RATE OF CONVERGENCE TO THE SEMI-CIRCULAR LAW.

F. GÖTZE[1] AND A. TIKHOMIROV[1,2]

University of Bielefeld[1]
Syktyvkar State University and the Mathematical
Department of IMM of the Russian Academy of Sciences[2]

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ABSTRACT. It is shown that the Kolmogorov distance between the expected spectral distribution function of a symmetric $n \times n$ matrix from a Wigner ensemble and the distribution function of the semi-circular law is of order $O(n^{-1/2})$. The bound is explicit and requires that the fourth moments of the entries of the matrix are uniformly bounded.

1. INTRODUCTION AND RESULTS.

Let $X_{ij}, 1 \leqslant i \leqslant j \leqslant n$ be independent random variables with $\mathbf{E}\, X_{ij} = 0$ and $\mathbf{E}\, X_{ij}^2 = 1$. Denote by $\lambda_1 \leqslant \ldots \leqslant \lambda_n$ the eigenvalues of the symmetric matrix

$$W = \Big(W(j,k)\Big)_{j,k=1}^{n}, \quad W(j,k) = n^{-\frac{1}{2}} X_{jk}, \text{ for } 1 \leqslant j \leqslant k \leqslant n,$$

and define its empirical distribution by

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{I}_{\{\lambda_k \leqslant x\}},$$

where $\mathbf{I}_{\{B\}}$ denotes the indicator of an event B. We consider the rate of convergence of the expected spectral distribution $\mathbf{E}\, F_n(x)$ to the distribution function of Wigner's semi-circular law. Let $g(x)$ and $G(x)$ denote the density and the distribution function of the standard semi-circular law, that is

$$g(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathbf{I}_{\{|x| \leqslant 2\}}, \quad G(x) = \int_{-\infty}^{x} g(u)du.$$

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Set

$$\Delta_n = \sup_x |\mathbf{E}\, F_n(x) - G(x)|.$$

By C (with an index or without it) we shall denote generic absolute constants, whereas $C(\cdot,\cdot)$ will denote positive constants depending on arguments.

Our main result is the following

Theorem 1.1. *Assume that X_{ij} satisfies the conditions above and that*

$$M_n := \sup_{1\leqslant j,k\leqslant n} \mathbf{E}\,|X_{jk}|^4 < \infty. \tag{1.1}$$

Then there exists an absolute constant $C>0$ such that

$$\Delta_n \leqslant C\sqrt{M}_n\, n^{-\frac{1}{2}}.$$

The investigation of the spectrum of high-dimensional random matrices has a long history. After the pioneering paper of Wigner (1958) a number of authors have studied the problem. We can refer to Arnold L. (1971), Bai (1993), Khorunzhy, Khoruzhchenko, Pastur (1996), Pastur, Figotin (1992), Girko (1998),

Voiculescu, Dykema, Nica (1991) and the survey Bai (1999a). Bai (1993) assuming $M = \sup\limits_n M_n < \infty$ proved that $\Delta_n \leqslant C(M)n^{-1/4}$. He noted in that paper that his result could be improved using higher moments by a similar approach up to an error bound of order $O(n^{-1/3+\eta})$, $\eta>0$. In Bai (1999) this bound was shown with $\eta = 0$ without assuming the existence of higher moments. In a survey Bai (1999a) announced a bound of order $O(n^{-1/2})$ assuming that the diagonal entries of W are i.i.d. with mean zero and $\mathbf{E}\,|X_{11}|^6 < \infty$ and the elements above the diagonal are i.i.d. with mean zero and $\mathbf{E}\,|X_{12}|^8 < \infty$.

In Girko(1998) states Theorem 1.1 as well. In this paper very differs approach based on the methods of steepest descent and properties Hermitian polynomials is used.

We shall prove our result using inequalities for the distance between distributions in terms of their Stieltjes transform which extend results of Bai (1993), Theorem 2.1. We shall make essential of the fact that the limiting Stiltjes transform, say $s(z)$ of $G(x)$, solves a quadratic equation, while avoiding certain problems in the approach of Bai (1993), (4.11), connected with the choice of branches of this equation.

Remark. *Consider the Hermitian matrix $W = (W_{lj}), 1\leqslant l,j\leqslant n$. Let*

$$W_{lj} = \frac{1}{\sqrt{n}}(X_{lj} + iY_{lj}), \quad 1\leqslant l<j\leqslant n, \text{ and } W_{ll} = X_{ll}, \quad 1\leqslant l\leqslant n.$$

Assume that $X_{lj}, Y_lj, 1\leqslant l,j\leqslant n$ are independent and $\mathbf{E}\,X_{lj} = \mathbf{E}\,Y_{lj} = 0$, $\mathbf{E}\,X_{lj}^2 = \mathbf{E}\,Y_{lj}^2 = 1/2$, for $l\neq j$ and $\mathbf{E}\,X_{ll}^2 = 1$. The conclusion of Theorem 1.1 still true for the Hermitian random matrix W too.

2. Inequalities for the distance between distributions via Stieltjes transforms.

In what follows we shall use the notation $\Im z$ and $\Re z$ for imaginary and really part of complex number z respectively.

