Characterizations of Generalized Polygons and Opposition in Rank 2 Twin Buildings

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Abstract

In this paper, we characterize the natural opposition relation on the set of flags of a generalized polygon. We also investigate when a certain relation on any rank 2 geometry of finite diameter is equivalent to the opposition relation in a generalized polygon. As a consequence we obtain a new definition of generalized polygons. Finally, we also characterize the opposition relation in twin trees, which are the analogues of polygons with infinite diameter.

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1 Introduction

One of the most fundamental concepts in the theory of generalized polygons is the opposition relation in both, the set of flags and the set of elements. For instance, the opposition relation on elements of a generalized polygon enables one to define the groups of projectivity via bijections of the "stars" of opposite elements. In fact, to define these bijections, one only uses the "1-twinning" property of opposition of flags, as introduced by Mühlherr in [5]. In the latter paper, the author characterizes twin buildings (a concept introduced

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by Ronan and Tits generalizing spherical buildings) as pairs of buildings endowed with a certain opposition relation that is a 2-twinning (which implies that it is a 1-twinning).

The question raised by Mühlherr whether twin buildings can even be characterized by just using the 1-twinning property first led both authors to constructing rank 2 counter-examples (which will be presented in detail in Subsection 5.2 below) and then to joint systematic investigations of the opposition relation and of 1-twinnings in spherical and in twin buildings. These investigations were published in [2], respectively will appear soon in [3]. In the present paper, we develop further our general notions and ideas in the rank 2 situation, where the one-dimensionality of the apartments permits some more elegant and unexpected characterizations. An important additional feature is that, though 1-twinnings originally were only considered in connection with buildings, we do not only consider generalized polygons here but also arbitrary (firm connected) rank 2 geometries. In this way one of our main results, namely a new characterization of generalized polygons amongst the rank 2 geometries (cf. Theorem 4.5), can be derived. Though the present paper emerged from twin building theory, it should be mentioned that the first four sections do not presuppose any knowledge of this theory but are written in the language of (rank 2) incidence geometry. We hope that this way our results and proofs are easier accessible to geometers.

We now describe a little more detailed the contents of the present paper. Restricted to the spherical case, one can read the main result of [5] as a characterization of the (natural) opposition relation of chambers "up to equivalence" (this notion of equivalence will be introduced, together with other basic concepts, in Section 2 below.) For rank 2 buildings, Mühlherr's result becomes void. In [3], we have characterized twin buildings, and consequently the opposition relation in spherical buildings, using the concept of a 1-twinning, together with one further axiom which boils down, roughly speaking, to require at least one twin apartment. In the case of generalized n-gons, this additional assumption can be replaced by some other natural condition, as we shall show in Section 3 (cf. Proposition 3.1). We also give a characterization of the opposition relation in generalized n-gons not referring to 1-twinnings (and being specific to the rank 2 situation) in Proposition 3.2. All the conditions (and some variants of them) used in one of these propositions, and among them in particular the 1-twinning property, make sense for arbitrary rank 2 geometries. So in Section 4, we do not only characterize the opposition relation by means of these conditions but, more importantly, we show that appropriate combinations of them are sufficient (and necessary) in order to force the rank 2 geometry to already be a generalized n-gon. As an application, we obtain our new definition of a generalized n-gon, and we show that the class of near $2n$-gons which are also dual near $2n$-gons coincides with the class of generalized $2n$-gons.

So the main purpose of the present paper is to state characterizations of the opposition relation in generalized polygons, and to characterize the generalized polygons themselves by means of that relation. In Section 5, we include twin trees in one of these characterizations (cf. Proposition 5.2), giving a nice combinatorial condition for a 1-twinning between two trees to yield a twin tree in the sense of Ronan and Tits [7]. In this context, the language of twin building theory certainly can not be avoided any longer (there are no "one copy models" for twin trees). As already claimed, the 1-twinning property alone does not characterize the opposition relation in twin buildings. Counter-examples of rank 2 proving this assertion are also discussed in Section 5 (cf. Proposition 5.4). Here generalized polygons (yielding some counter-examples already alluded to in [5]) and trees are treated simultaneously.

Using the classical language, it was necessary to introduce what we mean by a relation on flags to be equivalent to the (natural) opposition relation in a generalized polygon. We give a precise definition below, and we shall explain in our final Section 5.1 how this definition translates to the twin case, and how this twin case motivates that definition. On that occasion, we also state a generalization of one of our results to spherical buildings of arbitrary rank.

2 Definitions and preliminary results

2.1 Basic definitions

Definition 2.1 A *rank 2 pregeometry* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ consists of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, with $\mathcal{P} \cap \mathcal{L} = \emptyset$, and a set of *flags* $\mathcal{F} \subseteq \{\{p, L\} \mid p \in \mathcal{P} \text{ and } L \in \mathcal{L}\}$. It is called *thick* (*firm*) if every point and every line is contained in at least three (two) flags. If $\{p, L\} \in \mathcal{F}$, then we usually say that p and L *are incident*, or that p (respectively L) *is incident with* L (respectively p). For a given geometry Γ, the *incidence graph* $G(\Gamma) = (X(\Gamma), E(\Gamma))$ is the graph obtained from Γ by putting $X(\Gamma) = \mathcal{P} \cup \mathcal{L}$ (the set of vertices of $G(\Gamma)$) and $E(\Gamma) = \mathcal{F}$ (the set of edges of $G(\Gamma)$). We call Γ *connected* if the graph $G(\Gamma)$ is connected.

In this paper we will call a *(rank* 2*) geometry* any rank 2 pregeometry which is firm and connected. From now on, we assume that Γ is a geometry.

The *(local) diameter* of Γ is by definition the (local) diameter of $G(\Gamma)$ (in a certain vertex v; i.e., the distance in $G(\Gamma)$ from v to an element at maximal distance from v) and the *girth* of Γ is the girth of $G(\Gamma)$. Note that the girth, if finite, is always an even natural number (the girth is ∞ if there are no circuits); therefore one sometimes considers the *gonality* of Γ, which is half of the girth. When the diameter of Γ is finite and equal to the gonality of Γ, then we say that Γ is a *generalized polygon* (see [12]; these objects were introduced by Tits in [10]). This, in fact, is equivalent with saying that the simplicial complex with set of vertices $\mathcal{P} \cup \mathcal{L}$ and set of maximal simplices $\mathcal{F}$ is a spherical rank 2 building (for an explicit proof, see e.g. [14]).

A rank 2 geometry the incidence graph of which is an infinite tree without finite end points is equivalent to a non-spherical building of rank 2 and will be called a *tree* itself, or sometimes a *generalized* ∞*-gon* (but not polygon).

Corresponding to a geometry Γ there is also its *flag graph* $F(\Gamma)$. The vertices of $F(\Gamma)$ are the flags of Γ, and two flags are adjacent if they share exactly one common element. The distance function in $F(\Gamma)$ will be denoted by δ. When we talk about *adjacent flags*, then we mean adjacency in $F(\Gamma)$.

A *path* $(f_0, f_1, \ldots, f_k)$ of flags f_i, $i \in \{0, 1, \ldots, k\}$, is a sequence of flags such that f_{i-1} is adjacent with f_i, for all $i \in \{1, 2 \ldots, k\}$. Such a path is called *simple* if $f_{i-1} \cap f_{i+1} = \emptyset$ for all $i \in \{1, 2, \ldots, k-1\}$. It is called *minimal* if $\delta(f_0, f_k) = k$. It is called *closed* if $f_0 = f_k$. The *length* of the above path is by definition equal to k.

If Γ is a generalized n-gon, then the diameter of the flag graph $F(\Gamma)$ is equal to n and two flags at distance n are called *opposite*. The relation thus defined on the set of flags is called *the natural opposition relation* in Γ and we write f opp g to denote opposite flags f, g. We also write f^{opp} for the set of all flags opposite f in Γ.

Also, still for a generalized n-gon Γ, it is easily seen that the length of a simple closed path is at least $2n$. If it is exactly $2n$, then we call the set of flags of the path an *apartment* (by abuse of language) of Γ.

2.2 Opposition in generalized polygons

Below we will list a few properties of the natural opposition relation in a generalized polygon. These properties are *intrinsic* in the sense that they only depend on the structure of the sets f^{opp}, and not of $\{f\} \cup f^{\text{opp}}$. Using these properties to try to characterize the natural opposition relation, one also finds relations which are not exactly the natural opposition relation, but only modulo an automorphism. Hence it makes sense to have the following definition.

Definition 2.2 Let $\mathcal{O} \subseteq \mathcal{F} \times \mathcal{F}$ be a symmetric relation in the set of flags $\mathcal{F}$ of the generalized polygon $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$. Then we say that $\mathcal{O}$ is *equivalent to the natural opposition relation* if there is an automorphism α of Γ such that $f\mathcal{O}g$ if and only if f opp $\alpha(g)$. Note that we do not require that α is type preserving (so α may interchange the sets $\mathcal{P}$ and $\mathcal{L}$).

