The Hilbert metric and Gromov hyperbolicity

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1 Introduction

Let D be a bounded convex domain in $\mathbb{R}^n$ and let h be the Hilbert metric, which is defined as follows. For any distinct points $x, y \in D$ let x' and y' be the intersections of the line between x and y with ∂D closest to x and y respectively. Then

$$h(x,y) = \log \frac{xy' \cdot x'y}{xx' \cdot yy'}$$

where zw denotes the Euclidean distance $\|z - w\|$ between two points. For the basic properties of the distance h we refer to [Bus55] or [dlH93].

We will here give some sufficient conditions for the metric space (D, h) to be hyperbolic in the sense of Gromov . Namely, we show that a certain intersecting chords property implies Gromov hyperbolicity (Theorem 3.1). This intersecting chords property holds when the (Menger) curvature of any three points of the boundary of the domain is uniformly bounded from both above and below in a certain way (Proposition 2.2.1). Domains with C^2 boundary of everywhere nonzero curvature satisfy this condition as will be proved in section 4. Beardon showed in [Bea97] (see also [Bea99]) that a weaker intersecting chords property holds for any bounded strictly convex domain and he used this to establish some

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weak hyperbolicity results for the Hilbert metric. In section 6 we generalize his results to any bounded convex domain. It would be interesting to also understand what converse statements can be, for example what properties of ∂D does Gromov hyperbolicity of (D, h) imply? We give an argument that ∂D must be of class C^1 in section 5.

Some parts of the results in this paper are already known: Yves Benoist informed us that a convex domain with C^2 boundary is Gromov hyperbolic if the curvature of the boundary is everywhere nonzero. Benoist has also found examples of Gromov hyperbolic Hilbert geometries whose boundaries are C^1 but not C^2. In [Bea99] it is mentioned that C. Bell has proved an intersecting chords theorem in an unpublished work. We have however not found the present arguments in the literature.

Since the Hilbert distance can be defined in analogy with Kobayashi's pseudo-distance on complex spaces [Kob84], we would like to mention that Balogh and Bonk announced in [BB99] that the Kobayashi metric on any bounded strictly pseudoconvex domain with C^2 boundary is Gromov hyperbolic.

Note that metric spaces of this type are CAT(0) only in exceptional cases: Kelly and Straus proved in [KS58] that if (D, h) is nonpositively curved in the sense of Busemann then D is an ellipsoid and hence (D, h) is hyperbolic space. (This in particular answers a question later raised in [dlH93].) Compare this to the analogous situation for Banach spaces: a Banach space is CAT(0) if and only if it is a Hilbert space. Another category of results are of the type: large (infinite, cocompact, etc.) automorphism group and smooth boundary imply that the space is hyperbolic in the classical sense. The Hilbert metric has found many applications, typically using the fact that affine maps cannot increase Hilbert distances combined with the contraction mapping principle.

The work was mainly done during our stay at Bielefeld University. We thank this university for its hospitality.

2 Intersecting chords in convex domains

From elementary school we know that if c_1, c_2 are two chords in a circle and which intersect each other, then $l_1 l'_1 = l_2 l'_2$ where l_1, l'_1, and l_2, l'_2 denote the respective lengths of the segments the two chords are divided into. (It follows immediately from the similarity of associated triangles, see Fig. 1).

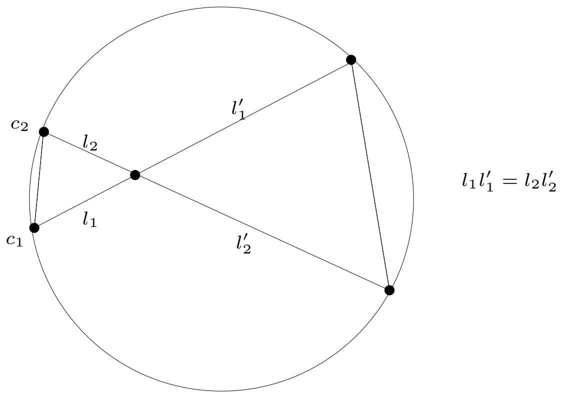

Figure 1: Intersecting chords theorem.

A generalization of this fact to any bounded strictly convex domain was given by Beardon in [Bea97] by an elegant argument using the Hilbert metric. He proved that if D is such a domain then for each positive δ there is a positive number $M = M(D, \delta)$ such that if c_1, c_2 are intersecting chords of D each of length at least δ divided by the point of intersection into segments of lengths l_1, l'_1, and l_2, l'_2 respectively, then

$$M^{-1} \leq \frac{l_1 l'_1}{l_2 l'_2} \leq M. \tag{1}$$

We say that a domain satisfies the intersecting chords property (ICP) if (1) holds for *any* two intersecting chords c_1 and c_2. It is easy to see that ICP may fail for a general strictly convex domain (at a curvature zero point or a "corner").

We show in this section that ICP holds for domains that satisfy a certain (non-differentiable) curvature condition. Domains with C^2 boundary of nonvanishing curvature satisfy this condition, see section 4.

