LATTICE POINT PROBLEMS AND VALUES OF QUADRATIC FORMS*

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Abstract. For d-dimensional ellipsoids E with $d \geq 5$ we show that the number of lattice points in rE is approximated by the volume of rE, as r tends to infinity, up to an error of order $\mathcal{O}(r^{d-2})$ for general ellipsoids and up to an error of order $o(r^{d-2})$ for *irrational* ones. The estimate refines earlier bounds of the same order for dimensions $d \geq 9$. As an application a conjecture of Davenport and Lewis about the shrinking of gaps between large consecutive values of $\mathbb{Q}[m]$, $m \in \mathbb{Z}^d$ of positive definite irrational quadratic forms $\mathbb{Q}$ of dimension $d \geq 5$ is proved. Finally, we provide explicit bounds for errors in terms of certain Minkowski minima of convex bodies related to these quadratic forms.

1. Introduction and results

Let $\mathbb{R}^d$, $1 \leq d < \infty$, denote the d-dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ defined by $|x|^2 = \langle x, x \rangle = x_1^2 + \dots + x_d^2$, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let $\mathbb{Z}^d$ denote the standard lattice of points with integer coordinates in $\mathbb{R}^d$. Consider the quadratic form

$$\mathbb{Q}[x] = \langle \mathbb{Q}x, x \rangle, \qquad \text{for} \quad x \in \mathbb{R}^d,$$

where $\mathbb{Q} : \mathbb{R}^d \to \mathbb{R}^d$ denotes a symmetric linear operator in $\mathrm{GL}(d, \mathbb{R})$ with eigenvalues, say, $q_1, \dots, q_d$. Write

$$q_0 = \min_{1 \leq j \leq d} |q_j|, \qquad q = \max_{1 \leq j \leq d} |q_j|.$$

We assume that the form is non-degenerate, that is, that $q_0 > 0$. Thus, without loss of generality we can and shall assume throughout that $q_0 = 1$, and hence $q \geq 1$. Define for $r \in \mathbb{R}$ the sets

$$E_r = \left\{ x \in \mathbb{R}^d : \mathbb{Q}[x] \leq r^2 \right\}.$$

If the quadratic form $\mathbb{Q}[x]$ is positive definite, then E_r is an ellipsoid.

Recall that a quadratic form $\mathbb{Q}[x]$ and the corresponding operator $\mathbb{Q}$ with non-zero matrix $\mathbb{Q} = (q_{ij})$, $1 \leq i, j \leq d$, is called *rational* if there exists a real number $\lambda \neq 0$ such that the matrix $\lambda\mathbb{Q}$ has integer entries only; otherwise it is called *irrational*.

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For any (measurable) set $B \subset \mathbb{R}^d$ let $\operatorname{vol} B$ denote the Lebesgue measure of B and $\operatorname{vol}_{\mathbb{Z}} B$ its lattice volume, that is the number of lattice points in $B \cap \mathbb{Z}^d$. We want to investigate the approximation of this lattice volume by the Lebesgue volume estimating the following *relative* lattice point rest of large ellipsoids $E_r + a$.

$$\Delta(r,a) \stackrel{\text{def}}{=} \Big| \frac{\operatorname{vol}_{\mathbb{Z}}(E_r + a) - \operatorname{vol} E_r}{\operatorname{vol} E_r} \Big|. \tag{1.1}$$

Theorem 1.5 below yields the following bounds for $\Delta(r,a)$

Theorem 1.1. *Assume that $\mathbb{Q}$ is positive definite and $d \geq 5$. Then*

$$\sup_{a \in \mathbb{R}^d} \Delta(r,a) = \mathcal{O}(r^{-2}), \tag{1.2}$$

as $r \to \infty$.

The estimate of Theorem 1.1 refines an explicit bound of order $\mathcal{O}(r^{-2})$ obtained by Bentkus and Götze (1997) for *arbitrary* ellipsoids and dimensions $d \geq 9$.

In the case of *rational* ellipsoids and $d \geq 5$ the bound $\mathcal{O}(r^{-2})$ is optimal. In the cases $2 \leq d \leq 4$ the error is of larger order. For balls and $d = 4$ Walfisz (1927, or 1957, p.95) established a lower bound $\Omega(r^{-2} \log\log r)$, for $d = 3$ Szegö (1926) has shown the lower bound $\Omega(r^{-2} \log^{1/2} r)$ and in the case of a circle, i.e. $d = 2$, Hardy (1916) proved a lower bound $\Omega(r^{-3/2} \log\log r)$.

For arbitrary ellipsoids Landau (1915) obtained the estimate $\mathcal{O}(r^{-2+2/(d+1)})$, $d \geq 2$. This result has been extended by Hlawka (1950) to convex bodies with smooth boundary and strictly positive Gaussian curvature. Hlawka's estimate has been improved by Krätzel and Nowak (1991, 1992) to $\mathcal{O}(r^{-2+\lambda})$, where $\lambda = 10/(6d+2)$, for $d \geq 8$, and $\lambda = 24/(14d+8)$, for $3 \leq d \leq 7$. For special ellipsoids a number of particular results is available. For example, the error bound $\mathcal{O}(r^{-2})$ holds for $d \geq 5$ and rational $\mathbb{Q}$, see Walfisz (1924), for $d \geq 9$, and Landau (1924), for $d \geq 5$). Jarnik (1928) proved the same bound for *diagonal* $\mathbb{Q}$ with arbitrary (non zero) real entries. For a detailed discussion, see the monograph of Walfisz (1957).

In case that $\mathbb{Q}$ is *irrational* Theorem 1.1 can be improved.

Theorem 1.2. *Assume that $\mathbb{Q}$ is positive definite and $d \geq 5$. Then*

$$\sup_{a \in \mathbb{R}^d} \Delta(r,a) = o(r^{-2}), \tag{1.3}$$

if and only if $\mathbb{Q}$ is irrational.

For *irrational* ellipsoids and dimension $d \geq 9$ the bound of Theorem 1.2 has been already proved in Bentkus and Götze (1999). For *diagonal* irrational $\mathbb{Q}$ of dimension $d \geq 5$ it extends the bound of order $o(r^{-2})$ of Jarnik and Walfisz (1930). They showed that the error $o(r^{-2})$ is optimal, that is, for any function ξ such that $\xi(r) \to \infty$, as $r \to \infty$, there exists an irrational diagonal form $\mathbb{Q}[x]$ such that

$$\limsup_{r \to \infty} \, r^2 \, \xi(r) \, \Delta(r,0) = \infty.$$

In Theorem 1.5 ii) an estimate of the remainder term in (1.3) in terms of certain Diophantine properties of $\mathbb{Q}$ will be given.

Lattice points in shells. Theorem 1.2 implies the following result

Corollary 1.3. *For $\eta > 0$ let $E_{r,\eta}$ denote the elliptic shell $\{x \in \mathbb{R}^d : r < \mathbb{Q}[x] \le r+\eta\}$. For any fixed $\eta > 0$*

$$\sup_{a\in\mathbb{R}^d} \frac{\operatorname{vol}_{\mathbb{Z}}(E_{r,\eta}+a)}{\operatorname{vol} E_{r,\eta}} = 1 + o(1),$$

as $r \to \infty$ holds uniformly in a provided that $d \ge 5$ and $\mathbb{Q}$ is irrational and positive definite.

Explicit bounds are given in Corollary 1.7 below. For indefinite quadratic forms $\mathbb{Q}$ an asymptotic approximation for number of lattice points in hyperbolic shells $\{x \in \mathbb{R}^d : a \le \mathbb{Q}[x] \le a+\eta\}$, intersected with boxes $|x|_\infty \le r$, where a and $\eta > 0$ are fixed and $r \to \infty$, has been proved by Eskin, Margulis and Mozes (1998). They proved a result like Corollary 1.3 for forms $\mathbb{Q}$ of signature (p,q) satisfying $\max(p,q) \ge 3$. This is a quantitative version of the well known Oppenheim problem concerning the distribution of values of $\mathbb{Q}[m]$, $m \in \mathbb{Z}^d$. In a seminal paper, Margulis (1986), established the Oppenheim conjecture, as stated by Davenport and Heilbronn that this set of values is dense in $\mathbb{R}$ provided that $d \ge 3$. For a detailed discussion of results on this problem by Oppenheim, Heilbronn and Davenport and others, see Margulis (1997).

In Bentkus and Götze (1999), explicit error bounds in the quantitative Oppenheim problem for the elliptic shells as well as for hyperbolic shells were proved for $d \ge 9$ by a common approach. The methods used in this paper for dimensions $d \ge 5$ do not seem to extend to the case of *general* indefinite forms of dimension $d \ge 5$. The methods used do apply though to forms with $\max(p,q) \ge 5$, the cases of reflections, where $\mathbb{Q}^2 = \mathrm{Id}$, $d \ge 5$, as well as to split forms like e.g. $\mathbb{Q} = \mathbb{Q}_1 - \mathbb{Q}_2$ where $\mathbb{Q}_1[x]$ depends on the first d_1 coordinates of $\mathbb{R}^d$ and $\mathbb{Q}_2[x]$ on the $d-d_2$, $d \ge 5$ remaining ones only. These results will be published somewhere else.

Gaps between values. Davenport and Lewis (1972) conjectured that the distance between successive values v_n of the quadratic form $\mathbb{Q}[x]$ on $\mathbb{Z}^d$ converges to zero as $n \to \infty$, provided that the dimension d is at least five and $\mathbb{Q}$ is irrational. Corollary 1.3 combined with Theorem 1.1 provides a complete solution of this problem.

For a vector $a \in \mathbb{R}^d$, let $0 \le v_1 \le v_2 \le \dots$ denote an enumeration of the values of $\mathbb{Q}[m-a]$, $m \in \mathbb{Z}^d$, in increasing order (that is without repeating equal values). Let

$$d(\tau;\mathbb{Q},a) = \sup\{v_{n+1} - v_n : \ v_n \ge \tau,\ n \in \mathbb{N}\}$$

denote the maximal gap between these values in the interval $[\tau,\infty)$. We have

Corollary 1.4. *Assume that $\mathbb{Q}$ is positive definite and $d \ge 5$. Then*

$$\sup_{a\in\mathbb{R}^d} d(\tau;\mathbb{Q},a) \to 0 \tag{1.4}$$

as $\tau \to \infty$, provided that $\mathbb{Q}$ is irrational. If both $\mathbb{Q}$ and a are rational then the set of values is confined to a lattice, that is $\inf_n \bigl(v_{n+1} - v_n \bigr) > 0$.

Answering a question of T. Esterman whether gaps must tend to zero for large dimensional positive forms, Davenport and Lewis (1972) proved the following: Assume that $d \ge d_0$ with some sufficiently large d_0. Let $\varepsilon > 0$. Suppose that $y \in \mathbb{Z}^d$ has a sufficiently large norm $|y|_\infty$. Then there exist $x \in \mathbb{Z}^d$ such that

$$\bigl| \mathbb{Q}[y+x] - \mathbb{Q}[x] \bigr| < \varepsilon. \tag{1.5}$$

Of course, (1.5) does not rule out the possibility of arbitrary large gaps between possible clusters of values $\mathbb{Q}[x]$, $x \in \mathbb{Z}^d$. The result of Davenport and Lewis was improved by Cook and Raghavan (1984). They obtained the estimate $d_0 \le 995$ and provided a lower bound for the number of solutions $x \in \mathbb{Z}^d$ of the inequality (1.5). See the reviews of Lewis (1973) and Margulis (1997).

In correspondence to the Oppenheim conjecture it seems likely that the Davenport–Lewis conjecture, that is (1.4), remains valid for dimensions $d = 3$ and $d = 4$ as well.