Of course, since $\mathcal{O}$ in the previous definition is symmetric, we deduce that f is opposite $\alpha(g)$ if and only if $\alpha(f)$ is opposite g, which happens if and only if $\alpha^2(f)$ is opposite $\alpha(g)$, since α is an automorphism. Noting that f is the unique chamber at distance n from f^{opp}, we see that f^{opp} determines f uniquely, and hence α has order at most 2.

Due to the results of [2], there is an easy criterion to decide whether a given relation is equivalent to the natural opposition relation. In the following lemma, we write $f^{\mathcal{O}} := \{g \in \mathcal{F} \mid g\mathcal{O}f\}$.

Lemma 2.3 *A symmetric relation $\mathcal{O} \subseteq \mathcal{F} \times \mathcal{F}$ on the set of flags of a thick generalized polygon Γ is equivalent to the natural opposition relation in Γ if and only if for every flag f there is some flag g such that $f^{\mathcal{O}} = g^{\mathrm{opp}}$.*

Proof. The "only if" part is clear. For the "if" part, the assumption enables us to define a map $\alpha : \mathcal{F} \to \mathcal{F}$ by $\alpha(f)^{\mathrm{opp}} = f^{\mathcal{O}}$. Then obviously, for any pair of flags f, g, we have that g opp $\alpha(f)$ implies that $g\mathcal{O}f$, which implies by the symmetry of $\mathcal{O}$ that f is opposite $\alpha(g)$. Now Corollary 5.5 of [2] implies that α can be extended to an automorphism of Γ of order at most 2, and we are done. □

2.3 1-twinnings in geometries

The main purpose of the present paper being analyzing the consequences of a 1-twinning for rank 2 geometries, we now define what we understand by 1-twinning.

Definition 2.4 Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a geometry as defined above. We consider a non-empty symmetric relation $\mathcal{O} \subseteq \mathcal{F} \times \mathcal{F}$, and we write $f\mathcal{O}g$ if $(f, g) \in \mathcal{O}$, for any two flags of Γ. Also, for any flag $f \in \mathcal{F}$, we denote $f^{\mathcal{O}} := \{g \in \mathcal{F} \mid g\mathcal{O}f\}$. Finally, for $f, g \in \mathcal{F}$, we put $\ell^*(f, g) := \min\{\delta(h, g) \mid h \in f^{\mathcal{O}}\}$. In words, $\ell^*(f, g)$ is the distance in the flag graph of g to the set $f^{\mathcal{O}}$. Below we shall see that $f^{\mathcal{O}}$ is always non-empty, and hence that $\ell^*(f, g)$ is a well defined natural number, for any pair of flags f, g.

We call $\mathcal{O}$ an *even* (respectively, *odd*) 1-*twinning of* Γ if, given $(f, g) \in \mathcal{O}$ and elements $x \in f$ and $y \in g$ of the same (respectively, different) type, then for any flag f' containing x there exists precisely one flag g' containing y such that $(f', g') \notin \mathcal{O}$.

The relation $\mathcal{O}$ is called a 1-*twinning* if it is an even or an odd 1-twinning. Two elements $x, y \in \mathcal{P} \cup \mathcal{L}$ are said to be of *$\mathcal{O}$-opposite type* if either $\mathcal{O}$ is an even 1-twinning and x and y are of the same type (i.e. $x, y \in \mathcal{P}$ or $x, y \in \mathcal{L}$), or $\mathcal{O}$ is an odd 1-twinning and x, y are not of the same type. We say that two simple paths $(f_0, f_1, \ldots, f_k)$ and $(g_0, g_1, \ldots, g_k)$ (of the same length) are *of $\mathcal{O}$-opposite type* if $f_{i-1} \cap f_i$ and $g_{i-1} \cap g_i$ are of $\mathcal{O}$-opposite type, for all $i \in \{1, 2, \ldots, k\}$.

The motivation for this definition comes from the fact that the natural opposition relation in generalized polygons (and more generally, in spherical buildings and twin buildings) is a 1-twinning. But also every symmetric relation equivalent to the natural opposition relation in a generalized polygon is a 1-twinning. The converse, however, is not true, as is proved by the counter-examples in Section 5 below. Part of the aim of the present paper is to give additional conditions under which a 1-twinning is equivalent to the natural opposition relation. We summarize the most important properties that we will encounter in the course of this paper in the following lemma.

Lemma 2.5 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a generalized n-gon, and let $\mathcal{O} \subseteq \mathcal{F} \times \mathcal{F}$ be a symmetric relation equivalent to the natural opposition relation. Then the following properties hold.*

(A) *$\mathcal{O}$ is a 1-twinning. More precisely, if the involution α corresponding to $\mathcal{O}$ is type preserving, then $\mathcal{O}$ is an even 1-twinning if n is even, and it is an odd 1-twinning if n is odd. If α does not preserve types, then it is the other way around.*

(B) *For every flag $f \in \mathcal{F}$ and every simple path $(f_0, f_1, \ldots, f_k)$ of Γ with $f_0, f_k \in f^{\mathcal{O}}$ and $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, k-1\}$, we automatically have that k is odd whenever $k < 2n$.*

(C) *For every flag f and every pair of flags $f', f'' \in f^{\mathcal{O}}$, there exists a simple path $(f' = f_0, f_1, \ldots, f_{2n} = f'')$ with $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$.*

(D) *For every flag f and every apartment Σ of Γ we have $\Sigma \cap f^{\mathcal{O}} \neq \emptyset$.*

Proof. (A) is obvious (as it is the original motivation to introduce 1-twinning, it is remarked by Mühlherr [5]). It follows directly from the fact that the gonality and the diameter of Γ is equal to n.

(B) will be proved more generally in Section 5, see Proposition 5.2. The interested reader, though, can prove (B) directly as an exercise.

(C) follows from considering minimal paths from f' and f'' to $\alpha(f)$ (which lies at distance n from both f' and f''); there are two kinds of such paths depending on the type of the intersection of $\alpha(f)$ with the unique adjacent element of the path. Combining different types of paths for the different flags f' and f'' yields (C).

(D) finally is a direct consequence of the fact that, if $(f_0, f_1, \ldots, f_n)$ is a simple path, then f_0 is opposite f_n (which follows immediately from the fact that the gonality of Γ is equal to n). Now one extends a minimal simple path connecting $\alpha(f)$ and any element of Σ suitably to a path of length n to obtain (D). $\square$

Property (C) will play an important role in Section 4. There it will also be shown, in the more general context of arbitrary rank 2 geometries, that (A) and (C) together already imply a stronger version of condition (D).

2.4 Some elementary properties of 1-twinnings

In this subsection we assume that $\mathcal{O}$ is a 1-twinning of a rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$. We will prove some elementary properties of $\mathcal{O}$, which will be used throughout the paper. The crucial observation is that Lemma 5.3 of [5] holds in this general rank 2 context. This is Lemma 2.7 below. We deduce some more properties after that.

Lemma 2.6 *Given $f, g \in \mathcal{F}$, we have*

(*i*) $f^{\mathcal{O}} \neq \emptyset$,

(*ii*) $\ell^*(f, g) = \ell^*(g, f)$.

Proof. Given a simple path $(h_0, h_1, \ldots, h_k)$ and a flag $h'_0 \in h_0^{\mathcal{O}}$, there exists a simple path $(h'_0, h'_1, \ldots, h'_k)$ of $\mathcal{O}$-opposite type such that $h_i \mathcal{O} h'_i$, for all $i \in \{0, 1, \ldots, k\}$. This immediately follows inductively by applying the definition of a 1-twinning. Now this elementary observation is applied twice.

(*i*) We choose an arbitrary pair $(h_0, h'_0) \in \mathcal{O}$ (which exists by the assumption that $\mathcal{O}$ is non-empty) and an arbitrary simple path $(h_0, h_1, \ldots, h_k)$ with $h_k = f$ (which must exist because Γ is connected). Then the above observation shows that $f^{\mathcal{O}}$ is non-empty.