In the next subsection we will clarify exactly the relation between the curvature of any triple of endpoints and the ratio as above that two intersecting line segments define.

2.1 Intersecting line segments

This subsection contains a piece of elementary geometry. Consider two line segments that intersect each other in one point, see Fig. 2. Each three of the four endpoints defines a circle going through these points. From elementary geometry we know that the radii of the four circles so obtained are

$$r_{\alpha 1} = \frac{c}{2\sin\alpha_1}, r_{\alpha 2} = \frac{d}{2\sin\alpha_2}, r_{\beta 1} = \frac{d}{2\sin\beta_1}, r_{\beta 2} = \frac{c}{2\sin\beta_2}.$$

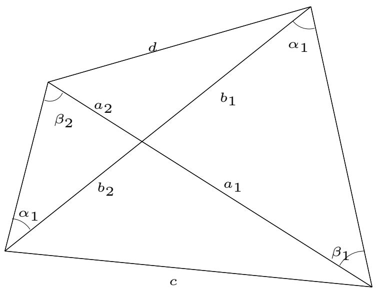

Figure 2: Two intersecting line segments.

Proposition 2.1.1 *In the notation above, the following equality holds:*

$$\frac{a_1 a_2}{b_1 b_2} = \frac{r_{\beta 1} r_{\beta 2}}{r_{\alpha 1} r_{\alpha 2}}.$$

Proof. By the sine law we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{\sin\alpha_1 \sin\alpha_2}{\sin\beta_1 \sin\beta_2} = \frac{2\sin\alpha_1}{c} \frac{c}{2\sin\beta_2} \frac{2\sin\alpha_2}{d} \frac{d}{2\sin\beta_1} = \frac{r_{\beta 1} r_{\beta 2}}{r_{\alpha 1} r_{\alpha 2}}.$$

□

2.2 The intersecting chords property and Menger curvature

Let $K(x, y, z)$ denote the (Menger) curvature of three distinct points which in a Euclidean space equals the reciprocal of the radius of the circle passing through these three points, cf. the previous subsection.

Proposition 2.2.1 *Let D be a bounded convex domain in $\mathbb{R}^n, n \geq 1$. Assume that there is a $C > 0$ such that*

$$\frac{K(x, y, z)}{K(x', y', z')} \leq C \tag{2}$$

for any two triples of distinct points in ∂D all lying in the same 2-dimensional plane. Then D satisfies the intersecting chords property.

Proof. Any two intersecting chords define a plane and by Proposition 2.1.1 we have

$$\frac{a_1 a_2}{b_1 b_2} = \frac{K_{\alpha 1} K_{\alpha 2}}{K_{\beta 1} K_{\beta 2}} \leq C^2. \tag{3}$$

□

2.3 Chords larger than δ

Proposition 2.3.1 *Let D be a bounded convex domain in $\mathbb{R}^n, n \geq 1$. Given δ such that the length of any line segment contained in ∂D is bounded from above by some $\delta' < \delta$. Then there is a constant $C = C(\delta) > 0$ such that if $x, y, z \in \partial D$, $xy \geq \delta$ we have*

$$C(\delta) \leq K(x, y, z) \leq \frac{1}{\delta}. \tag{4}$$

Proof. The angle $\alpha(x, y, v) := \angle_y(xy, v)$ is continuous in $x, y \in \mathbb{R}^n$ and $v \in UT_y(\partial D)$, the unit tangent cone at y. If $[x, y]$ does not lie in ∂D, then $0 < \alpha(x, y, v) < \pi$. The set

$$S = \{(x, y, v) \in \partial D \times \partial D \times UT_y(\partial D) : xy \geq \delta\} \tag{5}$$

is compact. Hence there is a constant $\alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha|(x, y, v) \leq \pi - \alpha_0 \tag{6}$$

for every $(x, y, v) \in S$. By the definition of the tangent cone and compactness there is an $\varepsilon > 0$ such that for any $y, z \in \partial D$, $0 < yz < \varepsilon$ there is $v \in UT_y(\partial D\ )$, for which

$$0 \leq \angle_y(yz, v) \leq \alpha_0/2. \tag{7}$$

The estimates (6),(7) imply the existence of $C > 0$ and the other inequality in 2.3.1 is trivial. □

Corollary 2.3.2 (Cf.[Bea99]) *Let D be a bounded convex domain such that any line segment in ∂D has length less than $\delta' < \delta$. Then intersecting chords property holds for any two chords each of length $\geq \delta$.*

Proof. This immediately follows from Propositions 2.3.1 and 2.1.1.

Remark 2.3.3 In view of this subsection it is clear that ICP implies restrictions on the curvature κ. We were however not able to establish the converse of Proposition 2.2.1.