Lemma 2.1. *Let F be a distribution function and let G denote the semi-circular distribution function. Denote their Stieltjes transforms by $f(z)$ and $g(z)$ respectively, where $z = u + iv$. Assume that $\int |F(x) - G(x)|dx < \infty$. Let $v > 0$ and a and ε be positive numbers such that*

$$\gamma = \frac{1}{\pi}\int_{|y|\leqslant a} \frac{1}{u^2+1}\,du > \frac{3}{4}, \tag{2.1}$$

and

$$\varepsilon > 2\,v\,a. \tag{2.2}$$

Given $\varepsilon > 0$ introduce the intervals $I_\varepsilon = [-2+\varepsilon, 2-\varepsilon]$ and $I'_\varepsilon = [-2+\frac{1}{2}\varepsilon, 2-\frac{1}{2}\varepsilon]$. Then there exist some positive constants $C_1(\gamma)$, $C_2(\gamma)$, $C_3(\gamma)$ depending on γ such that

$$\begin{aligned}\Delta(F,G) &:= \sup_x |F(x) - G(x)| \\ &\leqslant C_1(\gamma) \sup_{x\in I'_\varepsilon} |\Im\Big(\int_{-\infty}^{x} (f(z) - g(z))\,du\Big)| + C_2(\gamma)\,v + C_3(\gamma)\,\varepsilon^{3/2}.\end{aligned}$$

Proof. Note that

$$\sup_x |F(x) - G(x)| \leqslant 3 \sup_{x\in I_\varepsilon} |F(x) - G(x)| + 4G(-2+\varepsilon),$$

and $G(-2+\varepsilon) \leqslant C\varepsilon^{3/2}$. If (2.1) and (2.2) hold and $x \in I_\varepsilon$ then $x + va \in I'_\varepsilon$. Repeating the proof of Theorem 2.1, inequality (2.6) forward, in Bai (1993) (called B93 for short) we obtain the result. For the readers convenience we include these arguments here. First note that for each $x \in I'_\varepsilon$

$$\begin{aligned}\sup_{x\in I'_\varepsilon}\Big|\frac{1}{\pi}\Im\Big(\int_{-\infty}^{x} (f(z) - g(z))\,du\Big)\Big| &\geqslant \frac{1}{\pi}\Im\Big(\int_{-\infty}^{x} (f(z) - g(z))\,du\Big) \\ &= \frac{1}{\pi}\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} \frac{vd\big(F(y) - G(y)\big)}{(y-u)^2+v^2}\right]du \\ &= \frac{1}{\pi}\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} \frac{2(v(y-u)(F(y) - G(y))dy}{\big((y-u)^2+v^2\big)^2}\right]du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}(F(y) - G(y))\left[\int_{-\infty}^{x} \frac{2v(y-u)dy}{\big((y-u)^2+v^2\big)^2}\right]dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\big(F(x-vy) - G(x-vy)\big)dy}{y^2+1}.\end{aligned}$$

Since F is non increasing we have

$$\begin{aligned}\frac{1}{\pi}\int_{|y|<a} \frac{\big(F(x-vy) - G(x-vy)\big)dy}{y^2+1} &\geqslant \gamma\Big(F(x-va) - G(x-va)\Big) \\ &\quad - \frac{1}{\pi}\int_{|y|<a}\Big|G(x-vy) - G(x-va)\Big|dy \\ &\geqslant \gamma\Big(F(x-va) - G(x-va)\Big) \\ &\quad - \frac{1}{v\pi}\int_{|y|<va}\Big|G(x-y) - G(x-va)\Big|dy.\end{aligned}$$

Write $\Delta_\varepsilon(F,G)=\sup_{x\in I_\varepsilon}|F(x)-G(x)|$. Let $x_n\in I_\varepsilon$ such that $F(x_n)-G(x_n)\to\Delta_\varepsilon(F,G)$. Then $x=x_n+va\in I'_\varepsilon$. We have

$$\begin{aligned}
&\sup_{x\in I'_\varepsilon}\Big|\frac{1}{\pi}\Im\Big(\int_{-\infty}^{x}(f(z)-g(z))\,du\Big)\Big|\\
&\geqslant \gamma\Big(F(x_n)-G(x_n)\Big)-\frac{1}{v\pi}\sup_x\int_{|y|<2va}|G(x+y)-G(x)|dy-(1-\gamma)\Delta(F,G)\\
&\to \gamma\Delta_\varepsilon(F,G)-Cav-3(1-\gamma)\Delta_\varepsilon(F,G)-C\varepsilon^{3/2}\\
&\geqslant (4\gamma-3)\Delta_\varepsilon(F,G)-Cav-C\varepsilon^{3/2}.
\end{aligned}$$

Similar arguments may be used for the sequence $x_n\in I_\varepsilon$ such that $(F(x_n)-G(x_n))\to-\Delta_\varepsilon(F,G)$. This completes the proof. □

Lemma 2.2. *For any $V>v_0>0$ the following inequality holds*

$$\begin{aligned}
\sup_{x\in I'_\varepsilon}\Big|\int_{-\infty}^{x}\Im(f(z)-g(z))\,du\Big|\leqslant&\int_{-\infty}^{\infty}\big|(f(u+iV)-g(u+iV))\big|\,du\\
&+\sup_{x\in I'_\varepsilon}\Big|\Im\Big\{\int_{v_0}^{V}(f(x+iv)-g(x+iv))\,dv\Big\}\Big|.
\end{aligned}$$

Proof. Since the function $f(z)$ and $g(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write

$$\int_{-\infty}^{x}\Im(f(z)-g(z))\,du=\lim_{L\to\infty}\Im\int_{-L}^{x}(f(z)-g(z))du.$$

By Cauchy's integral theorem we have

$$\begin{aligned}
\int_{-L}^{x}(f(z)-g(z))du=&\int_{-L}^{x}(f(u+iV)-g(u+iV))du\\
&+\int_{v_0}^{V}(f(-L+iv)-g(-L+iv))dv-\int_{v_0}^{V}(f(x+iv)-g(x+iv))dv.
\end{aligned}$$

Denote by ξ (η) a random variables with distribution function $F(x)$ (resp. $G(x)$). Then we have

$$|f(-L+iv)|=\Big|\mathbf{E}\,\frac{1}{\xi+L-iv}\Big|\leqslant v^{-1}\mathbf{P}\left\{|\xi|>L/2\right\}+\frac{2}{L}.$$

