Majorization in Lattice Path Enumeration And Creating Order

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Abstract. Lattice paths are enumerated which first touch a periodic boundary at time n. Following a probabilistic method introduced by Gessel, for period length 2 formulae are obtained for a wide class of boundaries. This allows to give the generating function for paths not crossing or touching the diagonal $cx = 2y$ for odd c and to obtain a closed formula similar to the ballot numbers for the sum of the entries of two two-dimensionanal arrays related to these boundaries.
Via the theory of majorization bijections are obtained to further enumeration problems as ideals in the pushing order, ballot sequences, regular trees and further path models. Majorization also plays a role in a special model of creating order as introduced by Ahlswede, Ye, and Zhang.

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I. Introduction

We shall study combinatorial problems, e. g., the enumeration of lattice paths not crossing or touching a periodic boundary and the analysis of a model in creating order as introduced by Ahlswede, Ye, and Zhang [6], which involve the concept of majorization (see Marshall/Olkin [72]) or domination (as in the books by Narayana [76] and Mohanty [73]).

A sequence $a^m = (a_1, \dots, a_m)$ of nonnegative real numbers is *majorized* by $b^m = (b_1, \dots, b_m)$, denoted $a^m \preceq b^m$ if

$$\sum_{j=1}^{m} a_j = \sum_{j=1}^{m} b_j, \tag{i}$$

$$\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i} b_j \text{ for all } i = 1, 2, \dots m-1. \tag{ii}$$

i. e., all the partial sums formed by the initial segments of a^m are less or equal to the corresponding partial sums of b^m. If a^m and b^m are sequences of nonnegative integers, then we say that a^m is *dominated* by b^m if (i) and (ii) hold. Via domination a partial order is defined on all sequences of length m over the nonnegative integers.

Many problems under consideration can be formulated in terms of (possibly infinite) $\{0,1\}$–sequences

$$\underline{x} = (x_1, x_2, \dots) = (\underbrace{0, \dots, 0, 1}_{\mu_0}, \underbrace{0, \dots, 0, 1}_{\mu_1}, \underbrace{0, \dots, 0, 1}_{\mu_2}, \dots).$$

We shall describe the positions of the 1's in $\underline{x}$ by the sequence $(u_i)_{i=0,1,\dots}$, where the $(i+1)$–th 1 is found in position u_i and denote by $(\mu_i)_{i=0,1,2,\dots}$ the sequence of differences $\mu_i = u_i - u_{i-1}$ (with $\mu_0 = u_0$). In Coding Theory a number μ_i is called *run–length*, since it describes the size of a run, i. e., a block of consecutive 0's followed by a 1.

Especially useful are *ballot–type* sequences, which are $\{0,1\}$-sequences x^m of length m with a constant number of 1's such that in every initial segment $x^i = (x_1, \dots, x_i)$, $i = 1, \dots, m$, the number of 1's in x^i or – in the language of Coding Theory – the *weight* $wt(x^i)$, is at least a fraction Θ of i, i. e., $wt(x^i) \geq \Theta i$ for some Θ with $0 < \Theta < 1$.

It is well-known that for $\Theta = \frac{1}{2}$ ballot–type sequences are enumerated by the *Catalan numbers*

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

and that for $\Theta = \frac{1}{s}$ the *generalized Catalan numbers*

$$C_n^{(s)} = \frac{1}{sn+1}\binom{sn+1}{n} = \frac{1}{(s-1)n+1}\binom{sn}{n}.$$

are the counting function. (The notion "generalized Catalan numbers" as in [52] is not standard, for instance, in [49], pp. 344 – 350 it is suggested to denote the $C_n^{(s)}$ "Fuss numbers").

All ballot-type sequences with exactly $\lceil m\Theta \rceil$ many 1's can be obtained from the infinite binary sequence $\underline{y} = (y_1, y_2, \ldots)$ defined by

$$y_1 = 1, \qquad y_i = \begin{cases} 0, & \text{if } \frac{1}{i}(y_1 + \ldots + y_{i-1}) \geq \Theta \\ 1, & \text{else} \end{cases} \tag{1.1}$$

for all $i > 1$. For instance, if $\Theta = \frac{1}{2}$ it is $\underline{y} = (1, 0, 1, 0, \ldots)$. Any ballot-type sequence x^m of length m with $\lceil m\Theta \rceil$ many 1's now is obtained from the initial segment y^m of length m of $\underline{y}$, by *pushing* 1's in y^m to the left, i. e., if the positions of the 1's in y^m are $u_0, \ldots, u_{k-1}$ and the positions of the 1's in x^m are $v_0, \ldots, v_{k-1}$ then $v_j \leq u_j$ for all $j = 0, \ldots, k-1$. This pushing operation defines an order, which can equivalently be expressed in terms of domination (cf. Section III).

As shown in [76] and [73] the approach via domination of sequences is especially useful in the analysis of the number of lattice paths under restrictions on the boundary.

A path starting in the origin of the lattice $\{(s, t) : s, t \text{ integers}\}$ of pairs of integers here is a sequence of pairs (s_i, t_i) of nonnegative integers where $(s_0, t_0) = (0, 0)$ and (s_i, t_i) is either $(s_{i-1} + 1, t_{i-1})$ or $(s_{i-1}, t_{i-1} + 1)$. So, a particle following such a path can move either one step to the right, i. e. $s_i = s_{i-1} + 1$, or one step upwards, i. e. $t_i = t_{i-1} + 1$ in each time unit i. Further path models are discussed, e. g., in [37], pp. 126 – 128.

A one-to-one correspondence between a $\{0, 1\}$- sequence x^m and a path with m steps is obtained via the bijection $s_i = s_{i-1} + x_i = \sum_{j=1}^{i} x_j$ (with initial value $s_0 = 0$). So $s_1, \ldots s_m$ is just the vector of partial sums obtained from x^m. Observe that t_i is obtained from the complementary sequence $(1 - x_1, \ldots, 1 - x_m)$ in the same way.

Several methods for the enumeration of lattice paths are discussed in the book by Mohanty [73]. For the number of paths $N(\underline{u}, n)$ first touching the boundary $(0, u_0), (1, u_1), (2, u_2), \ldots$ in $(n-1, u_{n-1})$ (and not touching or crossing this boundary before) characterized by the infinite sequence $\underline{u} = (u_0, u_1, u_2, \ldots)$ there is a determinantal expression due to Kreweras [67] (as a special case of the same enumeration problem, when additionally there is also a lower boundary not allowed to be crossed), namely

$$N(\underline{u}, n) = \det \begin{pmatrix} \binom{u_{n-1}+1}{1} & 1 & 0 & \cdots & & 0 \\ \binom{u_{n-2}+1}{2} & \binom{u_{n-2}+1}{1} & 1 & 0 & \cdots & 0 \\ \binom{u_{n-3}+1}{3} & \binom{u_{n-3}+1}{2} & \binom{u_{n-3}+1}{1} & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \binom{u_1+1}{n-1} & \binom{u_1+1}{n-2} & \binom{u_1+1}{2} & \cdots & \binom{u_1+1}{1} & 1 \\ \binom{u_0+1}{n} & \binom{u_0+1}{n-1} & \binom{u_0+1}{n-2} & \cdots & \binom{u_0+1}{2} & \binom{u_0+1}{1} \end{pmatrix} \tag{1.2}$$

As pointed out in [76], p. 21, this immediately yields the recursion

$$N(\underline{u}, n) = \sum_{j=1}^{n} (-1)^{j-1} \binom{u_{n-j} + 1}{j} \cdot N(\underline{u}, n - j).$$

In principle, this settles the enumeration problem. However, one might be interested in an expression of closed form (e. g., as in [78]). For instance, if the boundary is given by the sequence $\underline{u} = (1, 2, 3, \ldots)$, then $N(\underline{u}, n)$ is the $n-th$ *Catalan number* and, more

generally, for $\underline{u} = (1+(s-1)\cdot n)_{n=0,1,2,\ldots}$ as counting function arise the *generalized Catalan numbers* $C_n^{(s)}$.
Note that this describes the case in which the sequence of differences $(u_n - u_{n-1})_{n=1,2,\ldots}$ is periodic with period length 1. In Section II we shall derive similar identities for period length 2, hereby following a probabilistic method introduced by Gessel [45], which allows to apply Lagrange inversion. For instance, it can be shown that if $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ are such that $u_{2i}^{(1)} = u_{2i}^{(2)} = s + ci$, $u_{2i+1}^{(1)} = s + \mu + ci$ and $u_{2i+1}^{(2)} = s + (c-\mu) + ci$, then

$$N(\underline{u}^{(1)}, 2n) + N(\underline{u}^{(2)}, 2n) = \frac{2}{(c+2)n+1}\binom{(c+2)n+1}{2n}. \tag{1.3}$$

Several combinatorial applications will be given in Section III.
First of all, it can be shown that by representing binary sequences x^m of constant weight as paths in a lattice as described above, the number of predecessors of such a sequence x^m in the left pushing order, which is important in the analysis of intersection theorems of Erdös–Ko–Rado type, can be determined by the above determinantal identity (with u_j being the position of the $(j+1)$-th 1 in x^m).
Another application is a new expression for the number of paths not crossing or touching the line $cx = 2y$ for odd c. This allows to analyze combinatorially the recurrence behaviour of the random walk on the line of a particle which in each time unit is allowed to move either c steps forward or 2 steps back.
As a last application, sequences y^m of the form (1.1) are studied. First the problem of enumerating all ballot–type sequences with $\lceil \Theta m \rceil$ many 1's is addressed, which for rational Θ can be approached by the results in Sections II and IV.
If the parameter Θ in the definition (1.1) of ballot-type sequences is irrational. the sequence $\underline{y}$ is obtained in a canonical way from a *Beatty sequence*. The Beatty sequence for the golden ratio $\Theta = \frac{\sqrt{5}-1}{2}$ plays an important role in the characterization of nonperiodic tilings. Since Beatty sequences are run-length-limited sequences, namely, the number of zeroes between two consecutive ones is either μ or $\mu+1$ for some μ, one might wonder if this property can be exploited in Coding Theory. For instance, the irregularity of these sequences might yield synchronization codes.

Further, an application of (1.3) in the analysis of two–dimensional arrays will be studied. For $i = 1, 2, \ldots$ let $\lambda_i^{(\nu)}$ denote the frequency of the number i in the sequences $\underline{u}^{(\nu)}$ describing two boundaries for $\nu = 1, 2$ and let $\lambda^{(\nu)} = (\lambda_1^{(\nu)}, \lambda_2^{(\nu)}, \ldots)$. Denoting by $\gamma^{(\nu)}(n,k)$ the number of paths from the origin to (n,k) not touching or crossing the boundary described by $\underline{u}^{(\nu)}$, in the case that $\underline{\lambda}^{(1)} = (\lambda, c-\lambda, \lambda, c-\lambda, \ldots)$ and $\underline{\lambda}^{(2)} = (c-\lambda, \lambda, c-\lambda, \lambda, \ldots)$ are both periodic with period length 2 it is

$$\gamma^{(1)}(n,k) + \gamma^{(2)}(n,k) = 2\cdot\binom{n+k}{k} - c\cdot\binom{n+k}{k-1} \tag{A.2}$$

which can be derived using (1.3). Further, (1.4) can be regarded as a generalization of the ballot numbers $\binom{n+k}{k} - \binom{n+k}{k-1}$.
A further generalization of the ballot numbers for boundaries described by sequences $\underline{\lambda}^{(\nu)}$, $\nu = 1, \ldots, d$ periodic with period length d is strongly conjectured.
From (1.4) results are immediate for the numbers $\alpha^{(\nu)}(n,k) = \gamma^{(\nu)}(n+k,k)$. Such a two-dimensional array $\alpha(n,k)$ had been found by Berlekamp [16] in the study of burst–error

correcting convolutional codes and thoroughly analyzed by Carlitz, Rosselle, and Scoville [29]

Two dimensional arrays may also yield fast recursive algorithms for the enumeration of trees. For instance the ballot numbers count the forests on n vertices consisting of $k+1$ trees. In Section V, a simple derivation is given for the same counting problem under the restriction that the degree of each vertex is at most a number d generalizing a well-known identity for $d = 2$, where the Motzkin numbers arise (in case $k = 0$).

A second application in the study of trees will further be given in Section V. There is a one–to–one correspondence between regular trees in which each inner vertex has degree exactly s and ballot–type sequences with parameter $\Theta = \frac{1}{s}$, s being a positive integer, which can hence be used in order to represent a regular tree, for instance when it has to be stored or used in a computer program. For every s, the set of all ballot–type sequences (taken over all $n = 1, 2, \ldots$) is prefix-free, hence Kraft's inequality must be fulfilled and it is well known that equality holds for $s = 2$. For $s > 2$ we shall show that $\frac{1}{2}$ is a lower bound for the expression in Kraft's inequality.

In Section VI, we shall present an ordering algorithm for binary sequences in the spirit of the pioneering paper by Ahlswede, Ye, and Zhang [6]. The difference to the original model is that a number $s > 1$ elements arrive at the organizer in each time unit, who has to put out $l < s$ elements in the same time unit. This has the effect that the size of the storage device, in which he can buffer the elements may be variable. If the memory can store every incoming element, the analysis reduces to a sequence of ballot-type with the difference that now $wt(x^i) \geq \frac{l}{s}$ only has to hold in those positions i divisible by s. This model yields the enumeration function (with "$\preceq$" the domination order)

$$a(0,t) = \sum_{(l,\ldots,l)\preceq(i_1,\ldots,i_t)} \binom{s}{i_1} \cdot \binom{s}{i_2} \cdots \binom{s}{i_t}$$

for the number $a(0,t)$ of sequences of input bits yielding an empty memory device at time t. For $l = 1$, the number of such sequences again is counted by the generalized Catalan numbers – but this time with index $t - 1$. This fact allows to analyze the stochastic process for the exhaustion of the memory device.

The matrix in (1.2) is a Hessenberg matrix of a special type (with $a_{i,i+1} = 1$ for all i) In Section VII, a formula for the determinants of such Hessenberg matrices is given. This is further motivated by the fact that several identities, occurring in the course of this chapter, can be expressed as determinant of a Hessenberg matrix. Special Hessenberg matrices are tridiagonal matrices, which for certain parameters are useful in the study of the above ordering process, when the size of the memory is limited.

II. Lattice Paths Never Touching a Given Boundary

II.1 GESSEL's PROBABILISTIC APPROACH

We saw already that the problems under consideration can be transformed into the equivalent problem of enumerating paths in an integer lattice from the origin $(0,0)$ to the point (n, u_n), which never touch any of the points (i, u_i), $i = 0, 1, \ldots, n-1$. In [45] Gessel introduced a general probabilistic method to determine the number of such paths, denoted by f_n , which he studied for the case that the sequence $(u_i)_{i=1,2\ldots}$ is periodic.

In this case the elements of the sequence $(u_i)_i$ are on the d lines

$$u_{di} = \mu_0 + ci \text{ and } u_{di+1} = \mu_0 + \mu_1 + ci, \quad \ldots \quad , u_{di+d-1} = \mu_0 + \mu_1 + \ldots + \mu_{d-1} + ci,$$

so $\mu_0 = u_0 > 0$, and $c := \mu_1 + \mu_2 + \ldots + \mu_d$, where $\mu_j = u_j - u_{j-1}$.

Gessel's probabilistic method is as follows. A particle starts at the origin $(0,0)$ and successively moves with probability p one unit to the right and with probability $q = 1-p$ one unit up. The particle stops if it touches one of the points (i, u_i).

The probability that the particle stops at (n, u_n) is $p^n q^{u_n} \cdot f_n$, which is $p^n q^{\mu_0+\ldots+\mu_j+cn}$ if $n \equiv j \bmod d$. Setting

$$f(t) = \sum_{n=0}^{\infty} f_n t^n = \sum_{j=0}^{d-1} t^j f^{(j)}(t^d)$$

(so $f^{(j)}(t) = \sum_{n=0}^{\infty} f_n^{(j)} t^{dn} = \sum_{n=0}^{\infty} f_{dn+j} t^n$ are the generating functions for thef_n's with indices congruent j modulo d), the probability that the particle eventually stops is

$$q^{u_0} f^{(0)}(p^d q^c) + pq^{u_1} f^{(1)}(p^d q^c) + pq^{u_2} f^{(2)}(p^d q^c) + \ldots + p^{d-1} q^{u_{d-1}} f^{(d-1)}(p^d q^c) = 1$$

where $u_j = \mu_0 + \ldots + \mu_j$.