Explicit bounds. In order to describe the explicit bounds we need to introduce some more notations. Let $|(x, y)|_\infty$ denote the maximum norm of a vector (x, y) in $\mathbb{R}^d \times \mathbb{R}^d$. For any $t > 0$ and $r \ge 2$ consider the norm F on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$F(x, y) \stackrel{\text{def}}{=} \bigl| \bigl(r\,(x + t\mathbb{Q}y)\,,\ y r^{-1} \bigr) \bigr|_\infty. \tag{1.6}$$

Introduce the so called Minkowski minima of the convex body $\{F \le 1\}$ as

$$M_{1,t} = \inf\bigl\{ F(m, n) \ :\ (m, n) \in (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus 0 \bigr\} \tag{1.7}$$

and define in general $M_{k,t}$ as the infimum of $\lambda > 0$ such that the set of lattice points with norm less than λ, that is $\bigl\{ (m, n) \in \mathbb{Z}^d \times \mathbb{Z}^d \ :\ F(m, n) < \lambda \bigr\}$, contains k linearly independent vectors. By definition we have $r\,M_{k,t} \ge 1$. Introduce for $d > 4$ and $r \ge 2$

$$\begin{aligned} \Gamma_{T,r} &= \inf\ \bigl\{ r^d M_{1,t} \cdots M_{d,t} \ :\ T^{-1/(d-4)} \le |t| \le T \bigr\} && (1.8) \\ \rho(r, \mathbb{Q}) &= \inf_{T \ge 1} \bigl\{ q^{3d/2-1} T^{-1/2} + q^{d+1} \Gamma_{T,r}^{-1/2+2/d} \bigl(\log(qT\Gamma_{T,r}) + 1 \bigr) + q^d r^{-d/2+2} \log r \bigr\}. \end{aligned}$$

For any fixed $T > 1$ and irrational $\mathbb{Q}$ it is shown in Lemma 3.11 that

$$\lim_{r \to \infty} \Gamma_{T,r} = \infty,$$

with a speed depending on the Diophantine properties of $\mathbb{Q}$. This implies that

$$\lim_{r \to \infty} \rho(r, \mathbb{Q}) = 0. \tag{1.9}$$

With these notations we may state the main result of this paper.

Theorem 1.5. *Let $\mathbb{Q}$ denote a positive definite d-dimensional quadratic form, normalised such that $q_0 = 1$. Assume that $d \geq 5$. Then the following estimates hold. There exist positive constants $c_j(d)$, $j = 1, 2$, depending on d only and such that, for any $r \geq 2$,*

$$\text{i)} \qquad \sup_{a \in \mathbb{R}^d} \left| \operatorname{vol}_{\mathbb{Z}}(E_r + a) - \operatorname{vol} E_r \right| \leq c_1(d)\, q^{d+1} (1 + \log q)\, r^{d-2},$$

and

$$\text{ii)} \qquad \sup_{a \in \mathbb{R}^d} \left| \operatorname{vol}_{\mathbb{Z}}(E_r + a) - \operatorname{vol} E_r \right| \leq c_2(d)\, \rho(r, \mathbb{Q})\, r^{d-2},$$

where $\lim_{r \to \infty} \rho(r, \mathbb{Q}) = 0$, if $\mathbb{Q}$ is irrational.

Note that the bound $\rho(r, \mathbb{Q}) r^{d-2}$ in Theorem 1.5 is at least of order $\mathcal{O}(r^{d/2} \log r)$. It may be indeed of this order since $r\, M_{j,t} \ll_d r$ shows that the maximal value of $\Gamma_{T,r}$ is of order $\mathcal{O}(r^d)$ and we may choose $T = \mathcal{O}(r^\alpha)$ with $\alpha > 0$ suffiently large. Note that an error bound of order $r^{d/2+\varepsilon}$ has been proved by Jarnik (1928) for *diagonal* $\mathbb{Q}$ for Lebesgue *almost all* coefficients a_j.

In the case of diagonal forms a more explicit description of the dependence on the coeffients of $\mathbb{Q}$ can be given.

Proposition 1.6. *Let $\mathbb{Q} = \operatorname{diag}(a_1, \dots, a_d)$, where w.l.g. $a_1 = 1 \leq a_2 \leq \ldots \leq a_d$. We have for all $T \geq 1$ and $d > 4$*

$$\Gamma_{T,r} = \inf\Big\{ \prod_{1 \leq j \leq d} f(n_j; t a_j, r) \,:\, t \in [T^{-1/(d-4)}, T],\ n_j \in \mathbb{Z},\ 0 < |n_j| \leq r \Big\}, \qquad (1.10)$$

where $f(n; x, r) = \max\{r^2 \|x n\|, |n|\}$ and $\|x\| = \min\{|x - n| \,:\, n \in \mathbb{Z}\}$.

Let

$$\rho_0(r, \mathbb{Q}) = \sup_{\tau \geq r} \rho(r, \mathbb{Q})$$

denote the monotone decreasing envelope of $\rho(r, \mathbb{Q})$. From the explicit bounds of Theorem 1.5 we obtain

Corollary 1.7. *For fixed η there exists positive constants $c_j(d, \eta)$, $j = 1, 2$ such that for any $\tau \geq r$ and $r \geq 2$*

$$\frac{\operatorname{vol}_{\mathbb{Z}}(E_{\tau,\eta} + a)}{\operatorname{vol} E_{\tau,\eta}} = 1 + \mathcal{R}, \qquad (1.11)$$

where $\mathcal{R}$ satisfies $|\mathcal{R}| \leq c(d)\, q^{d/2} \rho_0(r, \mathbb{Q})/\eta$. In particular the maximal gap $d(\tau, \mathbb{Q}, a)$ satisfies

$$d(\tau, \mathbb{Q}, a) \ll_d q^{d/2} \rho_0(r, \mathbb{Q}). \qquad (1.12)$$

For irrational $\mathbb{Q}$ the approximation error $\mathcal{R}$ in 1.12 still tends to zero for shrinking intervals $[r^2, r^2 + \eta(r)]$ of values $\mathbb{Q}[m]$ such that $\eta(r)/\rho_0(r, \mathbb{Q})$ tends to zero as $r \to \infty$. Note that these bounds do hold uniformly in a.

Basic Steps of Proof and Applications. The proof is based on a representation of $\mathrm{vol}_{\mathbb{Z}}(E_r)$ using Fourier inversion by an integral of the form

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{r^2 z}\,\theta(z)\,\frac{dz}{z}, \tag{1.13}$$

where $b = r^{-2}$ and $\theta(z) = \sum_{m\in\mathbb{Z}^d}\exp\{-z\,\mathbb{Q}[m]\}$ denotes the theta series of $\mathbb{Q}$. Approximating this series by an integral for arguments z near the singularity $z = 0$ results in part of the inversion integral (1.13) which is essentially the desired volume approximation. The integral over values of z in the complement of a neighborhood of 0 constitute the crucial part of the error. This representation has been used by Landau (1924) for diagonal rational forms.

Using methods from the geometry of numbers by Davenport (1958), who investigated indefinite forms, we show that that $r^{-d/2}\,|\theta(b+4t)|$ is bounded by a multiple of the quantity

$$\bigl(M_{1,t}\cdots M_{d,t}\bigr)^{-1/2} \le M_{1,t}^{-d/2}.$$

From here on the proof is based on arguments from metric number theory. The measure of the set of t in a bounded interval I such that $M_{1,t}$ is smaller than $\varepsilon > 0$ is shown to be of order ε^2 in Lemma 3.8, which allows by partial integration to bound

$$\int_I M_{1,t}^{-d/2}\,dt, \tag{1.14}$$

by a multiple of $\int_{r^{-1}}^{\infty}\varepsilon^{-d/2-1}\varepsilon^2\,d\varepsilon$. The latter integral is of order $r^{d/2-2}$, provided that $d > 4$. Thus $\int_I|\theta(b+it)|\,dt$ is of order r^{d-2} which indicates the order as well as the dimension dependence of the error.

This Lemma 3.8 in turn relies on a key observation in Lemma 3.6. For positive definite and symmetric $\mathbb{Q}$ two large Minkowski minima $M_{1,t}$ and $M_{1,t'}$, attained at vectors of length $\approx L$ in nearby directions, can occur for distances $t-t'$ only which are either 'small' (that is of order $\varepsilon/(Lr)$) or which are rather 'large' (of order L^{-2}). Thus, the set of t with $M_{1,t} < \varepsilon$ is contained in a union of 'small' intervals, separated by 'large' gaps. Hence at most $|I|/L^{-2}$ of them will fit into I. By definition of $M_{1,t}$, we have $L/r \le \varepsilon$ and hence the measure of the small intervals is bounded by $\varepsilon/(Lr)\,|I|/L^{-2} \le \varepsilon^2|I|$. This key observation implies the following result for multivariate Diophantine approximations, which may be of independent interest.

For a vector $a\in\mathbb{R}^d$ let $\|a\| \overset{\text{def}}{=} \inf_{m\in\mathbb{Z}^d}|a-m|_\infty$ denote the error of an integer approximation. Introduce for $\nu > 1$,

$$D(t,\nu) = \nu\,\min\bigl\{\|t\mathbb{Q}n\| \,:\, n\in\mathbb{Z}^d,\ 0 < |n|_\infty \le \nu\bigr\},$$

and let λ denote the Lebesgue measure. Then we have

Theorem 1.8. *Assume that $\mathbb{Q}$ is positive definite, symmetric and normalized such that $q_0 = 1$. Then there exists a constant $c(d) > 1$ depending on d only such that for any $r \ge 1$*

and any interval $[a,b]$ *satisfying* $0 < b-a < 1$ *the following inequalities hold*

$$\lambda\{t \in [a,b] \,:\, M_{1,t} \le \tau\} \;\le\; c(d)\big(q\,\tau^2\,(b-a) + \tau\, r^{-1}\big), \tag{1.15}$$

$$\sup_{t\in[a,b]} M_{1,t} \;\ge\; \min\{c(\mathbb{Q}), r\,(b-a)\}, \tag{1.16}$$

$$\sup_{t\in[a,b]} D(t,\nu) \;\ge\; \min\{c(\mathbb{Q})^2, \nu\,(b-a)/2\}, \tag{1.17}$$

for any $\nu \ge c(\mathbb{Q})$, *where* $c(\mathbb{Q}) = (2\,c(d)\,q)^{-1/2}$.

We expect that this result holds as well for *any* non degenerate operator $\mathbb{Q}$, that is without additional restrictions.

Using the bounds mentioned above it follows (by Lemma 3.5 that the minimum of $t \to |\theta(r^{-2}+it)|$ on *any* interval I of length r^{-1} is of order $r^{d/2}$, whereas the extremal value (e.g. at $t=0$) is of order r^d. That $r^{d/2}$ is the correct generic size is the background of the so called central limit theorem for the distribution of the values $t \to r^{-1/2}\,\theta(r^{-2}+it)$ for $d=1$. See Jurkat and van Horne (1982) as well as Marklof (1999).

The paper is organized as follows. In Section 2 we shall prove the main results—Theorems 1.1, 1.2 and 1.5, derive their corollaries. Section 3 collects Lemmas which yield bounds from the geometry of numbers, metric number theory for theta functions and integrals over theta functions used in the proofs of Theorems. We shall use the following notation.

By c with or without indices we shall denote generic absolute constants. We shall write $A \ll B$ instead of $A \le cB$. If a constant depends on a parameter, say d, then we write c_d or $c(d)$ and use $A \ll_d B$ instead of $A \le c_d B$. By $[B]$ we denote the integer part of a real number B.

The set of natural numbers is denoted as $\mathbb{N} = \{1, 2, \dots\}$, the set of integer numbers as $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Write $|x|_\infty = \max\limits_{1\le j\le d} |x_j|$. Throughout $\mathbb{I}\big\{A\big\}$ denotes the indicator function of an event A, that is, $\mathbb{I}\big\{A\big\} = 1$ if A occurs, and $\mathbb{I}\big\{A\big\} = 0$ otherwise. On right half plane $z \in \mathbb{C}$, $\operatorname{Re} z > 0$ we denote by $z^{1/2}$ the branch with $\operatorname{Re} z^{1/2} \ge 0$ and use $z^{d/2} = (z^{1/2})^d$.

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2. Proof of the Results.

Proof of Corollary 1.7. We have to prove (1.11) and (1.12). The proof of (1.12) reduces to proving that

$$\operatorname{vol}_{\mathbb{Z}}(E_{\tau,\eta} + a) > 0,$$

for $\eta \ge c(d)\,q^{3d/2}\,\rho_0(r,\mathbb{Q})$ with a sufficiently large constant $c(d)$. Using (1.11) it suffices to verify that $|\mathcal{R}| \le 1/2$, which is obviously fulfilled.