(*ii*) Here, we set $k = \ell^*(f, g)$ and we choose a simple path $(h_0, h_1, \ldots, h_k)$ such that $h_0 \mathcal{O} f$ and $h_k = g$. Applied to $h'_0 = f$, the above observation yields $\ell^*(g, f) \leq k = \ell^*(f, g)$ (since $h'_k \mathcal{O} h_k$). Similarly, $\ell^*(f, g) \leq \ell^*(g, f)$. $\square$

Lemma 2.7 *Let there be given two flags $f, g \in \mathcal{F}$ with $k = \ell^*(f, g)$ and two simple paths $(g_0, g_1, \ldots, g_k = g)$ and $(f = f_0, f_1, \ldots, f_k)$ of $\mathcal{O}$-opposite type. Then $g_0 \mathcal{O} f_0$ if and only if $g_k \mathcal{O} f_k$.*

Proof. We apply induction on k, the case $k = 0$ being trivial. So assume $k > 0$ and $g_0 \mathcal{O} f_0 = f$. By construction, $\ell^*(f, g_{k-1}) = k - 1$. Hence $f_{k-1} \mathcal{O} g_{k-1}$ by the induction hypothesis. But $(f_{k-1}, g_k) \notin \mathcal{O}$ since otherwise $\ell^*(g, f) \leq k - 1$ contradicting $\ell^*(g, f) = \ell^*(f, g) = k$ (using Lemma 2.6(*ii*)). So the 1-twinning condition yields $g_k \mathcal{O} f_k$ (recall that $g_{k-1} \cap g_k$ and $f_{k-1} \cap f_k$ are of $\mathcal{O}$-opposite type). Similarly $g_k \mathcal{O} f_k$ implies $g_0 \mathcal{O} f_0$ and the lemma is proved. $\square$

We will mainly use the above lemma in the following form.

Lemma 2.8 *Let f be any flag of Γ and let $(f_0, f_1, \ldots, f_k, \ldots, f_{2k})$ be a simple path with $f_i \in f^{\mathcal{O}}$ if and only if $i = 0$ or $i = 2k$, for all $i \in \{0, 1, \ldots, 2k\}$. Then $\ell^*(f_k, f) = k$, every flag g with $\delta(f_k, g) = k$ belongs to $f^{\mathcal{O}}$, and no flag g' with $\delta(f_k, g') < k$ belongs to $f^{\mathcal{O}}$.*

Proof. Clearly $\ell^*(f_k, f) \leq k$. But if $\ell := \ell^*(f_k, f) < k$, then Lemma 2.7 implies that either $f_{k+\ell}$ or $f_{k-\ell}$ would belong to $f^{\mathcal{O}}$ (depending on the type of the minimal simple path connecting f with $f_k^{\mathcal{O}}$). The rest of the lemma follows directly from Lemma 2.7. □

Finally, we prove a rather technical lemma, which we will need two times below, and so it is convenient to prove it here separately.

Lemma 2.9 *Let f and g be two adjacent flags of Γ and let $(f_0, f_1, \ldots, f_{2n})$ be a simple path with $f_0 = f_{2n}$, with $f_0 \in f^{\mathcal{O}} \setminus g^{\mathcal{O}}$ and with $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$. Then for exactly one $i \in \{1, 2, \ldots, 2n\}$ we have $f_i \in g^{\mathcal{O}}$, and this i necessarily belongs to the set $\{1, 2n-1\}$.*

Proof. Considering types and parity, we immediately see that $f_0 \cap f_1 \neq f_{2n-1} \cap f_0$. So, by symmetry, we may assume that the elements $f \cap g$ and $f_0 \cap f_1$ have $\mathcal{O}$-opposite types. Since $f_0 \in f^{\mathcal{O}} \setminus g^{\mathcal{O}}$, the definition of 1-twinning implies $g\mathcal{O}f_1$ (the symmetric argument gives $g\mathcal{O}f_{2n-1}$). Now suppose that $g\mathcal{O}f_i$, $i \in \{2, 3, \ldots, 2n-1\}$. If i is odd, then $f_{i-1} \cap f_i$ and $f \cap g$ have $\mathcal{O}$-opposite types, and the 1-twinning implies that one of f_{i-1}, f_i belongs to $f^{\mathcal{O}}$, a contradiction. If i is even, a similar argument shows that either f_i or f_{i+1} belongs to $f^{\mathcal{O}}$, likewise a contradiction. The lemma is proved. □

3 Characterizations of opposition in generalized polygons

Proposition 3.1 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a thick generalized n-gon. If $\mathcal{O}$ is a 1-twinning of Γ, and if there exists some flag $f \in \mathcal{F}$ and some simple path $(f_1, f_2, \ldots, f_{2n-1})$ in Γ such that $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$, then $\mathcal{O}$ is equivalent to the natural opposition relation in Γ.*

Proof. Let f and $(f_1, f_2, \ldots, f_{2n-1})$ be as in the statement of the proposition. We first show that $f^{\mathcal{O}} = f_n^{\mathrm{opp}}$. Note that by Lemma 2.6, the set $f^{\mathcal{O}}$ is not empty.

Put $k := \ell^*(f, f_n)$. If we had $k < n$, then Lemma 2.8 would imply that either f_{n+k} or f_{n-k} belongs to $f^{\mathcal{O}}$, hence $k = n$ and $f^{\mathcal{O}} \subseteq f_n^{\mathrm{opp}}$. But now Lemma 2.8 implies that $f_n^{\mathrm{opp}} \subseteq f^{\mathcal{O}}$. Hence $f^{\mathcal{O}} = f_n^{\mathrm{opp}}$.

Now we show that for any flag $f' \in \mathcal{F}$ there exists a simple path $(f'_1, f'_2, \ldots, f'_{2n-1})$ with $f'_i \notin f'^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$. By connectivity it suffices to show this for f' adjacent to f.

So let f' be adjacent to f. Using the fact that $\mathcal{O}$ is a 1-twinning, we easily see that there exists $g \in f^{\mathcal{O}} \setminus f'^{\mathcal{O}}$. Consider such g and let Σ be the unique apartment of Γ containing the opposite flags f_n and g. Let g' be the unique flag belonging to Σ such that $f \cap f'$ and $g \cap g'$ have $\mathcal{O}$-opposite types. Then Lemma 2.9 implies that path $(f'_1, f'_2, \ldots, f'_{2n-1})$ we are looking for can chosen to be the unique path of length $2n-2$ contained in $\Sigma \setminus \{g'\}$.

As a result we found for every flag $g \in \mathcal{F}$ a flag g' such that $g^{\mathcal{O}} = g'^{\mathrm{opp}}$. The proposition now follows from Lemma 2.3. □

The previous proposition characterizes the natural opposition relation in polygons as a 1-twinning together with an additional assumption which follows from property (C) in Lemma 2.5. A similar result can be shown by replacing the additional condition by (B) of Lemma 2.5. Since the latter also holds for twin trees, we state and prove this in Section 5, see Proposition 5.2. We leave it to the interested reader to formulate it for one generalized polygon. Note that Proposition 5.2 also holds in the non-thick case. In fact, using results from twin building theory, in particular the paper [3], the thickness assumption can be dispensed with in Proposition 3.1 (see Proposition 5.2 in [4] and the discussion at the end of Section 5.1). However, we preferred to present "twin free" proofs in this and the next section, the more so, since the main results of Section 4 do not require any thickness assumptions anyhow.

The next result characterizes the natural opposition relation in generalized polygons without the assumption of a 1-twinning. The conditions we assume here arise naturally in other characterizations, see Propositions 4.2 and 5.2 below.

Proposition 3.2 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a thick generalized n-gon. Let $\mathcal{O} \subseteq \mathcal{F} \times \mathcal{F}$ be a symmetric relation which satisfies conditions rm(B) and* (D) *of Lemma 2.5 as well as the following weakening of condition* (C).

(C*) *For every flag f, there exists a simple path $(f_0, f_1, \ldots, f_{2n})$ with $f_0, f_{2n} \in f^{\mathcal{O}}$, and $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$.*

Then $\mathcal{O}$ is equivalent to the natural opposition relation in Γ.

Proof. Let f be an arbitrary flag of Γ. Due to (C*), there exists a simple path $(f_0, f_1, \ldots, f_{2n})$ with $f_0, f_{2n} \in f^{\mathcal{O}}$ and $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$.

We first show that $f^{\mathcal{O}} \subseteq f_n^{\mathrm{opp}}$. So let $g \in f^{\mathcal{O}}$ and assume by way of contradiction that $g \notin f_n^{\mathrm{opp}}$. Let $(f_n, g_1, g_2, \ldots, g_k, g)$ be the unique simple path in Γ connecting f_n and g, with $k < n-1$. We choose g such that k is minimal, i.e., we assume that $g_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, k\}$. Put $g_0 = f_n$ and let $\ell \in \{0, 1, \ldots, k\}$ be maximal with respect to the property $g_\ell \in \{f_0, f_1, \ldots, f_{2n}\}$ (note that certainly g does not belong to the latter set) and let $m \in \{0, 1, \ldots, 2n\}$ be such that $g_\ell = f_m$. By symmetry we may assume that $m \leq n$. Then $(f_0, f_1, \ldots, f_{m-1}, g_{\ell+1}, g_{\ell+2}, \ldots, g_k, g)$ is a simple path with the property that $f_0, g \in f^{\mathcal{O}}$ and all other flags of that path — and there are exactly $m+k-\ell-1$ such — do not belong to $f^{\mathcal{O}}$. Similarly, $(f_{2n}, f_{2n-1}, f_m, g_{\ell+1}, g_{\ell+2}, \ldots, g_k, g)$ is a simple path with the property that $f_{2n}, g \in f^{\mathcal{O}}$ and all other flags of that path — and this time there are exactly $2n+k-m-\ell$ such — belong to $f^{\mathcal{O}}$. Note also that $2n+k-m-\ell < 2n-1$ since $\ell = n-m$. But $(2n+k-m-\ell)-(m+k-\ell-1) = 2n-2m+1$ is odd, which contradicts (B). Hence $f^{\mathcal{O}} \subseteq f_n^{\mathrm{opp}}$.