If p is sufficiently small, the particle will cross the boundary $(i, u_i)_{i=0,1,\cdots}$, or equivalently, enter the *forbidden area*, i. e. the lattice points behind this boundary, with probability 1. So for small p and with $t = pq^{c/d}$ it is

$$q(t)^{u_0} f^{(0)}(t^d) + p(t)q(t)^{u_1} f^{(1)}(t^d) + \cdots + p(t)^{d-1} q(t)^{u_{d-1}} f^{(d-1)}(t^d) = 1.$$

For p sufficiently small one may invert $t = p(1-p)^{c/d}$ to express p as a power series in t, namely $p = p(t)$. Then changing t to $\omega^j t$, $j = 1, \ldots, d-1$, where ω is a primitive d–th root of unity, yields the system of d equations

$$\begin{array}{ccccccccc}
q(t)^{u_0} f^{(0)}(t^d) & + & p(t)q(t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(t)^{d-1} q(t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1 \\
q(\omega t)^{u_0} f^{(0)}(t^d) & + & p(\omega t)q(\omega t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(\omega t)^{d-1} q(\omega t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1 \\
\vdots & + & \vdots & + \quad + & \vdots & = \vdots \\
q(\omega^{d-1}t)^{u_0} f^{(0)}(t^d) & + & p(\omega^{d-1}t)q(\omega^{d-1}t)^{u_1} f^{(1)}(t^d) & + \cdots + & p(\omega^{d-1}t)^{d-1} q(\omega^{d-1}t)^{u_{d-1}} f^{(d-1)}(t^d) & = 1
\end{array}.$$

Written in matrix form where $A = (p(\omega^i t)^j q(\omega^i t)^{u_j})_{i,j=0,\ldots,d-1}$ this is

$$A \cdot \begin{pmatrix} f^{(0)}(t^d) \\ f^{(1)}(t^d) \\ \vdots \\ f^{(d-1)}(t^d) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Denoting by $|A|$ the determinant of the matrix A, application of Cramer's rule yields the solution (if it exists)

$$f^{(j)}(t^d) = \frac{|A^{(j)}|}{|A|}, j = 0, \ldots, d-1$$

where $A^{(j)}$ is the matrix obtained by replacing the j-th row in A with the all-one vector. Observe that the determinants $|A|$ and $|A^{(j)}|$ are alternants, i. e., the corresponding quadratic matrices $(F_j(x_i))_{i,j=0,\ldots,d-1}$ are such that the rows correspond to variables x_i which are evaluated by a function F_j in row No. j. In [11] and [75] alternants are intensively discussed and algorithms are presented to evaluate such determinants if the functions F_j are monomials of the form $x^{a_0} \cdot \ldots \cdot x^{a_{j-1}}$ with positive exponents a_i as for Schur functions. Here we are almost in such a situation. The entries in the matrix A are of the form

$$p(\omega^i t)^j q(\omega^i t)^{\mu_0+\mu_1+\ldots+\mu_j} = q(\omega^i t)^{\mu_0} \cdot \prod_{l=1}^{j} (p(\omega^i t) q(\omega^i t)^{\mu_l})$$

$$= q(\omega^i t)^{\mu_0} \cdot \prod_{l=1}^{j} ((\omega^i t) q(\omega^i t)^{\mu_l - c/d}) = \omega^{ij} t^j \cdot q(\omega^i t)^{\mu_0} \prod_{l=1}^{j} q(\omega^i t)^{\mu_l - c/d}.$$

So the determinant $|A| = t^{\binom{d-1}{2}} \cdot |B|$ where

$$B = \left(\omega^{ij} q(\omega^i t)^{\mu_0 + \sum_{l=1}^{j} (\mu_l - c/d)}\right)_{i,j=0,\ldots,d-1}.$$

We shall study the period length $d = 2$ in the following subsection. For $d > 2$, however, these determinants seem hard to determine, since the factors ω^{ij} do not allow the standard approach – factoring out all differences $(x_i - x_j)$ – to work here. For $d = 2$ via Lagrange inversion there arise some nice identities, which might be generalizable to larger period lengths d. For instance, computer results suggest the following observation (which we shall prove for $d = 2$)

Observation 2.1: Let $\eta_1, \ldots, \eta_d$ be all greater than 0 with $\eta_1 + \ldots + \eta_d = c$. Further, let $f^{(i,0)}$ denote the function $f^{(0)}$ as above for the choice of parameters $(\mu_1^{(i)}, \ldots \mu_{d-1}^{(i)}) = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_d)$ for $i = 1, \ldots, d$. When $\mu_0^{(i)} = \mu_0$ is the same for all i, then

$$f^{(1,0)}(t^d) + f^{(2,0)}(t^d) + \ldots + f^{(d,0)}(t^d) = q(t)^{-\mu_0} + q(\omega t)^{-\mu_0} + \ldots + q(\omega^{d-1} t)^{-\mu_0}$$

$$= \sum_{n=0}^{\infty} \frac{d\mu_0}{(c+d)n + \mu_0} \binom{(c+d)n + \mu_0}{dn} t^{dn}.$$

We shall return to this observation in the Section IV, where a different approach yielding this identity as a special entry in a two–dimensional array will be discussed.

II.2 BOUNDARIES PERIODIC WITH PERIOD LENGTH 2

Let us denote $s = \mu_0$ and $\mu = \mu_1$. Then the boundary $(n, u_n)_{n=0,1,\ldots}$ is characterized by

$$u_{2i} = s + ci \text{ and } u_{2i+1} = s + \mu + ci, \tag{2.1}$$

Further, denoting $p(-t)$ by $\overline{p}(t)$ and similarly $q(-t)$ by $\overline{q}(t)$ and setting $g(t^2) = f^{(0)}(t^2)$ and $h(t^2) = f^{(1)}(t^2)$ (as in [45]) we obtain the two equations

$$q^s \cdot g(t^2) + p \cdot q^{s+\mu} \cdot h(t^2) = 1,$$

$$\overline{q}^s \cdot g(t^2) + \overline{p} \cdot \overline{q}^{s+\mu} \cdot h(t^2) = 1$$

which for $g(t^2)$ and $h(t^2)$ yield the solutions

$$g(t^2) = \frac{p^{-1}q^{-s-\mu} - \overline{p}^{-1}\overline{q}^{-s-\mu}}{p^{-1}q^{-\mu} - \overline{p}^{-1}\overline{q}^{-\mu}} = \frac{q^{c/2-\mu-s} + \overline{q}^{c/2-\mu-s}}{q^{c/2-\mu} + \overline{q}^{c/2-\mu}} \tag{2.2}$$

and

$$h(t^2) = \frac{q^{-s} - \overline{q}^{-s}}{t \cdot (q^{\mu-c/2} + \overline{q}^{\mu-c/2})}. \tag{2.3}$$

By Lagrange inversion (cf. e.g. [46], pp. 1032–1034) for any α it is

$$q^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha}{(c/2+1)n+\alpha} \binom{(c/2+1)n+\alpha}{n} \cdot t^n. \tag{2.4}$$

Actually, Gessel analyzed the case $\mu = \lambda, c = 2\lambda + 1$ for a positive integer λ, which arises in the enumeration of paths never touching or crossing the line $y = s - \frac{1}{2} + \frac{c}{2} \cdot x$. For the special case $s = 1$ he derived the following nice identity for the function $h(t^2)$.

Proposition 2.1 [45]: Let c be an odd positive integer, $s = 1$ and $\mu = \frac{c-1}{2}$. Then

$$h(t^2) = \frac{q^{-1/2} - \overline{q}^{-1/2}}{t} = \sum_{n=0}^{\infty} \frac{1}{(c+2)n+\mu+2} \binom{(c+2)n+\mu+2}{2n+1} t^{2n}.$$

So, the coefficients in the expansion of $h(t^2)$ have a similar form as the Catalan numbers. It is also possible to show that for these parameters

$$g(t^2) = \sum_{n=0}^{\infty} \frac{1}{(c+2)n+1} \binom{(c+2)n+1}{2n} t^{2n} - \frac{t^2}{2}[h(t^2)]^2$$

This is a special case of a more general result which we are going to derive now. Since we are going to look at several random walks in parallel, we shall introduce the parameters determining the restrictions as a superscript to the generating functions. So, $g^{(s,c,\mu)}$ and $h^{(s,c,\mu)}$ are the generating functions for even and odd n, respectively, for the random walk of a particle starting in the origin and first touching the boundary $(i, u_i)_{i=0,1,\dots}$ determined by the parameters s, c, and μ as in (2.1) in (n, u_n).

Proposition 2.2: Let s, c, μ be the parameters defined above with $0 \leq \mu < \frac{c}{2}$.

a)

$$g^{(s,c,\mu)}(t^2) + g^{(s,c,c-\mu)}(t^2) = q^{-s} + \overline{q}^{-s} = \sum_{n=0}^{\infty} \frac{2s}{(c+2)n+s} \binom{(c+2)n+s}{2n} t^{2n},$$

b)
$$g^{(s,c,c-\mu)}(t^2) - g^{(s,c,\mu)}(t^2) = t^2 \cdot h^{(s,c,\mu)}(t^2) \cdot h^{(c-2\mu,c,\mu)}(t^2).$$

Proof:

a) In order to derive the first identity observe that with $a = \frac{c}{2} - \mu$ it is

$$g^{(s,c,\mu)}(t^2) + g^{(s,c,c-\mu)}(t^2) = \frac{q^{a-s} + \overline{q}^{a-s}}{q^a + \overline{q}^a} + \frac{q^{-a-s} + \overline{q}^{-a-s}}{q^{-a} + \overline{q}^{-a}}$$

$$= \frac{(q^{a-s} + \overline{q}^{a-s})(q^{-a} + \overline{q}^{-a}) + (q^{-a-s} + \overline{q}^{-a-s})(q^a + \overline{q}^a)}{(q^a + \overline{q}^a)(q^{-a} + \overline{q}^{-a})}$$

$$= \frac{2q^{-s} + 2\overline{q}^{-s} + \overline{q}^{a-s}q^{-a} + q^{-a-s}\overline{q}^a + q^{a-s}\overline{q}^{-a} + \overline{q}^{-a-s}q^a}{2 + \overline{q}^a q^{-a} + q^a \overline{q}^{-a}} = q^{-s} + \overline{q}^{-s}.$$

Since by definition $\overline{q}(t) = q(-t)$, with Lagrange inversion it is

$$q^{-s} + \overline{q}^{-s} =$$

$$= \sum_{n=0}^{\infty} \frac{s}{(c/2+1)n+s} \binom{(c/2+1)n+s}{n} \cdot t^n + \sum_{n=0}^{\infty} \frac{s}{(c/2+1)n+s} \binom{(c/2+1)n+s}{n} \cdot (-t)^n$$

$$= \sum_{n=0}^{\infty} \frac{2s}{(c+2)n+s} \binom{(c+2)n+s}{2n} t^{2n}.$$

b) Let again $a = \frac{c}{2} - \mu$. Then

$$g^{(s,c,c-\mu)}(t^2) - t^2 \cdot h^{(s,c,\mu)}(t^2) \cdot h^{(c-2\mu,c,\mu)}(t^2) = \frac{q^{-a-s} + \overline{q}^{-a-s}}{q^{-a} + \overline{q}^{-a}} - t^2 \frac{q^{-s} - \overline{q}^{-s}}{t(q^{-a} + \overline{q}^{-a})} \cdot \frac{q^{-2a} - \overline{q}^{-2a}}{t(q^{-a} + \overline{q}^{-a})}$$

$$= \frac{q^{-a-s} + \overline{q}^{-a-s}}{q^{-a} + \overline{q}^{-a}} - \frac{(q^{-s} - \overline{q}^{-s})(q^{-a} - \overline{q}^{-a})}{(q^{-a} + \overline{q}^{-a})} = \frac{q^{-a}\overline{q}^{-s} + \overline{q}^{-a}q^{-s}}{q^{-a} + \overline{q}^{-a}} = \frac{q^{a-s} + \overline{q}^{a-s}}{q^a + \overline{q}^a} = g^{(s,c,\mu)}(t^2).$$

There is also a short combinatorial proof of the last identity. Since $\mu < c - \mu$, the forbidden area (i. e., the area on and behind the boundary) for the random walk of the particle determined by the parameters (s, c, μ) is completely contained in the forbidden area determined by the parameters $(s, c, c - \mu)$. So a path which is counted by the $n - th$ coefficient in $g^{(s,c,\mu)}(t^2)$ but not counted by the n-th coefficient in $g^{(s,c,c-\mu)}(t^2)$ must enter the forbidden area in one of the points $(i, s + \mu + ci)$ with odd i (for even i the boundaries coincide). Up to this entrance the path stays beneath the boundary determined by the parameters (s, c, μ) and there are h_i many possible such initial paths, where h_i is the i-th coefficient in $h^{(s,c,\mu)}(t^2)$. Then the path stays beneath the boundary determined by the parameters $(c - 2\mu, c, \mu)$ and there are h'_{n-i-1} many such paths from $(i, s + \mu + ci + 1)$ to $(n, s+cn)$ (n is even), where this time h'_{n-i-1} is the $(n-i-1)$-th coefficient of $h^{(c-2\mu,c,\mu)}(t^2)$. So the generating function of interest for the difference is $t^2 \cdot h^{(s,c,\mu)}(t^2) \cdot h^{(c-2\mu,c,\mu)}(t^2)$.

□

Similar identities can be derived for the case $s + \mu = c$.

Proposition 2.3: Let $c > 0$ be a positive integer, and $s + \mu = c$ with $s \geq \mu$. Then a)

$$h^{(s,c,c-s)}(t^2)+h^{(c-s,c,s)}(t^2)=\frac{1}{t^2}\cdot(p+\overline{p})=\sum_{n=1}^{\infty}\frac{2}{(c+2)n-1}\binom{(c+2)n-1}{2n}\cdot t^{2(n-1)}.$$

b) In the special case c odd, $s=\frac{c+1}{2}$ and $\mu=\frac{c-1}{2}$ it is

$$h^{(\frac{c+1}{2},c,\frac{c-1}{2})}(t^2)-h^{(\frac{c-1}{2},c,\frac{c+1}{2})}(t^2)=\left(g^{(\frac{c+1}{2},c,\frac{c-1}{2})}(t^2)\right)^2$$

where

$$g^{(\frac{c+1}{2},c,\frac{c-1}{2})}(t^2)=\frac{1}{t}\cdot(\overline{q}^{\frac{1}{2}}-q^{\frac{1}{2}})=\sum_{n=0}^{\infty}\frac{1}{(c+2)n+\frac{c+1}{2}}\binom{(c+2)n+\frac{c+1}{2}}{2n+1}\cdot t^{2n}.$$

Proof:
a) By (2.3)

$$h^{(s,c,c-s)}(t^2)+h^{(c-s,c,s)}(t^2)=\frac{q^{-s}-\overline{q}^{-s}}{t\cdot(q^{c/2-s}+\overline{q}^{c/2-s})}+\frac{q^{s-c}-\overline{q}^{s-c}}{t\cdot(q^{s-c/2}+\overline{q}^{s-c/2})}$$

$$=\frac{q^{-s}-\overline{q}^{-s}}{pq^{c-s}-\overline{pq}^{c-s}}+\frac{q^{s-c}-\overline{q}^{s-c}}{pq^{s}-\overline{pq}^{s}}=\frac{2(p+\overline{p})-q^s\overline{q}^{-s}(p+\overline{pq}^cq^{-c})-\overline{q}^sq^{-s}(\overline{p}+pq^c\overline{q}^{-c})}{p^2q^c+\overline{p}^2\overline{q}^c-q^s\overline{q}^{-s}(p\overline{pq}^c)-\overline{q}^sq^{-s}(p\overline{pq}^c)}$$

$$=\frac{(p+\overline{p})(2-q^s\overline{q}^{-s}(p/\overline{p})-\overline{q}^sq^{-s}(\overline{p}/p))}{t^2(2-q^s\overline{q}^{-s}(p/\overline{p})-\overline{q}^sq^{-s}(\overline{p}/p))}=\frac{p+\overline{p}}{t^2}$$

since $p^2q^c=\overline{p}^2\overline{q}^c=t^2$ by definition of t and since $\overline{p}(p+\overline{pq}^cq^{-c})=p(p+\overline{p})$.
b) With $s=\frac{c+1}{2}$ again by (2.3) as under a)

$$h^{(s,c,c-s)}(t^2)-h^{(c-s,c,s)}(t^2)=\frac{q^{-s}-\overline{q}^{-s}}{pq^{c-s}-\overline{pq}^{c-s}}-\frac{q^{s-c}-\overline{q}^{s-c}}{pq^{s}-\overline{pq}^{s}}$$

$$=\frac{\overline{pq}^{c-s}q^{-(c-s)}+pq^{c-s}\overline{q}^{-(c-s)}-\overline{pq}^sq^{-s}-pq^s\overline{q}^{-s}}{t^2(2-(pq^s)/(\overline{pq}^s)-(\overline{pq}^s)/(pq^s))}$$

$$=\frac{\overline{pq}^{\frac{c-1}{2}}q^{-\frac{c+1}{2}}(q-\overline{q})-pq^{\frac{c-1}{2}}\overline{q}^{-\frac{c+1}{2}}(q-\overline{q})}{t^2(2-(tq^{\frac{1}{2}})/(-t\overline{q}^{\frac{1}{2}})-(-t\overline{q}^{\frac{1}{2}})/(tq^{\frac{1}{2}}))}=\frac{q^{-\frac{1}{2}}\overline{q}^{-\frac{1}{2}}(q-\overline{q})(\overline{pq}^{\frac{c}{2}}q^{-\frac{c}{2}}-pq^{\frac{c}{2}}\overline{q}^{-\frac{c}{2}})}{t^2(2+q^{\frac{1}{2}}/\overline{q}^{\frac{1}{2}}+\overline{q}^{\frac{1}{2}}/q^{\frac{1}{2}})}$$

$$=\frac{q^{-\frac{1}{2}}\overline{q}^{-\frac{1}{2}}(q-\overline{q})(\overline{p}-p)}{t^2q^{-\frac{1}{2}}\overline{q}^{-\frac{1}{2}}(2q^{\frac{1}{2}}\overline{q}^{\frac{1}{2}}+q+\overline{q})}=\frac{(\overline{p}-p)^2}{t^2(q^{\frac{1}{2}}+\overline{q}^{\frac{1}{2}})^2}=\left(g^{(\frac{c+1}{2},c,\frac{c-1}{2})}(t^2)\right)^2$$

since $t=pq^{\frac{c}{2}}=-\overline{pq}^{\frac{c}{2}}$ and $p=1-q,\overline{p}=1-\overline{q}$ and by (2.2)

$$g^{(\frac{c+1}{2},c,\frac{c-1}{2})}(t^2)=\frac{q^{-c/2}+\overline{q}^{-c/2}}{q^{1/2}+\overline{q}^{1/2}}=\frac{p-\overline{p}}{t\cdot(q^{1/2}+\overline{q}^{1/2})}=\frac{\overline{q}-q}{t\cdot(q^{1/2}+\overline{q}^{1/2})}=\frac{1}{t}(\overline{q}^{1/2}-q^{1/2}).$$

$\square$

Further, several convolution identities for the generating functions can be derived. For instance:

Proposition 2.4:
a)

$$\left(g^{(s,c,\mu)}(t^2)+g^{(s,c,c-\mu)}(t^2)\right)\cdot h^{(s,c,\mu)}(t^2)=h^{(2s,c,\mu)}(t^2).$$

b)

$$g^{(c-2\mu,c,\mu)}(t^2)\cdot g^{(\mu,c,c-\mu)}(t^2)=g^{(c-\mu,c,\mu)}(t^2).$$

c) For $s_1+\mu_1+\mu_2=c$ it is

$$g^{(s_1,c,\mu_1)}(t^2)\cdot h^{(s_2,c,\mu_2)}(t^2)=h^{(s_2,c,s_1+\mu_2)}(t^2).$$

Especially, for odd c

$$g^{(1,c,\frac{c-1}{2})}(t^2)\cdot h^{(1,c,\frac{c-1}{2})}(t^2)=h^{(1,c,\frac{c+1}{2})}(t^2).$$

Proof: a) is immediate from the fact that $g^{(s,c,\mu)}(t^2)+g^{(s,c,c-\mu)})(t^2)=q^{-s}+\overline{q}^{-s}$ (Proposition 2.2a)) and b) is immediate, since the nominator of $g^{(c-2\mu,c,\mu)}(t^2)$ in (2.2) is at the same time denominator of $g^{(\mu,c,c-\mu)}(t^2)$. The nominator of $g^{(s_1,c,\mu_1)}(t^2)$ in c) by (2.2) is $q^{c/2-\mu_1-s_1}+\overline{q}^{c/2-\mu_1-s_1}$ and this is the term in brackets in the denominator in (2.3) of $h^{(s_2,c,\mu_2)}(t^2)=\frac{q^{-s_2}-\overline{q}^{-s_2}}{t(q^{\mu_2-c/2}+\overline{q}^{-\mu_2-c/2})}$. □

II.3 THE CASE $c=3$

Let us discuss the case $c=3$ a little closer and hereby illustrate the derived identities. The parameter choices $(s=1,\mu=1)$, $(s=1,\mu=2)$, and $(s=2,\mu=1)$ will be of interest in the combinatorial applications, we shall speak about later on. By application of the previous results, the generating functions for these parameters (after mapping $t^2\to x$) look as follows. Observe that they all can be expressed in terms of $a(x):=g^{(1,3,1)}(x)$ and $b(x):=g^{(1,3,2)}(x)$.