Let us prove (1.11). Consider an interval $(s, s+\eta]$ of values with $s = \tau^2$ and $\tau \geq r \geq 2$. We shall apply the bound of Theorem 1.5 which for $r \geq 2$ yields

$$\bigl|\mathrm{vol}_{\mathbb{Z}}(E_r + a) - \mathrm{vol}\, E_r\bigr| \ll_d q^d\, r^{d-2}\, \rho_0(r, \mathbb{Q}). \tag{2.1}$$

and writing $\tau_+ = \sqrt{\tau^2 + \eta}$ we get

$$\bigl|\mathrm{vol}_{\mathbb{Z}}\bigl((E_{\tau_+} + a) \setminus (E_\tau + a)\bigr) - \mathrm{vol}\bigl(E_{\tau_+} \setminus E_\tau\bigr)\bigr| \ll_d q^d\, \tau_+^{d-2}\, \bigl(\rho_0(\tau_+, \mathbb{Q}) + \rho_0(\tau, \mathbb{Q})\bigr) \tag{2.2}$$

The estimate (2.2) implies (1.11). Just note that $\rho_0(\tau, \mathbb{Q}) \leq \rho_0(r, \mathbb{Q})$, for $\tau \geq r$, divide both sides of (2.2) by $\mathrm{vol}\bigl(E_{\tau_+} \setminus E_\tau\bigr)$ and use

$$\mathrm{vol}\bigl(E_{\tau_+} \setminus E_\tau\bigr) = \bigl((s+\eta)^{d/2} - s^{d/2}\bigr)\ \mathrm{vol}\, E_1,$$

$$(s+\eta)^{d/2} - s^{d/2} \gg_d \int_{s+\eta/2}^{s+\eta} u^{-1+d/2}\, du \gg_d \eta\,(s+\eta/2)^{-1+d/2} \gg_d \eta\,(s+\eta)^{-1+d/2},$$

$$\mathrm{vol}\, E_1 = c\, \mathrm{vol}\bigl\{x \in \mathbb{R}^d :\ Q[x] \leq 1\bigr\} \geq \mathrm{vol}\bigl\{x \in \mathbb{R}^d :\ |x| \leq 1/\sqrt{q}\bigr\} = c_d\, q^{-d/2}.$$

Proof of Corollaries 1.4 and 1.3. It suffices to use (1.9), (1.11) and (1.12). □

Proof of Theorem 1.1 and 1.2. Since $\mathrm{vol}\, E_r \gg_d q^{-d/2}\, r^{d/2}$, Theorem 1.1 follows from Theorem 1.5 i). Assuming the irrationality of Q, the bound $o(r^{-2})$ is implied by Theorem 1.5 ii) and (1.9). The bound $o(r^{-2})$ in (1.3) implies as above $d(\tau, \mathbb{Q}, 0) \to 0$, as $\tau \to \infty$, which is impossible for rational Q. □

Proof of Theorem 1.8. The estimate (1.15) immediately follows from Lemma 3.9. This inequality ensures that there exists a $t \in [a, b]$ such that $M_{1,t} > \tau$ whenever $c(d)\bigl(q\tau^2(b-a) + \tau r^{-1}\bigr) < b - a$. This condition may be rewritten as $\tau < (1 - c(d)q\tau^2)(b-a)r$. The last inequality again (and hence $M_{1,t} > \tau$) follows from $\tau < \min\{c(\mathbb{Q}), r\,(b-a)/2\}$ since $\tau \leq c(\mathbb{Q})$ is equivalent to $1 - c(d)\,q\tau^2 \geq 1/2$ which proves (1.16).

By definition of $M_{1,t}$ the inequality $M_{1,t} > \tau \stackrel{\mathrm{def}}{=} \min\{c(\mathbb{Q}), r\,(b-a)/2\}$ implies that if $0 < |n|_\infty \leq \nu = \tau r$ then $\nu\|t\mathbb{Q}n\| > \tau^2$. Hence $D(t, \nu) \geq \tau^2 = \min\{c(\mathbb{Q})^2, \nu\,(b-a)/2\}$, since either $r\,(b-a)/2 > c(\mathbb{Q})$ and $\tau = c(\mathbb{Q})$ or $\tau = r\,(b-a)/2$ otherwise. This proves (1.17). □

Proof of Proposition 1.6 Suppose that the Minkowski minimum $M_{1,t}$ of the norm F, is attained at $(m, n) \in \mathbb{Z}^d \times \mathbb{Z}^d$, say. Note that with the notation of (1.10)

$$f(v; x, r) = \inf_{u \in \mathbb{Z}\setminus 0} \max\{r^2\,|u - x\,v|, |v|\}.$$

Let m_j resp. n_j denote the components of m resp. n. In case of a diagonal matrix $\mathbb{Q}$, (1.7) shows that

$$M_{1,t} \ = \ \min_{(m,n)\neq 0}\ \max_{1\leq j\leq d} f(n_j; t\,a_j, r), \tag{2.3}$$

does not change, when we replace for some suitable j the vector (m, n) by $a_1 = (u_j\, e_j, v_j\, e_j)$, where $u_j, v_j \in \mathbb{Z}$ and e_l, $l = 1, \ldots, e_d$ denotes the standard basis of $\mathbb{R}^d$. By the same argument any integer vector which is linearly independent of w_1 and where $M_{2,t}$ is attained may be replaced by a vector $w_2 = (u_{j'}e_{j'}, v_{j'}e_{j'})$ for some $j' \neq j$. Selecting in this way vectors $w_1, \dots, w_d$ we arrive at the representation

$$M_{1,t} \cdots M_{d,t} = f(v_1; t a_1, r) \cdots f(v_d; t, a_d, r), \tag{2.4}$$

thus proving Proposition 1.6. □

Proof of Theorem 1.5 We shall represent $\mathrm{vol}_{\mathbb{Z}}\,(E_r + a)$ for an ellipsoid E_r as an integral over the theta function $\theta(z)$ given by the series

$$\theta(z) = \sum_{m \in \mathbb{Z}^d} \exp\bigl\{-z\,\mathbb{Q}[m+a]\bigr\}, \quad z \in \mathbb{C}, \quad \mathrm{Re}\, z > 0, \tag{2.5}$$

which is absolutely convergent for $\mathrm{Re}\, z > 0$. The part of this integral near the singularity $z = 0$ will provide the approximation $\mathrm{vol}\, E_r$ while the remaining parts constitute the approximation error. More precisely, consider continuous approximations $V_w(r)$ of the (monotone) lattice point counting function $r \mapsto \mathrm{vol}_{\mathbb{Z}}(E_r + a)$ depending on some smoothing parameter $w \neq 0$. Let $J(w)$ denote the interval with endpoints r^2 and $r^2 + w \geq 0$ and define

$$V_w(r) = \frac{1}{|w|} \int_{J(w)} \mathrm{vol}_{\mathbb{Z}}\,(E_{\sqrt{v}} + a)\, dv,$$

and note that by monotonicity for $0 < w < r^2$

$$V_{-w}(r) \leq \mathrm{vol}_{\mathbb{Z}}(E_r + a) \leq V_w(r). \tag{2.6}$$

By Taylor's formula and by $\mathrm{vol}(E_r + a) = r^d\, \mathrm{vol}\, E_1$, we have, for $r \geq 1$, $0 < w < r^2$,

$$\frac{1}{w} \int_{J(\pm w)} \mathrm{vol}(E_{\sqrt{u}} + a)\, du = \mathrm{vol}\, E_r + w\, r^{d-2} R, \tag{2.7}$$

where $|R| \ll_d \mathrm{vol}\, E_1$. Note that $\mathrm{vol}\, E_1 \ll_d 1$ since $q_0 = 1$. Thus we obtain from (2.6) subtracting $\mathrm{vol}\, E_r$ and using (2.7)

$$\bigl|\mathrm{vol}_{\mathbb{Z}}\,(E_r + a) - \mathrm{vol}\ E_r\bigr| \leq \max_{\pm} \bigl|R_{\pm w}(r)\bigr| + c_d\, w\, r^{d-2}, \tag{2.8}$$

where

$$R_{\pm w}(r) = \frac{1}{w} \int_{J(\pm w)} \bigl(\mathrm{vol}_{\mathbb{Z}}(E_{\sqrt{v}} + a) - \mathrm{vol}\, E_{\sqrt{v}}\bigr)\, dv. \tag{2.9}$$

Once we have determined an upper bound of $\bigl|R_{\pm w}(r)\bigr|$ in terms of r and w we may choose an optimal value for $w > 0$. We shall use residue calculus or Fourier inversion to express $V_w(r)$ in terms of $\theta(z)$. Note first that

$$\frac{1}{2\pi i} \int_{b - i\infty}^{b + i\infty} \exp\{zT\}\, \frac{dz}{z^2} = \max\{T, 0\}, \tag{2.10}$$

for any $b > 0$, $T \in \mathbb{R}$, by standard residue calculus (complement $(b - i\infty, b + i\infty)$ by an infinite half circle in $\mathrm{Re}\, z \geq 0$ (resp. $\mathrm{Re}\, z \leq 0$) for $T < 0$ (resp. $T \geq 0$)). Thus, for $s > 0$, we get by interchanging integration and summation

$$
\begin{aligned}
\int_0^s \mathrm{vol}_{\mathbb{Z}}(E_{\sqrt{v}}+a)\,dv &= \sum_{m\in\mathbb{Z}^d}\int_0^s \mathbb{I}\{\mathbb{Q}[m+a]\le v\}\,dv \\
&= \sum_{m\in\mathbb{Z}^d} \max\{s-\mathbb{Q}[m+a],0\} \\
&= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\Big(\sum_{m\in\mathbb{Z}^d}\exp\{z\,(s-\mathbb{Q}[m+a])\}\Big)\frac{dz}{z^2} \\
&= \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{sz}\,\theta(z)\,\frac{dz}{z^2}\,, \qquad (2.11)
\end{aligned}
$$

since $\theta(z)$ is absolutely converging on the line of integration and $|e^{sz}\,\theta(z)| \le e^{sb}\theta(b)$ for all $z\in b+i\,\mathbb{R}$.

Replacing in the equalities above the summation over $\mathbb{Z}^d$ by integration over $\mathbb{R}^d$, we arrive in exactly the same way at the corresponding representation

$$
\int_0^s \mathrm{vol}(E_{\sqrt{v}}+a)\,dv = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\exp\{s\,z\}\,\theta_0(z)\,\frac{dz}{z^2}\,, \qquad (2.12)
$$

where $\theta_0(z)$ denotes the theta integral

$$
\theta_0(z) = \int_{\mathbb{R}^d}\exp\{-z\,\mathbb{Q}[x+a]\}\,dx = (\det\mathbb{Q})^{-1/2}\,\pi^{d/2}\,z^{-d/2}, \qquad (2.13)
$$

using the standard convention for $z^{d/2}$, see end of section 1. This allows to write (2.9) for $w>0$ using (2.11) and (2.12) with $s=r^2w$ respectively $s=r^2+w$ as

$$
R_w(r) = \frac{1}{2\pi i}\int_{b-i\,\infty}^{b+i\,\infty}\big(\theta(z)-\theta_0(z)\big)\,\exp\{r^2\,z\}\,\frac{\exp\{w\,z\}-1}{w\,z}\,\frac{dz}{z} \qquad (2.14)
$$

with a corresponding expression for $R_{-w}(r)$, $0<w<r^2$.

Obviously, the representation (2.11) is independent of $b>0$. We shall choose $b=r^{-2}$. Furthermore, introduce

$$
\varkappa_r = \pi\,r^{-1} \quad \text{and} \quad \varphi(z) = \frac{1}{2\pi i}\exp\{r^2\,z\}\,\frac{\exp\{w\,z\}-1}{w\,z^2}\,. \qquad (2.15)
$$

Consider the line segments $J_0 = \big[b-i\varkappa_r,\, b+i\varkappa_r\big]$ and $J_1 = (b+i\mathbb{R})\setminus J_0$. Then we may split

$$
\begin{aligned}
R_w(r) &= \int_{J_0}\big(\theta-\theta_0\big)(z)\,\varphi(z)\,dz - \int_{J_1}\theta_0(z)\,\varphi(z)\,dz + \int_{J_1}\theta(z)\,\varphi(z)\,dz \\
&= I_0 + I_1 + I_2, \qquad (2.16)
\end{aligned}
$$

say. Here I_0 and I_1 represent the difference between $\mathrm{vol}\,E_r$ and a sufficiently averaged version of $\mathrm{vol}_{\mathbb{Z}}\,E_r$, whereas I_2 controls the local fluctiations of this function. The bound of I_2 involves the crucial dimension dependent part of our arguments.

