Recall that the existence and then the value of the *Krull-Gabriel dimension* of an abelian category $\mathcal{A}$ is defined inductively as follows: The zero category has dimension -1, and $\mathcal{A}$ is said to have Krull-Gabriel dimension $n+1$ provided the full subcategory $\mathcal{A}_0$ of objects in $\mathcal{A}$ of finite length is non-zero, and $\mathcal{A}/\!\!/\mathcal{A}_0$ has Krull-Gabriel dimension n. (Here, $\mathcal{B} = \mathcal{A}/\!\!/\mathcal{A}_0$ is an abelian category with an exact functor $\eta\colon \mathcal{A} \to \mathcal{B}$ such that any exact functor ϕ from $\mathcal{A}$ to an arbitrary abelian category which sends all finite length objects to zero, factors uniquely via η.)

Proposition 5. *Let T be any ordered set. Then*

$$\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}Z)/\!\!/\,\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}Z)_0 \simeq \mathcal{D}(T\vec{\times}\mathbb{Z}).$$

Corollary 10. *Let $T = \mathbb{Z}^n$ with lexicographical ordering. Then $\mathcal{D}(T)$ has Krull-Gabriel dimension $n-1$.*

Proof of Theorem. We define an exact functor

$$\eta\colon \mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}Z)/\!\!/\,\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}Z)_0 \longrightarrow \mathcal{D}(T\vec{\times}\mathbb{Z})$$

as follows: Let $t, t' \in T$, and $x, x', y, y' \in \mathbb{Z}$, with $(t,x,y) \le (t',x',y')$. Define

$$\eta M((t,x,y),(t',x',y')) = \begin{cases} M((t,x),(t',x'-1)) & \text{in case } t < t', \\ & \text{or } t = t' \text{ and } x < x', \\ 0 & \text{in case } t = t' \text{ and } x = x'. \end{cases}$$

The indecomposable modules of finite length are the modules $M((t,x,y),(t,x,y'))$ with $y \le y'$, thus we see that η vanishes precisely on the modules of finite length. In order to see that η is exact, we may restrict to a grid category $\mathcal{G} \subset \mathcal{D}$. In order to see that the restriction of η to $\mathcal{G}$ is exact, it is sufficient to show that the almost split sequences in $\mathcal{G}$ are sent to exact sequences, see [B]. Thus, we have to consider only the exact sequences in $\mathcal{D}$ with indecomposable end terms, they are of the form $(*)$, and it is straight forward to check that these sequences remain exact when we apply η.

Note that two indecomposable objects of $\mathcal{D}(T\overrightarrow{\times}\mathbb{Z}\overrightarrow{\times}Z)$ which are not of finite length have images under η which are isomorphic if and only if they belong to the same Auslander-Reiten component.

Finally, consider any exact functor ϕ from $\mathcal{D}(T\overrightarrow{\times}\mathbb{Z}\overrightarrow{\times}Z)$ to an abelian category which sends all finite length modules to zero. But this means that under ϕ all the maps in a given Auslander-Reiten component are sent to isomorphisms. It follows easily that ϕ factors through η.

References

[B] M.C.R. Butler: Grothendieck groups and almost split sequences. In: Integral Representations an Applications (ed. K.W. Roggenkamp). Springer LNM 882 (1981)

[Ga] P. Gabriel: Auslander-Reiten sequences and representation-finite algebras. In: Representation Theory I (ed. V. Dlab and P. Gabriel). Springer LNM 831 (1980)

[Gei] Chr. Geiß: On components of type $\mathbb{Z}A_\infty^\infty$ for string algebras. Comm. Alg. 26 (1998), 749-758.

[Re] I. Reiten: Lectures at ICRA 9, Beijing 2000.

[RV] I. Reiten, M. Van den Bergh: Noetherian hereditary categories satisfying Serre duality. (Preprint)

[R1] C.M. Ringel: Finite-dimensional hereditary algebras of wild representation type. Math. Z. 161 (1978), 235-255.

[R2] C.M. Ringel: Infinite length modules. Some examples as introduction. In: Infinite Length Modules. Birkhäuser. Basel, Boston, Berlin 2000. p.1-73.

[R3] C.M. Ringel: Hereditary triangulated categories. Compositio Mathematica (To appear).

[Sch] J. Schröer. On the Krull-Gabriel dimension of an algebra Math. Z. 233 (2000), 287-303.

Claus Michael Ringel, Fakultät für Mathematik, Universität Bielefeld,
POBox 100 131, D-33 501 Bielefeld
ringel@mathematik.uni-bielefeld.de