Corollary 2.1:

$$a(x):=g^{(1,3,1)}(x)=\sum_{n=0}^{\infty}\frac{1}{5n+1}\binom{5n+1}{2n}x^n-\frac{x}{2}\cdot[h^{(1,3,1)}(x)]^2=1+2x+23x^2+377x^3+\ldots,$$

$$b(x):=g^{(1,3,2)}(x)=\sum_{n=0}^{\infty}\frac{1}{5n+1}\binom{5n+1}{2n}x^n+\frac{x}{2}\cdot[h^{(1,3,1)}(x)]^2=1+3x+37x^2+624x^3+\ldots,$$

$$g^{(2,3,1)}(x)=\sum_{n=0}^{\infty}\frac{1}{5n+2}\binom{5n+2}{2n+1}x^n=1+5x+66x^2+1156x^3+\ldots=a(x)\cdot b(x),$$

$$h^{(1,3,1)}(x)=\sum_{n=0}^{\infty}\frac{1}{5n+3}\binom{5n+3}{2n+1}x^n=1+7x+99x^2+1768x^3+\ldots=a(x)^2\cdot b(x),$$

$$h^{(1,3,2)}(x) = \sum_{n=1}^{\infty} \frac{1}{5n-1}\binom{5n-1}{2n} x^{n-1} - \frac{1}{2}[g^{(2,3,1)}(x)]^2 = 1+9x+136x^2+2504x^3+\ldots = a(x)^3 \cdot b(x),$$

$$h^{(2,3,1)}(x) \sum_{n=1}^{\infty} \frac{1}{5n-1}\binom{5n-1}{2n} x^{n-1} + \frac{1}{2} \cdot [g^{(2,3,1)}(x)]^2 = 2 + 19x + 293x^2 + 5332x^3 + \ldots$$

$$= h^{(1,3,2)}(x) + [g^{(2,3,1)}(x)]^2 = a(x)^3 \cdot b(x) + a(x)^2 \cdot b(x)^2$$

$$= (a(x) + b(x)) \cdot a(x)^2 \cdot b(x) = (g^{(1,3,1)}(x) + g^{(1,3,2)}(x)) \cdot h^{(1,3,1)}(x).$$

It is also possible to express all six functions in terms of either $a(x)$ or $b(x)$. In order to see this, two further convolution identities for $a(x) - 1$ and $b(x) - 1$ are useful.

Proposition 2.5: With $x = t^2$

a)

$$\frac{1}{x}(a(x) - 1) = a(x) \cdot h^{(2,3,1)}(x) = a(x)^4 \cdot b(x) + a(x)^3 \cdot b(x)^2,$$

b)

$$\frac{1}{x}(b(x) - 1) = (b(x) + 2a(x)) \cdot [g^{(2,3,1)}(x)]^2 = a(x)^2 \cdot b(x)^3 + 2a(x)^3 \cdot b(x)^2,$$

c)

$$b(x) = \frac{a(x)}{2}[(a(x) - 1) + \sqrt{(a(x) - 1)^2 + 4)}], \qquad a(x) = \frac{b(x)}{2(b(x) + 1)}[1 + \sqrt{4b(x) + 2}].$$

Proof: The formula in a) is equivalent to

$$(q^{-1/2} + \overline{q}^{-1/2} - q^{1/2} - \overline{q}^{1/2})(q^{-1/2} + \overline{q}^{-1/2}) = t \cdot (q^{-2} + \overline{q}^{-2})(q^{-1/2} + \overline{q}^{-1/2})$$

which can be easily verified taking into account that $t = pq^{3/2} = \overline{p} \cdot \overline{q}^{3/2}$. We omit the somewhat lengthy derivation of b). In order to prove c), observe that with a) and b) it is

$$\frac{1}{x}(b(x) - a(x)) = a(x)^2 \cdot b(x)^3 + a(x)^3 \cdot b(x)^2 - a(x)^4 \cdot b(x)$$

On the other hand, from the previous corollary we know that

$$\frac{1}{x}(b(x) - a(x)) = a(x)^4 \cdot b(x)^2$$

So we have the identity

$$a(x)^2 \cdot b(x) \cdot [b(x)^2 + a(x) \cdot b(x) - a(x)^2 - a(x)^2 \cdot b(x)] = 0$$

The term in brackets yields quadratic equations for $a(x)$ and $b(x)$, which can be solved as in c). $\square$

III. Combinatorial Applications

III.1 SIZE OF IDEALS IN THE PUSHING ORDER

We are going to consider $\{0,1\}$–sequences $x^m = (x_1, \ldots, x_m)$ of length m and weight $wt(x^m) = k$, i. e., $(x_1, \ldots, x_m)$ consists of k ones and $m-k$ zeros.
These can equivalently be regarded as k elementary subsets of the m-elementary set $[m] = \{1, \ldots, m\}$, namely via $x_i = 1$ exactly if i is contained in the subset corresponding to x^m.
On the set $\binom{[m]}{k}$ of all sequences of length m and constant weight k we define an order relation $\preceq_p$ in the following way. Let $\{u_0, \ldots, u_{k-1}\}$ and $\{u'_0, \ldots, u'_{k-1}\}$ denote the sets of positions of the 1's in the sequences x^m and y^m, respectively. Then $x^m \preceq_p y^m$ exactly if $u_r \leq u'_r$ for all $r = 0, \ldots, k-1$. So for all r the r-th 1 in x^m is not allowed to occur later than the r-th 1 in y^m. This can be interpreted in such a way that x^m can be obtained by "pushing" the 1's in y^m to the left (if x^m and y^m are written as row vectors).

The problem we are going to address is to determine the size of special upsets in this order, namely we are interested in the number

$$N(y^m) = |\{x^m \in \binom{[m]}{k} : x^m \preceq_p y^m\}|$$

preceding a given element $y^m \in \left[{m \atop k}\right]$.
This order plays an important role in the analysis of intersection theorems of Erdös-Ko-Rado type, e. g. [4], [5] or [19]. We shall see now that it can equivalently be defined in terms of domination (or majorization)
Obviously, via domination a partial order is defined on all sequences of length m over the nonnegative integers. On the set of binary sequences of length m this just yields the inverse of the "pushing" order $\preceq_p$ defined above, i. e., $y^m \preceq_p x^m$ since by condition (i) in the definition of majorization (see Introduction) the sequences x^m and y^m must have the same weight and by (ii) every 1 in x^m must precede its counterpart in y^m.
A second equivalent order in terms of domination is obtained by adding a final 1 to each sequence x^m and y^m, i. e., introducing $x_{m+1} = y_{m+1} = 1$ and interpreting the sequences $x^m = (\underbrace{0, \ldots, 0, 1}_{\mu_0}, \underbrace{0, \ldots, 0, 1}_{\mu_1}, \ldots, \underbrace{0, \ldots, 0, 1}_{\mu_k})$ and $y^m = (\underbrace{0, \ldots, 1}_{\lambda_0}, \underbrace{0, \ldots, 0, 1}_{\lambda_1}, \ldots, \underbrace{0, \ldots, 0, 1}_{\lambda_k})$ as two partitions $m+1 = \mu_0 + \mu_1 + \ldots + \mu_k = \lambda_0 + \lambda_1 + \ldots + \lambda_k$ of the integer $m+1$ into $k+1$ positive integers, where each summand λ_j or μ_j is just defined by the number of $\mu_j - 1$, resp. $\lambda_j - 1$, consecutive 0's preceding the i-th 1 in x^m and y^m, respectively. Then $x^m \preceq_p y^m$ exactly if $(\mu_0, \ldots, \mu_k)$ is predecessor of $(\lambda_0, \ldots, \lambda_k)$ in the domination order $\preceq$ (where now the elements are sequences of length k of positive integers, for several properties of this order cf. also [94], pp. 288-289).
In order to attack our enumeration problem, the sequences will be represented as a path in the lattice $\{(s,t) : s, t \text{ integers}\}$ of pairs of integers.
The one-to-one correspondence between the $\{0,1\}$- sequence x^m and a path with m steps is obtained via the bijection $s_i = s_{i-1} + x_i = \sum_{j=1}^{i} x_j$ (with initial value $s_0 = 0$) as in the introduction.
The fact that $x^m \preceq y^m$ in the lattice model translates to the property that the path obtained from x^m never crosses the path obtained from y^m. So, the path corresponding

to y^m is not crossed by any other path obtained from a sequence in the set $\{x^m \preceq y^m : x^m \in \left[{m \atop k}\right]$.
The size of upsets or even general intervals in the pushing order can of course be obtained as a determinant of the form (1.2) derived by Kreweras [67] for the equivalent lattice path problem as mentioned in the introduction (There are also determinantal identities for further lattice path problems, e. g., for non – touching paths as studied in [47] or [10]).
By our previous considerations, for periodic sequences one might obtain further results. When $x^m = (\underbrace{0,\ldots,1}_{s}, (\underbrace{0,\ldots,1}_{\mu}, \underbrace{0,\ldots,1}_{c-\mu})^m)$ with period length c and two 1's in positions μ and $c-\mu$ in each segment of the periodic part, the number of predecessors of x^m in this pushing order is the m–th coefficient of $g^{(s,c,\mu)}(x)$ and the m–th coefficient of $h^{(s,c,\mu)}(x)$ gives the number of predecessors of $x^m = (\underbrace{0,\ldots,1}_{s}, (\underbrace{0,\ldots,1}_{\mu}, \underbrace{0,\ldots,1}_{c-\mu})^{m-1}, \underbrace{0,\ldots,1}_{\mu})$.

III.2 LATTICE PATHS NOT TOUCHING THE DIAGONAL $cx = dy$

As pointed out before, as an example to illustrate his probabilistic approach Gessel in [45] analyzed half–integer slopes for odd c and $d = 2$ hereby counting paths starting in the origin and not touching the line $y = r + \frac{c}{2}x$ before (n, u_n). This line determines a boundary, which is given as in (2.1) by the parameters $s = r + \frac{1}{2}, \mu = \frac{c-1}{2}$ if r is a half-integer and $s = r, \mu = \frac{c+1}{2}$ if r is an integer. The number of paths first touching the line $y = r + \frac{c}{2}x$ in $(2n, u_{2n})$ then obviously is the n-th coeffcient of $g^{(s,c,\mu)}(x)$.
Observe that the original approach only works for $s > 0$, since for $s = 0$ the system of equations $g(t^2) + pq^\mu h(t^2) = 1, g(t^2) + \overline{pq}^\mu h(t^2) = 1$ does not yield a solution.
Several authors studied the number of paths starting in the origin and hereafter touching the line $cx = dy$ for the first time in (nd, nc) (the only intersections of the line with the integer lattice when c and d are coprime). In [73] on pp. 12 – 14 a recursive approach due to Bizley [20] is described. Namely, denoting by f_n the number of such paths to (nd, nc) it is

$$(c+d)f_1 = \binom{c+d}{d}, 2(c+d)f_2 = \binom{2(c+d)}{2d} - \binom{c+d}{d}f_1,$$

$$3(c+d)f_3 = \binom{3(c+d)}{3d} - \binom{2(c+d)}{2d}f_1 - \binom{c+d}{d}f_2, \ldots.$$

As an example, let us consider $c = 3$, and $d = 2$, where this recursion yields the numbers $f_1 = 2, f_2 = 19, f_3 = 293, \ldots$. These are just the coefficients in $h^{(2,3,1)}(x)$ studied in the previous section and this holds in wider generality.
Let us consider $d = 2$. Assume that the first step from the origin is to the right (by reversing the paths, i. e. mapping the path $(0,0), \ldots (nd, nc)$ to $(nd, nc), \ldots (0,0)$ the analysis for a first step upwards is analogous). Then, after this first step, the boundary is given by the parameters $s = \frac{c+1}{2}$ and $\mu = \frac{c-1}{2}$ where contrasting to the original model now $s = u_1$ (and not $s = u_0$). This has the effect that the generating function for the paths to (n, u_n) with even n now is $h^{(\frac{c+1}{2}, c, \frac{c-1}{2})}$. By our previous considerations hence
Theorem 3.1: The number of paths from the origin first touching the line $cx = 2y$ in $(2n, cn)$ and not crossing or touching this line before is the coefficient of $t^{2(n-1)}$ in

$$\sum_{n=1}^{\infty}\frac{1}{(c+2)n-1}\binom{(c+2)n-1}{2n}\cdot t^{2(n-1)}+\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{(c+2)n+\frac{c+1}{2}}\binom{(c+2)n+\frac{c+1}{2}}{2n+1}\cdot t^{2n}\right)^2.$$

III.3 RANDOM WALKS ON THE LINE WITH STEP SET $\{c,-d\}$

Consider a random walk on the real line of a particle starting in the origin and then moving successively either c steps to the right or d steps to the left, where d and c are coprime positive integers. If $d=c=1$ this random walk, of course, is folklore, cf. [39]. For $d=1$ and $c\geq 2$ an equivalent birth–and–death process had been introduced in [51]. The question whether the expected recurrence time exists for $p=\frac{d}{c}$ can be analyzed combinatorially, e. g. with the bounds from Dvoretzky and Motzkin [35] on the size of so called α–heads. An easy analysis for $d=2$ now is also possible with Theorem 3.1 by constructing a bijection to paths studied in the previous subsection III.2 and then applying Stirling's formula to the binomial coefficients in $g^{(\frac{c+1}{2},c,\frac{c-1}{2})}$ and $h^{(\frac{c+1}{2},c,\frac{c-1}{2})}$.

In order to study the recurrence behaviour one is interested in the proportion of positive paths within the set of all paths. A path as concatenation of single steps from $\{c,-d\}$ obviously can be represented as $(1,x_1),(2,x_1+x_2),\ldots,(m,\sum_{i=1}^{m}x_m)$ with $x_i\in\{c,-d\}$. Such a path is positive when all partial sums in the second coordinates are positive.
Note that this is a different path model as the one considered so far, but via the mapping $c\to 1,-d\to 0$ a one–to–one correspondence between these two models via sequences in the pushing order can be obtained. Left pushing of a 1 then corresponds to placing steps of size c earlier and steps of size $-d$ later in the path, which obviously increases the partial sums.
Let us again examine the example $c=3$ and $d=2$ a little closer. Here the positive path P obtained from the sum $3-2+3-2+3-2-2+3-2+3-2-2+3-2+3+\ldots$ is extremal in the sense that replacing any 3 by a -2 would yield a nonpositive path and that the same does not hold for any path obtained from P by pushing 3's to the left. P hence yields the minimal sequence $(1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,\ldots)$ whose initial segments are extremal in the pushing order.
Obviously, the first step must be c and hence w. l. o. g. we can omit the leading 1 yielding a sequence described by the parameters $s=2$ (since the leading segment is $0,1$) and $\mu=1$ (since the remaining part of the sequence is periodic as considered in Subsection III.1).
In our original path model (with steps from $\{0,1\}$), this just yields the path beneath the boundary $(n,u_n)_n$ as in Section III.2, where the paths were counted, which for the first time touch the diagonal $cx=2y$ at u_{2n}. But touching the boundary in this model is equivalent to the property that the corresponding path consisting of steps $\{c,-d\}$ becomes nonpositive. This can, of course, also occur in positions u_{2n+1} with odd index, which would yield the total path sum -1 (whereas in even positions this sum would be 0).

III.4 BALLOT AND BEATTY SEQUENCES

"Arrangements of m things of one sort and n things of another sort, under certain conditions of priority" had been studied by Whitworth [103] (see also [104]) in 1879. Regarding the "things of one sort" as votes for a candidate P_1 and the "things of another sort" as votes for candidate P_0 in a ballot may lead to the ballot problem, studied by Bertrand [18] and D. André [12] in 1887, i. e, the enumeration of ballots, in which there were always more votes for the winner P_1 than for P_0. Clearly, the sequence $(x_1, x_2, \ldots)$ of votes (where $x_j = \nu$ if the j–th vote was for candidate P_ν) then forms a sequence – usually called ballot-sequence – where the fraction of 1's in every initial segment $(x_1, x_2, \ldots, x_i)$ is *greater than* $\frac{1}{2}$.