Similarly,

$$|g(-L+iv)|\leqslant v^{-1}\mathbf{P}\left\{|\eta|>L/2\right\}+\frac{2}{L}.$$

This inequality implies that

$$\Big|\int_{v_0}^{V}(f(-L+iv)-g(-L+iv))dv\Big|\to 0\quad\text{as}\quad L\to\infty,$$

which completes the proof. □

Lemmas 2.1 and 2.2 together imply

Corollary 2.3. *There exist some absolute positive constants C_1, C_2 and C_3 such that for nay $V > v_0 > 0$ the following inequality holds*

$$\begin{aligned}\Delta(F,G) \leqslant & C_1\Big(\int_{-\infty}^{\infty} |(f(u+iV)-g(u+iV))|du \\ & + \sup_{x\in I'_\varepsilon}\big|\Im\Big\{\int_{v_0}^{V}(f(x+iu)-g(x+iu))du\Big\}\big|\Big) + C_2 v_0 + C_3\varepsilon^{3/2}.\end{aligned} \tag{2.3}$$

3. Auxiliary Lemmas

In order to make the paper self-contained we collect here some auxiliary Lemmas similar to those used in B93. In the following we shall denote by $\mathbf{I}$ with subscript or not the identical matrix.

Lemma 3.1. *Let $A = \big(a_{kj}\big)$ denote a non-degenerate matrix of order n and let A_k denote the major sub-matrix of order $n-1$. Assume that A_k is nonsingular too. Let $A^{-1} = \big(a^{jk}\big)$. Let α_k denote the vector obtained from the k-th row of A by deleting the kth entry and β_k the vector from the kth column by deleting the kth entry. Then we have*

$$a^{kk} = \frac{1}{a_{kk} - \alpha'_k A_k^{-1}\beta_k}.$$

Proof. Consider the obvious equality

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -CA^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} A & B \\ \mathbf{0} & D - CA^{-1}B\end{bmatrix}, \tag{3.1}$$

which implies

$$\det\begin{bmatrix} A & B \\ C & D\end{bmatrix} = \det\big(A\big)\det\big(D - CA^{-1}B\big). \tag{3.2}$$

Since

$$a^{kk} = \det\big(A_k\big)/\det\big(A\big),$$

applying equality (3.2) with $A = A_k$, $D = a_{kk}$, $C = \alpha_k$ and $B = \beta_k$ concludes the proof.

As a trivial corollary of a Sturmian separation theorem, see for example Bellman (1970), we formulate the following

Lemma 3.2. *Let A be a symmetric matrix and A_k its main sub-matrix. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ and $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$ denote the eigenvalues of A and A_k respectively. Then*

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n.$$

Lemma 3.3. *Let $z = u + iv$, and A be an $n \times n$ symmetric matrix. Then*

$$\Big|\operatorname{Tr}\big(A - z\mathbf{I}_n\big)^{-1} - \operatorname{Tr}\big(A_k - z\mathbf{I}_{n-1}\big)^{-1}\Big| \leqslant v^{-1}$$

Proof. We consider a nonsingular block matrix

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11}\mathbf{S}_{12} \\ \mathbf{S}_{21}\mathbf{S}_{22} \end{bmatrix}.$$

Then we have

$$\mathbf{S}^{-1} = \begin{bmatrix} \widetilde{\mathbf{S}}_{11}\widetilde{\mathbf{S}}_{12} \\ \widetilde{\mathbf{S}}_{21}\widetilde{\mathbf{S}}_{22} \end{bmatrix},$$

where

$$\begin{aligned}
\widetilde{\mathbf{S}}_{11} &= {\mathbf{S}_{11}}^{-1} + {\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big(\mathbf{S}_{22} - \mathbf{S}_{21}{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big)^{-1}\mathbf{S}_{21}{\mathbf{S}_{11}}^{-1}, \\
\widetilde{\mathbf{S}}_{12} &= -{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big(\mathbf{S}_{22} - \mathbf{S}_{21}{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big), \\
\widetilde{\mathbf{S}}_{21} &= -{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big(\mathbf{S}_{22} - \mathbf{S}_{21}{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big), \\
\widetilde{\mathbf{S}}_{22} &= \Big(\mathbf{S}_{22} - \mathbf{S}_{21}{\mathbf{S}_{11}}^{-1}\mathbf{S}_{12}\Big)^{-1}.
\end{aligned}$$

Applying this formula with $\mathbf{S}_{11} = A_k - z\mathbf{I}_{\,n-1}$, $\mathbf{S}_{21} = \alpha_k$, $\mathbf{S}_{12} = \alpha_k'$ and $\mathbf{S}_{22} = a_{kk} - z$ a direct calculation yields

$$\operatorname{Tr}(A - z\mathbf{I}_{\,n})^{-1} - \operatorname{Tr}(A_k - z\mathbf{I}_{\,n-1})^{-1} = \frac{1 + \alpha_k'(A_k - z\mathbf{I}_{\,n-1})^{-2}\alpha_k}{a_{kk} - z - \alpha_k'(A_k - z\mathbf{I}_{\,n-1})^{-1}\alpha_k}. \tag{3.3}$$

Let T be an orthogonal transformation which transforms A into diagonal form. Denote by $\mu_1 \leqslant \cdots \leqslant \mu_{n-1}$ the eigenvalues of A_k and let

$$(y_1, \dots, y_{n-1}) = \alpha_k' T'.$$

Then

$$\begin{aligned}
|1 + \alpha_k'(A_k - z\mathbf{I}_{\,n-1})^{-2}\alpha_k| &= \Big|1 + \sum_{l=1}^{n-1} y_l^2(\mu_l - z)^{-2}\Big| \\
&\leqslant 1 + \sum_{l=1}^{n-1} y_l^2\big((\mu_l - u)^2 + v^2\big)^{-1} \\
&\leqslant 1 + \alpha_k'\Big(\big(A_k - u\mathbf{I}_{\,n-1}\big)^2 + v^2\mathbf{I}_{\,n-1}\Big)^{-1}\alpha_k
\end{aligned}$$

Since for any matrices A, B such that $A^2 + B^2$ is non-degenerate

$$(A + iB)^{-1} = (A - iB)(A^2 + B^2)^{-1},$$

we can directly verified that

$$\Im\Big(a_{kk} - z - \alpha_k'\big(A - z\mathbf{I}_{\,n-1}\big)^{-1}\alpha_k\Big) = -v\Big(1 + \alpha_k'\Big((A - u\mathbf{I}_{\,n-1})^2 + v^2\mathbf{I}_{\,n-1}\Big)^{-1}\alpha_k\Big).$$

The last two relations together imply the result.

