Exact Structures on the Categories of Regular Modules

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Dedicated to Rüdiger Göbel on the occasion of his sixtieth birthday

Let $\mathcal{A}$ be a skeletally small additive category with split idempotents. A pair (i, d) of composable morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ is called *exact*, if i is a kernel of d and d a cokernel of i.

Let $\mathcal{E}$ be a class of exact pairs $X \xrightarrow{i} Y \xrightarrow{d} Z$ which is closed under isomorphisms. For $(i, d) \in \mathcal{E}$, the morphism i is called an *inflation* and d is called a *deflation*. The pair $(\mathcal{A}, \mathcal{E})$ is called an *exact category* if the following axioms are satiesfied:

E1 The identity morphism 1_0 of the zero object is a deflation.

E2 The composition of two deflations is a deflation.

E3 For each $f \in \mathcal{A}(Z', Z)$ and each deflation $d \in \mathcal{A}(Y, Z)$, there is a pullback

$$\begin{array}{ccc} Y' & \xrightarrow{d'} & Z' \\ \downarrow f' & & \downarrow f \\ Y & \xrightarrow{d} & Z \end{array}$$

where d' is a deflation.

E3' For each $f \in \mathcal{A}(X, X')$ and each inflation $i \in \mathcal{A}(X, Y)$, there is a pushout

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow f & & \downarrow f' \\ X' & \xrightarrow{i'} & Y' \end{array}$$

where i' is an inflation.

In [10] Keller shows that this system of axioms is equivalent to the axioms, given by Quillen [18].

If $(\mathcal{A}, \mathcal{E})$ is an exact category, an object P of $\mathcal{A}$ is called *projective*, if the induced map $(P, d): \mathcal{A}(P, Y) \to \mathcal{A}(P, Z)$ is surjective for all deflations $d: Y \to Z$. Injective objects are defined dually.

Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. This means that $\mathcal{I}(X, Y)$ is a subgroup of $\mathcal{A}(X, Y)$, for all $X, Y \in \mathcal{A}$ and a composition fgh is in $\mathcal{I}$, provided $g \in \mathcal{I}$. Note that the composition of morphism is written from the left to the right in this paper. The category $\mathcal{A}/\mathcal{I}$ has the same objects as $\mathcal{A}$ has and morphism spaces $(\mathcal{A}/\mathcal{I})(X,Y) = \mathcal{A}(X,Y)/\mathcal{I}(X,Y)$.

One of the topics studied in [4] is, when an ideal $\mathcal{I}$ induces an exact structure $\mathcal{E}/\mathcal{I}$ on the factor category $\mathcal{A}/\mathcal{I}$. A sufficient condition is shown in [4] and a necessary and sufficient condition is deduced in [5].

It turned out that results in [14] provide natural examples for such ideals, see also [5, (4)]. It is the aim of this paper, to present a direct proof of this result, in a more general situation than considered in [14].

Now let k be some algebraically closed field, and $\mathcal{H}$ be a wild hereditary connected k-category with a tilting object. In [8] Happel shows that in this case, up to derived equivalence, either $\mathcal{H}$ is equivalent to the category H-mod of finite dimensional modules for some connected wild hereditary algebra H, or $\mathcal{H} \cong \operatorname{coh} \mathbb{X}$, where $\mathbb{X}$ is a weighted projective line of wild type in the sense of Geigle and Lenzing [6]. Denote by $\mathcal{H}_w$ the category H-reg of regular H-modules if $\mathcal{H} \cong H$-mod, respectively the category $\operatorname{vec} \mathbb{X}$ of vector bundles, if $\mathcal{H} \cong \operatorname{coh} \mathbb{X}$. Then $\mathcal{H}_w$ is closed under images, extensions and Auslander-Reiten translations, and $\mathcal{H}$ is determined by $\mathcal{H}_w$. All Auslander-Reiten components of $\mathcal{H}_w$ are of type $\mathbb{Z}A_\infty$, and an indecomposable object $X \in \mathcal{H}$ is in $\mathcal{H}_w$ if and only if it is contained in an Auslander-Reiten component of type $\mathbb{Z}A_\infty$. The Auslander-Reiten translation $\tau_{\mathcal{H}}$ defines an equivalence on $\mathcal{H}_w$. Clearly $\mathcal{H}_w$ is an additive category with split idempotents, and the restriction of the functor $\operatorname{Ext}^1_{\mathcal{H}}$ to $\mathcal{H}_w$ defines an exact structure $\mathcal{E}_{\mathrm{Ext}}$ on $\mathcal{H}_w$.

If T is a tilting object in $\mathcal{H}$, the algebra $\operatorname{End}_{\mathcal{H}}(T)$ is called a *quasitilted* algebra of type $\mathcal{H}$, see [9]. A finite dimensional algebra B is called *piecewise hereditary* of type $\mathcal{H}$, if the derived categories of bounded complexes $\mathcal{D}^b(\mathcal{H})$ of $\mathcal{H}$, respectively $\mathcal{D}^b(B)$ of B-mod are equivalent as triangulated categories, see [7].