The ballot problem of Bertrand was generalized to the case that the number of votes for P_1 is always greater than $c-1$ times the number of votes for P_0, where $c \geq 3$ is an integer, by Barbier [14] in the same year 1887. This problem was solved by Aeppli [2] in 1923.

In our notation (cf. the definition in the introduction), sequences where this fraction is *not less than* Θ for any $0 < \Theta < 1$ are denoted as ballot–type sequences. Observe that contrasting to the original ballot problem now "greater than" is replaced by "not less than". In this setting ballot problems had been considered by Erdös and Kaplansky [38] for $\Theta = \frac{1}{2}$, by Motzkin [74] for $\Theta = \frac{1}{c}$ where $c \geq 2$ is an integer, and by Dvoretzky and Motzkin [35] for arbitrary (even irrational) Θ. Indeed these two variations are very closely related.

Most authors studying the ballot problem were rather interested in the probability that the number votes for P_1 are always greater than $\frac{1}{\Theta} - 1$ times the number of votes for P_0 during the counting. D. André derived the ballot numbers which give the the exact number of desired ballots. An excact formula is also obtainable if the paramater $c = \frac{1}{\Theta}$ is an integer, for instance by Sulanke's approach in Section IV or by application of the "cycle lemma" due to Dvoretzky and Motzkin [35] (cf. also [83] or [33]).

For noninteger c an exact formula is not known. However, as seen in Section IV, our results allow to generalize the ballot numbers (in a rather averaging fashion) also for rational numbers $\Theta = \frac{d}{z+d}$ for rational numbers with nominator greater than 1.

In the introduction we saw that ballot-type sequences with $\lceil m\Theta \rceil$ can be obtained from the initial segment y^m of length m of the ininite sequence

$$\underline{y} = (y_1, y_2, \ldots) = (1, \underbrace{0, \ldots, 0, 1}_{\mu_1}, \underbrace{0, \ldots, 0, 1}_{\mu_2}, \ldots)$$

defined by

$$y_1 = 1, \qquad y_i = \begin{cases} 0, & \text{if } \frac{1}{i}(y_1 + \ldots + y_{i-1}) \geq \Theta \\ 1, & \text{else} \end{cases}$$

by pushing 1's in y^m to the left. So the number $N(m, \Theta)$ of ballot-type sequences of length m with $\lceil m\Theta \rceil$ many 1's is the number of predecessors of y^m in the pushing order, which by the considerations in Section III.1 is just the number of $\{0,1\}$-sequences dominating y^m.

Let $\Theta = \frac{d}{c+d}$ be a rational number such that d and c are coprime integers. Then the sequence $(\mu_1, \mu_2, \mu_3, \ldots)$ will be periodic (with $\mu_0 = s = 1$ since by definition the sequence y^m must start with a 1) and the results from Sections II – IV may be applied. For instance, for $\Theta = \frac{2}{c+2}$, the generating functions for $N(m, \frac{2}{c+2})$ is $g^{(1,c,\frac{c-1}{2})}(t^2) + t \cdot h^{(1,c,\frac{c-1}{2})}(t^2)$

For irrational Θ, the sequence $(\mu_1, \mu_2, \mu_3, \ldots)$ is not periodic any more, which makes the analysis of the number $N(m, \Theta)$ much more difficult. In each case, a two-dimensional array can also be constructed in this case allowing a fast recursive procedure to obtain the number $N(m,\Theta)$ (cf. Section IV). Moreover an upper and lower bound on the number $N(m,\Theta)$ can be obtained from results by Dvoretzky and Motzkin [35].

In turn, for irrational Θ the sequence $\underline{y}$ because of its irregularity is more interesting compared to the periodic sequences for rational Θ.
Let Θ and γ be irrational numbers. A *Beatty sequence* is a set of the form $\{\lfloor n\cdot\Theta+\gamma\rfloor : n\in\mathbb{N}\}$, where $\lfloor x\rfloor$ denotes the integer part of the real number x.
Beatty sequences are named after Samuel Beatty, who in [15] demonstrated that for any irrational Θ every positive integer is contained either in $\{\lfloor n\Theta\rfloor : n\in\mathbb{N}\}$ or in $\{\lfloor n\frac{\Theta}{\Theta-1}\rfloor : n\in\mathbb{N}\}$, i. e., these two sets form a partition of the integers, hence

$$\mathbb{N}=\{\lfloor n\Theta\rfloor : n\in\mathbb{N}\}\,\dot\cup\{\lfloor n\frac{\Theta}{\Theta-1}\rfloor : n\in\mathbb{N}\} \tag{8.1}$$

where the union is disjoint. The two sequences in (8.1) are also said to be complementary. Beatty sequences found applications, for instance, in Number Theory [1] and the theory of semigroups [82]. Especially useful they turned out to be in the analysis of Wythoff's Nim game and the characterization of nonperiodic tilings as intensively discussed by Martin Gardner in [43], Ch. 1,2, and 8.
A sequence $\underline{y}$ as defined under (1.1) in the introduction for any irrational $0<\Theta<1$ corresponds in a natural way to the Beatty sequence $\lfloor n\Theta\rfloor : n\in\mathbb{N}=\{a_1,a_2,\ldots\}$, since obviously $a_n=\max\{m\in\mathbb{N}: m\le\Theta n\}$ and hence $a_n+1=\min\{m\in\mathbb{N}: m\ge\Theta n\}$. Thus $\underline{y}$ has 1's exactly in the positions $u_n=a_n+1$ for all n.
Let us take a closer look at the golden ratio, i. e., $\Theta=\frac{\sqrt{5}-1}{2}$. For this parameter, Beatty sequences have been studied, e. g., in [96]. The sequence $\underline{y}$ here looks as follows

$$1011010110110101101011011010110110101101011010110\ldots.$$

Gardner describes several properties of this sequence in Ch. II of [43]. It can be obtained as the limiting sequence of the set of sequences $\underline{r}^{(n)}$, where $\underline{r}^{(1)}=1$ and $\underline{r}^{(n)}$ is obtained from $\underline{r}^{(n-1)}$ by replacing every 1 by 10 and every 0 by 1 (cf. also the papers [31] and [32] by De Bruijn and the reference in them).
The irregularity of Beatty sequences could be of interest in synchronization problems. Observe that the sequence $\underline{y}$ for any irrational Θ is run–length limited, since the difference between two consecutive 1's in $\underline{y}$ is either a number μ_1 or μ_1+1.
As an example, for $\Theta=\frac{\sqrt{5}-1}{2}$, we obtain the sequence $\underline{y}'=(y_1',y_2',y_3',\ldots)$ of the form

$$21221212212212122121221221212212212122121221\ldots.$$

As described, if $y_i'=2$ in the last sequence $\underline{y}'$ then the next occurrence of 2 is either y_{i+2}' or y_{i+1}', so $\underline{y}'$ is a *quasi–Langford sequence* as defined in [36] with even a regularity in the description of the next occurrence. Namely, the sequence $\underline{y}'$ is again obtained as sequence of differences of two consecutive 2's in $\underline{y}'$.
By the prescription $y_i^{(1)}=1$ if the difference between the $(i+1)$–th 1 and the i–th 1 in $\underline{y}$ is μ_1+1 and $y_i^{(1)}=0$ if this difference is μ_1, we obtain a new $\{0,1\}$-sequence $\underline{y}^{(1)}$. This sequence $\underline{y}^{(1)}$ again is such that the differences between two consecutive 1's are either some μ_2 or μ_2+1 and again one gets a $\{0,1\}$-sequence $\underline{y}^{(2)}$ from $\{0,1\}$-sequence $\underline{y}^{(1)}$ in the same

way $\underline{y}^{(1)}$ was obtained from $\underline{y}$. Iterating this procedure yields a collection of sequences $(\underline{y}^{(n)})_{n=1,2,\ldots}$ with parameters $\mu_1, \mu_2, \ldots$ describing the possible differences between two consecutive 1's.
It is well known that Beatty sequences are closely related to the continued fraction expansion of Θ. For instance, $\mu_1, \mu_2, \ldots$ just seems to be this expansion - we couldn't find a reference.

Via $\underline{y}$ by definition the closest approximation to Θ by a rational number greater than Θ with denominator n is obtained. Namely if this fraction is $\frac{a_n}{n}$, then $y_i = a_{i+1} - a_i$.
In the same spirit, one might be interested in a $\{0,1\}$–sequence $\underline{x} = (x_1, x_2, \ldots)$ such that for all n the closest approximation to Θ with denominator n is $\frac{b_n}{n}$ with $x_i = b_{i+1} - b_i$.
The sequences $\underline{y}$ and $\underline{x}$ seem to be closely related. For instance, for $\Theta = \frac{\sqrt{5}-1}{2}$ it is

$$\underline{x} = 0101101101011011010 11\ldots.$$

IV. Two-Dimensional Arrays Generalizing the Ballot Numbers

IV.1 SULANKE's APPROACH

We saw that we have to enumerate lattice paths not touching a given boundary. This immediately yields a fast algorithm to determine these numbers recursively. Since the lattice paths arriving in (n,k) - by definition of the single steps - must pass either $(n,k-1)$ or $(n-1,k)$, the number $\beta(n,k)$ of paths from the origin $(0,0)$ to (n,k) obeys the recursion

$$\beta(n,k) = \beta(n,k-1) + \beta(n-1,k)$$

with initial values

$$\beta(0,0) = 1, \beta(n,u_n) = 0 \text{ for all } n.$$

The initial values just translate the fact that the boundary $(n,u_n), n = 0,1,2,\ldots$ cannot be touched.

For the boundary $(n,(s-1)n+1)$, $n = 0,1,2,\ldots$, the arising arrays have been studied under various aspects in Combinatorial Mathematics, Let us start with the case $s = 2$. Here the array looks as follows.

Example 1:

	0	1	2	3	4	5	6	...
0	1	0						
1	1	1	0					
2	1	2	2	0				
3	1	3	5	5	0			
4	1	4	9	14	14	0		
5	1	5	14	28	42	42	0	
⋮	⋮	⋮	⋮	⋮	⋮	⋮		

The entries here are the ballot numbers

$$\beta(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1} = \frac{n+1-k}{n+1}\binom{n+k}{k}$$

first presented by Bertrand [18], then derived by D. André [12] by application of the reflection principle. Especially, on the diagonal $n = k$ the Catalan numbers $\beta(n,n) = \frac{1}{2n+1}\binom{2n+1}{n}$ appear. The ballot numbers have further been studied, e. g. by MacMahon [70] and by Carlitz and Riordan in [28], cf. also [85].

From Sulanke's paper [97] can be obtained a simple method to derive the formula for the ballot numbers, which extends to further arrays defined by different boundaries. We extend the array beyond the boundary using the same recursion $\beta(n,k) = \beta(n,k-1) + \beta(n-1,k)$ and hence obtain negative entries $\beta(n,k)$ for $k > n$, namely

	0	1	2	3	4	5	6	...
−1	1	−1	−1	−1	−1	−1	−1	...
0	1	0	−1	−2	−3	−4	−5	...
1	1	1	0	−2	−5	−9	−14	...
2	1	2	2	0	−5	−14	−28	... .
3	1	3	5	5	0	−14	−42	...
4	1	4	9	14	14	0	−42	...
5	1	5	14	28	42	42	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮		

Observe that in the first row now $\beta(-1,k) = -1$ for all $k > 0$, from which $\beta(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1}$ is immediate.
More exactly, Sulanke [97] considered two–dimensional arrays β with $\beta(n,0) = 1$ for all n and $\beta(ck-1,k) = 0$ for all $k = 1,2,\ldots$.
For instance, when $c = 2$, the array looks as follows.

Example 2:

	0	1	2	3	4	5	6	...
−1	1	−2	−2	−2	−2	−2	−2	...
0	1	−1	−3	−5	−7	−9	−11	...
1	1	0	−3	−8	−15	−24	−35	...
2	1	1	−2	−10	−25	−49	−84	...
3	1	2	0	−10	−35	−84	−168	...
4	1	3	3	−7	−42	−126	−294	...
5	1	4	7	0	−42	−168	−462	...
6	1	5	12	12	−30	−198	−660	...
7	1	6	18	30	0	−198	−858	...
8	1	7	25	55	55	−143	−1001	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

which is an array defined for $d = 1$ by the recursion

$$\beta(n,k) = \beta(n,k-1) + \beta(n-1,k)$$

with initial values

$$\beta(n,0) = d \text{ for all } n \geq -1, \beta(-1,k) = -c \text{ for all } k \geq 1.$$

For any d it can be easily verified that

$$\beta(n,k) = d \cdot \binom{n+k}{k} - c \cdot \binom{n+k}{k-1} = \frac{d(n+1) - ck}{n+k+1}\binom{n+k+1}{k}.$$

IV.2 THE APPROACH OF BERLEKAMP AND CARLITZ/ROSSELLE/SCOVILLE

Berlekamp at the Waterloo Combinatorics Conference presented an algorithm for computing numbers of the form $\beta(n,k)$, which seemingly arose in the study of burst–error correcting convolutional codes [16]. This algorithm was thoroughly analyzed by Carlitz, Rosselle and Scoville in [29]. The idea is to consider a two-dimensional array with a recursion like in Pascal's triangle. This array can be obtained from β via $\alpha(n,k) = \beta(n+k,k)$, e. g. for the ballot numbers it looks like

$$
\begin{array}{r|rrrrrrrrrl}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\
\hline
-1 & 1 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & \dots \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
2 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\
3 & 1 & 2 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & \dots \\
4 & 1 & 3 & 2 & -2 & -3 & -1 & 0 & 0 & 0 & \dots \\
5 & 1 & 4 & 5 & 0 & -5 & -4 & -1 & 0 & 0 & \dots \\
6 & 1 & 5 & 9 & 5 & -5 & -9 & -5 & -1 & 0 & \dots \\
7 & 1 & 6 & 14 & 14 & 0 & -14 & -14 & -6 & -1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & &
\end{array}
.
$$

The recursion is hence

$$\alpha(n,k) = \alpha(n-1,k) + \alpha(n-1,k-1).$$

Actually, in [29] was considered the part of the array α consisting of positive entries, which are described by the conditions $\alpha(n,0) = 1$ for all n and $\alpha(\sigma(k)+1, k+1) = \alpha(\sigma(k), k)$. (Indeed, the array $d(k,j)$ in [29] was presented in a slightly different form. With n taking the role of j and by placing the elements of the k-th chain in the k-th column of our array α, the two arrays d and α are equivalent). It should be mentioned that the arrays, α and β are special cases of a more general model discussed by Sulanke in [97].
In order to describe the boundary $\sigma(k)$ from [29] in terms of our boundary $(u_m)_m$ we need some preliminaries. Let

$$\underline{u} = (u_0, u_1, u_2, \dots)$$

be the vector representing the boundary $(m, u_m)_{m=0,1,\dots}$ which is not allowed to be crossed or touched by a path in a lattice and let

$$\underline{\mu} = (\mu_1, \mu_2, \mu_3, \dots)$$

be the sequence of differences $\mu_i = u_i - u_{i-1}$. Let us denote

$$\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \dots) \tag{4.2}$$

where λ_i counts the frequency of the number i in $\underline{u}$ and let

$$\underline{v} = (v_0, v_1, v_2 \dots)$$

with $v_i = v_0 + \sum_{j=1}^{i} \lambda_j$.
By interchanging the roles of n and k (mapping $(n,k) \to (k,n)$), the pairs $(\underline{u}, \underline{\mu})$ and $(\underline{v}, \underline{\lambda})$ are somehow dual to each other. For instance, in Example 2 the subarray of positive values $\beta(n,k)$ is below the boundary $\underline{u} = (1,1,2,2,3,3,\dots)$ which is obtained from the condition that the path should never touch the line $2x = y$. On the other hand, $\underline{\lambda} = (2,2,2,\dots)$ determines the boundary for a path never touching the line $x = 2y + v_0$ just by reversing (going backwards) the path from $(0,0)$ to $(k, v_k = \lambda_k + \lambda_{k-1} + \dots \lambda_1 + v_0)$. In the analogous way, $\underline{u}$ and $\underline{\lambda}$ are related. Hence:

Proposition 4.1: The number of paths from the origin $(0,0)$ to (n,k), where $n = v_k = v_0 + \lambda_1 + \lambda_2 + \ldots \lambda_k$ never touching or crossing the boundary $(0,u_0), (1,u_1), \ldots$ is the same as the number of paths from the origin to the point (k,n) which never touch or cross the boundary $(0,v_0), (1, v_0+\lambda_k), \ldots, (k, v_0 + \lambda_k + \lambda_{k-1} + \ldots + \lambda_1)$.
With the above discussion, it can now be seen that

$$\sigma(k) = v_k + k - 1$$

Observe that we extend the array α by introducing the row $\alpha(-1,k)$. The reason is that in this row the numbers δ_k from [29] are contained. These numbers are defined recursively via

$$\delta_k = -\sum_{r=1}^{k} \binom{\sigma(k)}{r} \delta_{k-r} \tag{4.1}$$

with initial value $\delta_0 = 1$.
Reading out the numbers δ_k as entries $\alpha(-1,k)$ is a second method to derive the defining recursion. In [29] a different approach was chosen. Also it was derived that

$$\alpha(n,k) = \sum_{r=0}^{k} \binom{n+1}{r} \delta_{k-r}.$$

Of course, the array α can be defined for any boundary $\sigma(k), k = 1,2,\ldots$ or equivalently by the differences $B_k = \sigma(k) - \sigma(k-1)$ (this notation was used in [29]). In the special case that the difference B_k takes the constant value c, the entries $\alpha(n,k)$ were shown in [29] to be

$$\alpha(n,k) = \binom{n}{k} - c \cdot \binom{n}{k-1},$$

which can be regarded as a generalization of the ballot numbers, cf. also [52].