4. The bound of the first integral in (2.3).

We shall follow the notation of B93. Let

$$s(z) = -\frac{1}{2}(z - \sqrt{z^2-4}), \quad s_n(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} d\mathbf{E}\, F_n(x). \tag{4.1}$$

By definition of $F_n(x)$ we can write

$$s_n(z) = \mathbf{E}\Big(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\lambda_j - z}\Big) = \frac{1}{n}\mathbf{E}\ \mathrm{Tr}\, R(z) = \frac{1}{n}\sum_{j=1}^{n}\mathbf{E}\, R(j,j), \tag{4.2}$$

where $R(z) = \big(W - z\mathbf{I}_n\big)^{-1} = \big(R(j,k)\big)_{j,k=1}^{n}$.

Let $W(k)$ be the matrix obtained from W by deleting the kth row and kth column, and let $\alpha'(k) = \big(X_{1k}, \dots, X_{(k-1)k}, X_{(k+1)k}, \dots, X_{nk}\big)$. Set

$$\varepsilon_k = \frac{1}{\sqrt{n}}X_{kk} - \frac{1}{n}\alpha'(k)\Big(W(k) - z\mathbf{I}_{n-1}\Big)^{-1}\alpha_k + s_n(z), \tag{4.3}$$

where $\mathbf{I}_{n-1}$ denotes the $(n-1)\times(n-1)$ identity matrix. Introduce

$$\delta_n(z) = -\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,\varepsilon_k \frac{1}{(z+s_n(z))(z+s_n(z)-\varepsilon_k)}. \tag{4.4}$$

and the matrix

$$R_k = \Big(W(k) - z\mathbf{I}_{n-1}\Big)^{-1}.$$

By Lemma 3.1 and relations (4.2) and (4.3) we may write

$$R(j,j) = -\frac{1}{z+s_n(z)-\varepsilon_j} = -\frac{1}{z+s_n(z)} - \frac{\varepsilon_j}{(z+s_n(z))(z+s_n(z)-\varepsilon_j)}. \tag{4.5}$$

This implies that

$$s_n(z) = -\frac{1}{z+s_n(z)} + \delta_n(z). \tag{4.6}$$

To prove the Theorem 1.1 we shall use the result of Corollary 2.3. We start from the bound of the first integral on the right hand side in (2.3). We need some inequalities, which were proved in B93, but for the readers convenience we repeat the proof here.

Lemma 4.1. *Under condition of Theorem 1.1 for any $v > 0$ and for any $k = 1, \dots, n$ we have*

$$|\mathbf{E}\,\varepsilon_k| \leqslant \frac{C}{nv}. \tag{4.7}$$

Proof. Since $\alpha(k)$ and $W_n(k)$ are independent, the equalities (4.2) and (4.3) together imply

$$|\mathbf{E}\,\varepsilon_k| = \frac{1}{n}\Big|\mathbf{E}\Big[\ \mathrm{Tr}\Big(W_n - z\mathbf{I}_n\Big)^{-1} - \mathrm{Tr}\Big(W_n(k) - z\mathbf{I}_{n-1}\Big)^{-1}\Big]\Big|.$$

Using now Lemma 3.3 we obtain the inequality (4.7). $\square$

Write

$$M := M_n := \sup_{1\leqslant i \leqslant j \leqslant n}\mathbf{E}\,|X_{ij}|^4.$$

Lemma 4.2. *Assuming the conditions of Theorem 1.1 for any $v>0$ and for any $k=1,\dots,n$ we have*
$$\mathbf{E}\,|\varepsilon_k|^2\leqslant\frac{CM}{nv^2}. \tag{4.8}$$