Before estimating these integrals we derive some bounds for $\varphi(z)$. Using

$$\Big|\frac{\exp\{w(r^{-2}+it)\}-1}{w}\Big| \le \min\Big\{e\big|r^{-2}+it\big|,\ \frac{e+1}{w}\Big\}, \tag{2.17}$$

for $r^2 \ge w > 0,\ r \ge 1,$ we obtain

$$\big|\varphi(r^{-2}+it)\big| \ll \frac{1}{w|r^{-2}+it|^2}, \qquad \text{as well as} \tag{2.18}$$

$$\big|\varphi(r^{-2}+it)\big| \ll |r^{-2}+it|^{-1}. \tag{2.19}$$

Estimation of I_0. Inequality (2.19) together with Lemma 3.2 for $t \in J_0$ yields

$$\begin{aligned}\Theta_t &\stackrel{\text{def}}{=} \big|\big((\theta-\theta_0)\varphi\big)(r^{-2}+it)\big| \\ &\ll_d q^{d/2}\,|r^{-2}+it|^{-(d+2)/2}\exp\Big\{-\frac{\mu r^2}{1+r^4t^2}\Big\},\end{aligned}$$

where $\mu = \pi^2 q^{-1}/2$. Writing $|r^{-2}+it| = r^{-2}(1+r^4t^2)^{1/2}$, we may introduce the variable $v = (1+r^4t^2)^{-1}$ and the function $h(v) \stackrel{\text{def}}{=} v^{(d+2)/4}\exp\{-v\,\mu r^2\}$. The determination of the maximal value of h on $[0,\infty)$, which is $\ll_d (\mu r^2)^{-(d+2)/4}$ and is attained at $v_0 = \dfrac{d+2}{4\mu r^2}$, yields

$$\begin{aligned}\sup_{t\in J_0}\Theta_t &\ll_d q^{d/2}\,r^{d+2}\sup_{v\ge 0}h(v) \\ &\ll_d q^{d/2}\,r^{d+2}\,(\mu r^2)^{-(d+2)/4}.\end{aligned}$$

Integrating this bound over J_0, we obtain

$$I_0 = \int_{-\varkappa_r}^{\varkappa_r}\Theta_t\,dt \ll_d 2\varkappa_r\,q^{d/2}\,\mu^{-(d+2)/4}r^{d/2+1} \ll_d q^{(3d+2)/4}\,r^{d/2}. \tag{2.20}$$

Estimation of I_1. Using $q_0 = 1$, (2.13) and (2.19) to bound $\varphi(z)$, we have, for $r \ge 1$,

$$I_1 \ll_d \Big|\int_{J_1}\frac{\varphi(r^{-2}+it)}{(r^{-2}+it)^{d/2}}\,dt\Big| \ll_d \int_{\varkappa_r}^{\infty}\frac{dt}{t^{1+d/2}} \ll_d r^{d/2}, \tag{2.21}$$

using the symmetry in t around 0.

Collecting the bounds obtained so far, that is (2.8), (2.16), (2.20) and (2.21), we get

$$\big|\mathrm{vol}_{\mathbb{Z}}(E_r+a) - \mathrm{vol}\,E_r\big| \ll_d w\,r^{d-2} + q^{(3d+2)/4}\,r^{d/2} + I_2. \tag{2.22}$$

The estimate $|\varphi(z)| \ll_d \min\big\{1,\ (|z|w)^{-1}\big\}|z|^{-1}$ (see (2.18) and (2.19)) implies

$$\begin{aligned}I_2 &\ll_d \int_{|t|>\varkappa_r}\big|\theta(r^{-2}+it)\big|\min\Big\{1,\ \frac{1}{w\,|r^{-2}+it|}\Big\}\frac{dt}{|r^{-2}+it|} \\ &\ll_d \int_{|u|>\beta_r}\big|\theta\big(r^{-2}+i\pi u/2\big)\big|\,g(u)\,du,\end{aligned} \tag{2.23}$$

where

$$g(t) = \min\bigl\{1, (w\,|t|)^{-1}\bigr\}\,|t|^{-1}, \qquad \beta_r = 2\,r^{-1}. \tag{2.24}$$

Denote

$$G(a,b) \overset{\mathrm{def}}{=} \int_a^b g(t)\,dt, \qquad \text{for} \quad 0 < a < b \le \infty. \tag{2.25}$$

For $a \ge b > 0$ we define $G(a,b) = 0$. Note that

$$G(a,b) = \begin{cases} \log(b/a), & \text{for} \ \ a \le b \le w^{-1}, \\ -\log(w\,a) + 1 - (w\,b)^{-1}, & \text{for} \ \ a \le w^{-1} \le b, \\ (w\,a)^{-1} - (w\,b)^{-1}, & \text{for} \ \ w^{-1} \le a \le b. \end{cases} \tag{2.26}$$

The equality (2.26) and the definition of the function $G(a,b)$ imply the bound

$$G(a,b) \le \min\bigl\{\bigl|\log(w\,a)\bigr| + 1,\ \bigl|\log(b/a)\bigr|,\ (w\,a)^{-1}\bigr\} \qquad \text{for} \quad a, b > 0. \tag{2.27}$$

The upper bound of Lemma 3.5 for $|\theta(z)|$ in terms of Minkowski's successive minima $M_{j,t}$ (for the norm on $\mathbb{R}^d$ defined by (3.13) and (3.16) and related to $\mathbb{Q}$) now yields with $M(t) = M_{1,t} \cdots M_{d,t}$ and $q \ge 1$

$$I_2 \ll_d q^d\, r^{d/2} \int_{|t| > \beta_r} \frac{g(t)}{M(t)^{1/2}}\, dt = 2 q^d\, r^{d/2}\, I_3, \tag{2.28}$$

where

$$I_3 = \int_{\beta_r}^{\infty} \frac{g(t)}{M(t)^{1/2}}\, dt. \tag{2.29}$$

The last equality in (2.28) follows from the fact that the functions $g(\cdot)$ and $M(\cdot)$ are even (see (3.19)). Let

$$\gamma(a,b) = r^d \inf_{a \le t \le b} M(t), \qquad \text{for} \quad a \in \mathbb{R}. \tag{2.30}$$

Proof of Theorem 1.5(i). Applying Lemma 3.10 for the interval with endpoints $a = \beta_r$ and $b = \infty$, we get

$$\begin{aligned} I_3 \ \ll_d \ r^{d/2-2} \int_{\gamma_0}^{D_0} v^{-1/2+1/d} \Bigl(v^{1/d}\, q\, G\bigl(a_0(v^{1/d}), \infty\bigr) + g\bigl(a_0(v^{1/d})\bigr)\Bigr)\, \frac{dv}{v} \\ + G(\beta_r, \infty) \end{aligned} \tag{2.31}$$

with

$$\gamma_0 = \gamma(\beta_r,\, \infty), \quad D_0 = \max\bigl\{(2d)^{-d} r^d, \gamma_0\bigr\}, \quad a_0(v) = \max\bigl\{\beta_r, (2 q\, v\, d^{1/2})^{-1}\bigr\}. \tag{2.32}$$

Note that $\gamma_0 \ge 1$ by (3.15). Choosing $w = 1$, we get in view of (2.24), (2.27), (2.28), (2.31) and (2.32)

$$\begin{aligned} I_2 &\ll_d q^{d+1} r^{d/2} r^{d/2-2} \int_1^{D_0} v^{-1/2+2/d} \big(\log(q v^{1/d}) + 1\big) \frac{dv}{v} + q^d r^{d/2} (\log r + 1) \\ &\ll_d q^{d+1} (\log q + 1)\, r^{d-2}, \end{aligned} \tag{2.33}$$

provided that $d > 4$ and $r \ge 2$. In view of (2.22) this bound for I_2 yields

$$\big|\mathrm{vol}_{\mathbb{Z}}(E_r + a) - \mathrm{vol}\, E_r\big| \ll_d q^{d+1} (\log q + 1)\, r^{d-2},$$

thus proving Theorem 1.5(i).

Proof of Theorem 1.5(ii). In order to use nontrivial bounds for $\gamma(a, b)$ in the irrational case let us introduce further auxilliary parameters η, T such that $\beta_r \le \eta \le T$ with $T \ge 1$ which will be determined and optimized later. Thus we may split the integral I_3 in (2.29) which bounds I_2 in (2.28) into the parts

$$\begin{aligned} I_3 &= \Big(\int_{\beta_r}^{\eta} + \int_{\eta}^{T} + \int_{T}^{\infty}\Big) \frac{g(t)}{M(t)^{1/2}}\, dt \\ &= I_4 + I_5 + I_6, \qquad \text{say}. \end{aligned} \tag{2.34}$$

Define similarly to (2.32)

$$\gamma_1 = \gamma(\beta_r, \eta), \quad \gamma_2 = \gamma(\eta, T), \quad \gamma_3 = \gamma(T, \infty), \tag{2.35}$$

$$D_j = \max\big\{ (2d)^{-d} r^d,\, \gamma_j \big\}, \qquad j = 1, 2, 3, \tag{2.36}$$

$$a_1(u) = \max\big\{ \beta_r, f(u) \big\}, \quad a_2(u) = \max\big\{ \eta, f(u) \big\}, \quad a_3(u) = \max\big\{ T, f(u) \big\}, \tag{2.37}$$

where $f(u) = (2 q u d^{1/2})^{-1}$, $u > 0$. By (3.15) we have

$$\gamma_j \ge 1, \qquad j = 1, 2, 3. \tag{2.38}$$

Using (2.24) and (2.37), we see that

$$g(a_j(u)) \le 2 q u d^{1/2}, \qquad j = 1, 2, 3. \tag{2.39}$$

Applying Lemma 3.10 as above, consider first the interval with endpoints $a = \beta_r$ and $b = \eta$. Note that by Corollary 3.7 the quantity γ_1 (defined by (2.30) and (2.35)) satisfies

$$\gamma_1 \ge \delta \stackrel{\text{def}}{=} (d q \eta)^{-d}, \tag{2.40}$$

since $d \ge 5$, $\beta_r = 2 r^{-1}$ and $t r/2 \ge t r q^{-1}/2 \ge (q t r)^{-1}$ whenever $t > 2 r^{-1}$.

Lemma 3.10 yields in view of (2.26), (2.27), (2.39) and (2.40) the estimate

$$\begin{aligned} I_4 \;&\ll_d\; r^{d/2-2}\int_{\gamma_1}^{D_1} v^{-1/2+1/d}\big(v^{1/d} q\, G\big(a_1(v^{1/d}),\eta\big)+g\big(a_1(v^{1/d})\big)\big)\,\frac{dv}{v}+G(\beta_r,\eta)\\ &\ll_d\; q r^{d/2-2}\int_{\delta}^{D_1} v^{-1/2+2/d}\big(\big|\log(q v^{1/d}\eta)\big|+1\big)\,\frac{dv}{v}+G(\beta_r,\eta)\\ &\ll_d\; q^{d/2-1} r^{d/2-2}\eta^{d/2-2}+G(\beta_r,\eta), \end{aligned}\tag{2.41}$$

provided that $d>4$, using the change of variables $v=\delta u$ in the last inequality.

As for the term I_5 choose $a=\eta$, $b=T$. By Lemma 3.10 we obtain as above

$$\begin{aligned} I_5 \;&\ll_d\; r^{d/2-2}\int_{\gamma_2}^{D_2} v^{-1/2+1/d}\big(v^{1/d} q\, G\big(a_2(v^{1/d}),T\big)+g\big(a_2(v^{1/d})\big)\big)\,\frac{dv}{v}+G(\eta,T)\\ &\ll_d\; q r^{d/2-2}\int_{\gamma_2}^{D_2} v^{-1/2+2/d}\big(\big|\log(q v^{1/d}/w)\big|+1\big)\,\frac{dv}{v}+G(\eta,T)\\ &\ll_d\; q\, r^{d/2-2}{\gamma_2}^{-1/2+2/d}\big(\big|\log(q\gamma_2)\big|+\big|\log w\big|+1\big)+G(\eta,T). \end{aligned}\tag{2.42}$$

Finally for the term I_6 choose $a=T$ and $b=\infty$ and use (2.38) for $j=3$. Recall that we choose $T\ge 1$. Thus similarly as above using Lemma 3.10 and $G(a_3(v^{1/d}),\infty)\le G(T,\infty)\le T^{-1}w^{-1}$ and $g\big(a_3(v^{1/d})\big)\le T^{-2}w^{-1}$, we obtain (see (2.24), (2.27) and (2.37))

$$\begin{aligned} I_6 \;&\ll_d\; r^{d/2-2}\int_{1}^{D_3} v^{-1/2+1/d}\big(v^{1/d} q\, G\big(a_3(v^{1/d}),\infty\big)+g\big(a_3(v^{1/d})\big)\big)\,\frac{dv}{v}+G(T,\infty)\\ &\ll_d\; q r^{d/2-2}T^{-1}w^{-1}+G(T,\infty). \end{aligned}\tag{2.43}$$

Collecting (2.41)–(2.43), we get combining the terms $G(a,b)$ and using (2.34) and (2.38)

$$\begin{aligned} I_3 \;\ll_d\; r^{d/2-2}\Big\{q^{d/2-1}\eta^{d/2-2}+q{\gamma_2}^{-1/2+2/d}\big(\log(q\gamma_2)+|\log w|+1\big)+qT^{-1}w^{-1}\Big\}&\\ +G(\beta_r,\infty).& \end{aligned}\tag{2.44}$$

Hence, combining (2.22), (2.27)–(2.29) and (2.44) implies

$$\begin{aligned} \Delta_r \;&\stackrel{\text{def}}{=}\; \big|\operatorname{vol}_{\mathbb{Z}}(E_r+a)-\operatorname{vol}E_r\big|\\ &\ll\; r^{d-2}\Big\{w+q^{d+1}(Tw)^{-1}+q^{3d/2-1}\eta^{d/2-2}+q^{d+1}{\gamma_2}^{-1/2+2/d}\big(\log(q\gamma_2)+|\log w|+1\big)\Big\}\\ &\qquad +q^d r^{d/2}\big(1+\log r\big). \end{aligned}\tag{2.45}$$

By Lemma 3.11 for η, T fixed, we have $\gamma_2\to\infty$ for $r\to\infty$ and we may now choose the auxiliary parameters η, w and T to minimize the right hand side of (2.45) as follows. Let

$$T\ge 1,\quad w=T^{-1/2},\quad \eta=\max\big\{\beta_r,\,T^{-\alpha}\big\},\qquad \text{where } \alpha=\frac{1}{d-4},\tag{2.46}$$

provided that $d \geq 5$. In view of (2.35), (2.38), (2.45) and (2.46) the final bound for the lattice point remainder Δ_r in Theorem 1.5(ii) is given by

$$\begin{aligned}\Delta_r \quad \ll_d \quad r^{d-2} \inf_{T \geq 1} \Big\{ & q^{3d/2-1} T^{-1/2} \\ & + q^{d+1} \gamma\big(T^{-\alpha}, T\big)^{-1/2+2/d} \log\Big(q T \gamma\big(T^{-\alpha}, T\big) + 1\Big) \\ & + q^d r^{-2+d/2} \big(1 + \log r\big) \Big\},\end{aligned}$$

which proves Theorem 1.5(ii).