3 Hyperbolicity of Hilbert's metric

Let (Y, d) be a metric space. Given two points $z, w \in Y$, let

$$(z|w)_y = \frac{1}{2}(d(z, y) + d(w, y) - d(z, w))$$

be their Gromov product relative to y. We think of y as a fixed base point. The metric space Y is Gromov hyperbolic (or δ-hyperbolic) if there is a constant δ such that the inequality

$$(x|z)_y \geq \min\{(x|w)_y, (w|z)_y\} - \delta$$

holds for any three points x, z, w. By expanding the terms this inequality is equivalent to

$$d(x, z) + d(y, w) \leq \max\{d(x, y) + d(z, w), d(y, z) + d(x, w)\} + 2\delta. \tag{8}$$

Theorem 3.1 *Let D be a bounded convex domain in $\mathbb{R}^n$ satisfying the intersection chords property. Then the metric space (D, h) is Gromov hyperbolic.*

Proof. Suppose that intersecting chords property holds with a constant M. Let y be a fixed reference point and consider any other three points x, z, w in D. Set $A(u,v) = h(u,v) + h(w,y) - h(u,w) - h(v,y)$ for any two points u, v. By (8) we need to show that there is a constant δ independent of x, z, w such that

$$\min\{A(x,z), A(z,x)\} \leq 2\delta \tag{9}$$

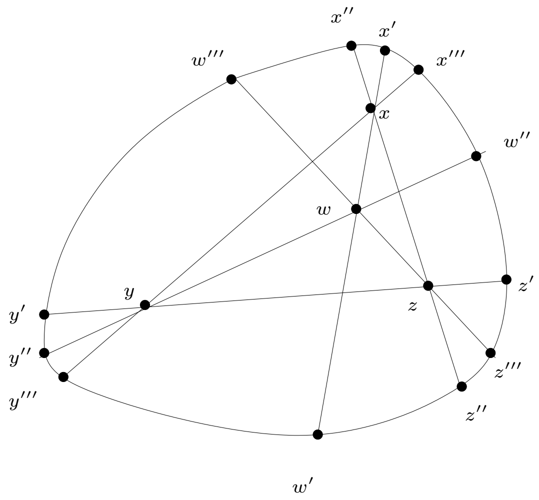

Figure 3: 4 points.

Now we have using the definition of h and the notation in the picture that

$$A(x,z) = \log\left(\frac{xz'' \cdot zx''}{xx'' \cdot zz''} \; \frac{wy'' \cdot yw''}{ww'' \cdot yy''} \; \frac{xx' \cdot ww'}{xw' \cdot wx'} \; \frac{zz' \cdot yy'}{zy' \cdot yz'}\right) =$$

(rearranging the members of the product)

$$= \log\left(\frac{xx' \cdot xz''}{xx'' \cdot xw'} \; \frac{yy' \cdot yw''}{yy'' \cdot yz'} \; \frac{zz' \cdot zx''}{zz'' \cdot zy'} \; \frac{ww' \cdot wy''}{ww'' \cdot wx'}\right) \leq$$

(using $\frac{xx'}{xx''} \leq M \frac{xz''}{xw'}$ and similar inequalities for the other fractions)

$$\leq M' + 2\log\left(\frac{xz''}{xw'} \; \frac{yw''}{yz'} \; \frac{zx''}{zy'} \; \frac{wy''}{wx'}\right)$$

$\leq$ (since y is fixed and zy', wy'' are bounded from above and below respectively)

$$\leq M'' + 2\log\left(\frac{xz''\cdot zx''}{xw'\cdot wx'}\right).$$

So (9) is equivalent to the boundedness of

$$\min\left\{\frac{xz''\cdot zx''}{xw'\cdot wx'}, \frac{zx''\cdot xz''}{zw''\cdot wz''}\right\}.$$

from above. By symmetry we can without loss of generality assume that $zz'' \leq xx''$. Now we have two cases:

Case 1. $xw \geq xx''$ or $zw \geq xx''$

If $xw \geq zw$ (so $xw \geq xx''$ for sure), then

$$\frac{xz''\cdot zx''}{xw'\cdot wx'} \leq \frac{(xz+zz'')(zx+xx'')}{(xw)^2} \leq \frac{(xw+wz+zz'')(zw+wx+xx'')}{(xw)^2} \leq \frac{(3xw)^2}{(xw)^2} \leq 9$$

When $zw \geq xw$, we estimate instead the other fraction (coming from switching x and z) in the same way.

Case 2. $xw \leq xx''$ and $zw \leq xx''$

$$\frac{xz''\cdot zx''}{xw'\cdot wx'} \leq (\text{chords at } x)\ \ M\ \frac{xx'\cdot zx''}{xx''\cdot wx'} \leq M\frac{xx'}{wx'}\frac{(xw+wz+xx'')}{xx''} \leq 3M$$

since $xx''\cdot xz'' \leq M(xx'\cdot xw')$. □

Remarks 3.2 Since the n-dimensional ball B^n obviously satisfies the condition of Proposition 2.2.1 with $C = 1$, this shows in particular the standard fact that (B^n, h), which is Klein's model of the n-dimensional hyperbolic space, is Gromov hyperbolic.

