A DUALITY BETWEEN COMPLEXES OF RIGHT AND LEFT MODULES

HENNING KRAUSE

Introduction

Recall that a module X over some associative ring R is endofinite if X has finite length when viewed as a module over its endomorphism ring $\operatorname{End}_R(X)$. Such modules have been studied for some while in representation theory, and most of what is known at present can be found in Crawley-Boevey's survey [3]. More recently, a somewhat different but related concept of endofiniteness attracted some interest. Let us consider three motivating examples. All of them are triangulated categories, and the first two are studied frequently in representation theory:

- the derived category $\mathcal{D}(\operatorname{Mod} R)$ of unbounded complexes of modules for a ring R,
- the stable module category $\underline{\operatorname{Mod}}\, kG$ of the group algebra for a finite group G,
- the stable homotopy category $\mathcal{S}$ of CW-spectra.

In each example there is a natural definition for an object to be endofinite.

- A complex $X \in \mathcal{D}(\operatorname{Mod} R)$ is *endofinite* if the homology $H^i(X)$ is a finite length module over $\operatorname{End}(X)$ for all $i \in \mathbb{Z}$.
- A module $X \in \underline{\operatorname{Mod}}\, kG$ is *endofinite* if X is an endofinite kG-module.
- A spectrum $X \in \mathcal{S}$ is *endofinite* if the stable homotopy group $\pi_i(X) = [S^i, X]$ is a finite length module over the ring $[X, X]$ of stable self maps for all $i \in \mathbb{Z}$.

In all three cases we are dealing with a so-called compactly generated triangulated category [16]. We will see that there is a very satisfactory theory of endofiniteness for such categories which spezializes in each example to the concept just defined.

Having developed a general theory of endofiniteness one might ask what this is good for. To answer this, recall that Crawley-Boevey used endofinite modules to give a conceptual definition of tame representation type for a noetherian algebra. It turns out that a similar approach leads to a new definition of tame representation type for the derived category of a noetherian algebra [5]. Another application is based on the decomposition theory for endofinite objects. Using Rickard's idempotent modules, one obtains a decomposition theory for thick subcategories of the stable category of finite dimensional representations for a finite group [10].

Returning to our examples, there is always attached a triangulated subcategory of 'finite' objects. In some interesting cases it is possible to classify all thick subcategories consisting of such finite objects [7, 14, 1]. These classifications are based on some infinite but endofinite objects, the most prominent examples being the Morava K-theories arising in stable homotopy theory. We know from the study of module categories that the global structure of the category of finite modules is controlled by the generic ones. Generic modules are by definition indecomposable endofinite but not finite, and it seems that such objects play a similar role for the global structure of the triangulated categories arising in representation theory or stable homotopy theory.

The aim in this paper is to establish a duality between endofinite complexes of right and left modules. This generalizes the duality between endofinite right and left modules introduced by Herzog and Crawley-Boevey [6, 3]. The duality is needed to show that the tameness definition in [5] for the derived category of a noetherian algebra is right-left symmetric.

The duality between complexes has an analogue in stable homotopy theory which is Brown-Comenetz duality. Brown and Comenetz defined a bijection $Y \mapsto Y^\vee$ between spectra having finite homotopy groups such that for any spectrum X

$$[X, Y^\vee]_{-*} \cong D\pi_*(X \wedge Y).$$

where $D = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and $X \wedge Y$ denotes the smash product of X and Y; see [2, 13]. For a finite spectrum F, this becomes

$$[TF, Y^\vee] \cong D[F, Y]$$

where TF denotes the Spanier-Whitehead dual of F satisfying $\pi_*(F \wedge X) \cong [TF, X]_*$ for all X. Spectra having finite homotopy groups are certainly endofinite, and we will see that the duality between endofinite complexes is induced by a duality between finite complexes in the same way as Brown-Comenetz duality is induced by the Spanier-Whitehead duality between finite spectra.