IV.3 GENERALIZATION OF THE BALLOT NUMBERS

When $d \geq 2$, the model studied in [97] is no longer valid, since the arrays contain rows with all entries different from 0. Observe that in each case the entries $\beta(ck-1, dk) = 0$, when d and c are coprime. However, the results in the previous section now allow us to derive similar identities for the case $d = 2$.

Theorem 4.1: Let $\gamma^{(1)}(n,k)$ denote the number of paths from the origin to (n,k) not touching or crossing the boundary $(m, u_m^{(1)})_m$ determined as defined above by $\underline{\lambda}^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots)$ and let $\gamma^{(2)}(n,k)$ denote the number of such paths where the boundary $(m, u_m^{(2)})_m$ is determined by $\underline{\lambda}^{(2)} = (\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots)$. If $\underline{\lambda}^{(1)} = (\lambda, c-\lambda, \lambda, c-\lambda, \ldots)$ and $\underline{\lambda}^{(2)} = (c-\lambda, \lambda, c-\lambda, \lambda, \ldots)$ are periodic with period length 2, then for all k and $n > \max\{\sum_{j=1}^{k} \lambda_j^{(1)}, \sum_{j=1}^{k} \lambda_j^{(2)}\}$ it is

$$\gamma^{(1)}(n,k) + \gamma^{(2)}(n,k) = 2 \cdot \binom{n+k}{k} - c \cdot \binom{n+k}{k-1}.$$

Proof: In order to prove the theorem we shall compare the array γ defined by $\gamma(n,k) = \gamma^{(1)}(n,k) + \gamma^{(2)}(n,k)$ with the array β where $\beta(n,k) = 2 \cdot \binom{n+k}{k} - c \cdot \binom{n+k}{k-1}$ and show that

$\gamma(n,k) = \beta(n,k)$ for all $n \geq \max\{\sum_{j=1}^k \lambda_j^{(1)}, \sum_{j=1}^k \lambda_j^{(2)}\}$. W. l. o. g. let $\sum_{j=1}^k \lambda_j^{(1)} \geq \sum_{j=1}^k \lambda_j^{(2)}$. Then we are done if we can show that $\gamma(\lambda_1 + \ldots + \lambda_k + 1, k) = \beta(\lambda_1 + \ldots + \lambda_k + 1, k)$ for all k, since both arrays from then on follow the same recursion. Namely, $\gamma(n,k) = \gamma(n,k-1) + \gamma(n-1,k)$, because $\gamma^{(\nu)}(n,k) = \gamma^{(\nu)}(n,k-1) + \gamma^{(\nu)}(n-1,k)$ for $\nu = 1,2$ and $\beta(n,k) = \beta(n,k-1) + \beta(n-1,k)$ was seen to hold even beyond the boundary.

So let us proceed by induction in k. The induction beginning for $k = 1$ and $k = 2$ is easily verified. Assume that for all $k = 1, 2, \ldots, 2K-2$ it is $\gamma(n,k) = \beta(n,k)$ whenever n is big enough as specified in the theorem.

Now observe that since the period length in $\underline{\lambda}^{(1)}$ and $\underline{\lambda}^{(2)}$ is 2, it is

$$\sum_{j=1}^{2K} \lambda_j^{(1)} = \sum_{j=1}^{2K} \lambda_j^{(2)} = cK.$$

This means that for $\nu = 1,2$ by the Proposition 4.1 $\gamma^{(\nu)}(cK+1, 2K)$ is the number of paths from the origin to $(cK+1, 2K)$ never touching the boundary $(0,1), (1, \lambda_{2K}^{(\nu)}+1), (2, \lambda_{2K}^{(\nu)} + \lambda_{2K-1}^{(\nu)} + 1), \ldots, (2K, \lambda_{2K}^{(\nu)} + \lambda_{2K-1}^{(\nu)} + \ldots + \lambda_1^{(\nu)} + 1)$.

These boundaries now are periodic with period length 2 as we studied before. The parameters as in (2.1) are $s = 1$, c and λ for $\nu = 1$ (or $c - \lambda$ for $\nu = 2$, respectively). The generating functions for the numbers of such paths are $g^{(s,c,\lambda)}(t^2)$ and $g^{(s,c,c-\lambda)}(t^2)$ as studied above and by Proposition 2.2

$$\gamma(cK+1, 2K) = \gamma^{(1)}(cK+1, 2K) + \gamma^{(2)}(cK+1, 2K) = \frac{2}{(c+2)K+1}\binom{(c+2)K+1}{2K}$$

$$= 2 \cdot \binom{(c+2)K}{2K} - c \cdot \binom{(c+2)K}{2K-1}$$

Now observe that also

$$\gamma(cK+1, 2K-1) = \gamma(cK+1, 2K) = \frac{2}{(c+2)K+1}\binom{(c+2)K+1}{2K}$$

because in both arrays $\gamma^{(1)}$ and $\gamma^{(2)}$ all paths from the origin to $(cK+1, 2K)$ must pass through $(cK+1, 2K-1)$. It is also clear that

$$\beta(cK+1, 2K-1) = \beta(cK+1, 2K) = \frac{2}{(c+2)K+1}\binom{(c+2)K+1}{2K}$$

Thus we found that in position $cK+1$ in each of the columns $2K-1$ and $2K$ the two arrays γ and β coincide. Since γ and β obey the same recursion under the boundary $(m, u_m^{(1)})_m$, the theorem is proven. $\square$

Corollary 4.1: Let $\gamma^{(1)}$ and $\gamma^{(2)}$ be defined as in the previous theorem. Arrays $\alpha^{(\nu)}$ for $\nu = 1,2$ are defined by $\alpha^{(\nu)}(n,k) = \gamma^{(\nu)}(n+k,k)$ for all n,k with $n \geq v_k + k$. The corresponding parameters $\delta^{(1)}(k)$ and $\delta^{(2)}(k)$ as defined under (4.1) fulfill for all $k \geq 1$.

$$\delta^{(1)}(k) + \delta^{(2)}(k) = (-1)^k \cdot (c+2).$$

Proof: Extend the array beyond the boundary by the recursion $\alpha(n,k) = \alpha(n-1,k) + \alpha(n-1,k-1)$ if $n + k < u_n$. As seen in the example for the ballot numbers, the numbers

$\delta^{(1)}(k) = \alpha^{(1)}(-1,k)$ and $\delta^{(2)}(k) = \alpha^{(2)}(-1,k)$ can be found as entries of row No. -1. in the arrays $\alpha^{(\nu)}$. $\square$

Example 3: For $d=2$, $c=3$ and $\mu=1$ the arrays $\alpha^{(1)}$ and $\alpha^{(2)}$ look as follows.

	0	1	2	3	4	...
-1	1	-2	0	7	-40	...
0	1	-1	-2	7	-33	...
1	1	0	-3	5	-26	...
2	1	1	-3	2	-21	...
3	1	2	-2	-1	-19	...
4	1	3	0	-3	-20	...
5	1	4	3	-3	-23	...
6	1	5	7	0	-26	...
7	1	6	12	7	-26	...
8	1	7	18	19	-19	...
9	1	8	25	37	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

	0	1	2	3	4	...
-1	1	-3	5	-12	45	...
0	1	-2	2	-7	33	...
1	1	-1	0	-5	26	...
2	1	0	-1	-5	21	...
3	1	1	-1	-6	16	...
4	1	2	0	-7	10	...
5	1	3	2	-7	3	...
6	1	4	5	-5	-4	...
7	1	5	9	0	-9	...
8	1	6	14	9	-9	...
9	1	7	20	23	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The sum array $\alpha = \alpha^{(1)} + \alpha^{(2)}$ hence is

	0	1	2	3	4	...
-1	2	-5	-5	-5	-5	...
0	2	-3	0	0	0	...
1	2	-1	-3	0	0	...
2	2	1	-4	-3	0	...
3	2	3	-3	-7	-3	...
4	2	5	0	-10	-10	...
5	2	7	5	-10	-20	...
6	2	9	12	-5	-30	...
7	2	11	21	7	-35	...
8	2	13	32	28	-28	...
9	2	15	45	60	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Via

$$\eta(n,k) = -\alpha(n+k, n+1)$$

we obtain the array

	0	1	2	3	4	5	6	7	8	...
-1	3	-2	-2	-2	-2	-2	-2	-2	-2	...
0	3	1	-1	-3	-5	-7	-9	-11	-13	...
1	3	4	3	0	-5	-12	-21	-32	-45	...
2	3	7	10	10	5	-7	-28	-60	-105	...
3	3	10	20	30	35	28	0	-60	-165	...
4	3	13	33	63	98	126	126	66	-99	...
5	3	16	49	112	210	336	462	528	429	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

with $\eta(0,k) = 3$, $\eta(n,-1) = -2$, and $\eta(n,k) = \eta(n,k-1) + \eta(n-1,k)$.
Proposition 4.2: The positive entries $\eta(n,k) > 0$ are the sum

$$\eta(n,k) = \eta^{(1)}(n,k) + \eta^{(2)}(n,k) + \eta^{(3)}(n,k)$$

where $\eta^{(\nu)}(n,k)$ enumerates the number of paths from the origin to (n,k) not touching or crossing the boundaries $(m, u_m^{(\nu)})_{m=0,1,\ldots}$ with sequences $u_m^{(\nu)}$ being periodic of period length 2 defined for $\nu = 1,2,3$ by

$$u_{2i}^{(1)} = 1+3i, u_{2i+1}^{(1)} = 2+3i, u_{2i}^{(2)} = 1+3i, u_{2i+1}^{(2)} = 3+3i, u_{2i}^{(3)} = 2+3i, u_{2i+1}^{(3)} = 3+3i.$$

Proof: Observe that the boundaries via $\underline{u}$ arise for the parameter $c = 3$ and the choices $(s = 1, \mu = 1)$ for $\nu = 1$, $(s = 1, \mu = 2)$ for $\nu = 2$, and $(s = 2, \mu = 1)$ for $\nu = 3$, respectively, which we studied intensively in Subsection II.3.
The proposition is easily verified, when for all k some n is found where $\eta(n,k) = \eta^{(1)}(n,k) + \eta^{(2)}(n,k) + \eta^{(3)}(n,k)$. In order to do so, observe that application of Corollary 2.1 yields

$$\eta(2j, 3j+1) = \eta^{(3)}(2j, 3j+1) = \frac{1}{5j+2}\binom{5j+2}{2j+1}$$

the j-th coefficient in $g^{(2,3,1)}(x)$ and

$$\eta(2j-1, 3j-1) = \eta^{(2)}(2j-1, 3j-1) + \eta^{(3)}(2j-1, 3j-1) = \frac{1}{5j-1}\binom{5j-1}{2j}$$

the sum of the j–th coefficients in $h^{(1,3,2)}$ and $h^{(2,3,1)}$.
Further, for all j it must be $\eta(2j-1, 3j) = 0$, since for all $\nu = 1,2,3$ it is $\eta^{(\nu)}(2j, 3j-1) = \eta^{(\nu)}(2j, 3j)$ (all paths to $(2j, 3j)$ must pass through $(2j, 3j-1)$). □

Unfortunately, this is the only array with $d > 2$ for which we could prove an identity similar as in Theorem 4.1 . We conjecture that such identities hold for every choice of d and c (see the open problem below). Actually, the analysis here was possible since the sequences $\underline{u}^{(\nu)}$ are periodic with period length 2 and this case was considered. The parameter d is the period length of the corresponding sequences $\underline{\lambda}^{(\nu)}$ in (4.2), which for $(d=3, c=2)$ are $\underline{\lambda}^{(1)} = (1,1,0,1,1,0,\ldots)$, $\underline{\lambda}^{(2)} = (1,0,1,1,0,1,\ldots)$, $\underline{\lambda}^{(3)} = (0,1,1,0,1,1,\ldots)$,

IV.4 OPEN PROBLEMS

1) Computer observations strongly suggest that the generalization of the ballot numbers holds for all positive integers d. More exactly, let $\underline{\lambda}^{(\nu)}$, $\nu = 1,\ldots,d$ be periodic sequences of period length d, such that the initial segment of length d in $\underline{\lambda}^{(\nu)}$ is a cyclic shift of order $\nu-1$ of the initial segment of $\underline{\lambda}^{(1)}$, i. e.

$$\underline{\lambda}^{(1)} = (\lambda_1, \lambda_2, \ldots \lambda_{d-1}, \lambda_d, \lambda_1, \lambda_2, \ldots, \lambda_{d-1}, \lambda_d, \lambda_1, \ldots),$$

$$\underline{\lambda}^{(2)} = (\lambda_2, \lambda_3, \ldots \lambda_d, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_d, \lambda_1, \lambda_2, \ldots), \ldots$$

$$\underline{\lambda}^{(d)} = (\lambda_d, \lambda_1, \ldots \lambda_{d-2}, \lambda_{d-1}, \lambda_d, \lambda_1, \ldots, \lambda_{d-2}, \lambda_{d-1}, \lambda_d, \lambda_1, \ldots),$$

Further, let the sequences $\underline{\lambda}^{(\nu)}$ describe the boundaries $\underline{u}^{(\nu)}$, $\nu = 1,\ldots,d$ as in (4.2), i .e., the points $(n, u_n)_{n=0,1,\ldots}$ are not allowed to be crossed or touched by paths enumerated in the arrays $\gamma^{(\nu)}(n,k)$, $\nu = 1,\ldots,d$

Conjecture 4.1: Whenever $n > \lambda_1^{(\nu)} + \ldots + \lambda_k^{(\nu)}$ for all $\nu = 1,\ldots,d$

$$\gamma^{(1)}(n,k) + \gamma^{(2)}(n,k) + \ldots + \gamma^{(d)}(n,k) = \beta(n,k)$$

where

$$\beta(n,0) = d, \qquad \beta(-1,k) = \lambda_1 + \ldots + \lambda_d, \qquad \beta(n,k) = \beta(n-1,k) + \beta(n,k-1).$$

We tried several approaches to solve this problem. Observation 1 from Section II.1 would follow from the conjecture. Vice versa, if Observation 1 would hold, the sum array $\gamma^{(1)} + \gamma^{(2)} + \ldots + \gamma^{(d)}$ would coincide with the array β on the columns $(n, r(\lambda_1 + \ldots + \lambda_d))_{n > r(\lambda_1 + \ldots + \lambda_d)}$ for all positive integers r.

A second approach was discussed in [99]. One could consider the arrays $\alpha^{(\nu)}$ defined by $\alpha^{(\nu)}(n,k) = \gamma^{(\nu)}(n+k,k)$. As mentioned before in Subsection IV.2, the entries $\alpha^{(\nu)}(-1,k)$ are just the numbers $\delta^{(\nu)}(k)$ discussed by Carlitz, Rosselle, and Scoville [29] arising in the analysis of the subarrays of positive entries in the $\gamma^{(\nu)}$'s. So, it would suffice to derive that for all $k \geq 1$

$$\delta^{(1)}(k) + \ldots + \delta^{(d)}(k) = (-1)^k(\lambda_1 + \ldots + \lambda_d)$$

The proof then would follow the lines of the derivation of the ballot numbers in [29] as pointed out in [99].

2) The array α we presented for the derivation of the generalized Catalan numbers $C_n^{(3)}$ corresponds on its positive values with the number of paths not touching or crossing the diagonal $y = 2x$. Analogously, one can consider the array with entries counting the number of paths not touching or crossing the diagonal $x = 2y$. Extending this array by allowing negative entries above the boundary we obtain

$$
\begin{array}{r|rrrrrrrc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\
\hline
-1 & 1 & -1 & 1 & -2 & 5 & -14 & 42 & \dots \\
0 & 1 & 0 & 0 & -1 & 3 & -9 & 28 & \dots \\
1 & 1 & 1 & 0 & -1 & 2 & -6 & 19 & \dots \\
2 & 1 & 2 & 1 & -1 & 1 & -4 & 13 & \dots \\
3 & 1 & 3 & 3 & 0 & 0 & -3 & 9 & \dots \\
4 & 1 & 4 & 6 & 3 & 0 & -3 & 6 & \dots \\
5 & 1 & 5 & 10 & 9 & 3 & -3 & 3 & \dots \\
6 & 1 & 6 & 15 & 19 & 12 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}.
$$

Observe that the first values

$$\delta(k) = \alpha(-1, k) = (-1)^k \cdot C_{k-1}$$

for $k = 1, \dots, 6$, which might hold for all k.

V. Ballot Sequences and Pascal-Like Triangles in the Enumeration of Trees

V.1 BALLOT SEQUENCES AND REGULAR TREES

It is well–known (e.g. [46]) that the Catalan number C_n counts the binary, regular, rooted trees on n inner vertices (where each inner vertex has exactly 2 successors). More generally, s–ary regular trees on n inner vertices, in which every inner vertex has exactly s successors, are enumerated by the generalized Catalan numbers $C_n^{(s)}$

There is a one–to–one correspondence between s–ary regular trees and ballot – type $\{0,1\}$–sequences $x^{sn} = (x_1, \dots, x_{sn})$ of weight (= number of 1's) $wt(x^{sn}) = n$ fulfilling the condition $wt(x_1, \dots, x_i) \geq \frac{i}{s}$ for all $i = 1, \dots, sn-1$.

This correspondence can be exploited to store regular trees, for instance, as context trees [105] in data compression, by assigning to them as codewords the ballot – type sequence. The codes thus obtained form a prefix code and for the set of codewords representing all binary trees there is equality in Kraft's inequality. We shall discuss this coding procedure now. For further methods of binary tree codings we refer to [71] or [61].

We shall take a closer look at the case $s = 2$. We introduce a code function $c_n : \mathcal{T}_n \longrightarrow \{0,1\}^{2n+1}$ on the set $\mathcal{T}_n$ of all rooted, binary trees on n inner vertices and $n+1$ terminal vertices. by assigning a 1 to every inner vertex and a 0 to every terminal vertex.