4 Intersecting chords theorem for convex C^2-domains

Theorem 4.1 *Let D be a bounded convex domain in $\mathbb{R}^n$. Suppose that the boundary ∂D is smooth of class C^2 and the curvature of ∂D is everywhere nonzero. Then there is a constant M such that if c_1, c_2 are intersecting chords of D, divided by the point of intersection into segments of lengths l_1, l_1', and l_2, l_2' respectively, then*

$$M^{-1} \leq \frac{l_1 l_1'}{l_2 l_2'} \leq M. \tag{10}$$

Remarks 4.2 The following sketch of the proof in dimension 2 looks the most attractive. Assume that D is a domain in $\mathbb{R}^2$ with C^2 boundary of nowhere vanishing curvature. It is a standard fact that the circle through three points converge to the osculating circle at p when the three points converge to p, see [BG88]. It is also easy to see that when two of the points collapse to a point q the circles converge to the circle through the third point and tangent to the tangent line at q. By compactness and continuity it follows that D satisfies the condition in Proposition 2.2.1 and thus it satisfies the intersection chords property. We have been unable to make this sketch rigorous, although in the paper [BM70] it is proven that for a curve in $\mathbb{R}^3$ of class C^3 the Menger curvature coincides with the classical one.

Proof of ICT in dimension 2. Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be a periodic arc length parameterization of the curve ∂D such that $\mathbf{r}$ is of class C^2 and $\mathbf{r}''$ does not vanish everywhere. Then for the curvature $\kappa(s)$ of ∂D at the point $\mathbf{r}(s)$ we have $|\kappa(s)| = |\mathbf{r}''(s)|$. Let $\kappa > 0$ be such that $\kappa \leq \mathbf{r}''(s) \leq \kappa^{-1}$ for $s \in \mathbb{R}$. By a ray in $D \cup \partial D$ we mean a segment joining a point of D (origin of the ray) with the point of ∂D (limit point of the ray.) Suppose that the origin of the ray R lies inside some osculating circle C. By a companion of R in O we call a ray R' of the interior of O such that $R \subset R'$ or $R' \subset R$. For a $\lambda > 0$ we write $R \sim_\lambda R'$ if $\lambda < |R|/|R'| < \lambda^{-1}$. By a projection of $p \in D$ onto ∂D we call any point $p' \in \partial D$ minimizing the distance function $x \mapsto |x - p|$. It is easy to see that the line, passing through p' and orthogonal to the segment $[p, p']$ is a supporting line for D that is all of D lies on one side of the line.

Lemma 4.3 *There is an $\varepsilon > 0$ and a constant $\lambda = \lambda(\varepsilon)$ such that for any ray $[p, q]$ of the length at most ε and any projection p' of p such that $|p - p'| < \varepsilon$ and any companion R' of R in the osculating circle at p' we have $R \sim_\lambda R'$.*

Proof. Choose $0 < \varepsilon < \kappa/2$ such that near any point $\mathbf{r}(s)$ the curve ∂D is a graph of a C^2 function $y = f(x), x \in (\varepsilon, \varepsilon)$ in a canonical Cartesian coordinate in the point $\mathbf{r}(s)$.

Recall that the osculating circle of ∂D at the point $\mathbf{r}(s)$ is a circle of radius $r = \frac{1}{|\mathbf{r}(s)|}$ and with a center $(0, r)$ in coordinates x, y. Since $f''(0) = \mathbf{r}''(s)$ ([Rut00], 7.8) we may assume, decreasing ε, that $1/2 \leq \frac{f''(0)}{f''(x)} \leq 3/2, x \in (\varepsilon, \varepsilon)$. Let R be the ray from y_0 to the point $p \in \partial D$ and suppose that the Euclidean length of R is not greater than $\varepsilon/2$. Then $p = (a, f(a))$ for some $a \in (\varepsilon, \varepsilon)$. The Euclidean ray, extending R, intersects the osculating circle in some point $q = (b, f(b)), b \in (\varepsilon, \varepsilon)$. Elementary geometry ensures that if $a \leq b$, then $\frac{1}{2} \leq a/b \leq 1$ and if $a \geq b$, then $1 \leq a/b \leq 3$. $\square$

Completing the proof. Let ε be the constant chosen as in Lemma 4.3. Consider a pair of chords c_1, c_2 of D, divided by the point of intersection p into the rays R_1, R_1' and R_2, R_2' of lengths l_1, l_1', and l_2, l_2' respectively. If $d(p, \partial D) \geq \varepsilon$ then all the chords are large enough and we can refer to the Proposition 2.3.1. If $d(p, \partial D) \leq \varepsilon$ then take the projection point p' of p onto ∂D as the origin for the canonical Cartesian coordinate system and let C be the osculating circle at p'. The rays R_1, R_1', R_2, R_2' by extending define the companion rays $\tilde{R}_1, \tilde{R}_1', \tilde{R}_2, \tilde{R}_2'$ of C. If one of the rays R_1, R_1', R_2, R_2', say R_1, is of length $\leq \varepsilon$ then by Lemma 4.3 $R \sim_\lambda R'$ with a universal constant λ. If one of the rays is of length $\geq \varepsilon$ then its length is bounded from both above and below by two positive constants. In any case the ratio $\frac{l_1 l_1'}{l_2 l_2'}$ is bounded from both above and below by two positive constants depending only on D.