If there existed a flag h in $f_n^{\mathrm{opp}} \setminus f^{\mathcal{O}}$, then the unique apartment of Γ containing both h and f_n would not contain any element of $f^{\mathcal{O}}$ (by the previous paragraph), and so this would contradict (D).

Hence $f^{\mathcal{O}} = f_n^{\mathrm{opp}}$ and the result follows from Lemma 2.3. □

4 Characterizations of generalized polygons

Aiming at a characterization of generalized polygons using flags at maximal distance in a geometry, we first characterize generalized polygons as the only geometries admitting a certain — for the time being abstract — opposition relation. Later on, we shall define a concrete relation in geometries and prove that, if it is a 1-twinning, then the conditions of the Proposition 4.2 are satisfied. We first need a lemma. Note that the condition on $\mathcal{O}$ we assume in addition to being a 1-twinning is precisely the property (C) of Lemma 2.5, which was stated there for generalized n-gons but also perfectly makes sense for arbitrary rank 2 geometries.

Lemma 4.1 *Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ be a (rank 2) geometry and let n be a positive integer. Suppose that $\mathcal{O}$ is a 1-twinning satisfying the condition* (C) *of Lemma 2.5.*

Then $\mathcal{O}$ also satisfies the following modification of property (D).

(D*) *For every flag f and for every simple path $(f_1, f_2, \ldots, f_{2n})$ in Γ, there exists $i \in \{1, 2, \ldots, 2n\}$ such that $f_i \in f^{\mathcal{O}}$.*

Proof. Let, by way of contradiction, the simple path $(g_1, g_2, \ldots, g_{2n})$ be a counterexample to (D*). By Lemma 2.6, the set $f^{\mathcal{O}}$ is nonempty. Suppose first that some flag $g \in f^{\mathcal{O}}$ exists such that $g_1 \cap g_2$ is not nearer to g than $g_1 \setminus g_2$ (in the incidence graph of Γ). Then we can extend the above simple path to a simple path $(g_k, g_{k+1}, \dots, g_{2n})$, with $k \leq 0$, such that $g_i \notin f^{\mathcal{O}}$, for all $i \in \{k+1, k+2, \dots, 2n\}$ and $g_k \in f^{\mathcal{O}}$. By shifting indices we may assume $k = 0$.

Now suppose that for all flags $g \in f^{\mathcal{O}}$, the element $g_1 \cap g_2$ is nearer to g than $g_1 \setminus g_2$. Put $\ell = \ell^*(f, g_1)$. If $\ell < 2n$, then Lemma 2.7 implies that $g_{1+\ell} \in f^{\mathcal{O}}$, a contradiction. Hence $\ell \geq 2n$ and, changing notation, we may again assume that there is a simple path $(g_0, g_1, \dots, g_{2n})$ with $g_i \in f^{\mathcal{O}}$ if and only if $i = 0$, for all $i \in \{0, 1, \dots, 2n\}$.

Now let $(g_0 = f_0, \dots, f_n, \dots, f_{2n} = g_0)$ be a simple path as in (C). An immediate check of types and parity yields that $f_0 \cap f_1 \neq f_{2n-1} \cap f_0$. By the definition of 1-twinning, one of f_1 or f_{2n-1} must be equal to g_1. Without loss of generality, we may assume $f_1 = g_1$. Note that from Lemma 2.8 it immediately follows that $\ell^*(f, f_n) = n$.

Now let $\ell \leq 2n - 1$ be maximal with respect to the property "$f_i = g_i$ for all $i \in \{0, 1, \dots, \ell\}$". If $\ell < n$, then applying Lemma 2.8 to the simple path $(f_n, \dots, .f_{\ell+1}, g_{\ell+1}, \dots, g_{2\ell+1})$ yields $g_{2\ell+1}\mathcal{O}f$, a contradiction. If $\ell \geq n$, then Lemma 2.8 implies that $g_{2n}\mathcal{O}f$, again a contradiction.

We conclude that (D*) holds. $\square$

Note that condition (C) of the previous lemma excludes the case $n = 1$, since by taking $f' = f''$, the path (f', f_1, f') can never be simple.

The thickness condition in the last sentence of the following proposition is not essential for the same reason which was mentioned below Proposition 3.1.

Proposition 4.2 $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ *be a (rank 2) geometry and let n be a positive integer. Suppose that $\mathcal{O}$ is a 1-twinning satisfying additionally* (C) *of Lemma 2.5. Then Γ is a generalized n-gon and, if Γ is thick, $\mathcal{O}$ is equivalent to the natural opposition relation in Γ.*

Proof. By the previous lemma, $\mathcal{O}$ also satisfies (D*) of that lemma. We will freely refer to that condition as (D*).

Let $f \in \mathcal{F}$ be an arbitrary flag of Γ. By Lemma 2.6, we may select a flag f_0 in $f^{\mathcal{O}}$. By (C), there is a simple path $\gamma = (f_0, f_1, \dots, f_n, \dots, f_{2n} = f_0)$ with $f_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \dots, 2n-1\}$. Again, an immediate check of parity and types implies that f_1 and f_{2n-1} cannot be adjacent or equal, or, in other words, $f_{2n-1} \cap f_0 \neq f_1 \cap f_0$.

We claim that, if $x, y \in f^{\mathcal{O}}$, and if $(x = g_0, g_1, \dots, g_n, \dots, g_{2n} = y)$ is a simple path with $g_i \notin f^{\mathcal{O}}$,for all $i \in \{1, 2, \dots, 2n-1\}$, then $g_n = f_n$.

Indeed, let $(x = h_0, h_1, \dots, h_n, \dots, h_{2n} = f_0)$ be a simple path in Γ with $h_i \notin f^{\mathcal{O}}$ for $i \in \{1, 2, \dots, 2n-1\}$ (this path exists by (D*). Since $f_{2n-1} \cap f_0 \neq f_1 \cap f_0$, we may without loss of generality assume that $h_{2n-1} \cap f_0 = f_{2n-1} \cap f_0$ (and then $h_{2n-1} = f_{2n-1}$ because

$\mathcal{O}$ is a 1-twinning). Let $j \geq 1$ be minimal with respect to the property $h_i = f_i$, for all $i \in \{j, j+1, \ldots, 2n\}$. If $j > n$, then the simple path $(h_1, h_2, \ldots, h_{j-1}, f_{j-1}, f_{j-2}, \ldots, f_1)$ contradicts (i). Hence $j \leq n$ and so $h_n = f_n$. Now one similarly shows that h_n is the middle element of any simple closed path of length $2n$ with extremities x no members of which lie in $f^{\mathcal{O}}$, except for x. And that middle element is in its turn equal to g_n by the same token. The claim follows.

So we have a well-defined map

$$\alpha : \mathcal{F} \to \mathcal{F} : f \mapsto f_n.$$

We show that α is surjective. It suffices to prove that every flag adjacent to f_n is in the image of α, since Γ is connected by assumption.

So let f^* be adjacent to f_n. Our first aim is to show that there is a simple closed path γ of length $2n$ containing f^* and f_n and such that exactly one element of that path belongs to $f^{\mathcal{O}}$.

We start by choosing a simple path $(g_0, g_1, \ldots, g_n)$ with $g_n = f_n$ and $g_{n-1} = f^*$ (by the firmness assumption, this is possible). By Lemma 2.8, $g_0 \in f^{\mathcal{O}}$ and $g_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, n-1\}$.

Next, we choose a simple path $(g_0 = g'_0, g'_1, \ldots, g'_{2n-1}, g'_{2n} = g_0)$, with $g'_i \notin f^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2n-1\}$, guaranteed by (C). We have shown above that $g'_n = f_n$. Hence we may assume that g_{n-1} is adjacent or equal to g'_{n-1} and consequently g'_{n+1} is not adjacent with nor equal to g_{n-1}. Hence $\gamma = (g_0, g_1, \ldots, g_n, g'_{n+1}, \ldots, g'_{2n-1}, g'_{2n} = g_0)$ is the required closed simple path.

By definition of 1-twinning, there exists now a flag g adjacent to f and such that $g'_{2n-1} \in g^{\mathcal{O}}$ and $g_0 \notin g^{\mathcal{O}}$. Lemma 2.9 implies that the only element of γ which belongs to $g^{\mathcal{O}}$ is g'_{2n-1}. Consequently $\alpha(g) = f_{n-1} = f^*$. We have shown that α is surjective.