Let A be a connected wild hereditary algebra. Let A-reg be the category of regular A-modules and denote by τ_A the Auslander-Reiten translation in A-mod. Let M be a nonzero regular A-module and denote by $\mathcal{O}(M) = \operatorname{add}(\tau_A^i M \mid i \in \mathbb{Z})$ the full subcategory of A-reg, defined by the τ_A-orbit of M. Denote by $\mathcal{I}_{\mathcal{O}(M)}$ the ideal in A-reg, consisting of maps f which factorise through $\mathcal{O}(M)$. For U and W regular A-modules, let $\operatorname{Ext}_{\mathcal{O}(M)}(W,U) \subset \operatorname{Ext}^1_A(W,U)$ consist of those short exact sequences $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ such that $(M',g)\colon \operatorname{Hom}_A(M',V) \to \operatorname{Hom}_A(M',W)$ is surjective for all $M' \in \mathcal{O}(M)$. This is equivalent to $(f,M')\colon \operatorname{Hom}_A(V,M') \to \operatorname{Hom}_A(U,M')$ is surjective for all $M' \in \mathcal{O}(M)$, see [1, IV.4.4].

It follows from [2, 4] that $\operatorname{Ext}_{\mathcal{O}(M)}$ not only defines an additive subfunctor of the Ext-functor on A-reg, but it also defines an exact structure $\mathcal{E}_{\mathcal{O}(M)}$ on A-reg. The exact pairs (i,d) correspond to the short exact sequences in $\operatorname{Ext}_{\mathcal{O}(M)}(Z,X)$. The modules in $\mathcal{O}(M)$ are exactly the projective, respectively injective, objects in the exact category $(A\text{-reg}, \mathcal{E}_{\mathcal{O}(M)})$.

The main result of this paper now can be formulated.

Theorem. *Let A be a connected wild hereditary algebra and M a quasi-simple regular A-module, such that the one-point extension $A[M]$ is piecewise hereditary of type $\mathcal{H}$. Then the category $(A\text{-reg}/\mathcal{I}_{\mathcal{O}(M)}, \mathcal{E}_{\mathcal{O}(M)}/\mathcal{I}_{\mathcal{O}(M)})$ is an exact category and it is equivalent to $(\mathcal{H}_w, \mathcal{E}_{\mathrm{Ext}})$, as exact category.*

It is shown in [10] that for an exact category $(\mathcal{A}, \mathcal{E})$ there exists an equivalence $F\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ a full extension closed subcategory of some abelian category $\mathcal{C}$, such that an exact pair (i, d) is in $\mathcal{E}$ if and only if $0 \to F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(d)} F(Z) \to 0$ is a short exact sequence in $\mathcal{B}$. Normally this abelian category $\mathcal{C}$ is constructed as a functor category. In the situation of the theorem, the abelian category $\mathcal{H}$ can be choosen for the exact pair $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(M)}, \mathcal{E}_{\mathcal{O}(M)}/\mathcal{I}_{\mathcal{O}(M)})$.

Some of the proofs in this paper are similar to those, given in [14]. For unexplained representation-theoretic terminology we refer to [20] or [1], for details on hereditary categories with tilting objects to [9].

1 Exact categories.

1.1 Let $(\mathcal{A}, \mathcal{E})$ be an exact category and $\mathcal{I}$ be an ideal in $\mathcal{A}$ (called a relation on $\mathcal{A}$ in [4, 5]).

Denote by $\mathcal{E}/\mathcal{I}$ the closure of the class $\{(\overline{i}, \overline{d}) \mid (i, d) \in \mathcal{E}\}$ under isomorphisms in $\mathcal{A}/\mathcal{I}$. Normally this class will not define an exact structure on $\mathcal{A}/\mathcal{I}$. In [5, 1.1] there are given necessary and sufficient conditions for $\mathcal{I}$, when $(\mathcal{A}/\mathcal{I}, \mathcal{E}/\mathcal{I})$ is an exact pair.

These conditions become especially handy in the following special case [5, 1.5]: let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$, closed under direct sums and direct summands and consisting of projective-injective objects only. Let $\mathcal{I}_{\mathcal{X}}$ be the ideal in $\mathcal{A}$, consisting of those morphisms $f\colon U \to V$, which factorise through $\mathcal{X}$. Then $(\mathcal{A}/\mathcal{I}_{\mathcal{X}}, \mathcal{E}/\mathcal{I}_{\mathcal{X}})$ is an exact category, if and only if $\mathcal{I}_{\mathcal{X}}$ satisfies the following two properties:

(A) If $d\colon Y \to Z$ is a deflation and $f\colon Z \to M$ a morphism with $df \in \mathcal{I}_{\mathcal{X}}$, then $f \in \mathcal{I}_{\mathcal{X}}$.

(B) If $i\colon X \to Y$ is an inflation and $g\colon M \to X$ a morphism with $gi \in \mathcal{I}_{\mathcal{X}}$, then $g \in \mathcal{I}_{\mathcal{X}}$.

Note that this is a rather restrictive condition. For example, if A is a connected self-injective algebra with $\operatorname{rad} A \neq 0$, and $\mathcal{X} = \operatorname{add} A$, then $\operatorname{Ext}_{\mathcal{X}} = \operatorname{Ext}^1_A$. Let P be indecomposable projective and consider the short exact sequence $0 \to \operatorname{rad} P \xrightarrow{i} P \xrightarrow{d} P/\operatorname{rad} P = S$. Then $d = d1_S \in \mathcal{I}_{\mathcal{X}}$, but $1_S \notin \mathcal{I}_{\mathcal{X}}$.

We want to return to the case, considered in the introduction. Let C be connected wild hereditary, let M be a regular C-module and consider the exact category $(C\text{-reg}, \mathcal{E}_{\mathcal{O}(M)})$. Normally the ideal $\mathcal{I}_{\mathcal{O}(M)}$ in C-reg will not satisfy the conditions (A) and (B) above. But there are exceptions.