***n*-Dimensional case.** It is enough to prove

Proposition 4.4 *Let D be a bounded convex domain in $\mathbb{R}^{n+1}, n \geq 2$. Suppose that the boundary ∂D is smooth of class C^2 and the Gauss curvature of ∂D is everywhere nonzero. Then there are positive constants $c < C$ depending only on D such that for any plane section S of ∂D and any two points $p_1, p_2 \in S$*

$$c < \left| \frac{\kappa(p_1)}{\kappa(p_2)} \right| < C,$$

where $\kappa(p)$ denotes the curvature at $p \in S$.

Proof. Let ρ be a C^2 defining function for D that is the following properties hold: $D = \{x \in \mathbb{R}^n : \rho(x) > 0\}, \mathbb{R}^n \setminus \overline{D} = \{x \in \mathbb{R}^n : \rho(x) < 0\}$ and gradient $\mathbf{n}(x) = \nabla \rho(x)$ is of the length 1 for all $x \in \partial D$. Then $\mathbf{n}(x)$ is a normal vector field on ∂D directed inside D.

Curvature. The curvature operator $\mathcal{W}_x : T_x\partial D \to T_x\partial D$ is defined by setting $\mathcal{W}_x(v)$ equal to the directions derivative of the normal in the direction of v, that is $\mathcal{W}_x(v) = \nabla_v \mathbf{n}(x) = v \cdot \nabla^2 \rho(x),\ v \in T_x\partial D$. The second fundamental form is the bilinear form II_x on the tangent space $T_x\partial D$ given by $II_x(v,w) = w \cdot \mathcal{W}_x(v) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_i} w_i v_j$, where v, w are tangent to ∂D at x. The value $II_x(v,v) = \kappa(v)$ on the unit tangent vector v is called the normal curvature of ∂D in the direction v. It is equal to the normal component of acceleration at x of every parameterized curve in ∂D passing through x with velocity v. Further insight into the meaning of normal curvature can be gained from normal sections. The normal section determined by the unit vector v at $T_x\partial D$ is an affine 2-plane Π_v passing through x and containing the vectors $v, \mathbf{n}$. There is neighbourhood of x in which the intersection C_v of normal section with ∂D 1 is a plane curve, and the curvature at x of this curve is equal to the normal curvature $\kappa(v)$ [Tho94].

The determinant $K(x)$ of $\mathcal{W}_x$ is called the Gauss curvature of ∂D at x. The n eigenvalues of $\mathcal{W}_x$ are called the principal curvatures of ∂D at x. By assumption and in view of compactness of ∂D there is a constant κ_D such that $\kappa_D < \kappa(v) < \kappa_D^{-1},\quad v \in T_x\partial D\ , x \in \partial D$.

Parameterization of plane sections. Let $p \in \partial D$, $\mathbf{n} = \mathbf{n}(p)$. For any choice of an orthonormal basis $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of $T_p\partial D$ and any choice of an angle $\theta, 0 \le \theta < \pi/2$ there is an affine 2-plane $\Pi_{\mathbf{e},\theta}$, containing p and spanned by the unit vectors $e_1, v = \sin\theta\ e_2 + \cos\theta\ \mathbf{n}$. Clearly θ is equal to the angle between $\Pi_{\mathbf{e},\theta}$ and $\mathbf{n}$ and to the angle between v and $\mathbf{n}$. The intersection $C_{\mathbf{e},\theta} = \Pi_{\mathbf{e},\theta} \cap \partial D$ is a C^2 plane curve in $\Pi_{v,w}$, [Tho94],thm.1,§12. It is easy to see that an arbitrary nondegenerate plane section of ∂D is of the form $C_{\mathbf{e},\theta}$.

Lemma 4.5 (Meussnier, 1776, [Kli78]).

$$\kappa(e_1) = \kappa \cos\theta,$$

where κ is a curvature of $C_{\mathbf{e},\theta}$ at p.