3. Lemmas

Lemma 3.1. *For* $\operatorname{Re} z > 0$, $a, b \in \mathbb{R}^d$ *and positive definite* $\mathbb{Q}$ *it holds*

$$\begin{aligned}&\sum_{m \in \mathbb{Z}^d} \exp\big\{-z\, \mathbb{Q}[m + a] + 2\pi i \langle m, b\rangle\big\} \\ &= \big(\det(\mathbb{Q}/\pi)\big)^{-1/2} z^{-d/2} \exp\big\{-2\pi i \langle a, b\rangle\big\} \sum_{n \in \mathbb{Z}^d} \exp\Big\{-\frac{\pi^2}{z} \mathbb{Q}^{-1}[n + b] - 2\pi i \langle a, n\rangle\Big\},\end{aligned}$$

where $\mathbb{Q}^{-1}[x]$ *denotes the quadratic form* $\langle \mathbb{Q}^{-1} x, x\rangle$, *defined by means of the positive definite operator* $\mathbb{Q}^{-1} : \mathbb{R}^d \to \mathbb{R}^d$.

Proof. See e.g. Fricker, p. 116 or Mumford, p. 189, (5.1) and p. 197, (5.9). □

Lemma 3.2. *For any* t *satisfying* $|t| \leq \varkappa_r$ (*defined in* (2.15)), *the following bound holds*

$$\big|(\theta - \theta_0)\,(r^{-2} + it)\big| \ll_d q^{d/2} |r^{-2} + it|^{-d/2} \exp\Big\{-\frac{\mu r^2}{1 + r^4 t^2}\Big\},$$

where $\mu = \pi^2 q^{-1}/2$.

Proof. By means of Poisson's formula for $\theta(z)$, $\operatorname{Re} z > 0$, see Lemma 3.1, we may write

$$\theta(z) = z^{-d/2} \big(\det(\mathbb{Q}\pi^{-1})\big)^{-1/2} \sum_{m \in \mathbb{Z}^d} \exp\big\{-z^{-1} \mathbb{Q}^{-1}[\pi m] - 2\pi i \langle m, a\rangle\big\}. \tag{3.1}$$

Note that the term with $m = 0$ in the series (3.1) is just $\theta_0(z)$ (see (2.13)). Thus we may write

$$\theta(z) - \theta_0(z) = \chi(z)\, z^{-d/2} \big(\det(\mathbb{Q}/\pi)\big)^{-1/2}, \tag{3.2}$$

where

$$\big|\chi(z)\big| \leq \sum_{m \in \mathbb{Z}^d \setminus 0} \exp\big\{-\operatorname{Re}(z^{-1})\, \mathbb{Q}^{-1}[\pi m]\big\}. \tag{3.3}$$

It is easy to see that for $z = r^{-2} + it$ with $|t| \le \varkappa_r = \pi r^{-1}$, $r \ge 1$, we have

$$\mathrm{Re}(z^{-1}) = \frac{r^2}{1 + r^4 t^2} \ge \frac{r^2}{1 + r^4 \varkappa_r^2} \ge c_0 = \frac{1}{1+\pi}. \tag{3.4}$$

Let $\mu = \dfrac{1}{2} \min\limits_{|x|=1} \mathbb{Q}^{-1}[\pi x] = \pi^2 q^{-1}/2$. Then (3.3) and (3.4) together imply

$$\bigl|\chi(z)\bigr| \le \exp\Bigl\{-\frac{\mu r^2}{1 + r^4 t^2}\Bigr\}\, \chi_0(z), \tag{3.5}$$

where

$$\chi_0(z) = \sum_{m \in \mathbb{Z}^d \setminus 0} \exp\bigl\{-c_0\,(\mathbb{Q}^{-1}[\pi m] - \mu)\bigr\}.$$

Note that for any $|x| \ge 1$, $\mathbb{Q}^{-1}[\pi x] - \mu \ge \mathbb{Q}^{-1}[\pi x]/2 \ge \dfrac{\pi^2}{2q}\,|x|^2$. Since for $\alpha > 0$

$$\sum_{m=1}^{\infty} \exp\{-\alpha m^2\} \le \int_0^\infty \exp\{-\alpha v^2\}\, dv \ll \alpha^{-1/2}, \tag{3.6}$$

we conclude (recall that $q \ge q_0 = 1$)

$$\chi_0(z) \le \sum_{m \in \mathbb{Z}^d \setminus 0} \exp\Bigl\{-\frac{\pi^2 c_0}{2q}\,|m|^2\Bigr\} \ll_d \bigl(1 + (2q/\pi^2 c_0)^{1/2}\bigr)^d \ll_d q^{d/2}. \tag{3.7}$$

Collecting the relations (3.2), (3.5) and (3.7), we obtain the desired bound. $\square$

Lemma 3.3. *Let $\theta(z)$ denote the theta function (2.5) depending on $\mathbb{Q}$ and a. For $r \ge 1$, $t \in \mathbb{R}$, the following bound holds*

$$\begin{aligned} \bigl|\theta(r^{-2} + it)\bigr| &\ll_d (\det \mathbb{Q})^{-1/4}\, r^{d/2}\, \psi(r,t)^{1/2}, \qquad \textit{where} \\ \psi(r,t) &= \sum_{m,n \in \mathbb{Z}^d} \exp\Bigl\{-\frac{r^2}{2}\,\mathbb{Q}^{-1}[\pi m - 2t\,\mathbb{Q} n] - \frac{2}{r^2}\,\mathbb{Q}[n]\Bigr\}. \end{aligned} \tag{3.8}$$

Note that the right hand side of this inequality is independent of $a \in \mathbb{R}^d$.

Proof. For any $x, y \in \mathbb{R}^d$ the equalities

$$\begin{aligned} 2\bigl(\mathbb{Q}[x] + \mathbb{Q}[y]\bigr) &= \mathbb{Q}[x+y] + \mathbb{Q}[x-y], \\ \bigl\langle \mathbb{Q}(x+y),\, x-y \bigr\rangle &= \mathbb{Q}[x] - \mathbb{Q}[y] \end{aligned} \tag{3.9}$$

hold. Rearranging $\theta(z)\,\overline{\theta(z)}$ and using (3.9), we would like to use $m+n$ and $m-n$ as new summation variables on a lattice. But both vectors have the same parity, i.e., $m + n \equiv m - n \mod 2$. Since they are dependent one has to consider the 2^d sublattices indexed by $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j = 0, 1$, for $1 \le j \le d$:

$$\mathbb{Z}^d_\alpha \stackrel{\text{def}}{=} \{m \in \mathbb{Z}^d : m \equiv \alpha \mod 2\},$$

where, for $m=(m_1,\dots,m_d)$, $m\equiv\alpha \mod 2$ means $m_j\equiv\alpha_j \mod 2,\ 1\le j\le d$. Thus writing

$$\theta_\alpha(z) \overset{\text{def}}{=} \sum_{m\in\mathbb{Z}^d_\alpha} \exp\bigl\{-z\,\mathbb{Q}[m+a]\bigr\},$$

one obtains $\theta(z)=\sum_\alpha \theta_\alpha(z)$ and hence by the Cauchy–Schwarz inequality

$$\bigl|\theta(z)\bigr|^2 \le\ 2^d \sum_\alpha \bigl|\theta_\alpha(z)\bigr|^2 \tag{3.10}$$

Note that the map

$$\begin{aligned} H:\mathbb{Z}^d_\alpha\times\mathbb{Z}^d_\alpha \quad &\to \quad \mathbb{Z}^d\times\mathbb{Z}^d,\\ (m,n) \quad &\longmapsto \quad \Bigl(\frac{m+n}{2},\,\frac{m-n}{2}\Bigr)\end{aligned}$$

is a bijection. Using (3.9) and the absolute convergence of $\theta_\alpha(z)$, we may rewrite $\theta_\alpha(z)\,\overline{\theta_\alpha(z)}$ as

$$\begin{aligned}\theta_\alpha(z)\,\overline{\theta_\alpha(z)} &= \sum_{m,n\in\mathbb{Z}^d_\alpha} \exp\Bigl\{-\frac{1}{2r^2}\,\bigl\{\mathbb{Q}[m+n+2a]+\mathbb{Q}[m-n]\bigr\}\\ &\qquad\qquad - it\bigl\langle\mathbb{Q}(m+n+2a),\,m-n\bigr\rangle\Bigr\}\\ &= \sum_{\overline{m},\overline{n}\in\mathbb{Z}^d} \exp\Bigl\{-\frac{2}{r^2}\,\bigl\{\mathbb{Q}[\overline{m}+a]+\mathbb{Q}[\overline{n}]\bigr\} - 4it\,\bigl\langle\mathbb{Q}(\overline{m}+a),\overline{n}\bigr\rangle\Bigr\}, \end{aligned} \tag{3.11}$$

where $\overline{m}=\dfrac{m+n}{2}$, $\overline{n}=\dfrac{m-n}{2}$, $\overline{m},\overline{n}\in\mathbb{Z}^d$. In this double sum fix $\overline{n}$ and sum over $\overline{m}\in\mathbb{Z}^d$ first. Using Lemma 3.1, we get

$$\begin{aligned}\theta(z,\overline{n}) &\overset{\text{def}}{=} \sum_{\overline{m}\in\mathbb{Z}^d} \exp\Bigl\{-\frac{2}{r^2}\,\mathbb{Q}[\overline{m}+a] - 4it\,\bigl\langle\mathbb{Q}(\overline{m}+a),\overline{n}\bigr\rangle\Bigr\}\\ &= \Bigl(\det\Bigl(\frac{2}{\pi r^2}\,\mathbb{Q}\Bigr)\Bigr)^{-1/2} \sum_{m\in\mathbb{Z}^d} \exp\Bigl\{-\frac{r^2}{2}\,\mathbb{Q}^{-1}[\pi m - 2t\,\mathbb{Q}\overline{n}] - 2\pi i\langle m,a\rangle\Bigr\}\end{aligned}$$

Thus

$$\bigl|\theta(z,\overline{n})\bigr| \le \Bigl(\det\Bigl(\frac{2}{\pi r^2}\,\mathbb{Q}\Bigr)\Bigr)^{-1/2} \sum_{m\in\mathbb{Z}^d} \exp\Bigl\{-\frac{r^2}{2}\,\mathbb{Q}^{-1}[\pi m - 2t\,\mathbb{Q}\overline{n}]\Bigr\}$$

and therefore by (3.10) and (3.11) we have

$$\bigl|\theta(z)\bigr|^2 \ll_d (\det\mathbb{Q})^{-1/2}\,r^d \sum_{m,\overline{n}\in\mathbb{Z}^d} \exp\Bigl\{-\frac{r^2}{2}\,\mathbb{Q}^{-1}[\pi m - 2t\,\mathbb{Q}\overline{n}] - \frac{2}{r^2}\,\mathbb{Q}[\overline{n}]\Bigr\},$$

which proves Lemma 3.3. $\square$

In the following we shall use some facts in the geometry of numbers (see Davenport 1958). Let $F : \mathbb{R}^d \to [0, \infty]$ denote a norm on $\mathbb{R}^d$, that is $F(\alpha x) = |\alpha| \, F(x)$, for $\alpha \in \mathbb{R}$, and $F(x+y) \le F(x) + F(y)$. The successive minima $M_1 \le \cdots \le M_d$ of F with respect to the lattice $\mathbb{Z}^d$ are defined as follows: Let $M_1 = \inf\bigl\{ F(m) : m \ne 0, m \in \mathbb{Z}^d \bigr\}$ and define M_k as the infimum of $\lambda > 0$ such that the set $\bigl\{ m \in \mathbb{Z}^d : F(m) < \lambda \bigr\}$ contains k linearly independent vectors. It is easy to see that these infima are attained, that is there exist linearly independent vectors $a_1, \dots, a_d \in \mathbb{Z}^d$ such that $F(a_j) = M_j$.