Definition. Let M be a quasi-simple regular module. The orbit $\mathcal{O}(M)$ is called *filtration-closed* if for any short exact sequence $0 \to U \to M' \to W \to 0$ with $M' \in \mathcal{O}(M)$ and $U, W \in C$-reg, the modules U and W are in $\mathcal{O}(M)$.

It is shown in [14] that modules M with this property exist. Considering a regular short exact sequence of the form $0 \to U \to M \to W \to 0$, it follows that M is elementary in the sense of [15], provided $\mathcal{O}(M)$ is filtration-closed. Consequently M is a quasi-simple brick and therefore $\operatorname{Hom}_C(M, \tau_C^{-r} M) = 0$ for all $r \geq 1$, see [12]. It will be shown that the conditions (A) and (B) are satisfied for the pair $(\mathcal{E}_{\mathcal{O}(M)}, \mathcal{I}_{\mathcal{O}(M)})$. For this another characterisation of filtration-closed orbits is needed.

Recall that for a regular C-module U and a class $\mathcal{X}$ of regular modules a morphism $f: X \to U$ is called a right $\operatorname{add}\mathcal{X}$-*approximation* of U, provided $X \in \operatorname{add}\mathcal{X}$ and for each $X' \in \operatorname{add}\mathcal{X}$ the map $(X', f): \operatorname{Hom}(X', X) \to \operatorname{Hom}(X', U)$ is surjective. Moreover f is called *right minimal*, if every $\alpha: X \to X$ with $\alpha f = f$ is an automorphism. A right minimal right $\operatorname{add}\mathcal{X}$-approximation is called a minimal right $\operatorname{add}\mathcal{X}$-approximation. Minimal left $\operatorname{add}\mathcal{X}$-approximations are defined dually. In general such approximations do not exist, but they clearly exist, provided $\mathcal{X}$ is finite. For example, if $U \neq 0$ and $M \neq 0$ are regular C-modules, neither right nor left $\mathcal{O}(M)$-approximations of U exist. Let now m be any integer. Since $\operatorname{Hom}_C(\tau_C^r M, U) = 0$ and $\operatorname{Hom}_C(U, \tau_C^{-r} M) = 0$, for $r \gg 0$ by [11], there exists a minimal right $\operatorname{add}(\tau^i M \mid i \geq m)$-approximation $\rho_U: \oplus_{i \geq m} \tau_C^i M^{a_i} \to U$ and a minimal left $\operatorname{add}(\tau_C^i M \mid i \leq m)$-approximation $\lambda_U: U \to \oplus_{i \leq m} \tau_C^i M^{b_i}$ of U. Of course, both direct sums are finite.

Lemma. *Let M be a quasi-simple regular C-module. Then there are equivalent:*

(a) *$\mathcal{O}(M)$ is filtration-closed.*

(b) *For any regular module U and each integer m, the minimal right $\operatorname{add}(\tau^i M \mid i \geq m)$-approximation ρ_U of U has preprojective kernel and the minimal left $\operatorname{add}(\tau_C^i M \mid i \leq m)$-approximation λ_U of U has preinjective cokernel.*

Proof. Note first, that the condition ρ_U has preprojective kernel is the same as saying $\tau_C^r \rho_U$ is injective, for $r \gg 0$. Dually, λ_U has preinjective cokernel if and only if $\tau_C^{-r} \lambda_U$ is surjective, for $r \gg 0$. Notice further, that for a nonzero regular module V there exists an integer r, only depending on $\dim V$, such that for each regular module R, all homomorphisms $\tau_C^i V \to R$ have regular kernels and all homomorphisms $R \to \tau_C^{-i} V$ have regular cokernel, provided $i \geq r$, see [15].

(a)$\Rightarrow$(b): If $\rho_U: \oplus_{i \geq m} \tau_C^i M^{a_i} \to U$ is a minimal right $\operatorname{add}(\tau_C^i M \mid i \geq m)$-approximation of U, then $\tau_C^r \rho_U: \oplus_{i \geq m} \tau_C^{i+r} M^{a_i} \to \tau_C^r U$ is a minimal right $\operatorname{add}(\tau_C^i M \mid i \geq m + r)$-approximation of $\tau_C^r U$, so we may assume that the kernel K of $\tau_C^r \rho_U$ is regular, for $r \gg 0$. Let $Q \in C$-reg be the image of $\tau_C^r \rho_U$ and consider the regular short exact sequence $0 \to K \to \oplus_{i \geq m} \tau_C^{i+r} M^{a_i} \to Q \to 0$.

Since $\mathcal{O}(M)$ is filtration-closed, we get $Q \in \mathcal{O}(M)$. Since it is a factor of $\oplus_{i \geq m} \tau_C^{i+r} M^{a_i}$ and $\operatorname{Hom}(M, \tau_C^{-i} M) = 0$, for $i > 0$ we get $Q \in \operatorname{add}(\tau_C^i M \mid i \geq m + r)$. It is checked easily that the inclusion $Q \to \tau_C^r U$ then is a right $\operatorname{add}(\tau_C^i M \mid i \geq m + r)$-approximation of $\tau_C^r U$. Since $\tau_C^r \rho_U$ is a minimal approximation, $K = 0$ follows. The second part of (b) is shown dually.