There is no closed simple path through f_n of length $k < 2n$, since, on one hand, such a path (which can be juxtaposed arbitrarily often) must contain an element of $f^{\mathcal{O}}$ by (D*), but on the other hand cannot contain such an element since this would mean $\ell^*(f_n, f) < n$, a contradiction. By surjectivity of α, we conclude that Γ has gonality n (note that (C) guarantees the existence of a closed path of length $2n$). Also, there are no flags at distance $n+1$ from f_n, since such a flag would be adjacent to a flag $f' \in f^{\mathcal{O}}$, and hence it would either itself belong to $f^{\mathcal{O}}$ (and then it has distance $\leq n$ from f_n by the fact that α is well-defined), or it is one of the two unique (by the 1-twinning) flags adjacent to f' on a path of length n from f' to f_n (and then it has distance $\leq n-1$ from f_n). Hence the local diameter at f_n is equal to n, and since α is surjective, the diameter of Γ is equal to n. We have shown that Γ is a (not necessarily thick) generalized n-gon.

If Γ is thick, it follows from Proposition 3.1 that $\mathcal{O}$ is equivalent to the natural opposition relation. $\square$

The following results characterize generalized polygons amongst all rank 2 geometries. In fact, they can be seen as weakenings of the so-called *gate property*, see [8].

Proposition 4.3 *Let Γ be a (rank 2) geometry with finite diameter, flag set $\mathcal{F}$ and diameter of the flag graph equal to n. For $f, f' \in \mathcal{F}$ we define $f\mathcal{O}f'$ if f and f' are at distance n in the flag graph of Γ, and we call two flags f and g opposite if $f\mathcal{O}g$. If $\mathcal{O}$ is a 1-twinning, then Γ is a generalized n-gon and $\mathcal{O}$ is the natural opposition relation.*

Proof. Let $f = \{x_1, x_2\}$ and $g = \{y_1, y_2\}$ be two opposite flags, where we assume that the type of y_i is $\mathcal{O}$-opposite the type of x_i, $i = 1, 2$. By the definition of 1-twinning, there are unique flags f_1 and g_1 containing x_1 and y_1, respectively, and at distance $n-1$ from g and f, respectively. Hence there are simple paths $(f = f_0, f_1, \ldots, f_n = g)$ and $(g = g_0, g_1, ..., g_n = f)$ such that $f_0 \cap f_1 = \{x_1\}$ and $g_0 \cap g_1 = \{y_1\}$. Also, the 1-twinning property implies that $f_{n-1} \cap f_n = \{y_2\}$ and $g_{n-1} \cap g_n = \{x_2\}$. If $g' = \{y'_1, y'_2\}$ is another flag opposite f, then with similar and obvious notation, we have paths $(f = f'_0, f'_1, \ldots, f'_n = g)$ and $(g' = g'_0, g'_1, \ldots, g'_n = f)$, with $f'_0 \cap f'_1 = \{x_1\}$, $g'_0 \cap g'_1 = \{y'_1\}$, $f'_{n-1} \cap f'_n = \{y'_2\}$ and $g'_{n-1} \cap g'_n = \{x_2\}$. To see now that condition (C) of Proposition 4.2 (see Lemma 2.5) is satisfied, we can consider the simple path $(g = f_n, \ldots, f_1, f_0 = f = g'_n, g'_{n-1}, \ldots, g'_0 = g')$. It follows now from Proposition 4.2 that Γ is a generalized n-gon. By the definition of $\mathcal{O}$, it has to be the natural opposition relation in Γ. $\square$

This proposition has some nice consequences. For the first one, we say that two vertices v, v' of a graph are at maximal distance if all vertices adjacent to v (respectively v') are not further away from v' (respectively v) than v (respectively v').

Corollary 4.4 *Let Γ be a (rank 2) geometry with finite diameter and with flag set $\mathcal{F}$. For $f, f' \in \mathcal{F}$ we define $f\mathcal{O}f'$ if f and f' are at maximal distance in the flag graph of Γ (i.e., the graph with vertex set $\mathcal{F}$ and adjacency is just adjacency of flags). If $\mathcal{O}$ is a 1-twinning of Γ, then Γ is a generalized polygon and $\mathcal{O}$ is the natural opposition relation.*

Proof. Let n be the diameter of the flag graph of Γ. We define a new relation $\mathcal{O}'$ on the set of flags by writing $f\mathcal{O}'g$ if f is at distance n from g in the flag graph of Γ. Obviously, $\mathcal{O}' \subseteq \mathcal{O}$. We now show that $\mathcal{O}'$ is a 1-twinning. To that end, we first show that, if $f = \{x_1, x_2\}$ and $g = \{y_1, y_2\}$ are flags at distance n, and if x_1 and y_1 have $\mathcal{O}$-opposite types, then any pair of flags $f' = \{x_1, x'_2\}$ and $g' = \{y_1, y'_2\}$ at distance $n-1$ from each other, are not at maximal distance. Suppose by way of contradiction that they are at maximal distance. Let $(f' = f_0, f_1, \ldots, f_{n-1} = g')$ be a minimal path. If $x_1 \in f_1$ and $y_1 \in f_{n-2}$, then f and g are also at distance $n-1$ from each other, a contradiction. If $x_1 \in f_1$ and $y'_2 \in f_{n-2}$, then both f and f_1 are not at maximal distance from g', hence, since $\mathcal{O}$ is a 1-twinning, they must coincide. But then f and g are at distance $n-1$ from each other, a contradiction. So we conclude that $x_2' \in f_1$ and $y_2' \in f_{n-2}$. But now x_2' and y_2' have $\mathcal{O}$-opposite type and the fact that f_1 is not at maximal distance from f_{n-2}, nor from g' contradicts $\mathcal{O}$ being a 1-twinning. Hence f' and g' cannot be at maximal distance.

Obviously, if $f' = \{x_1, x_2'\}$ and $g' = \{y_1, y_2'\}$ are at distance $n-2$ from each other, then they are not at maximal distance. Hence f' and g' are at maximal distance if and only if they are at distance n from each other. Hence $\mathcal{O}$ and $\mathcal{O}'$ coincide on the sets of flags containing x_1 and y_1, respectively. Thus $\mathcal{O}'$ is a 1-twinning, and the result follows from Proposition 4.3. □

The next result characterizes generalized polygons by means of the existence of projections from elements x onto elements y at "almost maximal" distance. This is a considerable weakening of the classical definition, where this is required for all elements x, y of non-maximal distance, and where also the paths of minimal length between x and y are required to be unique.

Theorem 4.5 *Let Γ be a (rank 2) geometry with finite diameter n. If for every pair of elements x, y of Γ, with x and y at distance $n-1$ from each other, there is a unique element x' incident with x and nearest to y (in the incidence graph), then Γ is a generalized n-gon.*

Proof. Given flags $f = \{x, x'\}$ and $g = \{y, y'\}$, where without loss of generality we assume $d(x, y') < d(x, y)$ (d denotes the distance function in the incidence graph of Γ), we observe that f and g are at distance n in the flag graph of Γ if and only if $d(x, y) = d(x', y') = n$. Hence our assumption implies in particular that there exist flags which are at distance n from each other (note that we also need our general assumption of firmness here). Therefore the diameter of the flag graph (which is always smaller than or equal to the diameter of the incidence graph) is equal to n. It is now clear that the assumption of the corollary implies that the relation $\mathcal{O}$ of Proposition 4.3 is an odd or even 1-twinning according to whether n is odd or even. Hence this proposition implies the corollary. □

The last corollary is about near $2n$-gons, introduced by Shult and Yanushka [9]. A *near $2n$-gon* is a (connected) rank 2 pregeometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ of diameter $2n$ such that (1) any two lines meet in at most one point, (2) every line is incident with at least 2 points, and (3) for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$ there is a unique point $p' \in \mathcal{P}$ on L nearest to p (in the incidence graph). Let us mention that it has been an open question since the introduction of near polygons whether a near polygon Γ which is also a dual near polygon is necessarily a generalized polygon. Noting that such Γ must necessarily be firm, the next corollary follows immediately from Theorem 4.5.

Corollary 4.6 *If Γ is a near $2n$-gon for some positive integer n, such that the dual of Γ is also a near $2n$-gon, then Γ is a generalized $2n$-gon.*

5 Twinnings of rank 2 buildings

5.1 A characterization theorem

In [3], we analyzed which 1-twinnings (in the original sense as introduced by Bernhard Mühlherr [5]) give rise to twinnings of buildings. Thereby we also obtained a new characterization of twin buildings. In this section we first use this result to prove a new — typically rank 2 — characterization (in particular yielding a new characterization of twin trees). Then we want to discuss some 1-twinnings of rank 2 buildings which are not twin buildings. These counterexamples motivated our analysis of 1-twinnings started in [3] and continued in the present paper.