Lemma 3.4. (Davenport 1958, Lemma 3)
Let $L_j(x) = \sum_{k=1}^d q_{jk} x_k$, $1 \le j \le d$, denote linear forms on $\mathbb{R}^d$ such that $q_{jk} = q_{kj}$, $j, k = 1, \dots, d$. Assume that $r \ge 1$ and let $\|v\|$ denote the distance of the number v to the nearest integer. Then the number of $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ such that

$$\|L_j(m)\| < r^{-1}, \quad |m_j| < r, \qquad \text{for all } \ 1 \le j \le d, \tag{3.12}$$

is bounded from above by $c_d (M_1 \cdots M_d)^{-1}$, where $c_d > 0$ denotes a constant depending on d only, $M_1 \le \cdots \le M_d$ are the first d of the $2d$ successive minima $M_1 \le \cdots \le M_{2d}$ of the norm $F : \mathbb{R}^{2d} \to [0, \infty)$ defined for vectors $y = (x, \overline{x}) \in \mathbb{R}^{2d}$, $x, \overline{x} \in \mathbb{R}^d$, $\overline{x} = (\overline{x}_1, \dots, \overline{x}_d)$, as

$$F(y) \stackrel{\mathrm{def}}{=} \max\bigl\{ r\,|L_1(x) - \overline{x}_1|, \dots, r\,|L_d(x) - \overline{x}_d|, \, r^{-1} |x|_\infty \bigr\}. \tag{3.13}$$

Moreover,

$$\frac{1}{2d} \le M_k M_{2d+1-k} \le (2\,d)^{2d-1}, \qquad 1 \le k \le 2d, \tag{3.14}$$

(see Davenport 1958, (20), p. 113).

Note that for some constant, say $c(d) > 0$, depending on d only

$$r^{-1} \le M_1 \le \cdots \le M_d \le c(d), \tag{3.15}$$

where the first inequality is obvious by $F(y) \ge r^{-1} |x|_\infty$. If here $x = 0$ then $\overline{x} \ne 0$ and $F(y) = r\,|\overline{x}|_\infty \ge r^{-1} |\overline{x}|_\infty \ge r^{-1}$. Finally, $M_d \ll_d 1$ follows from (3.14) for $k = d$.

In the following we shall consider linear forms

$$L_j(x) = \sum_{k=1}^d t\, q_{jk}\, x_k, \qquad 1 \le j \le d, \tag{3.16}$$

where $\mathbb{Q} = (q_{ij})$, $i, j = 1, \dots, d$, denotes the components of the positive definite matrix $\mathbb{Q}$ and where $t \in \mathbb{R}$ is arbitrary. Denote the corresponding successive minima of the norm $F(\cdot)$ defined by (3.13) and (3.16) for fixed t by $M_{j,t}$, $j = 1, \dots, d$. Thus, we can write

$$M_{j,t} = \bigl| L(m, n, t) \bigr|_\infty, \tag{3.17}$$

for some $m, n \in \mathbb{Z}^d$, where

$$L(m, n, t) = \bigl(r\,(m_1 - t\,(\mathbb{Q}n)_1), \dots, r\,(m_d - t\,(\mathbb{Q}n)_d), \, r^{-1} n_1, \dots, r^{-1} n_d \bigr). \tag{3.18}$$

It is easy to see from the definition that

$$M_{j,t} = M_{j,-t}, \qquad j = 1, \dots, d, \quad t \in \mathbb{R}. \tag{3.19}$$

Lemma 3.5. *Let* $r \ge 1$. *Then*

$$\big|\theta(r^{-2} + it\pi/2)\big| \ll_d q^d r^{d/2} (M_{1,t} \cdots M_{d,t})^{-1/2}.$$

Proof. By Lemma 3.3 we need to estimate the theta series $\psi(r, t\pi/2)$. Using the inequalities $\mathbb{Q}^{-1}[x] \ge q^{-1} |x|_\infty^2$ and $\mathbb{Q}[x] \ge |x|_\infty^2$, we have with $c_{\mathbb{Q}} = \min\{\pi^2/(2q), 2\}$

$$\psi(r, t\pi/2) \ll_d \sum_{m,n \in \mathbb{Z}^d} \exp\{-c_{\mathbb{Q}} \, |L(m,n,t)|_\infty^2\}, \tag{3.20}$$

where $L(m,n,t)$ is defined in (3.18). Let $H \overset{\text{def}}{=} \{(m,n) \in \mathbb{Z}^{2d} : |L(m,n,t)|_\infty < 1\}$. Now Lemma 3.4 may be restated for the forms (3.16) as

$$\# H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}. \tag{3.21}$$

In order to bound $\psi(r, t\pi/2)$, introduce for $k \overset{\text{def}}{=} (k_1, \dots, k_{2d}) \in \mathbb{Z}^{2d}$ the sets

$$B_k \overset{\text{def}}{=} \Big[k_1 - \frac{1}{2}, k_1 + \frac{1}{2}\Big) \times \cdots \times \Big[k_{2d} - \frac{1}{2}, k_{2d} + \frac{1}{2}\Big) \qquad \text{and}$$

$$H_k \overset{\text{def}}{=} \big\{(m,n) \in \mathbb{Z}^{2d} : L(m,n,t) \in B_k\big\}$$

such that $\mathbb{R}^{2d} = \bigcup_k B_k$. For any fixed $(m^*, n^*) \in H_k$ we have

$$(m - m^*, n - n^*) \in H \qquad \text{for any} \quad (m,n) \in H_k.$$

Hence we conclude for any $k \in \mathbb{Z}^{2d}$

$$\# H_k \le \# H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}. \tag{3.22}$$

Since $x \in B_k$ implies $|x|_\infty \ge |k|_\infty/2$, we obtain by (3.20) and (3.22)

$$\begin{aligned}
\psi(r, t\pi/2) &\ll_d \# H_0 + \sum_{k \in \mathbb{Z}^{2d} \setminus 0} \; \sum_{m,n \in \mathbb{Z}^{2d}} \mathbb{I}\big\{L(m,n,t) \in B_k\big\} \exp\{-c_{\mathbb{Q}} |k|_\infty^2/4\} \\
&\ll_d (M_{1,t} \cdots M_{d,t})^{-1} \sum_{k \in \mathbb{Z}^{2d}} \exp\{-c_{\mathbb{Q}} |k|_\infty^2/4\} \\
&\ll_d (M_{1,t} \cdots M_{d,t})^{-1} (c_{\mathbb{Q}}^{-1/2} + 1)^{2d},
\end{aligned}$$

using similar bounds as in (3.7). Some simple bounds together with Lemma 3.3 finally conclude the proof of Lemma 3.5. $\square$

Lemma 3.6. *Let* $(m,n), (m',n') \in \mathbb{Z}^{2d} \setminus 0$; $t, t' \in \mathbb{R}$ *and* $r \ge 1$. *Let* $M \overset{\text{def}}{=} |L(m,n,t)|_\infty$ *and* $M' \overset{\text{def}}{=} |L(m',n',t')|_\infty$. *Assume that* $\langle \mathbb{Q} n, n' \rangle > 0$ *and*

$$\max\{M, M'\} \le (4\,d)^{-1/2}. \tag{3.23}$$

Then either

$$\begin{aligned} &\text{i)} \quad |t-t'| \ \le \ \frac{d^{1/2}\max\{M,M'\}\,(|n|+|n'|)}{r\,\langle \mathbb{Q}n, n'\rangle} \\ \textit{or} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.24) \\ &\text{ii)} \quad |t-t'| \ \ge \ \langle \mathbb{Q}n, n'\rangle^{-1}/2, \end{aligned}$$

depending on

$$\Delta = \Delta(m,n;m',n') \stackrel{\text{def}}{=} \big|\,\langle n', m\rangle - \langle m', n\rangle\,\big| \tag{3.25}$$

being zero (*in case i*)) *or nonzero* (*in case ii*)). *In particular, assuming* $n = n'$ *and* (3.23) *the alternative (3.24 i) holds.*

Furthermore, assuming $(m,n) \in \mathbb{Z}^{2d}\setminus 0$ *and* $M = |L(m,n,t)|_\infty \le (4\,d)^{-1/2}$ *we have either*

$$\text{i)} \quad |t| \le \frac{2\,d\,M\,|n|}{r\,|\mathbb{Q}n|} \quad \textit{or} \quad \text{ii)} \quad |t| \ge \frac{1}{2\,|\mathbb{Q}n|}. \tag{3.26}$$

This means t, t' *resp.* $t, 0$ *have to be either 'near' to each other or 'far' apart.*

Proof. Note first that the condition (3.23) implies that $n, n' \ne 0$. Indeed, if $n = 0$, then (3.18) yields $M \ge r \ge 1$ that contradicts to (3.23).

Split $L(m,n,t) \in \mathbb{R}^{2d}$, see (3.18), into the first d and the second d components, i.e. $L(m,n,t) = (\widetilde{L}(m,n,t), r^{-1}n)$, with $\widetilde{L}(m,n,t) \in \mathbb{R}^d$. Let $J : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ denote the map $(x,y) \mapsto (y,-x)$, $x,y \in \mathbb{R}^d$, such that $J^2 = -\mathrm{Id}$. For $x \in \mathbb{R}^d$ let $|x|_1$ denote the l_1-norm of x. Then

$$\begin{aligned} W &\stackrel{\text{def}}{=} \big|\big\langle L(m,n,t), J\,L(m',n',t')\big\rangle\big| \\ &\le |\widetilde{L}(m,n,t)|_\infty\,|n'|_1\,r^{-1} + |\widetilde{L}(m',n',t')|_\infty\,|n|_1\,r^{-1} \\ &\le \overline{W} \stackrel{\text{def}}{=} \max\{M,M'\}\,r^{-1}\big(|n'|+|n|\big)\,d^{1/2}. \end{aligned} \tag{3.27}$$

On the other hand by definition and by symmetry of $\mathbb{Q}$

$$\begin{aligned} W &= \big|\,\langle m - t\mathbb{Q}n, n'\rangle - \langle n, m' - t'\mathbb{Q}n'\rangle\,\big| \\ &= \big|\,\big(\langle m, n'\rangle - \langle n, m'\rangle\big) - (t-t')\langle \mathbb{Q}n, n'\rangle\,\big|. \end{aligned} \tag{3.28}$$

Obviously, $\Delta \in \mathbb{N}_0$ (see (3.25)) and we have two cases

i) $\Delta = 0$: Here (3.28) implies $W = |t-t'|\,\langle \mathbb{Q}n, n'\rangle$ and in view of (3.27) we obtain

$$|t-t'| \le \max\{M,M'\}\,d^{1/2}\,\frac{|n'|+|n|}{r\,\langle \mathbb{Q}n, n'\rangle}.$$

ii) $\Delta \ge 1$: Here (3.28) and (3.27) together imply

$$|t-t'|\,\langle \mathbb{Q}n, n'\rangle \ge \Delta - W \ge \Delta - \overline{W} \ge \frac{1}{2},$$

since $W \le \overline{W} \le \max\{M, M'\}(M + M')\,d \le 2\,(\max\{M, M'\})^2 d < 1/2$ by assumption. Thus we obtain

$$|t - t'| \ge \langle \mathbb{Q}n, n' \rangle^{-1}/2.$$

This proves the first part of the Lemma.

Assuming $n = n'$ together with (3.23) implies $m = m'$ and hence $\Delta = 0$ or case (3.24 i).