1. Endofinite objects

Let $\mathcal{T}$ be a *compactly generated* triangulated category [16]. More precisely, $\mathcal{T}$ is a triangulated category and $\mathcal{T}$ has arbitrary coproducts. The translation functor is denoted by $\Sigma\colon \mathcal{T} \to \mathcal{T}$. An object X in $\mathcal{T}$ is called *compact* if for every coproduct $\coprod_i Y_i$ in $\mathcal{T}$ the canonical map $\coprod_i \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{C}$ the full subcategory of compact objects in $\mathcal{T}$ and observe that $\mathcal{C}$ is a triangulated subcategory of $\mathcal{T}$. For $\mathcal{T}$ being compactly generated the isomorphism classes of objects in $\mathcal{C}$ need to form a set, and $\operatorname{Hom}(C, X) = 0$ for all C in $\mathcal{C}$ implies $X = 0$ for every object X in $\mathcal{T}$.

Definition 1.1. An object X in $\mathcal{T}$ is called *endofinite* if for every compact object C in $\mathcal{T}$ the $\operatorname{End}(X)$-module $\operatorname{Hom}(C, X)$ has finite composition length.

Endofinite objects have nice decomposition properties. We recall the basic result which has been established in [10].

Theorem (Krause). *Let $\mathcal{T}$ be a compactly generated triangulated category. An endofinite object X in $\mathcal{T}$ has, up to isomorphism, a unique decomposition $X = \coprod_i X_i$ into indecomposable objects with local endomorphism ring.*

Proof. See [10, Theorem 1.2]. □

Next we describe a method to classify endofinite objects using certain ideals of maps in the category of compact objects. This approach has been developed in [11, 12]. An *ideal* $\mathfrak{I}$ in $\mathcal{C}$ consists of subgroups $\mathfrak{I}(X, Y)$ in $\operatorname{Hom}(X, Y)$ for every pair of objects X, Y in $\mathcal{C}$ such that for all ϕ in $\mathfrak{I}(X, Y)$ and all maps $\alpha\colon X' \to X$ and $\beta\colon Y \to Y'$ in $\mathcal{C}$ the composition $\beta \circ \phi \circ \alpha$ belongs to $\mathfrak{I}(X', Y')$. Given an additive functor $F\colon \mathcal{C} \to \mathcal{D}$, the ideal of maps ϕ in $\mathcal{C}$ satisfying $F\phi = 0$ is called the *annihilator* of F and is denoted by $\operatorname{Ann} F$.

Definition 1.2. An ideal $\mathfrak{I}$ in $\mathcal{C}$ is called *cofinite* if there exists a cohomological functor $F\colon \mathcal{C} \to \mathcal{A}$ into some abelian category $\mathcal{A}$ such that $\mathfrak{I} = \operatorname{Ann} F$ and FX has finite composition length for all $X \in \mathcal{C}$.

Recall that a functor $F\colon \mathcal{C} \to \mathcal{A}$ is *cohomological* if it sends triangles in $\mathcal{C}$ to exact sequences. The *annihilator* $\operatorname{Ann} X$ of an object X in $\mathcal{T}$ is the ideal $\operatorname{Ann}\operatorname{Hom}(-,X)|_{\mathcal{C}}$ in $\mathcal{C}$.

Theorem (Krause/Reichenbach). *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{C}$ be the full subcategory of compact objects in $\mathcal{T}$.*

(1) *An object X in $\mathcal{T}$ is endofinite if and only if $\operatorname{Ann} X$ is cofinite.*
(2) *The assignment $X \mapsto \operatorname{Ann} X$ induces a bijection between the set of isomorphism classes of indecomposable endofinite objects in $\mathcal{T}$ and the set of cofinite ideals in $\mathcal{C}$ which are maximal among all cofinite ideals different from $\mathcal{C}$.*

Proof. See Theorems 3.2 and 3.4 in [12]. □

In order to study endofinite object in a triangulated category, it is sometimes useful to pass to some appropriate abelian category. For later reference we include a lemma which illustrates this point of view. Let $\operatorname{mod}\mathcal{C}$ be the category of functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ having a presentation

$$\operatorname{Hom}(-,X) \longrightarrow \operatorname{Hom}(-,Y) \longrightarrow F \longrightarrow 0.$$