Proof. By the definition of ε_k we have the following inequality
$$\begin{aligned}\mathbf{E}\,|\varepsilon_j|^2\leqslant 2\Big(&\frac{1}{n}\mathbf{E}\,|X_{kk}|^2+\frac{1}{n^2}\mathbf{E}\,|\alpha'(k)R_k\alpha(k)-\operatorname{Tr}R_k|^2\\&+\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr}R_k-\operatorname{Tr}R|^2+\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr}R-\mathbf{E}\,\operatorname{Tr}R|^2\Big).\end{aligned}\tag{4.9}$$
By Lemma 3.3 we have
$$\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr}R_k-\operatorname{Tr}R|^2\leqslant\frac{1}{n^2v^2}. \tag{4.10}$$
To bound the last summand in the right hand side of (4.9) we repeat some arguments in B93. Let E_d denote the conditional expectation given $\{X_{ij}, d+1\leqslant i\leqslant j\leqslant n\}$. Introduce the $(n-2)\times(n-2)$ matrix $W(d,k)$ obtained from W by deleting of d-th and k-th rows and columns. Let $\alpha(d,k)$ denote the vector obtained from the d-th column of $\sqrt{n}W$ by deleting the d-th and k-th entries. Let
$$\begin{aligned}&\gamma_k(k)=0 \quad\text{ and for }\quad d\neq k\\&\gamma_d(k)=\mathbf{E}\,_{d-1}\operatorname{Tr}R_k-\mathbf{E}\,_{d}\operatorname{Tr}R_k=\mathbf{E}\,_{d-1}\sigma_d(k)-\mathbf{E}\,_{d}\sigma_d(k),\end{aligned}$$
where
$$\sigma_d(k)=\operatorname{Tr}R_k-\operatorname{Tr}R_d(k).$$
By Lemma 3.3
$$|\sigma_d(k)|\leqslant\frac{1}{v}.$$
This immediately implies that
$$|\gamma_d(k)|\leqslant\frac{2}{v}.$$
Since the random variables $\gamma_d(k)$ are uncorrelated for $d=1,\dots,n$ and $\operatorname{Tr}R_k-\mathbf{E}\,\operatorname{Tr}R_k=\sum\limits_{d=1}^{n}\gamma_d(k)$ we get
$$\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr}R_k-\mathbf{E}\,\operatorname{Tr}R_k|^2\leqslant\frac{4}{nv^2}. \tag{4.11}$$
Finally
$$\frac{1}{n^2}\mathbf{E}\,\Big|\alpha'(k)R_k\alpha(k)-\operatorname{Tr}R_k|^2\leqslant\frac{CM}{n^2}\mathbf{E}\,\|R_k\|^2.$$
We can write
$$\mathbf{E}\,\|R_k\|^2=n\int_{-\infty}^{\infty}\frac{1}{|x-z|^2}d\mathbf{E}\,F_n^{(k)}(x)\leqslant nv^{-2}.$$
This implies
$$\frac{1}{n^2}\Big(\mathbf{E}\,\Big|\alpha'(k)R_k\alpha(k)-\operatorname{Tr}R_k|^2\leqslant\frac{CM}{nv^2}. \tag{4.12}$$
The inequalities (4.9)–(4.12) conclude the proof. □

□

Lemma 4.3. *For any $v > 0$ the following inequalities hold*

$$|\delta_n(z)| \leqslant \frac{CM}{nv^3}|s_n(z)+z|^{-2} \tag{4.13}$$

and

$$|\delta_n(z)| \leqslant \frac{CM}{nv^5}. \tag{4.14}$$

Proof. The equalities (4.4) and (4.5) together imply that

$$\delta_n(z) = -\frac{1}{(s_n(z)+z)^2}\Big(\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,\varepsilon_k + \frac{1}{n}\sum_{k=1}^n \mathbf{E}\,\varepsilon_k^2 R(k,k)\Big). \tag{4.15}$$

Note that

$$\Im(z+s_n(z)) = \Im z + \Im s_n(z) \geqslant \Im z = v.$$

This implies that

$$|z+s_n(z)| \geqslant v. \tag{4.16}$$

In addition by the equality (3.3) we have

$$|\Im\Big(z+s_n(z)-\varepsilon_k\Big)| = v\Big(1+\alpha'(k)((W(k)-u\mathbf{I}_{n-1})^2+v^2\mathbf{I}_{n-1})^{-1}\alpha(k)\Big) \geqslant v. \tag{4.17}$$

The relations (4.15), (4.17) and Lemmas 4.1 and 4.2 together imply that

$$|\delta_n(z)| \leqslant C|s_n(z)+z|^{-2}\Big(\frac{1}{nv}+\frac{M}{nv^3}\Big).$$

Applying inequality (4.16) we get

$$|\delta_n(z)| \leqslant \frac{CM}{nv^5}. \tag{4.18}$$

. This completes the proof of the Lemma. □

It is well known that for $v > 0$, $s(z)$ satisfies the following equality

$$s(z) = -\frac{1}{z+s(z)}. \tag{4.19}$$

By the last equality and equality (4.6) we have

$$\begin{aligned} s_n(z)-s(z) &= -\frac{s(z)-s_n(z)}{(z+s_n(z))(s(z)+z)} + \delta_n(z) \\ &= \frac{(s(z)-s_n(z))s(z)}{(z+s_n(z))} + \delta_n(z). \end{aligned} \tag{4.20}$$

From (4.20) it follows that if $|s(z)|v^{-1} < 1$ then

$$|s(z)-s_n(z)| \leqslant \Big(1-\frac{|s(z)|}{v}\Big)^{-1}|\delta_n(z)|.$$

It is not difficult to check that for $|u|\geqslant 4$ and $v=1$
$$|s(z)|\leqslant\frac{1}{2}.$$
This implies that for $|u|\geqslant 4$ and $v=1$
$$|s_n(z)-s(z)|\leqslant 2|\delta_n(z)|. \tag{4.21}$$
For $|u|\leqslant 4$ and $v=1$ we rewrite the relation (4.6) in the form
$$s_n^2(z)+zs_n(z)+1=(s_n(z)+z)\delta_n(z).$$
use the equality (4.19) we arrive at
$$(s_n(z)-s(z))(s_n(z)+s(z)+z)=(s(z)+z)\delta_n(z)+(s_n(z)-s(z))\delta_n(z).$$
By (4.14) for $v=1$
$$|\delta_n(z)|\leqslant 1/4.$$
In addition, for $v=1$
$$|z+s_n(z)+s(z)|\geqslant\Im\Big(s_n(z)+s(z)+z\Big)\geqslant 1,$$
and
$$|z+s(z)|\leqslant 1+|z|\leqslant 6.$$
These inequalities imply that for $|u|\leqslant 4$ and $v=1$
$$|s_n(z)-s(z)|\leqslant 8|\delta_n(z)|. \tag{4.22}$$
Inequalities (4.18)–(4.22) together imply that for $v=1$ and any u
$$|s_n(z)-s(z)|\leqslant C|\delta_n(z)|. \tag{4.23}$$
Now we can bound the first integral in the right hand side of (2.3). Choose $V=1$. By (4.23)
$$\int_{-\infty}^{\infty}|s_n(u+iV)-s(u+iV)|du\leqslant C\int_{-\infty}^{\infty}|\delta_n(u+iV)|du. \tag{4.24}$$
The relations (4.13) and (4.6) together imply that
$$\begin{aligned}\int_{-\infty}^{\infty}|\delta_n(u+iV)|du&\leqslant\frac{C}{n}\int_{-\infty}^{\infty}|s_n(u+iV)+u+iV|^{-2}du\\&\leqslant\frac{C}{n}\Big(\int_{-\infty}^{\infty}|s_n(u+iV)|^2du+\int_{-\infty}^{\infty}|\delta_n(u+iV)|^2du\Big).\end{aligned} \tag{4.25}$$
For $n\geqslant 2C$ this implies that
$$\int_{-\infty}^{\infty}|\delta_n(u+iV)|du\leqslant\frac{2C}{n}\int_{-\infty}^{\infty}|s_n(u+iV)|^2du.$$
Repeating the arguments in B93, inequality (4.27), we get
$$\int_{-\infty}^{\infty}|s_n(z)|^2du\leqslant\mathbf{E}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x-u)^2+v^2}dudF_n(x)\leqslant v^{-1}. \tag{4.26}$$
The inequalities (4.24)–(4.26) imply that for $z=u+iV$ and $V=1$
$$\int_{-\infty}^{\infty}|s_n(z)-s(z)|du\leqslant\frac{C}{n} \tag{4.28}$$