For the remaining part of the Lemma let $t' = 0$, $m' = 0$ and let n' denote one of the standard basis vectors of $\mathbb{Z}^d$, that is $|n'| = 1$, $n' \in \mathbb{Z}^d$. Then $M' = r^{-1} \le M$ (see (3.15)) and choosing for n an appropriate basis vector n' such that $\langle \mathbb{Q}n, n' \rangle = |\mathbb{Q}n|_\infty$, we have $|\mathbb{Q}n|\,d^{-1/2} \le \langle \mathbb{Q}n, n' \rangle \le |\mathbb{Q}n|$. Hence (3.26) follows from (3.24) i), ii) by some easy estimations. □

Corollary 3.7. *Let $r \ge 1$ and $d \ge 4$. Then*

$$M_{1,t} \cdots M_{d,t} \ge d^{-d}\Big(\min\Big\{\frac{|t|\,r}{2},\ \frac{1}{q|t|\,r}\Big\}\Big)^d. \tag{3.29}$$

Proof. Let $M_{j,t} \le (4\,d)^{-1/2}$. By (3.26), $|\mathbb{Q}n| \ge |n|$, $|n| \ge q^{-1}|\mathbb{Q}n|$, $|n|_\infty \le r\,M_{j,t}$ and $2\,d^{1/2} \le d$ we have

$$\begin{aligned} &\text{i)} \qquad |t|\,r\,d^{-1} \le |t|\,r\,d^{-1}\,\frac{|\mathbb{Q}n|}{|n|} \le 2\,M_{j,t} \\ \text{or}& \\ &\text{ii)} \qquad \frac{1}{|t|} \le 2\,|\mathbb{Q}n| \le 2q\,|n| \le 2\,d^{1/2}q\,|n|_\infty \le q\,d\,r\,M_{j,t}, \end{aligned} \tag{3.30}$$

for appropriate $(m, n) \in \mathbb{Z}^{2d}$ depending on j such that $M_{j,t} = |L(m, n, t)|_\infty$. Note that if $M_{j,t} \ge (4\,d)^{-1/2}$, then $M_{j,t} \ge d^{-1}$ since $d \ge 4$. Combined with (3.30), this proves Corollary 3.7 since

$$\min\Big\{\frac{|t|\,r}{2},\ \frac{1}{q|t|\,r}\Big\} \le 1$$

(recall that $q \ge 1$). □

Lemma 3.8. *Let $[a, b] \subset \mathbb{R}$, $0 < a < b < \infty$. Define for $g \in C^1[a, b]$ such that $g \ge 0$, $g' \le 0$ on $[a, b]$,*

$$H_{a,b}(\tau) \stackrel{\text{def}}{=} \int_a^b \mathbb{I}\{M_{1,t} \le \tau\}\,g(t)\,dt.$$

Then, for all

$$a > r^{-1}, \qquad r^{-1} \le \tau \le (2\,d)^{-1}, \tag{3.31}$$

we have

$$H_{a,b}(\tau) \ \ll_d\ \overline{H}_{a,b}(\tau) \stackrel{\text{def}}{=} q\tau^2 \int_{a(\tau r)}^b g(t)\,dt + \frac{\tau}{r}\,g(a(\tau r)), \tag{3.32}$$

where $a(v) = \max\big\{a, (2qv\,d^{1/2})^{-1}\big\}$, provided that $a(\tau r) \le b$. In the case where $a(\tau r) > b$, we have $H_{a,b}(\tau) = 0$.

Proof. We shall show at first that the set of t from the interval $[a,b]$ such that $M_{1,t} \le \tau$ is contained in a finite union of sets of sufficiently small Lebesgue measure.

Let $u_j \in \mathbb{R}^d$, $|u_j| = 1$, $j = 1, \dots, K = K(d)$, denote vectors, such that the cones

$$C_j \stackrel{\rm def}{=} \big\{ u \in \mathbb{R}^d \,:\, \langle u, u_j \rangle \ge c\,|u| \big\}, \qquad j = 1, \dots, K, \qquad \text{where} \qquad c = \sqrt{3}/2,$$

cover $\mathbb{R}^d$. For $(m,n) \in \mathbb{Z}^{2d} \setminus 0$ and τ satisfying (3.31), define the sets

$$V_{m,n}(\tau) = \big\{ t \in [a,b] \,:\, |L(m,n,t)|_\infty \le \tau \big\}, \tag{3.33}$$

which by construction are closed intervals. Obviously, $V_{m,n}(\tau) = \varnothing$ for $n = 0$. Indeed, if $n = 0$ and $V_{m,n}(\tau) \ne \varnothing$, then $\tau \ge r \ge 2d > \tau$ (see (3.18) and (3.31)), a contradiction.

Furthermore, introduce for $1 \le j \le K$, $\nu \ge 1$

$$\begin{aligned} U_j(\tau,\nu) \;&=\; \bigcup_{(m,n)\in I_{j,\nu}} V_{m,n}(\tau), \qquad \text{where} &(3.34)\\ I_{j,\nu} \;&=\; \big\{ (m,n) \in \mathbb{Z}^{2d} \setminus 0 \,:\, \mathbb{Q}^{1/2}\, n \in C_j,\ \nu \le |n| < 2\nu \big\}. \end{aligned}$$

Since the sets $I_{j,\nu}$ cover Z^{2d} we obtain by definition of $M_{1,t}$

$$\{ t \in [a,b] \,:\, M_{1,t} \le \tau \} = \bigcup_{\nu,j}^{*} U_j(\tau,\nu), \tag{3.35}$$

where $\bigcup_{\nu,j}^{*}$ denotes the union over all $j = 1, \dots, K$ and $\nu = 2^l$, $l = 0, 1, \dots$ such that $\nu \le \tau\, r\, d^{1/2}$ since $\nu\, r^{-1} \le d^{1/2} M_{1,t} \le d^{1/2}\tau$.

Note that $U_j(\tau,\nu)$ is a union of a finite number of closed intervals, since $V_{m,n}(\tau) \ne \varnothing$ implies $|n|_\infty r^{-1} \le \tau$ and $r\,|m - t\mathbb{Q}n|_\infty \le \tau$, which implies that the vector $(m,n) \in \mathbb{Z}^{2d}$ belongs to a bounded set. Furthermore note that the sets $U_j(\tau,\nu)$ may have considerable overlap, since for any t there may exist corresponding vectors n of different lengths contained in several intervals $[\nu, 2\nu]$.

Obviously we have for any $v, v' \in C_j$ with $|v| = |v'| = 1, \langle v, v' \rangle \ge 1/2$. This is a consequence of

$$2 + 2\langle v, v' \rangle = |v + v'|^2 \ge \langle v + v', u_j \rangle^2 \ge (2c)^2 = 3.$$

Hence, it follows, for any $n, n' \in \mathbb{Z}^d \setminus 0$ with $\mathbb{Q}^{1/2}\, n,\ \mathbb{Q}^{1/2}\, n' \in C_j$ (recall that $q_0 = 1$),

$$4q\nu^2 \ge \langle \mathbb{Q}n, n' \rangle = \langle \mathbb{Q}^{1/2}\, n, \mathbb{Q}^{1/2}\, n' \rangle \ge \frac{1}{2}\, |\mathbb{Q}^{1/2}\, n|\, |\mathbb{Q}^{1/2}\, n'| \ge \nu^2/2. \tag{3.36}$$

We shall now show using Lemma 3.6 that $U_j(\tau,\nu)$ is contained in a union of 'small' intervals which are separated by 'large' gaps. Fix ν, j and τ. Let $t_0 = \inf\{ t \in U_j(\tau,\nu) \}$. Then $t_0 \in U_j(\tau,\nu)$ and there exist finitely many vectors $(m,n) \in \mathbb{Z}^{2d} \setminus 0$ such that

$$|L(m,n,t_0)|_\infty \le \tau, \quad \nu \le |n| < 2\nu \quad \text{and} \quad \mathbb{Q}^{1/2}\, n \in C_j.$$

Choose such a vector, say (m,n). For any $t \in U_j(\tau,\nu)$, we may again choose a vector, say $(m',n') \in I_{j,\nu}$, such that $t \in V_{m',n'}(\tau)$ and apply Lemma 3.6 since the condition (3.23) holds by (3.31) for the pair n, n'. Hence either the alternative (3.24 i) or (3.24 ii) in Lemma

3.6 applies to the pair $t_0; m, n$ and $t; m', n'$ depending on the integer $\Delta(m, n; m', n')$, see (3.25), being zero or non zero. Hence by (3.24 i) we have in view of (3.33), (3.36) and $|n|,\, |n'| \le 2\nu$ either

$$t - t_0 \le \varepsilon_\nu, \quad \text{where} \quad \varepsilon_\nu \overset{\text{def}}{=} \frac{4\,d^{1/2}\,\tau}{\nu\, r} \tag{3.37}$$

or by (3.33) and (3.36)

$$t - t_0 \ge \alpha_\nu, \ \text{ where } \ \ \alpha_\nu \overset{\text{def}}{=} \frac{1}{8\,q\,\nu^2}. \tag{3.38}$$

Thus we conclude

$$U_j(\tau, \nu) \subset [t_0, t_0 + \varepsilon_\nu] \cup [t_0 + \alpha_\nu, b]. \tag{3.39}$$

Write $t_1 = t_0 + \varepsilon_\nu$. If $[t_0 + \alpha_\nu, b]$ has positive length and

$$t_2 = \inf\{t \in U_j(\tau, \nu) \,:\, t \ge t_1,\, t \ge t_0 + \alpha_\nu\}$$

exists we continue the construction otherwise we stop. In the latter case we conclude $U_j(\tau, \nu) \subset [t_0, t_1]$, whereas in the first case we have $t_2 \in U_j(\tau, \nu)$ and

$$U_j(\tau, \nu) \subset [t_0, t_1] \cup [t_2, b]. \tag{3.40}$$

Replacing t_0 by t_2 and repeating this construction, we obtain a sequence

$$t_0, t_1, \dots, t_{2k}, t_{2k+1}, \dots \in U_j(\tau, \nu)$$

with $t_{2k} \le t_{2k+1} \le t_{2k+2},\ k = 0, 1, 2, \dots, \mu$ such that

$$t_{2k+1} - t_{2k} \le \varepsilon_\nu, \quad t_{2k+2} - t_{2k} \ge \alpha_\nu. \tag{3.41}$$

Hence there exists some $\mu = \mu(j, \nu)$ with $t_{2\mu} \le b$ and $t_{2\mu+2} > b$ (if that exists) such that

$$U_j(\tau, \nu) \subset \bigcup_{k=0}^{\mu} \,[t_{2k}, t_{2k+1}] \tag{3.42}$$

holds. Using for $t \in U_j(\tau, \nu)$ the relation (3.26) in Lemma 3.6, we may exclude there the case i) since in this case $0 \le t \le 2\,d\tau r^{-1} < r^{-1} < a$ by assumptions (3.31). This contradicts to $U_j(\tau, \nu) \subset [a, b]$ (see (3.33) and (3.34)). Note that $t \in V_{m,n}(\tau)$ and $t \in U_j(\tau, \nu)$ imply that $\nu \le |n| \le r\,\tau\,d^{1/2}$ (see (3.18) and (3.33)). Hence, by (3.26 ii), we may assume

$$t_0 \ge \max\{a, (2\,q\,|n|)^{-1}\} \ge \max\{a, (2\,q\,\tau\,r\,d^{1/2})^{-1}\} = a(\tau\,r).$$

With the help of the these results, (3.35) and (3.42) we can estimate the integral $H_{a,b}(\tau)$:

$$H_{a,b}(\tau) = \int_a^b \mathbb{I}\big\{M_{1,t} \le \tau\big\}\, g(t)\, dt \le {\sum_{j,\nu}}^{*} \int_{a(\tau r)}^{b} \mathbb{I}\big\{U_j(\tau, \nu)\big\}\, g(t)\, dt, \tag{3.43}$$

where $\sum_{j,\nu}^{*}$ denotes the sum over $j = 1, \dots, K$, and $\nu = 2^l,\ l = 0, 1, \dots,$ provided that $2^l \le \tau\,r\,d^{1/2}$.

In view of (3.41), (3.42) and the monotonicity of g we obtain

$$\begin{aligned} H_{a,b}(\tau) &\le \sum_{j,\nu}{}^{*} \sum_{k=0}^{\mu} \int_{t_{2k}}^{t_{2k+1}} g(t)\,dt \le \sum_{j,\nu}{}^{*} \sum_{k=0}^{\mu} g(t_{2k})\,(t_{2k+1}-t_{2k}) \\ &\le \sum_{j,\nu}{}^{*} \sum_{k=0}^{\mu} g(t_0 + k\,\alpha_\nu)\,\varepsilon_\nu, \end{aligned} \tag{3.44}$$

where the latter inequality holds by monotonicity of g. Again by the monotonicity of g we may separate the term corresponding to $k=0$ and estimate the remaining sum over k in (3.44) by an integral, that is

$$\begin{aligned} H_{a,b}(\tau) &\le \sum_{j,\nu}{}^{*} \Big(\int_{a(\tau r)}^{b} g(u)\,du\,\frac{\varepsilon_\nu}{\alpha_\nu} + \varepsilon_\nu\, g\big(a(\tau r)\big)\Big) \\ &\ll_d K\Big(q\,\big(\sum_{2^l \le \tau\, r\, d^{1/2}} 2^l\big)\,\frac{\tau}{r} \int_{a(\tau r)}^{b} g(u)\,du + \frac{\tau}{r}\, g\big(a(\tau r)\big)\Big), \end{aligned}$$

thus proving Lemma 3.8. □

For indicator functions g Lemma 3.8 reads as follows.