There are several possibilities to define such a code. In a breadth – first algorithm first the root is labelled, then to the vertices on the first level (from left to right) a 0 or a 1 is assigned, then we proceed with the second level, and so on.

However, for practical purposes a depth – first approach may be more suitable, i. e., after having labelled a vertex, we proceed with its successors (if this is possible, i. e., the vertex was an inner node). The algorithm, which we shall denote as *Tree Code* proceeds as follows.

Tree Code: Starting with the root, we label a vertex and proceed with its left successor if this is possible. If the left successor is already labelled, we proceed with the right successor. If both successors are labelled or if the vertex is a terminal node, we trace back the path to the root until we find a vertex whose right successor is not yet labelled and label this right successor. The algorithm stops, when we successfully check that the right successor of the root has a label.

So, for $n = 0, 1, 2, 3$ the possible trees are encoded as follows.

Let $\mathcal{C}_n$ denote the set of codewords which can be formed by this prescription. Further, we denote by $\mathcal{C} = \dot{\bigcup}_{n=0}^{\infty} \mathcal{C}_n$ the set of codewords for all possible binary, rooted trees.

Remarks:
1) Observe that from the algorithm "Tree Code" a recursive algorithm to label the vertices is obtained in the following way. Let a tree T on $2n+1$ vertices be obtained from a tree T^* on $2n-1$ vertices via appending two successors to a terminal node of T^* and let $c_{n-1}(T^*) = (x_1, \ldots, x_j = 0, \ldots, x_{2n-1})$ be the codeword for T^*, which has a 0 in position j which corresponds to the terminal node manipulated in the way described above. Then the codeword $c_n(T) = (x'_1, \ldots, x'_{2n+1})$ is obtained from $c(T^*)$ by replacing the 0 in position j by the sequence "100:, i. e.,
$x'_i = x_i$ for $i = 1, \ldots, j-1$, $x'_j = 1, x'_{j+1} = x'_{j+2} = 0$ and $x'_i = x'_{i-2}$ for $i = j+3, \ldots, n$
2) The (of course, well known) bijection between trees and ballot sequences is easily obtained by dropping the last digit (which is a 0) in all codewords in $\mathcal{C}_n$, This way we obtain a set of $\{0,1\}$ – sequences $(x_1, \ldots, x_{2n})$ of length $2n$ containing exactly n many 0's and 1's with the property that for all initial segments $(x_1, \ldots, x_i), i < 2n$ the number of 1's in this subsequence is always at least $\frac{i}{2}$.

It is also easily seen by Remark 1) that the set $\mathcal{C}$ is prefix free and hence the union of ballot sequences over all n (with an additional 0 yielding the above code) form a prefix code. Recall that for a prefix code it must hold Kraft's inequality. Kobayashi, Hoshi, and Morita [65] gave a combinatorial proof that the set $\mathcal{C}$ fulfills Kraft's inequality even with equality, i.e., if we denote by $L(T)$ the length of the codeword assigned to a tree T, then

$$\sum_T 2^{-L(T)} = \sum_{n=0}^{\infty} C_n 2^{-(2n+1)} = 1.$$

We shall derive this identity as a special case of a more general result for s–ary trees. The above coding procedure can be generalized in an obvious way (now replacing a 0 in the codeword $c_{n-1}(T^*)$ by a 1 followed by s zeros to obtain a codeword $c_n(T)$) to yield a procedure for representing s-ary trees by ballot-type sequences. Now there is no longer equality in Kraft's inequality if $s \geq 3$, however, with Kobayashi we could derive that $\sum_T 2^{-L(T)} > \frac{1}{2}$ for all s, see [64]. In order to see this, observe that the generalized Catalan number $C_n^{(s)}$ enumerates the trees on $sn+1$ vertices and that hence there are $C_n^{(s)}$ trees with codeword length $L(T) = sn+1$.
Proposition 5.1: For $s \geq 2$ it is

$$\frac{1}{2} < \sum_{n=0}^{\infty} C_n^{(s)} 2^{-(sn+1)} \leq 1,$$

with equality on the right hand side only for $s = 2$.

Proof: We shall first show the right–hand inequality in the theorem. In order to do so, observe that $\sum_{n=0}^{\infty} C_n^{(s)} 2^{-(sn+1)} = \sum_{n=0}^{\infty} C_n^{(s)} (\frac{1}{2})^n (\frac{1}{2})^{(s-1)n+1}$, which is just the probability that a particle moving in the integer lattice and never touching the boundary $(i, (s-1) \cdot i + 1)$ $i = 1, 2, \ldots$ will eventually stop in the model described in Section 2 for the special parameter choice $p = \frac{1}{2}$. This probability is 1, only if $p \leq \frac{1}{s}$, So Kraft's inequality is fulfilled with equality only if $s = 2$, i. e., if the counting function is determined by the Catalan numbers, and strict inequality must hold on the right-hand side for $s \geq 3$.

In order to show the inequality on the right–hand side, let $G_s(z) = \sum_{n=0}^{\infty} C_n^{(s)} z^n$ denote the generating function for the generalized Catalan numbers. Then obviously,

$$\sum_{n=0}^{\infty} C_n^{(s)} 2^{-(sn+1)} = \frac{1}{2} \cdot G_s(\frac{1}{2^s}).$$

It is well known that G_s fulfills the functional equation $G_s(z) = 1 + zG_s(z)^s$. This just means that $G_s(\frac{1}{2^s})$ is a root of

$$x^s - 2^s \cdot x + 2^s = (x-2) \cdot (x^{s-1} + 2 \cdot x^{s-2} + 4 \cdot x^{s-3} + \ldots + 2^{s-2}x - 2^{s-1})$$
$$= 2^s \cdot (y-1) \cdot (y^{s-1} + y^{s-2} + \ldots + y - 1) = 2^s \cdot (y-1) \cdot F_s(y)$$

by setting $x = 2y$ and defining $F_s(y) = y^{s-1} + y^{s-2} + \ldots + y - 1$.
Now, for $s = 2$ obviously 2 is the only root of $x^2 - 4x + 4$, which means that $G_2(\frac{1}{2^2}) = 2$ and hence Kraft's inequality must hold with equality. For $s > 2$ observe that $F_s(y)$ is monotonous on the positive reals with $F_s(1) = s - 2 > 0$ and $F_s(0) = -1 < 0$. Hence $F_s(y)$ can have only one positive root $y_0^{(s)}$, say. Since $\sum_{t=1}^{\infty} (\frac{1}{2})^t = 1$, obviously $y_0^{(s)} > \frac{1}{2}$, and the theorem is proven. □

Remarks:

1) Indeed $y_0^{(s)}$ quickly tends to $\frac{1}{2}$, e. g., for $s = 3$ the golden ratio $y_0^{(3)} = 0.61..$ is attained.

2) In [65] a prefix code representing s-ary trees is presented, for which equality holds in Kraft's inequality for all s.

3) Observe that we enumerated the *ordered*, regular, rooted trees, in which permutations of the successors of an inner vertex may yield different trees. The enumeration problem is much more difficult if these permutations are considered to yield isomorphic trees, i. e., we want to determine the number t_n of unordered trees on n inner vertices. In this case, a closed expression as for the Catalan numbers is not known. The asymptotic behaviour of t_n has been analyzed by Flajolet and Prodinger [41].

4) It might be interesting to find a one–to–one correspondence between ballot–type sequences in which every initial segment x^i has weight $w(x^i) \geq \frac{l}{s}$ and some class of trees for the rationals $\frac{l}{s}$ with l, s coprime integers and $l \geq 2$.

5) The connection between the enumeration of trees and forests to branching processes and random walks is studied, e. g., in the recent paper by Pitman [80].

V.2 PASCAL–LIKE TRIANGLES AND MOTZKIN NUMBERS

We saw that the Catalan numbers count the rooted, regular, binary trees. There is another application in the enumeration of trees, namely the Catalan number C_n also counts all rooted trees (with no restriction on the degree) on $n + 1$ vertices (cf. e.g. [58]).
This fact can be derived by interpreting the entries $b(n, k)$, which are just the ballot numbers (cf. Section 3), of the array obtained by the recursion

$$b(n+1, k) = \sum_{j=-1}^{\infty} b(n, k+j)$$

with initial values $b(0, 0) = 1, b(n, k) = 0$ for $k < 0$ and $k > n$, as number of forests consisting of exactly $k + 1$ trees on $n + 1$ vertices. Recall, that the Catalan numbers here arise as the entries $b(n, 0)$ of the triangle

	-1	0	1	2	3	4	5	6	...
0	0	1	0	0	0	0	0	0	...
1	0	1	1	0	0	0	0	0	...
2	0	2	2	1	0	0	0	0	...
3	0	5	5	3	1	0	0	0	...
4	0	14	14	9	4	1	0	0	...
5	0	42	42	28	14	5	1	0	...
6	0	132	132	90	48	20	6	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

which since $b(n,k) = \beta(n, n-k)$ can be regarded as the "inverted" two – dimensional array β studied in Section IV. This "inversion" has the advantage that now the columns are $(k+1)$-fold convolutions of column No. 0 $(b(n,0))_{n=1,2,\ldots}$.

This interpretation of the ballot numbers will occur as the extremal case $s = \infty$ of a more general result counting the forests consisting of trees in which there is a limitation on the degree, namely, a vertex is allowed to have at most s successors.

The number of trees with n vertices, in which every inner node has at most 2 successors, is the Motzkin number M_n [34] or total information number as in [90].

The Motzkin numbers were first introduced by Motzkin in [74]) in the analysis of an enumeration problem concerning the number of different triangulations of an n–gon, they arise as the enumeration function for another kind of trees [68] and have many further applications see [34], [8] and [94], pp. 238 – 239 .

The sequence $(M_n)_{n=0}^{\infty}$ starts with $1, 1, 2, 4, 9, 21, 51, \ldots$. Here a closed expression as for the Catalan numbers is not known, however in [34] it was also derived that the numbers M_n can be obtained as the entries $b^{(2)}(n,0)$ of the Pascal – like triangle defined by the recursion

$$b^{(2)}(n+1,k) = b^{(2)}(n,k-1) + b^{(2)}(n,k) + b^{(2)}(n,k+1)$$

with initial values $b^{(2)}(0,0) = 1, b^{(2)}(0,k) = 0$ for $k \neq 0$ and $b^{(2)}(n,-1) = 0$ for all n, so they define the triangle

	-1	0	1	2	3	4	5	6	...
0	0	1	0	0	0	0	0	0	...
1	0	1	1	0	0	0	0	0	...
2	0	2	2	1	0	0	0	0	...
3	0	4	5	3	1	0	0	0	... .
4	0	9	12	9	4	1	0	0	...
5	0	21	30	25	14	5	1	0	...
6	0	51	76	69	44	20	6	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

In the same spirit the numbers $M^{(s)}(n) = b^{(s)}(n,0)$ defined by the recursion

$$b^{(s)}(n+1,k) = b^{(s)}(n,k-1) + b^{(s)}(n,k) + b^{(s)}(n,k+1) + \ldots + b^{(s)}(n,k+s-1)$$

(with initial values as above) can be considered. For $s = 3$ this yields the following table

	-1	0	1	2	3	4	5	6	...
0	0	1	0	0	0	0	0	0	...
1	0	1	1	0	0	0	0	0	...
2	0	2	2	1	0	0	0	0	...
3	0	5	5	3	1	0	0	0	...
4	0	13	14	9	4	1	0	0	...
5	0	36	40	28	14	5	1	0	...
6	0	104	118	87	48	20	6	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Carlitz in [25] (cf. also [27]) in the study of a problem concerning lattice path enumeration analyzed the equivalent recursion $\beta^{(s)}(n+1,k) = \sum_{j=0}^{s} \beta^{(s)}(n,k-j), \beta^{(s)}(0,0) = 1, \beta^{(s)}(n,k) = 0$ for $k < 0$ and $k > n$, which just yields $\beta^{(s)}(n,k) = b^{(s)}(n,n-k)$. He could derive the generating function for the rows $(\beta^{(s)}(n,k))_{n=0,1,\dots}$ of the corresponding triangle. Sulanke [97], who studied a more general recurrence, which just yields $\beta^{(s)}(n,k)$ as a special case, gave $\beta^{(s)}(n,n-k) = b^{(s)}(n,k)$ as number of forests consisting of k trees, in which every vertex has at most s successors. We want to derive this identity using a functional equation due to Klarner [60] for the generating function of such trees.

Proposition 5.2 Let $b(n,k)$ denote the number of forests on $n+1$ vertices consisting of exactly $k+1$ trees, in which the degree of each vertex is from a given subset $\mathcal{D}$ of the natural numbers with $1 \in \mathcal{D}$. Then (with $b(0,0) = 1$ and $b(n,-1) = 0$ for all n)

$$b(n+1,k) = \sum_{d \in \mathcal{D}} b(n,k+d-2) \tag{1}$$

Proof: Let

$$T(x) = \sum_{n=0}^{\infty} b(n,0) \cdot x^{n+1}$$

be the generating function for these trees. Then clearly $T^k(x) = \sum_{n=0}^{\infty} b(n,k)x^{n+1}$ is the generating function for the forests consisting of exactly $k+1$ such trees. In [60] it was derived that $T(x)$ obeys the functional equation

$$T(x) = x + x \sum_{d \in \mathcal{D}, d \neq 1} T^{d-1}(x).$$

Then $T^k(x) = T(x) \cdot T^{k-1}(x) = x \cdot T^{k-1}(x) + x \sum_{d \in \mathcal{D}, d \neq 1} T^{k+d-2}(x)$, which just yields the recurrence for the numbers $b(n,k)$.

Now when each vertex has maximum degree $s+1$, then of course recursion (3) arises and hence

Corollary 5.1 ([97]): The numbers $b^{(s)}(n,k)$ count the ordered forests on $n+1$ vertices consisting of exactly $k+1$ trees, in which each vertex has degree at most $s+1$. Especially, $b^{(s)}(n,0)$ is the number of trees on $n+1$ vertices in which every vertex has degree at most $s+1$.

Remarks:

1) Although the proposition is immediate from Klarner's recursion, we couldn't find a reference to this method of proof, which makes use of the fact that the rows $(b(n,k))_{n=0,1,\ldots}$ of the triangle defined by (6) are $(k+1)$ – fold convolutions of row number 0. In a series of papers Hoggat and Bicknell et. al. intensively studied such convolution arrays, e. g. [53], [54]. They are also a special case of the Riordan group studied in [92], where the generating function of the k-th row is $g(x) \cdot f(x)^k$ with $g(x) = 1 + g_1x + g_2x^2 + \ldots$ and $f(x) = x + f_2x^2 + f_3x^3$. To see this, choose $f(x) = T(x)$ and $g(x) = \frac{1}{x}T(x)$.

2) Further two - dimensional arrays in which the Catalan numbers occur as entries have been studied e. g. by Aigner [9] and Shapiro [91].

VI. Generalized Catalan Numbers and Creating Order

VI.1 CREATING ORDER

Ahlswede, Ye, and Zhang [6] introduced the following model for creating order in sequence spaces. We are given a box containing (a fixed number) β balls labelled by letters from an alphabet of size α. In each time unit a person $\mathcal{O}$ – denoted as organizer – takes out one ball of the box which is replaced by a new ball thrown into the box by a second person $\mathcal{I}$. The aim of the organizer is to reduce the space of possible output sequences. As a measure for the efficiency of the ordering process the number of possible output sequences and the entropy of the output space have been studied (cf. also [7], [57], [101]). A related model in which the output sequence is regarded as a message from person $\mathcal{I}$ to a decoder $\mathcal{D}$ was considered by several authors studying the permuting channel, e. g. [3], [63], and [79].

Here we are going to discuss a multi–user version of the original model for creating order. Now there are $s \geq 2$ persons $\mathcal{I}_1, \dots, \mathcal{I}_s$, say, throwing balls labelled either 0 or 1 into the box. We shall present the model using a slightly different terminology. In each time unit s sources $\mathcal{I}_1, \dots, \mathcal{I}_s$ produce one bit each. These s bits arrive at an organizer who in the same time unit has to choose l bits for output, where $1 \leq l \leq s$. These output bits may be among the s arriving bits or may be taken from some memory device (the box), in which the bits not used so far may be stored. The organizer follows a simple strategy: if it is possible the output must be a 1. So if one of the arriving bits is a 1, the organizer will put out a 1 for sure. The bits not used for output he may store in the memory device. If all the s sources produce a 0, then the organizer will take a look at the memory. If there is still a 1 contained he will put out a 1, otherwise he must put out a 0.

We assume that in the beginning the memory device is empty. At some point it may occur that no further 1 can be stored, since the device is full of 1's (a 0 may be replaced by a 1). In this case there is a maximum size or capacity of M bits which cannot be superceded. Of theoretical interest is also the not very realistic model, in which the memory device can store all incoming (hence infinitely many) bits.