(b)$\Rightarrow$(a). Let $M' = \oplus_{i=m}^n \tau_C^i M^{c_i}$ be in $\mathcal{O}(M)$ and $0 \to U \xrightarrow{f} M' \xrightarrow{g} W \to 0$ be a short exact sequence with U and W regular. Let $\rho_W: \oplus_{i \geq m} \tau_C^i M^{a_i} \to W$ be a minimal right $\operatorname{add}(\tau_C^i M \mid i \geq m)$-approximation of W. By some τ_C-shift ρ_W becomes injective, hence we may already assume that ρ_W is injective. Since ρ_W is an approximation, there exists a morphism $h: M' \to \oplus_{i \geq m} \tau_C^i M^{a_i}$ with $g = h \rho_W$. Consequently ρ_W is surjective, hence an isomorphism that is $W \cong \oplus_{i \geq m} \tau_C^i M^{a_i}$. Dually one shows $U \in \mathcal{O}(M)$.

1.2 Proposition. *Let M be a quasi-simple regular C-module such that $\mathcal{O}(M)$ is filtration-closed. Let $\mathcal{E}_{\mathcal{O}(M)}$ be the exact structure on C-reg, induced by $\mathcal{O}(M)$. Then the pair $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(M)}, \mathcal{E}_{\mathcal{O}(M)}/\mathcal{I}_{\mathcal{O}(M)})$ is an exact category.*

Proof. More will be shown: if $0 \to X \xrightarrow{i} Y \xrightarrow{d} Z \to 0$ is any short exact sequence in C-reg (not only a short exact sequence from the subfunctor $\mathrm{Ext}_{\mathcal{O}(M)}$), then the conditions (A) and (B) of [5], mentioned in 1.1 hold.

Let V be a regular C-module, $h: Z \to V$ such that $dh \in \mathcal{I}_{\mathcal{O}(M)}$. Let m be maximal with $\mathrm{Hom}_C(Y, \tau_C^r M) = 0$ for all $r < m$ and let $\rho_V: \oplus_{i \geq m} \tau_C^i M^{a_i} \to V$ be a minimal right $\mathrm{add}(\tau^i M \mid i \geq m)$-approximation of V with preprojective kernel P. Since dh factorises through $\mathcal{O}(M)$, it factorised through ρ_V. Hence we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{i} & Y & \xrightarrow{d} & Z & \to & 0 \\ & & \downarrow f & & \downarrow h' & & \downarrow h & & \\ 0 & \to & P & \longrightarrow & \oplus_{i \geq m} \tau_C^i M^{a_i} & \xrightarrow{\rho_V} & V & & \end{array}$$

Since X is regular and P is preprojective, we get $f = 0$. Consequently the map h factorises through ρ_V, so it is in $\mathcal{I}_{\mathcal{O}(M)}$. This proves condition (A). Dually one shows (B).

2 Wild hereditary categories.

2.1 Let $\mathcal{H}$ be a wild hereditary connected k-category with a tilting object and with Grothendieck group $\mathrm{K}_0(\mathcal{H})$ of rank at least 3.

Let $X \in \mathcal{H}_w$ be quasi-simple with $\mathrm{Ext}^1_{\mathcal{H}}(X, X) = 0$ and let $0 \to \tau_{\mathcal{H}} X \to Z \to X \to 0$ be the Auslander-Reiten sequence ending in X. Then Z is indecomposable, since X is quasi-simple.

Denote by $X^\perp$ the *right perpendicular* category of X. By definition, $X^\perp$ is the full subcategory of $\mathcal{H}$, defined by the objects M with $\mathrm{Hom}(X, M) = \mathrm{Ext}(X, M) = 0$. Since $\mathrm{Ext}(X, X) = 0$ and $\mathcal{H}$ is hereditary, $X^\perp$ is again a hereditary category with tilting object. Since X is quasi-simple in $\mathcal{H}_w$, we get more: $X^\perp \cong C$-mod, where C is a connected wild hereditary algebra, see [21, 17]. To simplify notation, we will identify these two categories. For example we write τ_C for the relative Auslander-Reiten translation in $X^\perp$. Consequently C-reg is the additive closure of the indecomposable objects M in $X^\perp$ with $\tau_C^i M \neq 0$ for all integers i. Note that $Z \in C$-reg [3, 19] and C-reg is contained in $\mathcal{H}_w$.

We will keep and use this notation in the remaining part of the paper. For example Z always denotes the middle term of the Auslander-Reiten sequence, ending in X. Basic for this section is the following:

Theorem. *There exists a full and dense functor $F: C$-reg $\to \mathcal{H}_w$ with the following properties:*

(a) $F\tau_C \cong \tau_{\mathcal{H}} F$.

(b) $\ker F = \mathcal{I}_{\mathcal{O}(Z)}$, *the ideal of morphisms, which factorise through the τ_C-orbit of Z.*

This result first was shown in [3] for $\mathcal{H} = H$-mod and then in [19] for $\mathcal{H} = \mathrm{coh}\,\mathbb{X}$. Consequently, the functor F induces an equivalence $\overline{F}: C\text{-reg}/\mathcal{I}_{\mathcal{O}(Z)} \xrightarrow{\cong} \mathcal{H}_w$. Thus the exact structure $\mathcal{E}_{\mathrm{Ext}}$ on $\mathcal{H}_w$, induced by $\mathrm{Ext}_{\mathcal{H}}$ induces an exact structure on C-reg$/\mathcal{I}_{\mathcal{O}(Z)}$ via the equivalence $\overline{F}$. This exact structure will be desribed in terms of C-reg.

2.2 We will use in this paper the following description of the functor F, given in [13, 19]: For $M \in C$-reg we denote by $\rho_M : A_M = \tau_{\mathcal{H}} X^a \oplus_{i=1}^r \tau_C^i Z^{a_i} \to M$ the minimal right $\mathrm{add}(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximation of M, which exists, since $\mathrm{Hom}_C(\tau_C^i Z, M) = 0$ for $i \gg 0$. Dually let $\lambda_M : M \to B_M = X^b \oplus_{j=1}^t \tau_C^{-j} Z^{b_j}$ be the minimal left $\mathrm{add}(X, \tau_C^{-j} Z \mid j > 0)$-approximation of M.