5. An improved bound for $\delta_n(z)$.

To prove the result of Theorem 1.1 we need the following

Lemma 5.1. *Under the conditions of Theorem 1.1 there exists an absolute positive constant C such that for any $1 \geqslant v \geqslant C\sqrt{M}n^{-\frac{1}{2}}$ and $u \in [-2, 2]$ the following inequality holds*

$$\Im(z + \delta_n(z)) > 0, \quad \text{where } z = u + iv.$$

For some positive constants $a_1 < a_2$ we shall use the following condition

$$a_1 \leqslant |s_n(z) + z| \leqslant a_2. \tag{5.1}$$

Lemma 5.2. *Assume that (5.1) holds. Then there exist some positive constants $C_1(a_1, a_2)$ and $C_2(a_1, a_2)$ depending on a_1 and a_2 such that for $u \in [-2, 2]$ and $1 \geqslant v \geqslant C_1(a_1, a_2)\sqrt{M}n^{-\frac{1}{2}}$*

$$|\delta_n(z)| \leqslant \frac{C_2(a_1, a_2)}{nv}.$$

To prove the Lemmas 5.1, 5.2 we need some additional results. Recall that the matrix $W(d, k)$ is obtained from W by deleting the d-th and k-th rows and columns, and $\alpha(d, k)$ denotes the vector obtained from the d-th column of $\sqrt{n}W$ by deleting the d-th and k-th entries. Following B93 set

$$R_d(k) = \Big(W(d, k) - z\mathbf{I}_{n-2}\Big)^{-1}, \quad s_{nd}(z) = \frac{1}{n}\mathbf{E}\ \mathrm{Tr}\, R_d, \quad s_{nd(k)}(z) = \frac{1}{n}\mathbf{E}\ \mathrm{Tr}\, R_d(k),$$

and

$$\varepsilon_d(k) = \frac{1}{\sqrt{n}}X_{dd} - \frac{1}{n}\alpha'(d, k)R_d(k)\alpha(d, k) + s_{nd}(z).$$

Lemma 5.3. *Assume that (5.1) holds. Then there exist some positive absolute constant C_1 and positive constant $C_2(a_1, a_2)$ depending on a_1 and a_2 such that for $u \in [-2, 2]$ and $1 \geqslant v \geqslant C_1 n^{-\frac{1}{2}}$*

$$\mathbf{E}\,|\varepsilon_d(k)|^2 \leqslant \mathbf{E}\,|\varepsilon_d|^2 + \frac{C_2(a_1, a_2)}{nv}.$$

Proof. We have the obvious equalities

$$\mathbf{E}\,|\varepsilon_d(k)|^2 = |\mathbf{E}\,\varepsilon_d(k)|^2 + \mathbf{E}\,|\varepsilon_d(k) - \mathbf{E}\,\varepsilon_d(k)|^2 \tag{5.2}$$

and

$$\mathbf{E}\,|\varepsilon_d|^2 = |\mathbf{E}\,\varepsilon_d|^2 + \mathbf{E}\,|\varepsilon_d - \mathbf{E}\,\varepsilon_d|^2. \tag{5.3}$$

By Lemma 4.1

$$\max\{|\mathbf{E}\,\varepsilon_d(k)|, |\mathbf{E}\,\varepsilon_d|\} \leqslant \frac{1}{nv}. \tag{5.4}$$

Furthermore,

$$\begin{aligned}\mathbf{E}\,|\varepsilon_d(k) - \mathbf{E}\,\varepsilon_d(k)|^2 = \frac{1}{n} + \frac{1}{n^2}\mathbf{E}\,|\alpha'(d, k)R_d(k)\alpha(d, k) - \mathrm{Tr}\, R_d(k)|^2 \\ + \frac{1}{n^2}\mathbf{E}\,|\,\mathrm{Tr}\, R_d(k) - \mathbf{E}\ \mathrm{Tr}\, R_d(k)|^2.\end{aligned} \tag{5.5}$$

Similarly,

$$\mathbf{E}\,|\varepsilon_d-\mathbf{E}\,\varepsilon_d|^2=\frac{1}{n}+\frac{1}{n^2}\mathbf{E}\,|\alpha'(d)R_d\alpha(d)-\operatorname{Tr}R_d|^2$$
$$+\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr}R_d-\mathbf{E}\,\operatorname{Tr}R_d|^2. \tag{5.6}$$

Note that

$$\mathbf{E}\,|\alpha'(d)R_d\alpha(d)-\operatorname{Tr}R_d|^2=\sum_{j\neq d}\mathbf{E}\,X_{dj}^4|R_d(j,j)|^2+2\sum_{\substack{i\neq j,\\ i\neq d,j\neq d}}|R_d(i,j)|^2$$