This is an abelian category and every cofinite ideal $\mathfrak{I}$ defines a Serre subcategory

$$\mathcal{S}_{\mathfrak{I}} = \{F \in \operatorname{mod}\mathcal{C} \mid F \cong \operatorname{Im}\operatorname{Hom}(-,\phi) \text{ for some } \phi \in \mathfrak{I}\}.$$

We denote by $\operatorname{mod}\mathcal{C}/\mathcal{S}_{\mathfrak{I}}$ the corresponding abelian quotient category and observe that $\mathfrak{I} = \operatorname{Ann} F$ for the compsite $F = Q \circ Y$ of the Yoneda functor $Y\colon \mathcal{C} \to \operatorname{mod}\mathcal{C}$ with the quotient functor $Q\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{C}/\mathcal{S}_{\mathfrak{I}}$.

Lemma 1.3. *Let $X \in \mathcal{T}$ be endofinite and $\mathfrak{I} = \operatorname{Ann} X$. Let $\mathcal{A} = \operatorname{mod}\mathcal{C}/\mathcal{S}_{\mathfrak{I}}$ and $C \in \mathcal{C}$.*

(1) $\operatorname{length}_{\operatorname{End}(X)} \operatorname{Hom}(C,X)$ *equals the composition length of* $\operatorname{Hom}(-,C)$ *in* $\mathcal{A}$.
(2) *Suppose X is indecomposable and let $S \in \mathcal{A}$ be a simple object. Then*

$$\operatorname{End}(X)/\operatorname{rad}\operatorname{End}(X) \cong \operatorname{End}(S).$$

Proof. (1) See Lemma 4.3 in [12].

(2) The assertion follows from the fact that X can be identified with the injective envelope of S in some appropriate Grothendieck category containing $\mathcal{A}$; see the proof of Theorem 3.4 in [12]. □

2. Examples

Let us return to our examples from the introduction. A basic reference for the derived category of a module category is [9] and we refer to [4, 13] for the stable homotopy category of spectra. It is easy to specify in each case the compact objects, at least up to isomorphism:

- the perfect complexes are the compact objects in $\mathcal{D}(\operatorname{Mod} R)$,
- the finitely generated modules are the compact objects in $\underline{\operatorname{Mod}}\, kG$,
- the finite spectra are the compact objects in $\mathcal{S}$.

Recall that a complex is *perfect* if it is isomorphic to a bounded complex of finitely generated projective modules. In each example we have a distinguished compact object which generates the triangulated category in some appropriate sense:

- the ring R, viewed as a complex concentrated in degree 0,
- the semi-simple module $kG/\operatorname{rad} kG$,
- the sphere spectrum $S = S^0$.

The following lemma explains why our formal definition of endofiniteness agrees in each example with the more intuitive one given in the introduction.

Lemma 2.1. *Let $\mathcal{T}$ be a compactly generated triangulated category and let C be a compact object such that* $\mathrm{Hom}(\Sigma^i C, X) = 0$ *for all* $i \in \mathbb{Z}$ *implies* $X = 0$. *Then* $X \in \mathcal{T}$ *is endofinite if and only if* $\mathrm{Hom}(\Sigma^i C, X)$ *has finite length over* $\mathrm{End}(X)$ *for all* $i \in \mathbb{Z}$.

Proof. Let $\mathcal{C}_0 = \{\Sigma^i C \mid i \in \mathbb{Z}\}$ and define inductively $\mathcal{C}_n$ to be the class of objects Z which fit into a triangle $X \to Y \to Z \to \Sigma X$ with $X, Y \in \mathcal{C}_{n-1}$. Then one can show that the compact object in $\mathcal{T}$ are precisely the direct factors of objects in $\bigcup_{n\geq 0} \mathcal{C}_n$; see [15]. Now one uses the exactness of $\mathrm{Hom}(-, X)$ to prove the assertion of the lemma. □

It is interesting to notice that the full subcatgory of endofinite objects in a compactly generated triangulated category is completely determined by the full subcategory of compact objects. In fact, Theorem 1.2 in [10] shows that endofinite objects are pure-injective in some appropriate sense, and Corollary 1.15 in [11] shows that the full subcategory of pure-injectives is determined by the compact objects.