Since $\mathrm{Hom}(\tau_C^i Z, \tau_C^{-j} Z) = \mathrm{Hom}(\tau_{\mathcal{H}} X, \tau_C^{-j} Z) = \mathrm{Hom}(\tau_{\mathcal{H}} X, X) = 0$, for $i, j > 0$, we get $0 = \rho_M \lambda_M$ for all $M \in C$-reg. The description of F in [13, 19] is as follows.

Proposition. *Let M be in C-reg. Then the following hold.*

(a) *The minimal right $\mathrm{add}(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximation $\rho_M : A_M \to M$ of M is injective with cokernel $h(M)$.*

(b) *The minimal left $\mathrm{add}(X, \tau_C^{-j} Z \mid j > 0)$-approximation $\lambda_M : M \to B_M$ of M is surjective with kernel $e(M)$.*

(c) *The following commutative diagram has exact rows and columns*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A_M & \xrightarrow{\rho'} & e(M) & \longrightarrow & F(M) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A_M & \xrightarrow{\rho_M} & M & \longrightarrow & h(M) & \to & 0 \\
 & & & & \downarrow\lambda_M & & \downarrow\lambda' & & \\
 & & & & B_M & = & B_M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The morphism $\rho' : A_M \to e(M)$ is a minimal right $\mathrm{add}(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximation of $e(M)$, and $\lambda' : h(M) \to B_M$ is a minimal left $\mathrm{add}(X, \tau_C^{-j} Z \mid j > 0)$-approximation of $h(M)$.

Clearly the assignments $M \mapsto h(M)$, $M \mapsto e(M)$ and consequently $M \mapsto F(M)$ are functorial. But it is not obvious from this description of F, how the properties of F follow.

For $M \in C$-reg one has $\mathrm{Hom}(\tau^r M, \tau_C^{-j} Z) = 0$ for $r \gg 0$ and all $j \geq 0$. Since $\mathrm{Hom}(\tau_C^r M, X) \cong \mathrm{Hom}(\tau_C^r M, Z)$, we get $B_{\tau_C^r M} = 0$ for $r \gg 0$, hence $h(\tau_C^r M) = F(\tau_C^r M) \cong \tau_{\mathcal{H}}^r F(M)$ for $r \gg 0$ follows. Dually one shows $e(\tau_C^{-r} M) = F(\tau_C^{-r} M) \cong \tau_{\mathcal{H}}^{-r} F(M)$ for $r \gg 0$. This fact will be used frequently.

2.3 Let $\eta : 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in C-reg. Then the image $F(\eta) : 0 \to F(U) \xrightarrow{F(f)} F(V) \xrightarrow{F(g)} F(W) \to 0$ normally will not be exact. But it still is a complex.

Lemma. *Let $\eta : 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in C-reg. Then $F(f) : F(U) \to F(V)$ is a mono with cokernel in $\mathcal{H}_w$ and $F(g) : F(V) \to F(W)$ is an epi with kernel in $\mathcal{H}_w$.*

Proof. Consider first the case that $\mathrm{Hom}(U, \tau_C^{-j} Z) = \mathrm{Hom}(V, \tau_C^{-j} Z) = \mathrm{Hom}(W, \tau_C^{-j} Z) = 0$ for all $j \geq 0$. Then $F(\eta) = h(\eta)$. Let $\rho_M : A_M \to M$ be the minimal right add$(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximation for any $M \in C$-reg. We get the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & A_V & \longrightarrow & A_W & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & U & \xrightarrow{f} & V & \xrightarrow{g} & W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \longrightarrow & F(V) & \xrightarrow{F(g)} & F(W) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Q & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where K is the kernel of $F(g)$. Clearly is $F(g)$ epimorphic. If $\mathcal{H}_w = \operatorname{vec}\mathbb{X}$, then $K \in \mathcal{H}_w$, since $\operatorname{vec}\mathbb{X}$ is closed under kernels. If $\mathcal{H}_w = H$-reg, then K has no nonzero preinjective direct summand, since it is a submodule of $F(V)$. By the snake lemma, there exists an epi $A_W \to Q$. Since A_W is regular in H-mod, Q has no nonzero preprojective direct summand. Since K is an extension of a factor module of the regular module U by Q, it finally has to be regular. Hence $F(g)$ is an epi with kernel in $\mathcal{H}_w$, provided $F(\eta) = h(\eta)$. The general case follows from $\tau_{\mathcal{H}}^{-r} F \tau_C^r \cong F$ and the fact that τ_C is an equivalence in C-reg, respectively $\tau_{\mathcal{H}}$ is an equivalence in $\mathcal{H}_w$.

Dually one shows the assertion for $F(f)$.

Remark. From this lemma it follows easily that $\mathcal{O}(Z)$ is filtration closed in C-reg. Indeed, let $\eta\colon 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in C-reg with $V \in \mathcal{O}(Z)$. Then $F(V) = 0$, hence $F(U) = F(W) = 0$ that is $U, W \in \mathcal{O}(Z)$. But a stronger property is shown in [13, 19] and will be used: the module Z is *orbital elementary*. It means that each short exact sequence $\eta\colon 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ in C-reg with $V \in \mathcal{O}(Z)$ splits.