In the following, we assume that we are given two generalized n-gons $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, where we allow $n = \infty$ (see Subsection 2.1). The notion of a 1-twinning of a pair of generalized n-gons specializes Mühlherr's concept of a k-twinning of a pair of buildings (see [5] and also [3]).

Definition 5.1 A non-empty symmetric relation $\mathcal{O} \subseteq (\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+)$ is called a 1-*twinning* of (Γ_+, Γ_-) if the following condition holds.

(1Tw) For any two flags $(f_+, f_-) \in \mathcal{O}$, for any two elements $x_+ \in f_+$ and $x_- \in f_-$ of the same type, for any $\epsilon \in \{+,-\}$ and and any flag $f'_\epsilon \ni x_\epsilon$, there exists a unique flag $f'_{-\epsilon} \ni x_{-\epsilon}$ such that $(f'_\epsilon, f'_{-\epsilon}) \notin \mathcal{O}$.

Our definition of a 1-twinning in a single geometry as given in Section 2 is of course motivated by the above definition. Note that in the present context we can always require that "being of $\mathcal{O}$-opposite type" is equal to "being of the same type" since we can always replace Γ_- with its dual if necessary.

We shall again use the notations $f_+\mathcal{O}f_-, f_\epsilon^{\mathcal{O}}, \ \ell^*(f_\epsilon, g_{-\epsilon})$ ($\epsilon \in \{+,-\}$, and we always assume that objects with a subscript ϵ belong to Γ_ϵ). We shall also freely use the obvious analogies of Lemma 2.6 and Lemma 2.7 in our present situation (cf. also Section 5 of [5], and Section 2 of [3]).

In the following we always assume that $\mathcal{O}$ is a 1-twinning of (Γ_+, Γ_-). The main result of [3], namely Theorem 3.5 of that paper, is a new characterization of twin buildings. In the present situation, it yields a necessary and sufficient condition for the triple $(\Gamma_+, \Gamma_-, \mathcal{O})$ to "be" a (rank 2) twin building, meaning that $\mathcal{O}$ induces a (then uniquely determined) codistance on (Γ_+, Γ_-) in the sense of the original definition given in Tits [13]. This condition reads as follows.

(TA) There exists an $\epsilon \in \{+,-\}$ and a flag $f_\epsilon \in \mathcal{F}_\epsilon$ such that for any flag $f_{-\epsilon} \in f_\epsilon^{\mathcal{O}}$, there exists an apartment $\Sigma_{-\epsilon}$ of $\Gamma_{-\epsilon}$ satisfying $\{f_{-\epsilon}\} = f_\epsilon^{\mathcal{O}} \cap \Sigma_{-\epsilon}$.

Due to special features of the rank 2 case, we are able to prove a new characterization of rank 2 twin buildings now. It is inspired by (B) of Lemma 2.5.

Proposition 5.2 *The* 1*-twinning* $\mathcal{O}$ *yields a twin building* $(\Gamma_+, \Gamma_-, \mathcal{O})$ *if and only if the following condition is satisfied for some* $\epsilon \in \{+, -\}$ *and some* $f_\epsilon \in \mathcal{F}_\epsilon$.

(B*) *For any simple path* $(f_0, f_1, \ldots, f_k)$ *in* $\Gamma_{-\epsilon}$ *satisfying* $f_0, f_k \in f_\epsilon^{\mathcal{O}}$, $f_i \notin f_\epsilon^{\mathcal{O}}$, *for all* $i \in \{1, 2, \ldots, k-1\}$, *and* $k < 2n$, *the number* k *is necessarily odd.*

Proof. First assume that (B*) holds and suppose $\epsilon = +$ (the case $\epsilon = -$ being similar). Let $f_- \in f_+^{\mathcal{O}}$ be given. We want to construct an apartment Σ_- of Γ_- such that $\Sigma_- \cap f_+^{\mathcal{O}} = \{f_-\}$. To this end, we inductively construct simple paths $(f_- = g_0, g_1, \ldots, g_m)$ such that $\ell^*(f_+, g_i) = i$, for all $i \in \{0, 1, \ldots, m\}$. Whenever $m < n$, we shall show that there is a longer simple path $(g_0, g_1, \ldots, g_{m+1})$ with $\ell^*(f_+, g_{m+1}) = m + 1$.

Let $x_- \in g_m$ be different from $g_{m-1} \cap g_m$. For all flags $g_- \ni x_-$, we obviously have $\ell^*(f_+, g_-) \in \{m-1, m, m+1\}$. Now $\ell^*(f_+, g_-) = m - 1$ would imply the existence of a simple path $(g_- = g_{m+1}, g_{m+2}, \ldots, g_{2m})$ with $g_{2m} \in f_+^{\mathcal{O}}$. Note that $g_- \cap g_{m+2} \neq \{x_-\}$ since $\ell^*(f_+, g_{m+2}) = m - 2$. Hence $(g_0, g_1, \ldots, g_m, g_{m+1}, \ldots, g_{2m})$ would be a simple path contradicting (B*). So $\ell^*(f_+, g_-) \in \{m, m+1\}$, for all flags $g_- \ni x_-$. Choose a simple path $(f_+ = f'_0, f'_1, \ldots, f'_m)$ of the same type as $(g_0, g_1, \ldots, g_m)$. The Lemma 2.7 (or rather its appropriate analogue) implies $f'_m \mathcal{O} g_-$, for all $g_- \ni x_-$ with $\ell^*(f_+, g_-) = m$. But since $\mathcal{O}$ is a 1-twinning there must exist a $g_{m+1} \ni x_-$ satisfying $(f'_m, g_{m+1}) \notin \mathcal{O}$ and hence $\ell^*(f_+, g_{m+1}) = m + 1$ by the previous argument. Now we distinguish two cases.

If $n = \infty$ (and hence Γ_- is a tree), we construct by the above procedure a "doubly infinite" path $(\ldots, g_{-1}, g_0 = f_-, g_1, g_2, \ldots)$ such that $g_{-1} \cap g_0 \neq g_0 \cap g_1$ and $\ell^*(f_+, g_j) = |j|$, for all integers j. This obviously yields the desired apartment Σ_- of Γ_-.

If $n \neq \infty$, we first construct a simple path $(f_- = g_0, g_1, \ldots, g_n)$ with $\ell^*(g_+, g_i) = i$, for all $i \in \{0, 1, \ldots, n\}$. Then there exists a unique apartment $\Sigma_- = \{g_0, g_1, \ldots, g_n, g_{n+1}, \ldots, g_{2n-1}\}$ of Γ_- containing this path. Since $\ell^*(f_+, g_n) = n$, we have that $g_i \mathcal{O} f_+$ is only possible for the flag $g_0 = f_-$ of Σ_-.

Next we assume that $(\Gamma_+, \Gamma_-, \mathcal{O})$ is a twin building. We want to prove that for all $f_+ \in \mathcal{F}_+$ condition (B*) (with $\epsilon = +$) holds. Assume by way of contradiction that there is a simple path $(f_0, f_1, \ldots, f_{2\ell})$, $\ell < n$, in Γ_- such that $f_0, f_{2\ell} \in f_+^{\mathcal{O}}$ and $f_i \notin f_+^{\mathcal{O}}$, for all $i \in \{1, 2, \ldots, 2\ell - 1\}$. Since twin buildings have "sufficiently many twin apartments", there exists an apartment Σ_- of Γ_- such that $f_\ell \in \Sigma_-$ and $\Sigma_- \cap f_+^{\mathcal{O}}$ consists of a single flag (cf. for instance Lemma 2(*ii*) of [1]). On the other hand, Lemma 2.7 (in the form of Lemma 2.8), together with the above assumptions, firstly implies that $\ell^*(f_\ell, f_+) = \ell$. But then the same lemma secondly implies that the apartment Σ_- contains two different flags $h, h' \in f_+\mathcal{O}$, both at distance ℓ from f_ℓ, contradicting the choice of Σ_-. $\square$

5.2 Some counterexamples

In this section, we will construct 1-twinnings of rank 2 buildings which are not twinnings. We use the notation of the previous section. In particular, a generalized ∞-gon is just a tree without finite end points.

We make the following observation. Let $(\Gamma_+, \Gamma_-, \mathcal{O})$ be a 1-twinning of two generalized n-gons, n possibly infinite. Then for every point p_+ of Γ_+, the set of points of Γ_- not opposite p_+ is a *geometric hyperplane* $\mathcal{H}_{p_+}$ of Γ_-, i.e., a set of points such that any line L either meets $\mathcal{H}_{p_+}$ in exactly 1 point, or every point of L is contained in $\mathcal{H}_{p_+}$. Also, if $p_- \in \mathcal{H}_{p_+}$, then $p_+ \in \mathcal{H}_{p_-}$ (with obvious notation). Similarly, for every line L_ϵ of Γ_ϵ. $\epsilon \in \{+,-\}$, there is a unique dual geometric hyperplane $\mathcal{H}_{L_\epsilon}$ in $\Gamma_{-\epsilon}$. Hence, if we want 1-twinnings which are not twinnings, then we have to look for suitable sets of geometric hyperplanes, i.e., we must find a set of hyperplanes and dual hyperplanes which intersect in the right way. The easiest way to make the intersections right is simply to choose all (dual) geometric hyperplanes disjoint (and it follows that no geometric hyperplane will contain the set of points of a line; hence, for $n \neq \infty$, all geometric hyperplanes are distance-2-ovoids in the sense of Chapter 7 of [14]).