Note that in our third example the category of finite spectra is far more accessible than the category of all spectra. Let me recommend Freyd's exposition [4] as an excellent introduction for algebraists into stable homotopy theory. This article does not mention the category of all spectra but explains the category of finite spectra (and we have just seen that this is sufficient to understand all endofinite spectra).

In all three examples there are certain endofinite objects which are not compact in general, but very important for understanding the full subcategory of all compact objects. Of particular interest is the collection of *thick* subcategories. These are, by definition, the full triangulated subcategories which are closed under direct factors. In fact, a complete classification of all thick subcategories consisting of compact objects has been achieved in the following cases:

- for $\mathcal{D}(\mathrm{Mod}\, R)$ if R is commutative noetherian [7, 16],
- for $\underline{\mathrm{Mod}}\, kG$ if G is a p-group (p a prime) [1, 8],
- for the category $\mathcal{S}_p$ of p-local spectra (p a prime) [7].

Recall that a spectrum X is *p-local* if $\pi_i(X)$ is a p-local abelian group for all $i \in \mathbb{Z}$. Let us list the endofinite objects which are relevant for the classification of the thick subcategories:

- the field of fractions of $R/\mathfrak{p}$, viewed as a complex concentrated in degree 0, for each prime ideal $\mathfrak{p} \subset R$,
- the module $K(\mathfrak{p})$ constructed in [8] for each homogeneous prime ideal $\mathfrak{p}$ of the cohomology ring $H^*(G, k)$,
- the Morava K-theory $K(n)$ for each $n \geq 0$.

It seems to be an interesting project to study the characteristic properties of these objects. More precisely, what distinguishes the objects listed above from other endofinite objects? For example, the complexes of the form $R/\mathfrak{p}$ are, up to shift, precisely those having minimal endolength. It is conceivable that an analysis of appropriate endofinite objects leads to further classifications of thick subcategories.

3. Duality for modules

Let R be an associative k-algebra over some commutative ring k. We denote by $\mathrm{Mod}\, R$ the category of (right) R-modules and $\mathrm{mod}\, R$ denotes the full subcategory of finitely presented R-modules. The left modules over R are identified with the right modules over the opposite ring R^{op}. We fix a minimal injective cogenerator I for $\operatorname{Mod} k$ and denote by $D = \operatorname{Hom}_k(-, I)$ the corresponding functor $\operatorname{Mod} k \to \operatorname{Mod} k$.

Recall from [3] that an R-module X is *endofinite* if the composition length of X viewed as a module over its endomorphism ring $\operatorname{End}_R(X)$ is finite. The *endolengh* of X is by definition

$$\operatorname{endol} X = \operatorname{length}_{\operatorname{End}(X)} X.$$

Note that every endofinite module decomposes esssentially uniquely into indecomposable endofinite modules.

Theorem (Crawley-Boevey, Herzog). *There exists a bijection $X \mapsto X^\vee$ between the isomorphism classes of indecomposable endofinite right and left R-modules. This bijection has the following properties (where X denotes an indecomposable endofinite R-module):*

(1) $X^{\vee\vee} \cong X$.
(2) *DX is a coproduct of copies of $X^\vee$.*
(3) $\operatorname{endol} X^\vee = \operatorname{endol} X$.
(4) $\Delta(X^\vee) \cong \Delta(X)^{\mathrm{op}}$, *where* $\Delta(X) = \operatorname{End}_R(X)/\operatorname{rad}\operatorname{End}_R(X)$.
(5) $X^\vee \in \operatorname{mod} R^{\mathrm{op}}$ *if and only if* $X \in \operatorname{mod} R$, *provided that R is a noetherian algebra.*

Proof. See Theorem 4.10 in [6] and 6.2 in [3]. □

Note that (2) completely determines the map $X \mapsto X^\vee$. Next we extend this bijection between indecomposables to a bijection between the isomorphism classes of all endofinite modules as follows. Let X and Y be endofinite R-modules and fix decompositions $X = \coprod_i X_i$ and $Y = \coprod_j Y_j$ into indecomposable modules. We define $X^\vee = \coprod_i X_i^\vee$. The modules X and Y are *equivalent* and we write $X \sim Y$ if for every i and j there exist i' and j' such that $X_i \cong Y_{i'}$ and $Y_j \cong X_{j'}$.