Lemma 3.9. *Let λ denote the Lebesgue measure. There exists a constant $c(d)$ depending on d only such that for any $r \ge 1$, $\tau > 0$ and any interval $[a,b]$ with $b > a$*

$$I(\tau) \overset{\text{def}}{=} \lambda\{t \in [a,b] \,:\, M_{1,t} \le \tau\} \;\le\; c(d)\big(q\tau^2\,(b-a) + \tau\, r^{-1}\big).$$

Proof. Repeating the proof of Lemma 3.8 till (3.42), note that we made no restrictions on a in Lemma 3.9. Hence skipping the arguments between (3.42) and (3.43) we get with the same notations as above similar as in (3.43) and (3.44) using (3.42)

$$\begin{aligned} I(\tau) &\le \sum_{j,\nu}{}^{*} \lambda\big([a,b] \cap U_j(\tau,\nu)\big) \\ &\le \sum_{j,\nu}{}^{*} \sum_{k=0}^{\mu} (t_{2k+1}-t_{2k}) \le \sum_{j,\nu}{}^{*} \varepsilon_\nu\,(\mu+1). \end{aligned} \tag{3.45}$$

Either we have $t_2 \le b$ and hence $(\mu+1)\,\alpha_\nu \le \sum_{k=0}^{\mu}(t_{2k+2}-t_{2k}) \le b-a$ which implies $\mu+1 \le (b-a)/\alpha_\nu$ or $t_2 > b$ and only the interval $[t_0,t_1]$ intersects $[a,b]$. In any case we get

$$I(\tau) \le \sum_{j,\nu}{}^{*} \varepsilon_\nu \max\Big\{\frac{b-a}{\alpha_\nu}, 1\Big\} \ll_d q\,\tau^2\,(b-a) + \tau\, r^{-1}, \tag{3.46}$$

which proves Lemma 3.9. □

Lemma 3.10. *Let $M(t) = M_{1,t} \cdots M_{d,t}$, $\gamma = \gamma(a,b) = r^d \inf_{a \le t \le b} M(t)$ and introduce*

$$D = \max\bigl\{ (2d)^{-d} r^d,\, \gamma \bigr\} \qquad \text{and} \qquad G(a,b) = \int_a^b g(t)\, dt, \tag{3.47}$$

for $0 < a < b \le \infty$ and let $g(t)$ and $a(v)$ be as in Lemma 3.8. For $a > b$ we define $G(a,b) = 0$. Then

$$\begin{aligned} I_{a,b} &\overset{\text{def}}{=} \int_a^b \frac{g(t)}{M(t)^{1/2}}\, dt \\ &\ll_d r^{d/2-2} \int_\gamma^D v^{-1/2+1/d} \Bigl(v^{1/d} q\, G\bigl(a(v^{1/d}), b\bigr) + g\bigl(a(v^{1/d})\bigr) \Bigr) \frac{dv}{v} + G(a,b). \end{aligned} \tag{3.48}$$

Proof. Write $\overline{\gamma} = \inf_{a \le t \le b} M(t)$ and $c_d = (2d)^{-d}$. If $\overline{\gamma} \ge c_d$, then $I_{a,b} \ll_d G(a,b)$ and (3.48) is obvious. Let $\overline{\gamma} < c_d$. Define

$$J_{a,b}(v) \overset{\text{def}}{=} \int_a^b g(t)\, \mathbb{I}\bigl\{ M(t) \le v \bigr\}\, dt \tag{3.49}$$

for $0 < a < b$. Then $M_{j,t} \le M_{d,t} \ll_d 1$, for $j = 1, \dots, d$, by Lemma 3.4 which implies $M(t) \le \bar{M}$ for all t, and some $\bar{M}$ depending on d only. Writing for $t \in [a,b]$

$$M(t)^{-1/2} = \int_{\overline{\gamma}}^{\bar{M}} \varepsilon^{-1/2}\, d\mathbb{I}\bigl\{ M(t) \le \varepsilon \bigr\}.$$

Fubini's theorem implies

$$I_{a,b} = \int_{\overline{\gamma}}^{\bar{M}} \varepsilon^{-1/2}\, dJ_{a,b}(\varepsilon).$$

Splitting the integral $I_{a,b}$ into the part where $\varepsilon \le c_d$ and its complement, we obtain

$$I_{a,b} \le \int_{\overline{\gamma}}^{c_d} \varepsilon^{-1/2}\, dJ_{a,b}(\varepsilon) + c_d^{-1/2} \int_a^b g(t)\, dt. \tag{3.50}$$

By partial integration

$$I_{a,b} \le \frac{1}{2} \int_{\overline{\gamma}}^{c_d} \varepsilon^{-3/2} J_{a,b}(\varepsilon)\, d\varepsilon + 2 c_d^{-1/2} G(a,b). \tag{3.51}$$

Furthermore, $M(t) \ge (M_{1,t})^d \ge r^{-d}$ (see (3.15)) implies together with Lemma 3.8

$$J_{a,b}(\varepsilon) \le H_{a,b}(\varepsilon^{1/d}) \le \overline{H}_{a,b}(\varepsilon^{1/d}), \qquad \text{for } \varepsilon \le c_d. \tag{3.52}$$

Thus we conclude by using (3.51), (3.52), Lemma 3.8 and the change of variable $v = r^d \varepsilon$

$$\begin{aligned} I_{a,b} &\ll_d \int_{\overline{\gamma}}^{c_d} \varepsilon^{-1/2+1/d} \Bigl(\varepsilon^{1/d} q\, G\bigl(a(r\varepsilon^{1/d}), b\bigr) + r^{-1} g\bigl(a(r\varepsilon^{1/d})\bigr) \Bigr) \frac{d\varepsilon}{\varepsilon} + G(a,b) \\ &\ll_d r^{d/2-2} \int_\gamma^D v^{-1/2+1/d} \Bigl(v^{1/d} q\, G\bigl(a(v^{1/d}), b\bigr) + g\bigl(a(v^{1/d})\bigr) \Bigr) \frac{dv}{v} + G(a,b) \end{aligned}$$

which proves (3.48). This completes the proof of Lemma 3.10. $\square$

Lemma 3.11. *Let* $0 < a < b < \infty$. *Then*

$$\limsup_{r\to\infty} \inf_{t\in[a,b]} \big(r\,M_{1,t}\big)\cdots\big(r\,M_{d,t}\big) = \infty$$

if and only if $\mathbb{Q}$ *is irrational.*

Proof. Assume

$$\inf_{t\in[a,b]} \big(r\,M_{1,t}\big)\cdots\big(r\,M_{d,t}\big) < K < \infty \tag{3.53}$$

for all $r \geq r_0 \geq 1$. The inequality (3.53) together with (3.15) implies by definition

$$r^{-1} \leq M_{j,t} \leq K\,r^{-1} \tag{3.54}$$

for any $j = 1,\ldots,d,\ r \geq r_0$ and some $t = t(r) \in [a,b]$.

Let $(m_j, n_j) \in \mathbb{Z}^{2d}$, denote linearly independent vectors such that

$$F\big((m_j, n_j)\big) = |L\,(m_j, n_j, t)|_\infty = M_{j,t} \tag{3.55}$$

for all $1 \leq j \leq 2d$. These equalities together with (3.54) imply

$$|n_j|_\infty \ll K, \qquad |m_j|_\infty \ll K + bqK, \qquad 1 \leq j \leq d. \tag{3.56}$$

For $j = 1,\ldots,d$, and $r = r_l \to \infty$ as $l \to \infty$ select a sequence of of $2d$-tuples of vectors $(m_j^{(l)}, n_j^{(l)}) \in \mathbb{Z}^{2d}\setminus 0$, and points $t = t_l \in [a,b]$ satisfying (3.55) and hence (3.56). Then

$$\Big|\sum_{k=1}^{d} t_l\,q_{pk}\,n_{jk}^{(l)} - m_{jp}^{(l)}\Big| \ll \frac{1}{r_l^2}, \qquad 1 \leq j \leq d, \quad 1 \leq p \leq d. \tag{3.57}$$

Here $n_{jk}^{(l)}$ and $m_{jp}^{(l)}$ are the coordinates of the vectors $n_j^{(l)}$ and $m_j^{(l)}$ respectively. The inequalities (3.56) guaranty that the sequence of d-tuples of vectors $\{m_1^{(l)},\ldots,m_d^{(l)}\}$ in $\mathbb{Z}^d$ is repeating infinitely often as $l \to \infty$. Choosing an appropriate subsequence, we may assume that $m_j^{(l)} = m_j$ are independent of l. Similarly we may assume that $n_j^{(l)} = n_j$ are independent of l. Passing to the limit as $l \to \infty$ in (3.57) along a subsequence and using $t_l \to t_0$, $a \leq t_0 \leq b$, it follows that the numbers $t_0\,q_{p1},\ldots,t_0\,q_{pd}$ satisfy

$$\sum_{k=1}^{d} n_{jk}\,t_0\,q_{pk} = m_{jp}, \qquad 1 \leq j \leq d, \tag{3.58}$$

for all $1 \leq p \leq d$. Below we shall prove that the vectors $n_1,\ldots,n_d$ are linearly independent. Therefore the system (3.58) has the unique solution $t_0\,q_{p1},\ldots,t_0\,q_{pd}$, which obviously has to be *rational* by Cramer's rule, for all $1 \leq p \leq d$.

To conclude the proof we have to show that the vectors $n_1,\ldots,n_d \in \mathbb{Z}^d$ are linearly independent in $\mathbb{Z}^d$ (or, equivalently, in $\mathbb{R}^d$). If $n_1,\ldots,n_d$ are linearly dependent then there exist $v_j \in \mathbb{R}$ not all equal to zero such that $|v_j| < 1$ and $\sum_{j=1}^{d} v_j\,n_j = 0$. Let us prove that there exist integers $u_1, \dots, u_d$ not all equal to zero such that $|u_j| \ll 1$ and $x \overset{\text{def}}{=} \sum_{j=1}^{d} u_j n_j = 0$. By the multivariate Dirichlet approximation (see, for example, Cassels 1959, Section V.10), for any $N \in \mathbb{N}$ there exist $u_j \in \mathbb{Z}$ and an integer $0 < k \le N$ such that

$$\left| v_j - \frac{u_j}{k} \right| < \frac{1}{k N^{1/d}}, \qquad \text{for all } 1 \le j \le d. \tag{3.59}$$

The inequalities $|v_j| < 1$, $0 < k \le N$ and (3.59) yield $|u_j| \le 2N$. Since the vectors n_j have integer coordinates and $|n_j|_\infty \ll 1$, the equality $\sum_{j=1}^{d} v_j n_j = 0$ together with (3.59) implies $x = \sum_{j=1}^{d} u_j n_j = 0$, for sufficiently large $N \ll 1$. Hence $|u_j| \ll 1$.

For the vector (m, x) with $m \overset{\text{def}}{=} \sum_{j=1}^{d} u_j m_j$ we have (see (3.18), (3.54) and (3.55))

$$F\big((m,x)\big) \le \sum_{j=1}^{d} |u_j| F\big((m_j, n_j)\big) \ll r^{-1} \tag{3.60}$$

since $|u_j| \ll 1$ and $F\big((m_j, n_j)\big) \asymp r^{-1}$. Using (3.18), (3.55) and $x = 0$, we have

$$F\big((m,x)\big) = F\big((m,0)\big) \ge r \left| \sum_{j=1}^{d} u_j m_{jp} \right|, \qquad \text{for all } 1 \le p \le d. \tag{3.61}$$

Combining (3.60) and (3.61), using that r is sufficiently large and that $\sum_{j=1}^{d} u_j m_{jp}$ are integers, we conclude that $\sum_{j=1}^{d} u_j m_{jp} = 0$, for all $1 \le p \le d$. In other words, $m = \sum_{j=1}^{d} u_j m_j = 0$, which together with the assumption $x = \sum_{j=1}^{d} u_j n_j = 0$ means that the vectors $(m_j, n_j) \in \mathbb{Z}^{2d}$, $1 \le j \le d$, are linearly dependent, a contradiction. □

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