Observe that contrasting to the original model for creating order the size of the memory device (or box) now may vary in time. A natural question is: how much influence does the maximum size of the memory have on the behaviour of the sequence of bits arranged by the organizer? Of course, in the strategy considered the organizers aim is to produce the all–one sequence and we shall study how well he can manage to achieve this goal. As a new measure for the influence of the memory we consider the expected value of the first occurrence of a 0 in this sequence. We shall denote this expectation (if it exists) as

$$E_0 = \sum_{t=1}^{\infty} t \cdot \text{Prob(first 0 at time t)}.$$

Further, we shall denote the sequences of bits produced by the sources $\mathcal{I}_i$ by

$$\underline{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots),$$

The results below are mainly derived from properties of the sequence

$$\underline{z} = (z_1, \dots z_s, z_{s+1}, \dots, z_{2s}, z_{2s+1}, \dots) = (x_1^{(1)}, \dots, x_1^{(s)}, x_2^{(1)}, \dots, x_2^{(s)}, x_3^{(1)}, \dots)$$

which is obtained by merging the sequences $\underline{x}^{(i)}, i = 1, \ldots, s$ into one sequence $\underline{z}$, where the bits in the positions $\equiv i \bmod s$ are those produced by $\mathcal{I}_i$.
We assume that all s sources are memoryless and independently produce their outputs, where the probability for producing a 1 for each source is the same $P(X = 1) = p$ (and hence the probability for a 0 is $P(X = 0) = 1 - p$). The ordering procedure then defines a random walk, namely, if X_t denotes the random variable for the size of the memory at time unit t, then

$$\text{Prob}(X_{t+1} = m + j - l | X_t = m) = p^j \cdot (1-p)^{s-j}, j = 0, \ldots, s.$$

We shall be able to analyze the case $l = 1$ output bit per time unit Clearly an empty memory at time t here is a necessary condition for the occurrence of the first 0 at time unit $t+1$, so we are interested in the probability that the memory is empty in some time unit t and $X_i \geq 0$ for all $i = 1, \ldots, t-1$.
We denote by $a(m,t)$ the number of sequences $\underline{z}$ produced by the s sources leading to the all-one sequence as output with actual memory size (= number of 1's in the device) m at time $t-1$. So the combinatorial analysis reduces to determine the number $a(0, t-1)$

VI.2 EVERY INCOMING BIT CAN BE STORED

Recall that there are two necessary conditions for the occurence of the first 0 at time unit t: 1) there are no further 1's in the memory device after the t–th bit has been put out by the organizer, 2) up to time t the all-one sequence has been arranged by the organizer. In case that the memory device can store every incoming bit, these conditions can be translated into conditions required from the sequence $\underline{z}$, namely

$$\text{wt}(z_1, \ldots, z_{st}) = l \cdot t, \tag{6.1}$$

$$\text{wt}(z_1, \ldots, z_{si}) \geq l \cdot i \text{ for } i = 1, \ldots, t-1. \tag{6.2}$$

As usual, here the weight $\text{wt}(z^{si})$ of a $\{0,1\}$–vector z^{si} denotes the number of 1's in z^{si}. By condition (6.1) no 1's can be left in the device, since t 1's have arrived at the organizer and t 1's have been used for output. The second condition (6.2) assures that at all time units before it was possible to put out a 1 (by the same argumentation).
For the analysis of the numbers $a(0,t)$ the concept of domination (or majorization) defined in the introduction comes into play.

Proposition 6.1: When in each time unit s bits arrive at the organizer who in the same time unit has to put out a fixed number $l < s$ bits (using the strategy which prefers a 1 towards a 0), then the number $a(0,t)$ of sequences fulfilling (6.1) and (6.2) is

$$a(0,t) = \sum_{(l,\ldots,l) \preceq (i_1,\ldots,i_t)} \binom{s}{i_1} \cdot \binom{s}{i_2} \cdots \binom{s}{i_t}.$$

Proof: For $j = 1, \ldots, t$ the binomial coefficient $\binom{s}{i_j}$ is the number of possible ways in which exactly i_j bits can arrive at the organizer in time unit j. In order to assure that the memory is exhausted at time t, i. e. (i') holds, it must be $i_1 + \ldots + i_t = l \cdot t$ and in order to guarantee (ii') for all $j < t$, it must hold $i_1 + \ldots + i_j \geq l \cdot i$. This just means that the sequence $(i_1, \ldots, i_t)$ dominates the sequence $\underbrace{(l, \ldots, l)}_{t}$. $\square$

We shall see that for $l = 1$ outgoing bit in our model for creating order the expectation E_0 can be determined, since the underlying stochastic process can be reduced to a random walk on the line, where in each unit one can move either 1 step forward or $s-1$ steps back such that finally the same counting function - the generalized Catalan numbers - arises.
If there are $l \geq 2$ ($\gcd(l, s) = 1$) outgoing bits per time unit, the counting functions for the nonnegative paths in the corresponding lattice and the exhaustion of the memory, respectively, are different. Let, e. g., $s = 5$ and $l = 2$, then the numbers $a(0, t)$ of sequences fulfilling (6.1) and (6.2) for $s = 5$ and $l = 2$ start with $a(0, 1) = 10$, $a(0, 2) = 155$, $a(0, 3) = 2335$.
On the other hand, we know from the considerations in the previous sections that for the parameter choice $s = 5$ and $l = 2$, the generating function for the nonnegative paths is $g^{(1,3,1)}(x) = 1 + 2x + 23x^2 + 377x^3 + \ldots$.
For the random walk on the line, usually there are different counting functions for the nonnegative paths corresponding to the walks in which the single steps are from $\{l, -(s-l)\}$ or $\{-l, s-l\}$, respectively. For instance, for $s = 5$ and $l = 3$ the generating function for the walk with steps from $\{3, -2\}$ is $g^{(1,3,2)}(x) = 1 + 3x + 37x^2 + 624x^3 + \ldots$.
In contrast, the counting function for the exhaustion of the memory with the parameters s, l and $s, s-l$ are the same, since $(l, l, \ldots, l) \preceq (i_1, i_2, \ldots, i_t)$ exactly if $(s-l, s-l, \ldots, s-l) \preceq (s - i_t, s - i_{t-1}, \ldots, s - i_1)$.
As mentioned before, for the choice $l = 1$ the number $a(0, t)$ of all sequences of length s fulfilling the conditions (6.1) and (6.2) are well known in Combinatorial Theory, these are just the generalized Catalan numbers.

Proposition 6.2: For $t = 1, 2, \ldots$ the number of sequences $\underline{z}$ of length st, which fulfill (6.1) and (6.2), is the $(t+1)$–th generalized Catalan number $C_{t+1}^{(s)} = \frac{1}{(s-1)(t+1)}\binom{s(t+1)}{t+1}$.

Proof: As a generalization of the ballot theorem, Motzkin derived in [74] that $C_{t+1}^{(s)}$ is the number of $\{0, 1\}$ – sequences $\underline{y} = (y_1, \ldots, y_{s(t+1)})$ consisting of exactly $t+1$ many 1's and $(s-1)(t+1)$ many 0's such that

$$\mathrm{wt}(y_1, \ldots, y_i) \geq \frac{i}{s} \text{ for } i = 1, \ldots, s(t+1). \tag{6.3}$$

(So condition (6.1) must hold for every every initial segment and not only for those segments of size divisible by s.) Now, since (6.3) holds for $t = 1$ each such sequence $\underline{y}$ must begin with a 1 and it must end with $s-1$ consecutive 0's, because otherwise (6.3) would be violated for $i = st + 1$.
Now we add a leading 1 and $s-1$ final 0's to a sequence $\underline{z}$ with the properties (6.1) and (6.2) in order to obtain a sequence $\underline{y}$ fulfilling (6.3). To see this, let $\underline{y} = (y_1, y_2, \ldots, y_{s(t+1)})$ with $y_1 = 1, y_j = z_{j-1}$ for $j = 2, \ldots, st+1$ and $y_j = 0$ for $j = st+2, \ldots, s(t+1)$, where $\underline{z} = (z_1, \ldots, z_{st})$ fulfills (6.1) and (6.2).
Then for each initial segment $(y_1, \ldots, y_i) = (1, z_1, \ldots, z_{i-1})$ where $i - 1 = s \cdot m + k, k < s$ it is $\mathrm{wt}(1, z_1, \ldots, z_{i-1}) \geq m + 1$, since by (6.2) $\mathrm{wt}(z_1, \ldots, z_{sm}) \geq m$. So

$$\frac{1}{i}\mathrm{wt}(y_1, \ldots, y_i) \geq \frac{m+1}{sm+k+1} \geq \frac{1}{s}$$

since $k + 1 \leq s$.
On the other hand, if $\underline{z}$ is a sequence consisting of exactly t many 1's and $(s-1)t$ many 0's, which does not fulfill (6.2), let m be the first number with $\mathrm{wt}(z_1, \ldots, z_{sm}) \leq m - 1$.

Then $\mathrm{wt}(1, z_1, \ldots, z_{sm}) \leq m$ and hence $(y_1, \ldots, y_{s(t+1)})$ defined as above cannot fulfill (6.3) since

$$\frac{1}{sm+1}\mathrm{wt}(y_1, \ldots, y_{sm+1}) = \frac{1}{sm+1}\mathrm{wt}(1, z_1, \ldots, z_{sm}) \leq \frac{m}{sm+1} < \frac{1}{s}.$$

So there is a one–to–one correspondence between sequences $\underline{z}$ with the properties (6.1) and (6.3) and sequences $\underline{y}$ fulfilling (6.3). $\square$

With Proposition 1, the following identity for the generalized Catalan numbers is immediate.

Corollary 6.1:

$$C_{t+1}^{(s)} = \sum_{(1,\ldots,1)\preceq(i_1,\ldots,i_t)} \binom{s}{i_1} \cdot \binom{s}{i_2} \cdots \binom{s}{i_t}.$$

Theorem 6.1: Let there be s identical sources $\mathcal{I}_1$, ..., $\mathcal{I}_s$ producing one bit each per time unit with $\mathrm{Prob}(X = 1) = p$, $\mathrm{Prob}(X = 0) = 1 - p$. If the memory device can store every incoming bit, then the expected value for the occurence of the first 0 in the sequence arranged by an organizer (if he puts out a 1 if possible) is

$$E_0 \begin{cases} = \infty, & p = \frac{1}{s} \\ = \frac{sp}{1-sp}, & p < \frac{1}{s} \end{cases}.$$

Proof: Recall that the sources $\mathcal{I}_1, \ldots, \mathcal{I}_s$ each produce a 1 with probability p. Hence with probability $(1-p)^s$ only 0's arrive at the organizer at time unit $t+1$. In this case he has to put out a 1 if the memory device is empty, which happens by the preceding discussion with probability $a(0,t) \cdot p^t(1-p)^{(s-1)t}$. So the probability that the first 0 is put out at time $t+1$ is

$$a(0,t) \cdot p^t(1-p)^{(s-1)t} \cdot (1-p)^s = \frac{1}{p} \cdot C_{t+1}^{(s)} p^{t+1}(1-p)^{(s-1)(t+1)+1}$$

which yields the probability generating function

$$G(z) = \sum_{t=0}^{\infty} \mathrm{Prob}(\text{first 0 at time } t) \cdot z^t = \frac{1}{p}\sum_{t=1}^{\infty} C_t^{(s)} p^t (1-p)^{(s-1)t+1} \cdot z^{t-1}.$$

Now it is known (see [73], p. 129), that for $p < \frac{1}{s}$ the numbers $q_t = C_t^{(s)} p^t (1-p)^{(s-1)t+1}, t = 0, 1, 2, \ldots$ yield a probability distribution $(q_1, q_2, \ldots)$ on the nonnegative integers with expected value $\frac{p}{1-sp}$. This distribution has probability generating function

$$H(z) = \sum_{t=0}^{\infty} q_t \cdot z^t = \sum_{t=0}^{\infty} C_t^{(s)} p^t (1-p)^{(s-1)t+1} \cdot z^t$$

and with the above remarks, of course

$$H(1) = 1 \text{ and } H'(1) = \frac{p}{1-sp}.$$

Now observe that

$$G(z) = \frac{1}{p} \cdot \frac{1}{z} \cdot [H(z) - (1-p)].$$

Obviously, $G(1) = \frac{1}{p} \cdot [1-(1-p)] = 1$ and hence we have a probability distribution whose expected value is obtained via the derivative

$$G'(z) = \frac{1}{p} \cdot [\frac{1}{z} H'(z) - \frac{1}{z^2}(H(z) - (1-p))]$$

as

$$E_0 = G'(1) = \frac{1}{p} \cdot [H'(1) - (H(1) - (1-p))] = \frac{1}{p} \cdot [\frac{p}{1-sp} - p] = \frac{1}{1-sp} - 1 = \frac{sp}{1-sp}.$$

In [65] it is shown that $\sum_{t=0}^{\infty} C_t^{(s)} \cdot \left[\frac{(s-1)^{s-1}}{s^s}\right]^t = \frac{s}{s-1}$. Hence for $p = \frac{1}{s}, 1-p = \frac{s-1}{s}$

$$G(1) = (s-1) \cdot \sum_{t=1}^{\infty} C_t^{(s)} (\frac{1}{s})^t (\frac{s-1}{s})^{(s-1)t} = (s-1) \cdot (\frac{s}{s-1} - 1) = 1$$

and hence again we have a probability distribution. The expected value here does not exist by application of Stirling's formula. $\square$

Remarks:

1) Obviously, Proposition 1 can be extended to a model, in which the number of incoming and outgoing bits per unit may vary in time.

2) It should be mentioned that the Catalan numbers occur as the enumeration function in a related ordering model, namely in sorting of permutations using a stack, which can be regarded as a memory device, since the elements can be stored in the stack and then will be retrieved according to the rule "first in - last out". The Catalan numbers just count the number of permutations which can be sorted when one stack is allowed, as already shown by Knuth [62]. There has been lot of recent interest in two-stack sortable permutations, e. g., in [102], [106], and [22].

3) Recall that the analysis was carried out via properties of the sequence $\underline{z}$ obtained by merging the input sequences $\underline{x}^{(i)}$, $i = 1, \ldots, s$, into one sequence. Such merging procedures also play a role in interleaving codes applied to correct burst errors, cf. e. g. [21], or in codes with the "identifiable parents property" as in [56].

VI.3 CHEBYCHEV POLYNOMIALS AND CREATING ORDER

The above ordering process with limitations on the size of the memory device can be analyzed with the help of the Chebyshev polynomials $(t_n(x))_{n=1,2,\ldots}$ and the Chebychev polynomials of the second kind $(u_n(x))_{n=1,2,\ldots}$ (cf. e. g. [30], p. 228), where

$$t_n(x) = \frac{n}{2} \cdot \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i}{n-i} \binom{n-i}{i} (2x)^{n-2i} = \det T_n(x)$$

with

$$T_n(x) = \begin{pmatrix} x & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 2x & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 2x & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 2x \end{pmatrix},$$

$$u_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (2x)^{n-2i} = \det U_n(x)$$

with

$$U_n(x) = \begin{pmatrix} 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2x \end{pmatrix}.$$

Proposition 6.3: If $\mathcal{I}_1$ and $\mathcal{I}_2$ are memoryless, symmetrical sources, then the expectation E_0 for the occurrence of the first 0 in the sequence of outputs obtained from the ordering procedure in the previous section when at most M 1's can be stored in the memory device is

$$E_0 = \frac{1}{4} \sum_{i=1}^{M+1} c_i (1 - \frac{\lambda_i^{(M+1)}}{4})^{-2}$$

where $\lambda_i^{(M+1)} = 4 \cdot \sin^2 \frac{i\pi}{2M+1}, i = 1, \dots, M+1$ are the eigenvalues of the matrix $(M+1) \times (M+1)$ matrix

$$A_{M+1} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 3 \end{pmatrix},$$

(with initial matrices $A_1 = (3)$, $A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$) and the numbers $c_i, i = 1, \dots, M+1$ are appropriate constants.

Proof: The changes in the memory size can be expressed by the recursion formulae for the numbers $a(m,t)$ defined in the previous section

$$\begin{pmatrix} a(0,t) \\ \vdots \\ a(M,t) \end{pmatrix} = A_{M+1} \cdot \begin{pmatrix} a(0,t-1) \\ \vdots \\ a(M,t-1) \end{pmatrix}.$$

In [88] it is shown that

$$u_n(x) + u_{n-1}(x) = \det V_n(x),$$

where

$$V_n(x) = \begin{pmatrix} 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2x+1 \end{pmatrix}.$$

Observe that the transition matrix $A_n = V_n(1)$ occurs for the special value $x = 1$. and the eigenvalues of $V_n(1)$ are $4 \cdot \sin^2 \frac{i\pi}{2n+1}, i = 1, \dots, n$ (cf. [88]). □

The Fibonacci numbers F_n occur in the analysis of the expected value E_0 for maximum memory size $M = 1$. Obviously, if there is no memory ($M = 0$), then the number $a(0,t) = 3^t$, where 3 is also the largest eigenvalue of $A_1 = (3)$. In this case $E_0 = \sum_{t=1}^{\infty} t\cdot(\frac{3}{4})^{t-1}\cdot\frac{1}{4} = 4$ is just the expected value of the geometric distribution with parameter $\frac{1}{4}$, since with probability $\frac{1}{4}$ two 0's arrive at t.

Corollary 6.2: For maximum memory size $M = 1$ the numbers $a(0,t)$ and $a(1,t)$ are

$$a(0,2t) = 5^t \cdot F_t,\ a(1,t) = 5^t \cdot F_{t+1},\ a(0,2t+1) = 5^t \cdot (2F_t + F_{t+1}),$$

$$a(1,2t+1) = 5^t \cdot (F_t + 3F_{t+1}).$$

Proof: For the transition matrices $A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ it is $A_2^2 = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, from which $A_2^{2t} = 5^t \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^t = 5^t \cdot \begin{pmatrix} F_t & F_{t+1} \\ F_{t+1} & F_{t+2} \end{pmatrix}$ by a well–known property of the Fibonacci numbers. $\square$

Finally, let us take a closer look at the matrices A_n. Observe that these matrices can be obtained as submatrices of the squares of matrices B_{2n+1}, where

$$B_m = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 2 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

with $B_1 = (0)$ and $B_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

More exactly, letting $B_{2n+1}^2 = (b_{ij})_{i,j=1,\dots 2n+1}$, by an appropriate enumeration of rows and columns

$$B_{2n+1}^2 = \begin{pmatrix} A_n & 0 \\ 0 & C_{n+1} \end{pmatrix}$$

where

$$A_n = (b_{ij})_{i,j} \text{ even}, C_{n+1} = (b_{ij})_{i,j} \text{ odd} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2 & 2 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix},$$

with $C_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $C_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

Analogously, we can consider the matrices

$$\overline{A}_n = \begin{pmatrix} 2 & 1 & \dots & 0 & 0 \\ 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & \dots & 1 & 2 \end{pmatrix}, \overline{B}_n = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \overline{C}_n = \begin{pmatrix} 1 & 1 & \dots & 0 & 0 \\ 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix},$$

where analogously it can be seen (with $\overline{B}_n^2 = (\bar{b}_{ij})_{i,j=1,\ldots,n})$)

$$\overline{B}_{2n+1}^2 = \begin{pmatrix} \overline{A}_n & 0 \\ 0 & \overline{C}_{n+1} \end{pmatrix}. \text{ where } \overline{A}_n = (\bar{b}_{ij})_{i,j} \text{ even and } \overline{C}_{n+1} = (\bar{b}_{ij})_{i,j} \text{ odd}.$$

The matrices B_n and $\overline{B}_n$ are well known as transition matrices for random walks on the line (cf. [59] and [93], pp. 238–240) and the matrices A_n and $\overline{A}_n$ have an application in geometry (cf. [88]). Think of the vertices $P_1, \ldots, P_N$ of a regular N–gon drawn on a unit circle. Then the eigenvalues of $A_{\frac{N-1}{2}}$ (if N is odd) or $\overline{A}_{\frac{N-2}{2}}$ (if N is even) give the squares of the different distances $\overline{P_i, P_j}, i, j = 1, \ldots, N$.