2.4 As in the first section we consider in C-reg the subfunctor $\operatorname{Ext}_{\mathcal{O}(Z)}$ of Ext_C^1 and the induced exact category $(C\text{-reg}, \mathcal{E}_{\mathcal{O}(Z)})$. Since $\mathcal{O}(Z)$ is filtration closed, we get by 1.2 an exact factor category $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(Z)}, \mathcal{E}_{\mathcal{O}(Z)}/\mathcal{I}_{\mathcal{O}(Z)})$. The restriction of the functor $\operatorname{Ext}_{\mathcal{H}}$ to $\mathcal{H}_w$ defines an exact category $(\mathcal{H}_w, \mathcal{E}_{\operatorname{Ext}})$. The main result of the paper is a consequence the following more special version

Theorem. (a) *Let $\eta\colon 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in C-reg. Then $F(\eta)$ is exact, if and only if $\eta \in \mathcal{E}_{\mathcal{O}(Z)}$.*

(b) *Let η' be a short exact sequence in $\mathcal{H}_w$. Then there exists a short exact sequence η in C-reg, with $F(\eta) = \eta'$.*

(c) *The exact categories $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(Z)}, \mathcal{E}_{\mathcal{O}(Z)}/\mathcal{I}_{\mathcal{O}(Z)})$ and $(\mathcal{H}_w, \mathcal{E}_{\operatorname{Ext}})$ are equivalent as exact categories.*

Let us show, how the main result from the introduction follows from this theorem:

(a) If $A[M]$ is piecewise hereditary of type $\mathcal{H}$, then $A[\tau_A^r M]$ is quasitilted of type $\mathcal{H}$, for $r \gg 0$, see Lache [16, Theorem 1]. Hence we may assume that $A[M]$ already is quasitilted of type $\mathcal{H}$.

(b) In this case, there exists a tilting $\mathcal{H}$-object $T = X \oplus P$ with $\mathrm{End}_{\mathcal{H}} = A[M]$, where $X \in \mathcal{H}_w$ is quasi-simple and P is preprojective in the right perpendicular category $X^\perp = C$-mod, see for example [9].

(i) If P is the minimal projective generator in $X^\perp = C$-mod, then $\mathrm{End}(T) = C[Z]$ and this case is considered in the theorem above.

(ii) In the general case, let $A = \mathrm{End}(P)$. The tilting functor $\mathrm{Hom}_C(P, -): C\text{-mod} \to A\text{-mod}$ then induces an equivalence $\mathrm{Hom}_C(P, -): C\text{-reg} \to A\text{-reg}$ with $\mathrm{Hom}_C(P, Z) = M$, see [9]. This shows the conclusion.

2.5 We start with some preparation for the proof of part (a) of the theorem. Let $\eta: 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in C-reg. As in the proof of 2.3 we may assume that $\mathrm{Hom}(U, \tau_C^{-j} Z) = \mathrm{Hom}(V, \tau_C^{-j} Z) = \mathrm{Hom}(W, \tau_C^{-j} Z) = 0$ for all $j \geq 0$. Denote by $\rho_U: A_U \to U$, $\rho_V: A_V \to V$ and $\rho_W: A_W \to W$ the corresponding minimal right $\mathrm{add}(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximations. Then we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta_A: & 0 & \to & A_U & \xrightarrow{f'} & A_V & \xrightarrow{g'} & A_W & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta: & 0 & \to & U & \xrightarrow{f} & V & \xrightarrow{g} & W & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
F(\eta): & 0 & \to & F(U) & \xrightarrow{F(f)} & F(V) & \xrightarrow{F(g)} & F(W) & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

The columns of this diagram and the middle row η are exact, whereas the rows η_A and $F(\eta)$ in general are just complexes. Clearly $F(\eta)$ is is short exact sequence if and only if η_A is a short exact sequence.

Lemma. *Let*

$$\mu: 0 \to \tau_{\mathcal{H}} X^a \oplus Z_1 \xrightarrow{f} \tau_{\mathcal{H}} X^b \oplus Z_2 \xrightarrow{g} \tau_{\mathcal{H}} X^c \oplus Z_3 \to 0$$

be a short exact sequence in $\mathcal{H}_w$ *with* $Z_i \in \mathrm{add}(\tau_C^i Z \mid i > 0)$ *for all* i. *Then* μ *splits.*

Proof. Since $\mathcal{O}(Z) \subset C\text{-reg} \subset X^\perp = {}^\perp\tau_{\mathcal{H}} X$, we get $f = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ and $g = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$. Application of the functor $(-, \tau_{\mathcal{H}} X) = \mathrm{Hom}(-, \tau_{\mathcal{H}} X)$ then gives the short exact sequence

$$(*) \quad 0 \to (\tau_{\mathcal{H}} X^c, \tau_{\mathcal{H}} X) \xrightarrow{(a_2, \tau_{\mathcal{H}} X)} (\tau_{\mathcal{H}} X^b, \tau_{\mathcal{H}} X) \xrightarrow{(a_1, \tau_{\mathcal{H}} X)} (\tau_{\mathcal{H}} X^a, \tau_{\mathcal{H}} X) \to 0.$$

Since $\tau_{\mathcal{H}} X$ has not self-extensions, a_1 is a mono and $b = a + c$, by $(*)$.