Proposition 5.3 *Let Γ_ϵ be a generalized n-gon, with $n \geq 2$. Suppose that there is a non-trivial partition Π_ϵ of the points set of Γ_ϵ into disjoint geometric hyperplanes, and a partition Π'_ϵ of the line set of Γ_ϵ into dual geometric hyperplanes. Suppose also that there are pairings $b : \Pi_+ \longleftrightarrow \Pi_-$ and $b' : \Pi'_+ \longleftrightarrow \Pi'_-$. For an element x_ϵ of Γ_ϵ, we denote by H_{x_ϵ} the unique (dual) geometric hyperplane of Π_ϵ (Π'_ϵ) containing x_ϵ. For chambers $\{p_\epsilon, L_\epsilon\}$ of Γ_ϵ, we define*

$$\{p_+, L_+\}\mathcal{O}\{p_-, L_-\} \Longleftrightarrow H^b_{p_+} \neq H_{p_-} \text{ and } H^{b'}_{L_+} \neq H_{L_-}.$$

Then $\mathcal{O}$ satisfies Condition (1Tw) *of* Definition 5.1 *and hence defines a* 1*-twinning of* (Γ_+, Γ_-) *such that* $(\Gamma_+, \Gamma_-, \mathcal{O})$ *is not a twin building.*

Proof. Obviously, to prove (1Tw) of Definition 5.1, all we have to show is that, up to point-line duality and plus-minus duality, if for the lines L_+ of Γ_+ and L_- of Γ_- the sets $H^{b'}_{L_+}$ and H_{L_-} are not the same, then for every point p_+ of L_+, there is a unique point p_- of L_- belonging to $H^b_{p_+}$. The existence of p_- follows from the fact that $H^b_{p_+}$ is a geometric hyperplane. If p_- were not unique, then every point of L_- would belong to $H^b_{p_+}$ and hence L_- would not meet any other member of Π_-, a contradiction.

Now the function ℓ^* associated to the 1-twinning $(\Gamma_+, \Gamma_-, \mathcal{O})$ satisfies $\ell^*(f_+, g_-) < 3$ for all flags f_+, g_-, and hence one sees that $(\Gamma_+, \Gamma_-, \mathcal{O})$ can never have twin apartments. $\square$

We now discuss for different values of n how the previous proposition gives rise to explicit (counter-)examples. We restrict ourselves to the thick case, use the notation of the previous proposition, and denote by (s_ϵ, t_ϵ) be the *order* of Γ_ϵ (i.e., each line is incident with $s_\epsilon + 1$ points and each point with $t_\epsilon + 1$ lines; both numbers can be infinite).

For generalized digons, the only geometric hyperplanes are the points themselves; hence there is a unique way to form partitions of geometric hyperplanes and dual geometric hyperplanes of Γ_+ and Γ_- and we thus obtain 1-twinnings of generalized digons (with necessarily the same number of points and the same number of lines because of the pairings b and b'). But it is easy to see that these 1-twinnings are always equivalent to the natural opposition relation, so Proposition 5.3 does not provide counter-examples in this case.

For projective planes, the only geometric hyperplanes are the sets of points incident with a line, and the full point set. Hence there is no way to choose disjoint ones. Consequently Proposition 5.3 is neither in this case of any use.

Now suppose $n \geq 4$ and note that, if Π_ϵ and Π'_ϵ both exist for $\epsilon \in \{+, -\}$, then necessary and sufficient for the existence of the pairings b and b' is obviously that $s_+ = s_-$ and $t_+ = t_-$. In the finite case, there are no generalized polygons known admitting partitions of geometric hyperplanes and dual geometric hyperplanes other than the quadrangles $T^*(O)$ of Tits (see [6]) and hence the latter admit 1-twinnings which are not twinnings. In the infinite case, every generalized n-gon with $n > 3$ admits such partitions by transfinite induction. For the sake of simplicity, let us explain this in the case $s_+ = s_- = t_+ = t_- = |\mathbb{N}|$. Let us construct a partition Π_+ of the point set of Γ_+ into distance-2-ovoids, or briefly *ovoids*. Let $\{p_i \,|\, i \in \mathbb{N}\}$ be the set of points of Γ_+ and let $\{L_j \,|\, j \in \mathbb{N}\}$ be the set of lines of Γ_+. Our task is to construct a countable set $\{O_i \,|\, i \in \mathbb{N}\}$ of disjoint ovoids O_i covering the whole point set of Γ_+. Suppose we already constructed all ovoids O_i with $i < k$, for certain $k \in \mathbb{N}$ (this includes $k = 0$, which corresponds to the first step of the induction). We now construct O_k, also by induction. Let m be minimal with respect to the property "$p_m \notin O_0 \cup \ldots \cup O_{k-1}$" (for $k = 0$, just set $m = 0$) and let m' be minimal with respect to the property "$L_{m'}$ is not incident with p_m". Then we put $\Theta_{m'} = \{p_m\}$. Suppose now we have constructed a (finite) set Θ_ℓ of mutually non-collinear points such that every line L_i with $i < \ell$ is incident with a unique element of Θ_ℓ, but no element of Θ_ℓ is incident with L_ℓ, for some $\ell \in \mathbb{N}$. Then we can choose a point x on L_ℓ outside $O_1 \cup \cdots \cup O_{k-1}$ and not collinear with any element of Θ_ℓ. If ℓ' is the smallest positive integer with the property that $L_{\ell'}$ is not incident with any element of $\Theta_\ell \cup \{x\}$, then we put $\Theta_{\ell'} = \Theta_\ell \cup \{x\}$. Now we define O_k as the union of all the Θ_ℓ, and the construction is complete.

For trees, the construction is even simpler. Let T_+, T_- be two isomorphic semi-homogeneous trees with valencies a and b, more precisely, all vertices in T_+ and T_- of the first type are supposed to have valency b (and we call such vertices *points*) and all vertices of the second type to have valency a (and we call these vertices *lines*). Let I and J be two index sets with cardinalities a and b, respectively. For any vertex in T_+ or T_-, denote by $N(x)$ its set of neighbors. Now choose a function f_1 (which can be very easily constructed) from the set of all points in T_+ and T_- to I such that the restriction of f_1 to any set $N(L)$, where L is a line, is a bijection onto I. Analogously, construct a function f_2 from the set of all lines to J such that the restriction to any $N(p)$, where p is a point, is a bijection onto J. The $+$ and $-$ parts of the fibers of f_1 (f_2) are the (dual) geometric hyperplanes in T_+, T_- we wanted.

For projective planes and generalized digons, we can construct 1-twinnings which are not twinnings in the following way. Consider a pair of projective planes or generalized digons (Γ_+, Γ_-) of the same order q (possibly infinite). Let Π_ϵ, $\epsilon \in \{+, -\}$, be a partition of the flag set of Γ_ϵ such that every point and every line of Γ_ϵ is contained in exactly one element of each member of Π_ϵ. Then $|\Pi_+| = |\Pi_-| = q + 1$ and we can choose a bijection b from Π_+ to Π_-. If we define $f_+ \mathcal{O} f_-$, for two flags f_+, f_- in Γ_+, Γ_-, respectively, if the image under b of the class of Π_+ containing f_+ does not contain f_-, then we have a 1-twinning $(\Gamma_+, \Gamma_-, \mathcal{O})$ which is clearly not a twinning. It is again easy to construct such partitions for generalized digons and for infinite projective planes. In the finite projective plane case, we consider for Γ_+ and Γ_- the Desarguesian projective plane of order q. There, such a partition can be defined as the set of orbits in the flag set under a Singer cycle. For the plane of order 2, this was exactly the counterexample alluded to in [5], attributed there to the second author.

Note that the previous construction also works for generalized n-gons with $n > 3$.

The previous discussions prove the following result.

Proposition 5.4 *For every $n \in \mathbb{N} \cup \{\infty\}$, $n \geq 2$, there exists a 1-twinning $\mathcal{O}$ of a pair (Γ_+, Γ_-) of generalized n-gons such that $(\Gamma_+, \Gamma_-, \mathcal{O})$ is not a twin building.*

Taking for Γ_+ and Γ_- isomorphic generalized polygons in the above examples, and identifying them by an arbitrarily chosen isomorphism, we have also proved the following (see also Proposition 6.4 below)

Corollary 5.5 *For every $n \in \mathbb{N}$, $n \geq 2$, there exists a generalized polygon Γ and a 1-twinning which is not equivalent to the natural opposition relation.*

6 What about spherical buildings of higher rank?

In the general theory of buildings, a flag in a generalized polygon would be called a *chamber*. An immediate consequence of Proposition 3.1 is now the following result.