Pinching θ. In view of Meussnier's lemma and since $\kappa(e_1)$ is positively pinched it is enough to prove that the variation of the $\cos\theta$ along any 2-dimensional section of ∂D is uniformly bounded in the following sense. There are positive constants $c < C$ depending only on D such that the ratio $\frac{\theta_1}{\theta_2} \in [c, C]$ for any two angles at the points of the section. Compactness arguments show that it is enough to bound the ratio for small sections, i. e. for section of diameter $\leq \varepsilon$ where ε is some positive constant. We will choose ε in several steps. We switch to an alternative point of view onto plane sections. Given a point $p \in \partial D$, we can choose the coordinate axis of $\mathbb{R}^{n+1}$ so that the origin O of the coordinates is at p and the x_0 axis is directed along the positive normal of S at p. It follows that in a neighbourhood of p the surface ∂D can be represented in the form $x_0 = f(x_1, x_2, \ldots, x_n), x = (x_1, x_2, \ldots, x_n) \in U \subset \mathbb{R}^n$, where U is an open set and f is a C^2 function with $f(0) = 0, f_{x_i}(0) = 0, i = 1, \ldots, n$. We have a 2-dimensional section $C_\varepsilon : f(x_1, x_2, 0, \ldots, 0) = \varepsilon$. From this point of view the angle $\theta = \theta(q)$ equals to the angle between $\mathbf{n}(q)$ and plane $\Pi = \mathbb{R}e_1 + \mathbb{R}e_2$. Thus we need to bound the variation of the angle $\theta = \angle(\mathbf{n}(q), \mathbf{n}(p)), q \in C_\varepsilon$. Clearly

$$\cos \angle(\mathbf{n}(q), \Pi) = \frac{1}{\sqrt{1 + f_{x_1}^2 + \cdots + f_{x_n}^2}} \sqrt{f_{x_1}^2 + f_{x_2}^2}$$

Pinching the variation of a gradient. Since the denominator in formula above can be made uniformly close to 1 in the neighbourhood of p, we reduce the question to bounding the variation of function $\sqrt{f_{x_1}^2 + f_{x_2}^2}$ on C_ε. Using the notations $x = x_1, y = x_2, f(x, y) = f(x_1, x_2, 0, \ldots, 0)$ we reformulate the problem as the problem to bound the variation of the gradient $|\nabla f| = \sqrt{f_x^2 + f_y^2}$ when q runs through C_ε. By rotation in the plane x, y we may assume the x and y axis along the direction of maximal principal curvature. Thus, $f_{xy}(0, 0) = 0$ and the principal curvatures in p are $a = f_{xx}(0, 0), b = f_{yy}(0, 0)$, $\kappa_D < a, b < \kappa_D^{-1}$. The second fundamental form of ∂D at p applied to the vector $(u, v) \in \mathbb{R}^2$ becomes, in this case $II_0(u, v) = \frac{1}{2}(au^2 + bv^2)$. By developing $f(x, y)$ into a Taylor's expansion about the origin, and taking into account that $f_x(0, 0) = 0 = f_y(0, 0)$, we obtain $f(x, y) = \frac{1}{2}(ax^2 + by^2) + r$, where r vanishes at $(0, 0)$ together with all its derivatives up to second order. We have $\nabla f = (ax, by) + \nabla r := \ell + \nabla r$. Since a, b are pinched there are positive constants c, C, depending only on D such that $c < \frac{|\ell(x,y)|}{\sqrt{x^2+y^2}} < C$. Next $|\nabla r| = |(r_x, r_y)| = |(r_{xx}(x')x, r_{yy}(y')y)|$ for some $x' \in (0, x), y' \in (0, y)$. Hence, decreasing ε, we assume that $|\nabla r| \leq \frac{c}{2}\sqrt{x^2+y^2}$ if $\sqrt{x^2+y^2} < \varepsilon$. We conclude that $c/2 < \frac{|\nabla f(x,y)|}{\sqrt{x^2+y^2}} < 2C$ if $\sqrt{x^2+y^2} < \varepsilon$.

Pinching the variation of x^2+y^2 on the plane section. By the previous equality it remains to show the boundedness of the variation of the function x^2+y^2 on C_ε.

Decrease ε so that $|r_{xx}|, |r_{xy}|, |r_{yy}| < \frac{1}{2}\kappa_D$ in the ε-neighbourhood of p uniformly with respect to $p \in \partial D$. By Taylor's formula $r(x,y) = d^2 r_{(\alpha x, \alpha y)}(x,y)$ for some $\alpha \in (0,1)$ and hence $|r(x,y)| < \frac{1}{4}\kappa_D(x^2+y^2)$. Hence $\frac{1}{4}\kappa_D(x^2+y^2) < f(x,y) < \frac{5}{4}\kappa_D(x^2+y^2)$. Since $f(x,y) = \varepsilon$ on the curve C_ε we conclude from the last equation that the variation of x^2+y^2 is bounded on C_ε – this proves the last claim and the proposition too. $\square$

5 Consequences of Gromov hyperbolicity for the shape of boundary

Proposition 5.1 *Let D be an bounded convex domain in $\mathbb{R}^n, n \geq 1$ and let h be a Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is strictly convex, that is it does not contain a (nondegenerate) segment.*

This can be proven by following the proof of N. Ivanov [Iva97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus ≥ 2) are not Gromov hyperbolic. The proof makes use the Gromov's exponential divergence criterion.

Theorem 5.2 *Let D be an bounded convex domain in $\mathbb{R}^2$, and let h be a Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is smooth of class C^1.*

Proof. First, by the previous result, D is strictly convex. Let $y = f(x), x \in (-a,a)$ be an equation of ∂D near some point. Then f is strictly convex and hence one-sided derivatives $f'_-(x), f'_+(x)$ exist and are strictly increasing on $(\varepsilon, \varepsilon)$, [RV73], §11.