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & Z_1 & \xrightarrow{c_1} & Z_2 & \xrightarrow{c_2} & Z_3 & \to & 0 \\
 & & \downarrow e_1 & & \downarrow e_2 & & \downarrow e_3 & & \\
0 & \to & \tau_{\mathcal{H}} X^a \oplus Z_1 & \xrightarrow{f} & \tau_{\mathcal{H}} X^b \oplus Z_2 & \xrightarrow{g} & \tau_{\mathcal{H}} X^c \oplus Z_3 & \to & 0 \\
 & & \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\
0 & \to & \tau_{\mathcal{H}} X^a & \xrightarrow{a_1} & \tau_{\mathcal{H}} X^b & \xrightarrow{a_2} & \tau_{\mathcal{H}} X^c & \to & 0
\end{array}$$

where the e_i are the canonical inclusions and the p_i are the canonical projections. We know already that a_1 is a mono and a_2 is an epi. From $b = a + c$ it therefore follows that the third row is a short exact sequence. Consequently the first row is a short exact sequence, too. Since $\tau_{\mathcal{H}} X$ has no self-extensions, the third row is a split short exact sequence. The first row splits, since Z is orbital elementary in C-reg, see [13, 19]. Consequently also μ is a split short exact sequence, by the shape of f and g.

2.6 We have a functorial isomorphism $\phi\colon \operatorname{Hom}_{\mathcal{H}}(\tau_{\mathcal{H}} X, -)_{|X^\perp} \to \operatorname{Hom}_C(Z, -)$. For $f \in \operatorname{Hom}_{\mathcal{H}}(\tau_{\mathcal{H}} X, M)$ with $M \in C$-mod, $\overline{f} = \phi(f)\colon Z \to M$ is determined by $e\overline{f} = f$, where $e\colon \tau_{\mathcal{H}} X \to Z$ is the irreducible map. From this fact immediately follows:

Lemma. *Let $Z_1 \in \operatorname{add}(\tau_C^i Z \mid i > 0)$ and $M \in C$-reg. Let $\rho = (f, g)^t\colon \tau_{\mathcal{H}} X^a \oplus Z_1 \to M$ be a morphism and $\overline{f}\colon Z^a \to M$ be induced by f. Let $\overline{\rho} = (\overline{f}, g)^t\colon Z^a \oplus Z_1 \to M$. Then ρ is a minimal right $\operatorname{add}(\tau_{\mathcal{H}} X, \tau_C^i Z \mid i > 0)$-approximation if and only if $\overline{\rho}$ is a minimal right $\operatorname{add}(\tau_C^i Z \mid i \geq 0)$-approximation of M .*

2.7 The proof of part (a) now will be finished. Still we assume that the modules U, V and W only map trivially to $\tau_C^{-j} Z$, for $j \geq 0$.

(A) Assume $F(\eta)$ is exact. Then, by 2.5, η_A is a split short exact sequence and we have the following diagram

$$\begin{array}{cccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta_A\colon & 0 & \to & A_U & \xrightarrow{f'} & A_V & \xrightarrow{g'} & A_W & \to & 0 \\
 & & & \downarrow & & \downarrow \rho_V & & \downarrow \rho_W & & \\
\eta\colon & 0 & \to & U & \xrightarrow{f} & V & \xrightarrow{g} & W & \to & 0
\end{array}$$

Since g' is a split epi, there exists $h\colon A_W \to V$ with $\rho_W = hg$. By 2.6 this implies that the minimal right $\operatorname{add}(\tau_C^i Z \mid i \geq 0)$-approximation $\overline{\rho_W}$ of W has a factorisation $\overline{\rho_W} = \overline{h} g$. Consequently all morphisms $u\colon Z_1 \to W$ with $Z_1 \in \operatorname{add}(\tau_C^i Z \mid i \geq 0)$ factorise through g. Since $\operatorname{Hom}_C(U, \tau_C^{-j} Z) = 0$ for all $j \geq 0$, all morphisms $v\colon Z_2 \to W$ with $Z_2 \in \mathcal{O}(Z)$ factorise through g by [1, IV.4.4], hence $\eta \in \operatorname{Ext}_{\mathcal{O}(Z)}(W, U)$.

(B) Assume $\eta \in \operatorname{Ext}_{\mathcal{O}(Z)}(W, U)$. We will show that η_A is a split short exact sequence. Let $\overline{\rho_V}\colon \overline{A_V} \to V$ and $\overline{\rho_W}\colon \overline{A_W} \to W$ be the minimal right $\operatorname{add}(\tau_C^i Z \mid i \geq 0)$-approximation of V, induced by ρ_V, respectively ρ_W, and let $g_2\colon \overline{A_V} \to \overline{A_W}$ be induced by $g\colon V \to W$, that is $\overline{\rho_V} g = g_2 \overline{\rho_W}$.

Since $\eta \in \operatorname{Ext}_{\mathcal{O}(Z)}(W, U)$, there exist a map $h'\colon \overline{A_W} \to V$ with $h' g = \overline{\rho_W}$. Since $\overline{\rho_V}$ is an $\operatorname{add}(\tau_C^i Z \mid i \geq 0)$-approximation of V, $h' = h \overline{\rho_V}$ holds, with $h\colon \overline{A_W} \to \overline{A_V}$. Since

$\overline{\rho_W}$ is a right minimal map and $\overline{\rho_W} = hg_2\overline{\rho_W}$, the morphism g_2 is a split epi with section $\overline{s}$. Consequently also $g'\colon A_V \to A_W$ is a split epi, with a section s. Clearly $f'\colon A_U \to A_V$ is a mono. It is easy to check that $(\rho_U f, s\rho_V)^t\colon A_U \oplus A_W \to V$ is a right $\mathrm{add}(\tau_{\mathcal{H}}X, \tau_C^i Z \mid i > 0)$-approximation of V, which finally implies that η_A is a split short exact sequence. Therefore $F(\eta)$ is exact.