Corollary. *There exists a bijection $X \mapsto X^\vee$ between the isomorphism classes of endofinite right and left R-modules. This bijection has the following properties (where X denotes an endofinite R-module):*

(1) $X^{\vee\vee} \cong X$.
(2) $DX \sim X^\vee$.
(3) $\operatorname{endol} X^\vee = \operatorname{endol} X$.

4. Duality for complexes

We fix again an associative k-algebra R and keep the notation from Section 3. Consider the derived category $\mathcal{D}(\operatorname{Mod} R)$ of unbounded complexes of R-modules. The full subcategory of bounded complexes of finitely presented Λ-modules is denoted by $\mathcal{D}^b(\operatorname{mod} R)$. We identify the homotopy category $\mathcal{K}^b(\operatorname{proj} R)$ of finitely generated projective Λ-modules with the full subcategory of perfect complexes. Recall that a complex is *perfect* if it is isomorphic to a bounded complex of finitely generated projective R-modules. Note that $\mathcal{D}(\operatorname{Mod} R)$ is a compactly generated triangulated category and that the perfect complexes are precisely the compact objects in $\mathcal{D}(\operatorname{Mod} R)$. We have for every complex X and every $i \in \mathbb{Z}$

$$H^i X = \operatorname{Hom}(R, \Sigma^i X)$$

where R is identified with the corresponding complex concentrated in degree 0. The *homology endolength* of a complex X is by definition the vector

$$\text{h-endol}\, X = (n_i)_{i \in \mathbb{Z}} \quad \text{with} \quad n_i = \operatorname{length}_{\operatorname{End}(X)} H^i(X).$$

Lemma 4.1. *A complex X is an endofinite object in $\mathcal{D}(\operatorname{Mod} R)$ if and only if $H^i(X)$ has finite composition length as $\operatorname{End}(X)$-module for every $i \in \mathbb{Z}$.*

Proof. Suppose that $H^i(X)$ has finite length over $\operatorname{End}(X)$ for all i. The shifted copies of R generate the triangulated subcategory of perfect complexes. Using the fact that $\operatorname{Hom}(-, X)$ is a cohomological functor, it follows that $\operatorname{Hom}(P, X)$ has finite length over $\operatorname{End}(X)$ for every perfect complex P. Thus X is endofinite since compact objects and perfect complexes coincide in $\mathcal{D}(\operatorname{Mod} R)$. $\square$

Lemma 4.2. *Let X be an endofinite complex in $\mathcal{D}(\operatorname{Mod} R)$. Then $H^i(X)$ is an endofinite R-module for all $i \in \mathbb{Z}$.*

Proof. $H^i(X)$ is an R-$\operatorname{End}(X)$-bimodule which has finite length over $\operatorname{End}(X)$. It follows that $H^i(X)$ has finite length over $\operatorname{End}_R(H^i(X))$. $\square$

To compare complexes of right and left modules we shall use a duality for the category of perfect complexes. Indeed, the equivalence

$$(\operatorname{proj} R)^{\mathrm{op}} \longrightarrow \operatorname{proj} R^{\mathrm{op}}, \quad X \mapsto TX = \operatorname{Hom}_R(X, R)$$

induces an equivalence

$$T\colon \mathcal{K}^b(\operatorname{proj} R)^{\mathrm{op}} \longrightarrow \mathcal{K}^b(\operatorname{proj} R^{\mathrm{op}}).$$

We have also a functor between complexes of right and left modules which is induced by $D\colon \operatorname{Mod} R \to \operatorname{Mod} R^{\mathrm{op}}$. Given a complex X of R-modules, we obtain a complex DX of R^{op}-modules by defining $(DX)^* = D(X^{-*})$.