We prove that $f'_-(0) = f'_+(0)$. Suppose not, then by choosing appropriate Cartesian coordinates we may assume that $f'_-(0) < 0, f'_+(0) > 0$. For each sufficiently small ε build an ideal triangle $\Delta = \Delta(\varepsilon)$ in D with one vertex 0 and two other vertices corresponding to the intersection of the line $y=\varepsilon$ with ∂D . We assert that the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero. Namely we show that the Hilbert distance between the point $P=(0,\varepsilon)$ and any point Q of the side $[0,B]$ tends to ∞. Let $f'_+(0)=\tan\alpha, 0<\alpha<\pi/2$. Let $x_1<x_2$ be the points such that $f(x_1)=\varepsilon$ and $f'_+(0)x_2=\varepsilon$. Then

$$PQ \geq \varepsilon\cos\alpha = f(x_1)\cos\alpha. \tag{11}$$

Let O, R be the intersection points of the line PQ with ∂D . We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1)-f'_+(0)x_1}{f'_+(0)} \tag{12}$$

and hence combining the last two inequalities

$$\frac{PQ}{QR} \geq \frac{f'_+(0)f(x_1)\cos\alpha}{f(x_1)-f'_+(0)x_1} = \tag{13}$$

$$= \frac{f'_+(0)\cos\alpha}{1-f'_+(0)\frac{x_1}{f(x_1)}} \to \infty \text{ when } x_1\to 0. \tag{14}$$

It follows that

$$h(P,Q) = \ln\left(1+\frac{PQ}{OP}\right)\left(1+\frac{PQ}{QR}\right) \to \infty \text{ when } x_1 \to 0$$

and hence the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero.

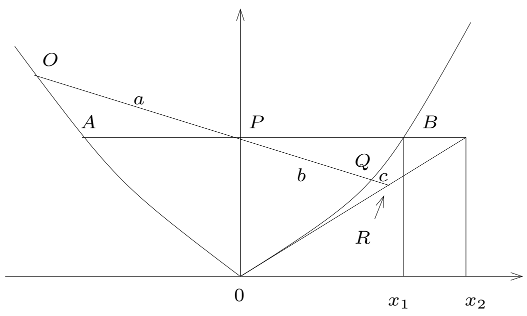

Figure 4: Hyperbolicity implies C^1.

It remains to show that f' is continuous. By [RV73],§14 we have

$$\lim_{x\to x_0+} f'_+(x) = f'_+(x_0)$$

$$\lim_{x\to x_0-} f'_+(x) = f'_-(x_0)$$

From this we conclude that f'_+ is continuous in x_0 since $f'_+(x_0) = f'_-(x_0)$. But $f'(x_0) = f'_+(x_0)$ hence f' is also continuous in x_0.

6 Non-strictly convex domains

This section owes much of its existence to [Bea97] and [Bea99]. We extend some of the results obtained in those papers to arbitrary bounded convex domains. In addition to Beardon's arguments, we take advantage of Proposition 2.2.1 in the present paper and an argument in [Kar99].

For some collection of open sets $\{U_\alpha\}$ a bounded convex domain D we let

$$S = S(\{U_\alpha\}) = \partial D \ \times \partial D \ \setminus \bigcup_\alpha (U_\alpha \cap \partial D \ ) \times (U_\alpha \cap \partial D \ ).$$

Since S is compact, following the proof of Proposition 2.3.1 we get

Lemma 6.1 *Let D be a bounded convex domain. Given neighborhoods U_α around every set α of the type $[x, z]$ (a point or a line segment) contained in ∂D . Then there exists a constant $C > 0$ such that for any $(x, y) \in S$ and any other point z (different from x, y) in ∂D*

$$C^{-1} \le K(x, y, z) \le C.$$

□

As an immediate consequence of Proposition 2.2.1 and Lemma 6.1 we get

Proposition 6.2 *Let D be a bounded convex domain. Given neighborhoods U_α around every set α of the type $[x, z]$ (a point or a line segment) contained in ∂D . Then there exists a constant $M > 0$ such that for any two intersecting chords, with two pairs of endpoints in S,*

$$M^{-1} \le \frac{l_1 l'_1}{l_2 l'_2} \le M,$$

where l_i, l'_i are the lengths of the segments as usual.