2.8 We show part (b) of the theorem. Let $\eta'\colon 0 \to U' \xrightarrow{f'} V' \xrightarrow{g'} W' \to 0$ be a short exact sequence in $\mathcal{H}_w$. Choose elements U, V and W in C-reg, all of them without direct summands in $\mathcal{O}(Z)$, which are mapped under F to U', V' and W'. Choose further $f\colon U \to V$ and $g\colon V \to W$ with $F(f) = f'$ and $F(g) = g'$. From $F(fg) = 0$ it follows that $fg \in \mathcal{I}_{\mathcal{O}(Z)}$. Hence there exists $Z_1 \in \mathcal{O}(Z)$ with $a\colon U \to Z_1$ and $b\colon Z_1 \to W$, such that $\kappa\colon U \xrightarrow{(f,a)} V \oplus Z_1 \xrightarrow{(g,b)^t} W$ is a complex. Modulo some τ_C-shift we way assume that the kernel of any homomorphism from U, respectively $V \oplus Z_1$, to a regular C-module is regular [15], and we will assume this.

(A) Let $U \xrightarrow{p} U_0 \xrightarrow{e} V \oplus Z_1$ be an epi-mono factorisation of (f,a). Since $\ker p$ is regular, $F(p)$ is an epi, by 2.3. Since $F(f) = F((f,a)) = F(p)F(e)$ is a mono, also $F(p)$ is a mono, hence $F(p)$ is an isomorphism. Since $p\colon U \to U_0$ is surjective and U has no nonzero direct summand from $\mathcal{O}(Z)$, the morphism p is an isomorphism and therefore (f,a) is injective.

(B) Let $V \oplus Z_1 \xrightarrow{p} W_0 \xrightarrow{e} W$ be an epi-mono factorisation of $(g,b)^t$ and let K be the regular kernel of p We get the following commutative diagram

$$\begin{array}{ccccccccc} & & U & \longrightarrow & V \oplus Z_1 & \longrightarrow & W & & \\ & & \downarrow i & & \| & & \uparrow e & & \\ 0 & \to & K & \longrightarrow & V \oplus Z_1 & \xrightarrow{p} & W_0 & \to & 0 \end{array}$$

where the first row is the complex κ and the second row is a short exact sequence η. Application of the functor F to this diagram gives

$$\begin{array}{ccccccccc} 0 & \to & U' & \xrightarrow{f'} & V' & \xrightarrow{g'} & W' & \to & 0 \\ & & \downarrow F(i) & & \| & & \uparrow F(e) & & \\ 0 & \to & F(K) & \longrightarrow & V' & \xrightarrow{F(p)} & F(W_0) & \to & 0 \end{array}$$

Since f' is a mono, so is $F(i)$. Since g' is an epi, so is $F(e)$.

Denote by $|M|$ for $M \in \mathcal{H}_w$ either its dimension $\dim M$, if $\mathcal{H}_w = H$-reg, or its rank $\mathrm{rk}\, M$ if $\mathcal{H}_w = \mathrm{vec}\,\mathbb{X}$. In both cases $|-|$ is additive on short exact sequences, and $|N| = 0$ for a subobject N of $M \in \mathcal{H}_w$ implies $N = 0$. Since η is a short exact sequence in C-reg, 2.3 implies $|V'| \geq |F(K)| + |F(W_0)|$. Since $F(i)$ is a mono, respectively $F(e)$ is an epi, we get $|U'| \leq |F(K)|$, respectively $|W'| \leq |F(W_0)|$. Finally $|V'| = |U'| + |W'|$ holds, since η' is a short exact sequence.

Consequently we get $|F(W_0)| = |W'|$, that is $F(e)$ is an isomorphism. Again it follows from the choice of W that e is an isomorphism, that is $(g,b)^t$ is surjective. Therefore we may assume that $W = W_0$ and $e = 1_W$. From this and from $|U'| = |F(K)|$ it follows that $F(i)$ is an isomorphism. Therefore $F(\eta)$ is a short exact sequence, isomorphic to η'. Note that this implies that i is a split mono and $K \cong U \oplus Z'$ for some $Z' \in \mathcal{O}(Z)$.

This finishes the proof of part (b). Part (c) of the theorem follows from (a) and (b).

2.9 We close with an example. Let C be the path-algebra of the quiver

$$\begin{tikzcd}[column sep=small]
 & & & \circ \arrow[d] & & & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

Let M, respectively N, be the unique indecomposable C-modules with

$$\underline{\dim} M = \begin{pmatrix} & & & 1 & & & & \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \underline{\dim} N = \begin{pmatrix} & & & 1 & & & & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

The one-point extension $C[M]$ is a wild canonical algebra. It is quasitilted of type $\mathrm{coh}\mathbb{X}$, where $\mathbb{X}$ s a weighted projective line of weight type $(2, 4, 5)$. Hence the exact category $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(M)}, \mathcal{E}_{\mathcal{O}(M)}/\mathcal{I}_{\mathcal{O}(M)})$ is equivalent to $(\mathrm{vec}\,\mathbb{X}, \mathcal{E}_{\mathrm{Ext}})$.
The one-point extension $C[N]$ is tilted of type H-mod, where H is the path-algebra of the quiver

$$\begin{tikzcd}[column sep=small]
 & & \circ \arrow[d] & & & & & & \\
\circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

Therefore $(C\text{-reg}/\mathcal{I}_{\mathcal{O}(N)}, \mathcal{E}_{\mathcal{O}(N)}/\mathcal{I}_{\mathcal{O}(N)})$ is equivalent to $(H\text{-reg}, \mathcal{E}_{\mathrm{Ext}})$.

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