Lemma 4.3. *Let $X \in \mathcal{D}(\operatorname{Mod} R)$.*

(1) *$H^*(DX) \cong DH^{-*}(X)$.*
(2) *$\operatorname{Hom}(TP, DX) \cong D\operatorname{Hom}(P, X)$ for $P \in \mathcal{K}^b(\operatorname{proj} R)$.*

Proof. The first isomorphism is clear since $D\colon \operatorname{Mod} R \to \operatorname{Mod} R^{\mathrm{op}}$ is exact. Now let us construct a map $D\operatorname{Hom}(P, X) \to \operatorname{Hom}(TP, DX)$. To this end denote by $\operatorname{Hom}_{\mathrm{Ch}}(P, X)$ the maps in the category of chain complexes and let $\operatorname{Hom}_{\mathrm{Ch}}(P, X) \to \operatorname{Hom}(P, X)$ be the canonical map into the set of maps in $\mathcal{D}(\operatorname{Mod} R)$. For R-modules Q and Y with Q finitely generated projective we have an isomorphism $D\operatorname{Hom}_R(Q, Y) \to \operatorname{Hom}_{R^{\mathrm{op}}}(TQ, DY)$ which induces a map $D\operatorname{Hom}_{\mathrm{Ch}}(P, X) \to \operatorname{Hom}_{\mathrm{Ch}}(TP, DX)$. Thus we get a map

$$\phi_{PX}\colon D\operatorname{Hom}(P, X) \to D\operatorname{Hom}_{\mathrm{Ch}}(P, X) \to \operatorname{Hom}_{\mathrm{Ch}}(TP, DX) \to \operatorname{Hom}(TP, DX).$$

This becomes the isomorphism in (1) if $P = \Sigma^i R$ for some i. Using that $\operatorname{Hom}(T-, DX)$ and $D\operatorname{Hom}(-, X)$ are both cohomological and that R generates $\mathcal{K}^b(\operatorname{proj} R)$, we conclude that ϕ_{PX} is an isomorphism for all P in $\mathcal{K}^b(\operatorname{proj} R)$. $\square$

Lemma 4.4. *Suppose that R is a noetherian k-algebra. Then the following are equivalent for $X \in \mathcal{D}(\operatorname{Mod} R)$:*

(1) *X is endofinite and belongs to $\mathcal{D}^b(\operatorname{mod} R)$.*
(2) *X is a bounded complex of finite length R-modules.*
(3) *DX is endofinite and belongs to $\mathcal{D}^b(\operatorname{mod} R^{\mathrm{op}})$.*
(4) *DX is a bounded complex of finite length R^{op}-modules.*

Proof. By definition, k is noetherian and R is finitely generated as a k-module. We use that an R-module has finite length if and only if it has finite length over k.

(1) $\Rightarrow$ (2) Suppose that $X \in \mathcal{D}^b(\operatorname{mod} R)$ is endofinite and fix $i \in \mathbb{Z}$. Then $H^i(X)$ has finite length over $\operatorname{End}_R(H^i(X))$ by Lemma 4.2 and this is a noetherian k-algebra. Thus $H^i(X)$ has finite length over k and therefore finite length over R for all i.

(2) $\Rightarrow$ (1) Suppose that $H^i(X)$ has finite length over R for all i. It follows that $H^i(X)$ has finite length over k and therefore over $\operatorname{End}(X)$ for all i. Thus X is endofinite.

(3) $\Leftrightarrow$ (4) This follows from the equivalence of (1) and (2) by symmetry.