Corollary 6.1 *Let Γ be a thick generalized n-gon. If $\mathcal{O}$ is a* 1*-twinning of* Γ*, and if there exists some chamber $f \in \mathcal{F}$ and some apartment Σ in Γ such that $\Sigma \cap f^{\mathcal{O}}$ is a singleton, then $\mathcal{O}$ is equivalent to the natural opposition relation in Γ.*

This corollary can directly be generalized to spherical buildings if the occurring notions are appropriately defined in this more general context. In the following, Δ is always a spherical building (considered as a simplicial complex) and $\mathcal{C}$ its set of chambers. Chambers $c, d \in \mathcal{C}$ at maximal (gallery) distance from each other are called *opposite*, and we again use the notation c opp d here and call it the *natural opposition relation in* Δ. We also choose an index set I of cardinality the rank of Δ and a type function (also called numbering) type: $\Delta \to 2^I$ as introduced in Subsection 3.8 of [11].

Definition 6.2 Let $\mathcal{O} \subseteq \mathcal{C} \times \mathcal{C}$ be a non-empty symmetric relation. Given a permutation $\pi : I \to I$ of order at most 2, we say that $\mathcal{O}$ is a 1*-twinning of* Δ *with respect to* π if for any pair of chambers $(c, d) \in \mathcal{O}$, any two vertices $x \in c, y \in d$ such that $\mathrm{type}(x) = \pi(\mathrm{type}(y))$ and any chamber c' containing the panel $c \setminus \{x\}$, there exists precisely one chamber d' containing $d \setminus \{y\}$ which satisfies $(c', d') \notin \mathcal{O}$. The permutation of I corresponding to the natural opposition relation (which is of course a 1-twinning) will be denoted by ω.

We say that $\mathcal{O}$ is *equivalent to the natural opposition relation in* Δ if there exists a (not necessarily type-preserving) automorphism α of Δ such that $c\mathcal{O}d$ if and only c opp $\alpha(d)$ for any $c, d \in \mathcal{C}$ (then α is of order at most 2, as was explained in Subsection 2.2).

Now by modifying the proof of Proposition 3.1 appropriately in the special situation described in Corollary 6.1 above (here the flag f_n of that proof can alternatively be described as the unique flag in Σ opposite the unique flag $f' \in \Sigma$ satisfying $f'\mathcal{O}f$), the following generalization to arbitrary spherical buildings is easily obtained (a different proof is indicated at the end of this section).

Corollary 6.3 *Let $\mathcal{O}$ be a* 1*-twinning of the thick spherical building Δ with respect to the permutation π of I. If there exist a chamber c and an apartment Σ of Δ such that $c'\mathcal{O}c$ for precisely one chamber c' in Σ, then $\mathcal{O}$ is equivalent to the natural opposition relation in Δ.*

Note that this corollary implies that π necessarily induces an automorphism of the diagram associated to Δ (which was not presupposed!).

Finally, we discuss the connection between relations $\mathcal{O}$ equivalent to the natural opposition relation in Δ and twinnings of two copies of Δ. So let a non-empty symmetric relation $\mathcal{O} \subseteq \mathcal{C} \times \mathcal{C}$ be given. We take two copies Δ_+, Δ_- of Δ (and $\mathcal{C}_+, \mathcal{C}_-$ of $\mathcal{C}$); more formally we are considering isomorphisms (of simplicial complexes) $\beta_+ : \Delta_+ \to \Delta$, $\beta_- : \Delta_- \to \Delta$. Now we associate a symmetric relation $\mathcal{O}'$ on $\mathcal{C}_+ \times \mathcal{C}_- \cup \mathcal{C}_- \times \mathcal{C}_+$ to $\mathcal{O}$ by declaring

$$c_+\mathcal{O}'d_- :\Longleftrightarrow \beta_+(c_+)\mathcal{O}\beta_-(d_-) \Longleftrightarrow: d_-\mathcal{O}'c_+.$$

We want to decide when the triple $(\Delta_+, \Delta_-, \mathcal{O}')$ is a twin building (in the sense of the characterization of twin buildings derived in [3]). Due to the fact that in twin building theory types are always chosen such that "opposite types" means the same as "equal types" (and hence no permutation of types occurs in the 1-twinning definition (1Tw)), the type functions on Δ_+ and Δ_- will have to be chosen suitably in the following. We shall assume that the index set I is the same for all the three type functions on Δ, Δ_+ and Δ_- and that β_+ is type-preserving. Then the type function on Δ_- will be defined according to our respective requirements (so that β_- usually is not type-preserving). For instance, if $\mathcal{O}$ is a 1-twinning with respect to a permutation π, then $(\Delta_+, \Delta_-, \mathcal{O}')$ satisfies (1Tw) if $\mathrm{type}(\beta_-(x)) = \pi(\mathrm{type}(x))$ for any $x \in \Delta_-$. Now we can formulate the following result.

Proposition 6.4 *The relation $\mathcal{O}$ is equivalent to the natural opposition relation in Δ if and only if the triple $(\Delta_+, \Delta_-, \mathcal{O}')$ introduced above is, for a suitable choice of type functions on Δ_+ and Δ_-, a twin building.*

Proof. Assume that $\mathcal{O}$ is equivalent to the natural opposition relation and that α is the corresponding automorphism of Δ. Let π_α be the permutation of I satisfying $\mathrm{type}(\alpha(x)) = \pi_\alpha(\mathrm{type}(x))$ for any $x \in \Delta$, and denote by π the composite $\pi := \pi_\alpha\omega$. Hence $\mathcal{O}$ is in particular a 1-twinning of Δ with respect to π. So setting $\mathrm{type}(x_+) := \mathrm{type}(\beta_+^{-1}(x_+))$ for $x_+ \in \Delta_+$ and $\mathrm{type}(x_-) := \pi(\mathrm{type}(\beta_-^{-1}(x_-)))$ for $x_- \in \Delta_-$, $\mathcal{O}'$ becomes a 1-twinning of (Δ_+, Δ_-). Given chambers $c_+\mathcal{O}'d_-$, we easily find an apartment Σ_- such that a chamber e_- of Σ_- satisfies $e_-\mathcal{O}'c_+$ if and only if $e_- = d_-$ (simply take $\Sigma_- := \beta_-^{-1}\alpha(\Sigma)$, where Σ is the apartment of Δ containing $\beta_+(c_+)$ and $\alpha\beta_-(d_-)$). Hence by Theorem 3.5 of [3], $(\Delta_+, \Delta_-, \mathcal{O}')$ is a twin building.

Now suppose that $(\Delta_+, \Delta_-, \mathcal{O}')$ is a twin building. Then there exists an isomorphism $\theta : \Delta_+ \to \Delta_-$ (inducing the permutation ω on I) such that any two chambers $c_+, d_+ \in \mathcal{C}_+$ are opposite in Δ_+ if and only if $c_+\mathcal{O}'\theta(d_+)$. This is "folklore" in twin building theory (cf. Proposition 1 in [13]); technically θ can be described as the map assigning to each $x_+ \in \Delta_+$ its "coprojection" onto the empty simplex in Δ_- (cf. Section 4 of chapter I in [1]). Recalling the definition of $\mathcal{O}'$, we obtain for any two chambers c, d of Δ that $c\mathcal{O}d$ holds if and only if $\beta_+^{-1}(c)$ and $\theta^{-1}\beta_-^{-1}(d)$ are opposite in Δ_+, hence if and only if c and $\beta_+\theta^{-1}\beta_-^{-1}(d)$ are opposite in Δ. Using the automorphism $\alpha := \beta_+\theta^{-1}\beta_-^{-1}$ of Δ, we see that $\mathcal{O}$ is equivalent to the natural opposition relation in Δ. □

Hence, loosely speaking, we can say that a relation on the set of chambers of a spherical building Δ is equivalent to the natural opposition relation if by doubling the building, this relation becomes the opposition relation of a twin building. The translation provided by Proposition 6.4 allows to apply results about twin buildings in order to characterize (up to equivalence) the opposition relation in spherical buildings. In the situation of Corollary 6.3 for instance, we first create, as described above, a 1-twinning of a pair of copies of Δ, we then apply Corollary 3.6 of [3] in order to see that this 1-twinning yields a twin building and we finally deduce from Proposition 6.4 that the relation $\mathcal{O}$ we started with has to be equivalent to the natural opposition relation in Δ. This reasoning even works if Δ is not thick. For similar reasons, the thickness assumption may be dropped in the Propositions 3.1 and 4.2, as we already remarked earlier.

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