This implies that

$$\mathbf{E}\,|\alpha'(d)R_d\alpha(d)-\operatorname{Tr}R_d|^2\leqslant CM\mathbf{E}\,\operatorname{Tr}|R_d|^2. \tag{5.7}$$

Similarly,

$$\mathbf{E}\,|\alpha'(d,k)R_d(k)\alpha(d,k)-\operatorname{Tr}R_d(k)|^2\leqslant CM\mathbf{E}\,\operatorname{Tr}|R_d(k)|^2. \tag{5.8}$$

The relations (5.2)–(5.8) imply that

$$\Big|\mathbf{E}\,|\varepsilon_d(k)|^2-\mathbf{E}\,|\varepsilon_d|^2\Big|\leqslant\frac{C}{n^2v^2}+\frac{CM}{n^2}\Big(\mathbf{E}\,\operatorname{Tr}|R_d|^2+\mathbf{E}\,\operatorname{Tr}|R_d(k)|^2\Big)$$
$$+\frac{1}{n^2}\Big|\mathbf{E}\,|\operatorname{Tr}R_d-\mathbf{E}\,\operatorname{Tr}R_d|^2-\mathbf{E}\,|\operatorname{Tr}R_d(k)-\mathbf{E}\,\operatorname{Tr}R_d(k)|^2\Big|. \tag{5.9}$$

Denote by λ_j^d (respectively $\lambda_j^{d,k}$) and by $F_n^d(x)$ (respectively $F_n^{d,k}(x)$) the eigenvalues and the spectral distribution function of the matrices $W(d)$ (respectively $W(d,k)$). With this notation we have

$$\mathbf{E}\,\frac{1}{n}\operatorname{Tr}|R_d|^2=\int_{-\infty}^{\infty}\frac{1}{|x-z|^2}d\,\mathbf{E}\,F_n^d(x)=\frac{\Im s_{nd}(z)}{v}\leqslant\frac{|z+s_{nd}(z)|}{v}. \tag{5.10}$$

Similarly,

$$\mathbf{E}\,\frac{1}{n}\operatorname{Tr}|R_d(k)|^2=\frac{n-2}{n}\frac{\Im s_{nd(k)}(z)}{v}\leqslant\frac{|z+s_{nd(k)}(z)|}{v}. \tag{5.11}$$

The inequalities (5.10), (5.11), (5.1) together imply

$$\frac{1}{n^2}\big(\mathbf{E}\,\operatorname{Tr}|R_d|^2+\mathbf{E}\,\operatorname{Tr}|R_d(k)|^2\big)\leqslant\frac{C(a_1,a_2)}{nv}. \tag{5.12}$$

Finally note that

$$\frac{1}{n^2}\Big|\mathbf{E}\Big|\operatorname{Tr}R_d-\mathbf{E}\,\operatorname{Tr}R_d\Big|^2-\mathbf{E}\Big|\operatorname{Tr}R_d(k)-\mathbf{E}\,\operatorname{Tr}R_d(k)\Big|^2\Big|$$
$$\leqslant\frac{2}{n^2}\mathbf{E}\Big|\operatorname{Tr}R_d-\operatorname{Tr}R_d(k)\Big|\Big|\operatorname{Tr}R_d+\operatorname{Tr}R_d(k)\Big|$$
$$\leqslant\frac{C}{n^2v}\mathbf{E}\Big|\operatorname{Tr}R_d+\operatorname{Tr}R_d(k)\Big|. \tag{5.13}$$

The last inequality follows from Lemma 3.3. It is obvious that

$$\frac{1}{n}\mathbf{E}\,|\operatorname{Tr} R_d| \leqslant \frac{1}{n}|\mathbf{E}\ \operatorname{Tr} R_d| + \Big(\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d|^2\Big)^{\frac{1}{2}}. \tag{5.14}$$

Following B93 we introduce the random variables $\gamma_k(k) = 0$, and for $k \neq d$

$$\gamma_k(d) = \mathbf{E}_{\,k-1} \operatorname{Tr} R_d - \mathbf{E}_{\,k} \operatorname{Tr} R_d = \mathbf{E}_{\,k-1}\sigma_k(d) - \mathbf{E}_{\,k}\sigma_k(d),$$

where

$$\sigma_k(d) = \operatorname{Tr} R_d - \operatorname{Tr} R_d(k).$$

By Lemma 3.3

$$|\sigma_k(d)| \leqslant v^{-1}. \tag{5.15}$$

Since

$$\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d = \sum_{k=1}^{n} \gamma_k(d) \tag{5.16}$$

and random variables $\gamma_k(d)$, $k = 1, \dots n$ are uncorrelated, we have

$$\mathbf{E}\,\Big|\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d\Big|^2 \leqslant \sum_{k=1}^{n} \mathbf{E}\,|\gamma_k(d)|^2. \tag{5.17}$$

Inequalities (5.14)–(5.17) together imply

$$\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d|^2 \leqslant \frac{C}{nv^2} \tag{5.18}$$

From (5.19) it follows that for $v \geqslant Cn^{-1/2}$

$$\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d|^2 \leqslant C. \tag{5.19}$$

Similarly,

$$\frac{1}{n^2}\mathbf{E}\,|\operatorname{Tr} R_d(k) - \mathbf{E}\ \operatorname{Tr} R_d(k)|^2 \leqslant C. \tag{5.20}$$

By (5.1) and (4.1) we have

$$\begin{aligned} &|s_n(z)| \leqslant |s_n(z) + z| + |z| \leqslant a_2 + |z| \leqslant a_2 + 3, \\ &|\delta_n(z)| \leqslant |\frac{1}{z + s_n(z)}| + |s_n(z)| \leqslant C_1^{-1} + C_2 + |z| \leqslant a_1^{-1} + a_2 + 3, \end{aligned} \tag{5.21}$$

and by Lemma 3.3 we have

$$\max\Big\{\big|s_{nd}(z) - s_n(z)\big|, \big|s_{nd(k)}(z) - s_n(z)\big|\Big\} \leqslant \frac{C}{nv} \tag{5.22}$$