$\square$

Lemma 6.3 *Let D be a bounded convex domain. Let $\{x_n\}, \{y_n\}$ be two sequences of points in D. Assume that $x_n \to \overline{x} \in \partial D$, $y_n \to \overline{y} \in \overline{D}$ and $[\overline{x}, \overline{y}] \nsubseteq \partial D$. Let x'_n and y'_n denote the endpoints of the chord through x_n and y_n as usual. Then x'_n converges to $\overline{x}$ and y'_n converges to the endpoint $\overline{y}'$ of the chord defined by $\overline{x}$ and $\overline{y}$ different from $\overline{x}$.*

Proof. Cf. Lemma 5.3. in [Bea97]. Every limit point of chord endpoints must belong to the line through $\overline{x}$ and $\overline{y}$. In addition, in the case of x'_n for example, any limit point must lie in the halfline from $\overline{x}$ not containing $\overline{y}$. At the same time every limit point must belong to the boundary of D, and the statement follow since the line through $\overline{x}$ and $\overline{y}$ intersect ∂D in $\overline{x}$ and $\overline{y}$. $\square$

Theorem 6.4 *Let D be a bounded convex domain. Let $\{x_n\}$ and $\{z_n\}$ be two sequences of points in D. Assume that $x_n \to \overline{x} \in \partial D$, $z_n \to \overline{z} \in \partial D$ and $[\overline{x}, \overline{z}] \nsubseteq \partial D$. Then there is a constant $K = K(x, z)$ such that for the Gromov product $(x_n|z_n)_y$ relative to some fixed point y in D we have*

$$\limsup_{n\to\infty} (x_n|z_n)_y \leq K.$$

Proof. Pick the collection U_α suitably (depending on $\overline{x}$ and $\overline{y}$, so that

$$(\overline{x}, \overline{z}), (\overline{x}, \overline{y}'), (\overline{z}, \overline{y}'') \in S$$

for the limits of the endpoints $\overline{x}\ \overline{z}\ \overline{y}'\overline{y}''$. By Lemma 6.3 we thus have for all large enough n that

$$(\overline{x}_n{}''', \overline{z}_n{}'''), (\overline{x}_n{}', \overline{y}_n{}'), (\overline{z}_n{}'', \overline{y}_n{}'') \in S,$$

see the picture for the notation. Moreover, there is a $\delta > 0$ such that $x_n z'''_n$ and $z_n x'''_n$ are both larger than δ.

Because y is fixed and Proposition 6.2 we then have for all large enough n that

$$(x_n|z_n)_y = \frac{1}{2} \log \left(\frac{x_n y'_n \cdot y x'_n}{x_n x'_n \cdot y y'_n} \ \frac{y z''_n \cdot z_n y''_n}{y y''_n \cdot z_n z''_n} \ \frac{x_n x'''_n \cdot z_n z'''_n}{x_n z'''_n \cdot z_n x'''_n} \right)$$

$$\leq C + \frac{1}{2}\log\left(\frac{x_n x_n'''}{x_n x_n' \cdot x_n z_n'''} \quad \frac{z_n z_n'''}{z_n z_n'' \cdot z_n x_n'''}\right)$$

$$\leq C' + \log\left(\frac{1}{x_n z_n'''} \quad \frac{1}{z_n x_n'''}\right) \leq C' - 2\log\delta =: K$$

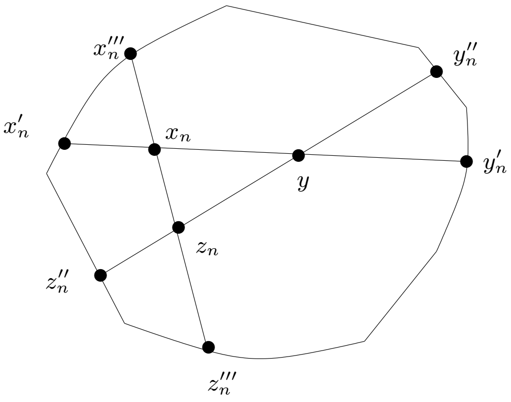

Figure 5: Weak hyperbolicity.

Corollary 6.5 *Let D be a bounded convex domain and $\phi : D \to D$ be a map which does not increase Hilbert distances (e.g. ϕ may be an isometry). Then either the orbit $\{\phi^n(y)\}_{n=1}^{\infty}$ is bounded or there is a limit point y of the orbit such that for any other limit point x of the orbit it holds that $[x, y] \subset \partial D$.*

Proof. This is easily proved using the arguments in [Kar99], compare with Proposition 5.1 in that paper. □

Remark 6.6 The content of Theorem 6.4 is that (D, h) satisfies a weak notion of hyperbolicity. This property should be compared with Gromov hyperbolicity, especially with the fact for Gromov hyperbolic spaces that two sequences converge to the same point of the boundary if and only if their Gromov product tends to infinity. Theorem 6.4 can be used as in [Kar99], Theorem 8 to the study of random walks on the automorphism group of D, and it is also likely to be useful for analyzing commuting nonexpanding maps or isometries of (D, h).

Remarks 6.7 We suggest that the same picture might hold for the classical Teichmüller spaces and also more general Kobayashi hyperbolic complex spaces. Hilbert geodesic rays from a point y that terminate in a line segment contained in the boundary may correspond to the Teichmüller geodesic rays defined by Jenkins-Strebel differentials that H. Masur considered when demonstrating the failure of CAT(0) for the Teichmüller space of Riemann surfaces of genus $g \geq 2$. The complement of the union of all line segments in the boundary in ∂D may correspond to the uniquely ergodic foliation points on the Thurston boundary of Teichmüller space.

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