(2) $\Leftrightarrow$ (4) Use that $D\colon \operatorname{Mod} R \to \operatorname{Mod} R^{\mathrm{op}}$ induces a duality between the finite length modules over R and R^{op}. ☐

Theorem 4.5. *There exists a bijection $X \mapsto X^\vee$ between the isomorphism classes of indecomposable endofinite complexes in $\mathcal{D}(\operatorname{Mod} R)$ and $\mathcal{D}(\operatorname{Mod} R^{\mathrm{op}})$. This bijection has the following properties (where X denotes an indecomposable endofinite complex in $\mathcal{D}(\operatorname{Mod} R)$):*

1. *$X^{\vee\vee} \cong X$.*
2. *DX is a coproduct of copies of $X^\vee$.*
3. *If $\operatorname{h-endol} X = (n_i)_{i\in\mathbb{Z}}$, then $\operatorname{h-endol} X^\vee = (n_{-i})_{i\in\mathbb{Z}}$.*
4. *$\Delta(X^\vee) \cong \Delta(X)^{\mathrm{op}}$, where $\Delta(X) = \operatorname{End}(X)/\operatorname{rad}\operatorname{End}(X)$.*
5. *$H^i(X^\vee) \sim H^{-i}(X)^\vee$ in $\operatorname{Mod} R^{\mathrm{op}}$ for all $i \in \mathbb{Z}$.*
6. *$X^\vee \in \mathcal{D}^b(\operatorname{mod} R^{\mathrm{op}})$ if and only if $X \in \mathcal{D}^b(\operatorname{mod} R)$, provided that R is a noetherian algebra.*

Remark. (1) The map $X \mapsto X^\vee$ is completely determined by the fact that DX is a coproduct of copies of $X^\vee$.
(2) The bijection between indecomposable endofinite complexes specializes to the bijection between indecomposable endofinite right and left R-modules if one identifies modules with complexes concentrated in degree 0.

Proof. In order to define the bijection we use the description of indecomposable endofinite objects via cofinite ideals in the category of compact objects. Clearly, the equivalence $T\colon \mathcal{K}^b(\operatorname{proj} R)^{\mathrm{op}} \to \mathcal{K}^b(\operatorname{proj} R^{\mathrm{op}})$ induces a bijection between the cofinite ideals in $\mathcal{K}^b(\operatorname{proj} R)$ and $\mathcal{K}^b(\operatorname{proj} R^{\mathrm{op}})$. For an indecomposable endofinite complex X in $\mathcal{D}(\operatorname{Mod} R)$ we define $X^\vee$ to be the unique indecomposable endofinite complex in $\mathcal{D}(\operatorname{Mod} R^{\mathrm{op}})$ with $\operatorname{Ann} X^\vee = T \operatorname{Ann} X$.

(1) $T^2 = \mathrm{Id}$ implies $\operatorname{Ann} X^{\vee\vee} = \operatorname{Ann} X$. Thus $X^{\vee\vee} \cong X$.

(2) The isomorphism $\operatorname{Hom}(TP, DX) \cong D\operatorname{Hom}(P, X)$ is functorial in P and implies therefore $\operatorname{Ann} DX = T\operatorname{Ann} X$. By definition, $T\operatorname{Ann} X = \operatorname{Ann} X^\vee$. Thus Theorem 3.7 in [12] implies that DX is a coproduct of copies of $X^\vee$ since $\operatorname{Ann} DX = \operatorname{Ann} X^\vee$.

(3) Let $\mathcal{C} = \mathcal{K}^b(\operatorname{proj} R)$ and $\mathcal{C}' = \mathcal{K}^b(\operatorname{proj} R^{\mathrm{op}})$. Then T induces an equivalence $\overline{T}\colon (\operatorname{mod} \mathcal{C})^{\mathrm{op}} \to \operatorname{mod} \mathcal{C}'$ by sending $\operatorname{Hom}(-, X)$ to $\operatorname{Hom}(-, TX)$. There is a Serre subcategory $\mathcal{S}_{\mathfrak{I}}$ in $\operatorname{mod} \mathcal{C}$ which corresponds to $\mathfrak{I} = \operatorname{Ann} X$. We have $\overline{T}\mathcal{S}_{\mathfrak{I}} = \mathcal{S}_{\mathfrak{J}}$ where $\mathfrak{J} = T\mathfrak{I} = \operatorname{Ann} X^\vee$. Therefore $\overline{T}$ induces an equivalence

$$\widehat{T}\colon (\operatorname{mod} \mathcal{C}/\mathcal{S}_{\mathfrak{I}})^{\mathrm{op}} \longrightarrow \operatorname{mod} \mathcal{C}'/\mathcal{S}_{\mathfrak{J}}.$$