Since we couldn't find the following identities for the characteristic polynomials of the matrices under discussion in literature, we shall present them as our final proposition.

Proposition 6.4: The characteristic polynomials of the matrices we consider are

$$\chi_{\overline{B}_n}(\lambda) = u_n(-\frac{\lambda}{2}) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-i} \binom{n-i}{i} \lambda^{n-2i}, \tag{6.4}$$

$$\chi_{B_n}(\lambda) = 2 \cdot t_n(-\frac{\lambda}{2}), \tag{6.5}$$

$$\chi_{A_n}(\lambda^2) = \frac{(-1)^{n+1}}{\lambda} \chi_{B_{2n+1}}(\lambda),\ \chi_{\overline{A}_n}(\lambda^2) = \frac{(-1)^{n+1}}{\lambda} \chi_{\overline{B}_{2n+1}}(\lambda), \tag{6.6}$$

$$\chi_{C_{n+1}}(\lambda) = -\lambda \chi_{A_n}(\lambda),\ \chi_{\overline{C}_{n+1}}(\lambda) = -\lambda \chi_{\overline{A}_n}(\lambda), \tag{6.7}$$

$$\mid \frac{1}{\lambda} (\chi_{B_{2n+1}}(\lambda))^2 \mid = \mid \chi_{B_{2n+1}^2}(\lambda^2) \mid, \quad \mid \frac{1}{\lambda} (\chi_{\overline{B}_{2n+1}}(\lambda))^2 \mid = \mid \chi_{\overline{B}_{2n+1}^2}(\lambda^2) \mid. \tag{6.8}$$

Proof: By definition, (6.4) follows from the fact that $B_n = U_n(0)$.

Using the following well–known properties of the Chebyshev polynomials (cf. [88] and [86], p. 9)

$$t_n(x) = 2x \cdot t_{n-1}(x) - t_{n-2}(x), u_n(x) = 2x \cdot u_{n-1}(x) - u_{n-2}(x),$$

$$2 \cdot t_n(x) = u_n(x) - u_{n-2}(x),$$

(6.5) follows via

$$\chi_{B_n}(\lambda) = -\lambda \cdot \chi_{\overline{B}_{n-1}}(\lambda) - 2 \cdot \chi_{\overline{B}_{n-2}}(\lambda)$$
$$= -\lambda \cdot u_{n-1}(\frac{\lambda}{2}) - 2 \cdot u_{n-2}(\frac{\lambda}{2}) = u_n(\frac{\lambda}{2}) - \cdot u_{n-2}(\frac{\lambda}{2}).$$

The characteristic polynomials in (6.6) can be derived using the recurrence $\chi_{A_n}(\lambda) = (2-\lambda) \cdot \chi_{A_{n-1}}(\lambda) - \chi_{A_{n-2}}(\lambda)$ which is the same as for $\chi_{\overline{A}_n}$, only the initial values differ: $\chi_{A_1}(\lambda) = 3 - \lambda,\ \chi_{A_2}(\lambda) = \lambda^2 - 5\lambda + 5$, whereas $\chi_{\overline{A}_1}(\lambda) = 2 - \lambda,\ \chi_{\overline{A}_2}(\lambda) = \lambda^2 - 4\lambda + 3$.

(6.7) can be obtained using $\chi_{C_n}(\lambda) = (2-\lambda) \cdot \chi_{T_{n-1}(1)}(\lambda) - 2 \cdot \chi_{T_{n-2}(1)}(\lambda)$ and $\chi_{\overline{C}_n}(\lambda) = (1-\lambda) \cdot \chi_{T_{n-1}(1)}(\lambda) - \chi_{T_{n-2}(1)}(\lambda)$.

Finally (6.8), follows from (6.6) and (6.7). □

Let us conclude with some reflection about the matrices $U_n(x)$ and $V_n(x)$. As pointed out, they describe the transitions for certain elementary random walks of a particle moving

on the set $\{0, 1, \ldots, n\}$ with absorption at 0 and either reflection (for $V_n(x)$) or absorption (for $U_n(x)$) at n, where the particle moves one step forward or one step back, each with probability $\frac{1}{x+2}$, or remains in its position with probability $\frac{x}{x+2}$. More exactly, the largest eigenvalue $\lambda_{U_n(x)}$ or $\lambda_{V_n(x)}$, respectively, determines the asymptotic behaviour of the number of positive paths that start and end in 0 but never return to 0 in between.
Letting n tend to infinity, the limiting process does not have a boundary n at which the particle is reflected or absorbed, any more. The counting functions $a_x(m)$ for this limiting random walk might be of interest. Here $a_x(m)$ denotes the number of positive paths consisting of m steps

It is well known that the Catalan numbers arise as $a_0(m)$, the Motzkin numbers as $a_1(m)$ and the Catalan numbers with even indices as $a_2(m)$. It might be interesting to study the numbers $a_x(m)$ for integers $x \geq 3$.
For instance, is there some regularity in the sequence of their generating functions $C_x(z)$, say? It is $C_0(z) = 1 + zC_0(z)^2$, $C_1(z) = 1 + zC_1(z) + z^2C_1(z)^2$ and $C_2(z)^2 = 1 + 4zC_2(z)^4$.
In order to analyze if the expected recurrence time exists, the asymptotic behaviour must be determined. Since the limiting random walk will return to the origin with probability 1, it must be $a_x(m) \sim (x+2)^m$ – the largest eigenvalues of $U_n(x)$ or $V_n(x)$ tend to $(x+2)$ for $n \to \infty$.
However, for the Catalan and Motzkin numbers, the asymptotics can be determined more exactly, namely $C_k \sim \sqrt{\frac{1}{\pi}}k^{-1/2}2^k$ for the Catalan numbers and $M_k \sim \sqrt{\frac{3}{4\pi}}k^{-3/2}3^k$ (cf. [95]) for the Motzkin numbers. What can be said for $x \geq 3$?

VII. Hessenberg Matrices and Chebychev Polynomials

VII.1 THE DETERMINANT OF A HESSENBERG MATRIX

In this section we shall present an explicit (not recursive) formula for the determinant of a Hessenberg matrix (in normalized form), which we could not find in literature, and point out by recalling several examples its use in various fields of mathematics, as Geometry, Number Theory, Probability Theory, and Combinatorics. Especially, it is related to generating functions of the form $H(z) = \frac{1}{1-G(z)}$.

A *Hessenberg matrix* $H_n = (a_{i,j})_i, j = 1, \ldots n$ is a quadratic matrix with $a_{i,j} = 0$ for $j > i+1$. We present the concept of the Hessenberg matrix in a normalized form , i.e. $a_{i,i+1} = 1$ for $i = 1, \ldots, n$. Usually arbitrary coefficients $a_{i,i+1}$ on the upper side diagonal are allowed (even 0, in which case the normalization does not work). Rearranging the indices such that the entries now are enumerated according to the column and the diagonal yields

$$H_n = \begin{pmatrix} a_0^{(1)} & 1 & 0 & \cdots & & 0 \\ a_0^{(2)} & a_1^{(1)} & 1 & 0 & \cdots & 0 \\ a_0^{(3)} & a_1^{(2)} & a_2^{(1)} & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ a_0^{(n-1)} & a_1^{(n-2)} & a_2^{(n-3)} & \cdots & a_{n-2}^{(1)} & 1 \\ a_0^{(n)} & a_1^{(n-1)} & a_2^{(n-2)} & \cdots & a_{n-2}^{(2)} & a_{n-1}^{(1)} \end{pmatrix}.$$

Further, by H'_n is denoted the matrix obtained from H_n by replacing the 1's on the upper side diagonal with -1's, i.e. $a_{i,i+1} = -1$ for $i = 1, \ldots, n$.

Hessenberg matrices play an important role in numerical mathematics, because their determinants can be calculated very fast by the following recursion (for the normalized case with $d_n = \det(H_n)$)

$$\det(H_n) = d_n = \sum_{t=1}^{n} (-1)^{t-1} a_{n-t}^{(t)} d_{t-1}.$$

Since there also exist fast algorithms, which allow to transform a given quadratic matrix to a Hessenberg matrix, the evaluation of the characteristic polynomial of an arbitrary quadratic matrix can be reduced to the same problem for an equivalent Hessenberg matrix (cf. [42], pp. 251 – 258).

Special determinants of this form are discussed in [75] under the name recurrents and in the theory of continued fractions as continuants.

By analyzing the above recursion or by evaluating the determinant by the last row, the following proposition is immediate.

Proposition 7.1: The determinant of a Hessenberg matrix H_n defined as above is

$$\det(H_n) = \sum_{k=1}^{n} (-1)^k \sum_{t_1+\ldots+t_k=n} a_0^{(t_1)} a_{t_1}^{(t_2)} a_{t_1+t_2}^{(t_3)} \cdots a_{t_1+\ldots+t_{n-1}}^{(t_n)},$$

$$\det(H'_n) = \sum_{k=1}^{n} \sum_{t_1+\ldots+t_k=n} a_0^{(t_1)} a_{t_1}^{(t_2)} a_{t_1+t_2}^{(t_3)} \cdots a_{t_1+\ldots+t_{n-1}}^{(t_n)}.$$

Remark. It can also be verified that $\det(H_n)$ equals the permanent of the matrix H'_n.

VII.2 SOME APPLICATIONS

This determinant is quite interesting, since many formulae, which occur in various fields of mathematics, have this special form, for instance, n-fold summation (Example 1) and n-fold integration (Example 2) of the 1. The first example was already presented in the introduction.

Example 1 (e. g. ,[50]).

$$\sum_{x_0=0}^{a_0}\sum_{x_1=x_0}^{a_1}\cdots\sum_{x_{n-1}=x_{n-2}}^{a_{n-1}} 1 = \det(\binom{a_i+1}{i-j+1})_{i,j=1,\dots,n}$$

is the number of sequences of nonnegative integers $(b_0,\dots,b_{n-1})$ dominated by the sequence $(a_0, a_1-a_0\dots,a_{n-1}-a_{n-2})$. Another application of Hessenberg matrices in the theory of majorization can be found in [23].

Example 2 (cf. [76], p. 82).

$$\int_0^{a_0}\int_{x_1}^{a_1}\int_{x_2}^{a_2}\cdots\int_{x_{n-1}}^{a_{n-1}} dx_n dx_{n-1}\cdots dx_2 dx_1 = \sum_{k=1}^{n}(-1)^k \sum_{t_1+\ldots+t_k=n} \frac{a_0^{t_1}}{t_1!}\frac{a_{t_1}^{t_2}}{t_2!}\frac{a_{t_1+t_2}^{t_3}}{t_3!}\cdots\frac{a_{t_1+\ldots+t_{n-1}}^{t_n}}{t_n!}$$

which is the determinant of a Hessenberg matrix H_n with $a_i^{(j)} = \frac{a_i^j}{j!}$ Observe that for the special choice $a_i = 1$ for all i this is just the volume of an n–dimensional simplex. For arbitrary choice of the $a_i > 0$ the volume of a more general n–dimensional body is obtained.

For the special choice $a_i^{(j)} = b_j$ for all i, i.e. on each diagonal all the elements are the same, we have

$$\det(H_n) = \begin{pmatrix} b_1 & 1 & 0 & \cdots & & 0 \\ b_2 & b_1 & 1 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_1 & 1 \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 \end{pmatrix} = \sum_{k=1}^{n}(-1)^k \sum_{t_1+\ldots+t_k=n} b_{t_1}\cdots b_{t_n}. \quad (7.1)$$

Example 3. As already pointed out above, if we set $a_i = 1$ for all i in Example 2, we obtain the volume $\frac{1}{n!}$ of an n–dimensional simplex, which can be represented as such a determinant with $b_j = \frac{1}{j!}$.

Example 4. By choosing $b_j = \frac{1}{(j+1)!}$ in (7.1) we obtain the following identity for the Bernoulli numbers B_n recursively defined by $B_0 = 1, B_{n+1} = -\frac{1}{n+2}\cdot\sum_{i=0}^{n}\binom{n+2}{i}\cdot B_i$

$$B_n = (-1)^{n-1}(2n)!\cdot\det(H_n).$$

This example is taken from the book [75], where in Chapter 21 many more examples for determinants of Hessenberg matrices of the form (7.1) (under the name recurrents) are presented.

The determinant of a Hessenberg matrix of the form (7.1) also arises in Combinatorial Theory (enumeration of trees [89] and lattice paths [73], theory of partitions) and Probability Theory (renewal theory, random walks) because of its connection to generating functions of the form $H(z) = \frac{1}{1-G(z)}$ (or $H(z) = \frac{G(z)}{1-G(z)}$ which we shall use later on).

Let there be given two sequences $(h_n)_{n=1,2,\ldots}$ and $(g_n)_{n=1,2,\ldots}$, where the elements b_n are obtained from the first sequence by

$$h_n = \sum_{k=1}^{n} \sum_{t_1+\ldots+t_k=n} g_{t_1} \cdots g_{t_n}$$

It is easy to verify the inversion

$$g_n = \sum_{k=1}^{n} (-1)^k \sum_{t_1+\ldots+t_k=n} h_{t_1} \cdots h_{t_n}$$

which is of the form (7.1). Now if

$$G(z) = g_1 z + g_2 z^2 + g_3 z^3 + \ldots$$

is the generating function for the g_i's then the generating function $H(z) = h_1 z + h_2 z^2 + h_3 z^3 + \ldots$ for the coefficients $h_i, i = 1, 2, 3, \ldots$ is obtained from $G(z)$ via

$$H(z) = \sum_{k=1}^{\infty} G(z)^k = \frac{G(z)}{1 - G(z)}.$$

It is clear that $g_n = \det(H_n)$, when H_n is defined in the form (7.1) (with h_i taking the role of b_i).

Such generating functions play an important role in the analysis of random walks, when $G(z)$ is the generating functions for the steps of a random walk and $H(z)$ is the generating function for the paths arising from this set of steps (cf. [44]). Also, in the renewal equation (e. g., [39], p. 272) are involved generating functions $H(z)$ and $G(z)$ related as above.

Example 5. The standard example in this context are the Catalan numbers $C_j = \frac{1}{j+1}\binom{2j}{j}, j = 0, 1, \ldots$.

For instance, if $g_j = C_j$ is the j–th Catalan number, then $h_i = \binom{2i}{i}$.

It can also be shown that if $g_j = C_{j-1}$ is chosen as the $(j-1)$–th Catalan number (hence $g_1 = C_0 = 1$ and $g_2 = C_1 = 1$), then $h_i = C_i$ is the i-th Catalan number, which follows immediately from the well – known functional equation $G(z) = 1 + zG(z)$ for the generating function of the Catalan numbers.

Example 6. For $g_j = \binom{j+l-1}{l}$ where $l \in \{1, 2, 3, \ldots\}$ it is $h_i = \sum_t (\binom{(l+1)i-lt}{t})$.

The last formula arises as the solution of "Simon Newcomb's Problem" in the theory of partitions (e. g. [13]). Observe that for $l = 1$, i. e. $a_j = j$, the even Fibonacci numbers $b_1 = 1, b_2 = 3, b_3 = 8, \ldots$ occur.

Example 7. A special Hessenberg matrix is a tridiagonal matrix $H_n = (a_{ij})_{i,j=1,\ldots n}$ with $a_{i,j} = 0$ for $|i - j| > 1$, whose determinant d_n is usually evaluated by exploiting the three – term recurrence

$$d_n = a_{n,n} \cdot d_{n-1} - a_{n,n-1} \cdot a_{n-1,n} \cdot d_{n-2}.$$

A three – term recurrence occurs, for instance, in the theory of continued fractions and of orthogonal polynomials, which can hence be expressed as determinants, e. g. the continuant corresponding to a continued fraction (cf. [77], p. 11) is a determinant of a matrix H'_n (with -1's on the upper side diagonal). A very useful property of a sequence $(P_n)_n$ of orthogonal polynomials is that their eigenvalues interlace. This extends to Hermitian Hessenberg matrices. For a recent result on interlacing properties in the context of Hessenberg matrices see [24].

The explicit form of the Hessenberg determinant, e. g., allows to calculate directly (without using the above recurrence) the identities from the previous chapter for the Chebyshev polynomials $(t_n(x))_{n=1,2,\dots}$ and the Chebychev polynomials of the second kind $(u_n(x))_{n=1,2,\dots}$ (cf. e. g. [30], p. 228)

$$t_n(x) = \frac{n}{2} \cdot \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i}{n-i} \binom{n-i}{i} (2x)^{n-2i} = \det T_n(x)$$

with

$$T_n(x) = \begin{pmatrix} x & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2x \end{pmatrix},$$

$$u_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} (2x)^{n-2i} = \det U_n(x)$$

with

$$U_n(x) = \begin{pmatrix} 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2x \end{pmatrix}.$$

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