Lemma 1.3 implies that the length of the $\operatorname{End}(X)$-module $H^i(X)$ equals the length of $\operatorname{Hom}(-, \Sigma^{-i} R)$ in $\operatorname{mod} \mathcal{C}/\mathcal{S}_{\mathfrak{I}}$. Using the equivalence $\widehat{T}$, the assertion follows.

(4) We keep the notation from part (3) and denote by S the unique simple object in $\operatorname{mod} \mathcal{C}/\mathcal{S}_{\mathfrak{I}}$. Applying again Lemma 1.3, we get

$$\Delta(X)^{\mathrm{op}} \cong \operatorname{End}(S)^{\mathrm{op}} \cong \operatorname{End}(\widehat{T}S) \cong \Delta(X^\vee).$$

(5) Taking homology preserves coproducts. Therefore (2) implies $H^i(X^\vee) \sim H^i(DX)$. The duality for endofinite modules implies $DH^{-i}(X) \sim H^{-i}(X)^\vee$. Thus $H^i(X^\vee) \sim H^{-i}(X)^\vee$ by Lemma 4.3.

(6) Follows from Lemma 4.4 and (2). ☐

We have seen that the duality between endofinite complexes of right and left modules is induced by the equivalence $T\colon \mathcal{K}^b(\operatorname{proj} R)^{\mathrm{op}} \to \mathcal{K}^b(\operatorname{proj} R^{\mathrm{op}})$. More generally, let $\mathcal{T}$ and $\mathcal{T}'$ be compactly generated triangulated categories and suppose we have an equivalence $T\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$ for the categories of compact objects in $\mathcal{T}$ and $\mathcal{T}'$. Then T induces a bijection $X \mapsto X^\vee$ between the isomorphism classes of indecomposable endofinite objects in $\mathcal{T}$ and $\mathcal{T}'$ such that $\operatorname{Ann} X^\vee = T \operatorname{Ann} X$. Let us analyse this duality for the remaining examples from the introduction.

- For $\mathcal{T} = \underline{\mathrm{Mod}}\, kG$ and $\mathcal{T}' = \underline{\mathrm{Mod}}\, kG^{\mathrm{op}}$, the corresponding duality between endofinite objects is, up to projective direct factors, the duality between endofinite right and left kG-modules from Section 3.
- For the stable homotopy category $\mathcal{T} = \mathcal{S} = \mathcal{T}'$, Spanier-Whitehead duality gives an equivalence $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ for the category $\mathcal{C}$ of finite spectra. The corresponding duality for endofinite spectra is the classical Brown-Comenetz duality mentioned in the introduction.

5. Generic complexes

We consider the bounded derived category $\mathcal{D}^b(\operatorname{Mod} R)$ of modules over a noetherian algebra R. The finiteness condition arising in Lemma 4.4 motivates the following definition. A complex $X \in \mathcal{D}^b(\operatorname{Mod} R)$ is called *generic* if X is indecomposable and endofinite but not isomorphic to an object in $\mathcal{D}^b(\operatorname{mod} R)$. In [5], generic complexes are used to make the following definition.

Definition 5.1. A noetherian algebra R is called *generically derived tame* if for every family $\mathbf{n} = (n_i)_{i\in\mathbb{Z}}$ of natural numbers there are at most finitely many isomorphism classes of generic objects X in $\mathcal{D}^b(\operatorname{Mod} R)$ such that $\text{h-endol} = \mathbf{n}$.

We have now the following consequence of Theorem 4.5.

Corollary 5.2. *A noetherian algebra R is generically derived tame if and only if R^{op} is generically derived tame.*

References

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Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`