Inequalities (5.1)and (5.22) together imply that

$$\begin{aligned} &\max\Big\{\frac{1}{n}|\mathbf{E}\ \operatorname{Tr} R|, \frac{1}{n}|\mathbf{E}\ \operatorname{Tr} R_d|, \frac{1}{n}|\mathbf{E}\ \operatorname{Tr} R_d(k)|\Big\} \\ &\qquad = \max\Big\{|s_n(z)|, |s_{nd}(z)|, |s_{nd(k)}(z)|\Big\} \leqslant C(a_1, a_2) \end{aligned} \tag{5.23}$$

From the relations (5.13), (5.14) and (5.19)–(5.23) it follows that

$$\frac{1}{n^2}|\mathbf{E}\,|\operatorname{Tr} R_d - \mathbf{E}\ \operatorname{Tr} R_d|^2 - \mathbf{E}\,|\operatorname{Tr} R_d(k) - \mathbf{E}\ \operatorname{Tr} R_d(k)|^2| \leqslant \frac{C(a_1, a_2)}{nv}. \tag{5.24}$$

The inequalities (5.9), (5.12) and (5.24) together complete the proof. □

Now we shall improve the bound (4.8) for $\mathbf{E}\,|\varepsilon_d|^2$.

Lemma 5.4. *Assume that condition* (5.1) *holds. Then there exist a positive absolute constants* C_1 *and positive constants* $C_2(a_1, a_2)$ *depending on* a_1 *and* a_2 *such that for any* $1 \geqslant v \geqslant C_1 n^{-1/2}$, $u \in [-2, 2]$ *we have*

$$\mathbf{E}\,|\varepsilon_d|^2 \leqslant \frac{C_2(a_1, a_2)M}{nv}, \quad d = 1, \dots, n.$$

Proof. We start from the following equality

$$\mathbf{E}\,|\varepsilon_d|^2 = \frac{1}{n} + \frac{1}{n^2}\mathbf{E}\,|\alpha(d)' R_d \alpha(d) - \operatorname{Tr} R_d|^2 \\ + \mathbf{E}\,\Big|\frac{1}{n}\operatorname{Tr} R_d - \frac{1}{n}\mathbf{E}\,\operatorname{Tr} R_d\Big|^2 + |\mathbf{E}\,\varepsilon_d|^2. \tag{5.25}$$

Since the random matrix R_d and the random vector $\alpha(d)$ are independent, we have

$$\frac{1}{n^2}\mathbf{E}\,|\alpha(d)' R_d \alpha(d) - \operatorname{Tr} R_d|^2 \leqslant \frac{2M}{n^2}\mathbf{E}\,\operatorname{Tr}|R_d|^2. \tag{5.26}$$

By inequalities (5.1) and (5.10) inequality (5.26) yields

$$\frac{1}{n^2}\mathbf{E}\,|\alpha(d)' R_d \alpha(d) - \operatorname{Tr} R_d|^2 \leqslant \frac{2C(a_1, a_2)M}{nv}. \tag{5.27}$$

For the sake of completeness we repeat here some arguments of B93. Introduce the random variables $\gamma_k(d)$ such that $\gamma_k(k) = 0$ and for $d \neq k$

$$\gamma_k(d) = \mathbf{E}\,_{k-1}\sigma_k(d) - \mathbf{E}\,_k\sigma_k(d),$$

where

$$\sigma_k(d) = \operatorname{Tr} R_d - \operatorname{Tr} R_d(k),$$

and $\mathbf{E}\,_k$ denotes the conditional expectation given $\big\{X_{ij}, k+1 \leqslant i \leqslant j \leqslant n\big\}$. It is obvious that

$$\mathbf{E}\,\Big|\frac{1}{n}\operatorname{Tr} R_d - \frac{1}{n}\mathbf{E}\,\operatorname{Tr} R_d\Big|^2 \leqslant \frac{1}{n^2}\sum_{k=1}^n \mathbf{E}\,|\gamma_k(d)|^2. \tag{5.28}$$

By definition of $\sigma_k(d)$ and Lemma 3.3 we get

$$\sigma_k(d) = -\frac{1 + \frac{1}{n}\alpha'(k,d) R_k(d)^2 \alpha(k,d)}{z + s_{nk(d)} - \varepsilon_{k(d)}}, \tag{5.29}$$

where $\alpha(k,d)$ is the vector obtained from the kth column of W by deleting kth and dth entries. We can represent $\sigma_k(d)$ in the form

$$\sigma_k(d) = \sigma_k^{(1)}(d) + \sigma_k^{(2)}(d) + \sigma_k^{(3)}(d) \tag{5.30}$$

where

$$\sigma_{k(d)}^{(1)} = -\frac{1 + \frac{1}{n}\operatorname{Tr}(R_k^2(d))}{z + s_{nk(d)}(z)},$$

$$\sigma_k^{(2)}(d) = -\frac{\varepsilon_k(d)\sigma_k(d)}{z + s_{nk(d)}(z)},$$

$$\sigma_k^{(3)}(d) = -\frac{\frac{1}{n}\alpha'(k,d)R_k^2(d)\alpha(k,d) - \frac{1}{n}\operatorname{Tr}(R_k^2(d))}{z + s_{nk(d)}(z)}.$$

Similar to (5.27) we obtain

$$\frac{1}{n^2}\mathbf{E}\,|\alpha'(k,d)R_k^2(d)\alpha(k,d) - \operatorname{Tr} R_k^2(d)|^2 \leqslant \frac{2M}{n^2}\mathbf{E}\,\operatorname{Tr}|R_k^2(d)|^2. \tag{5.31}$$

Let $R_k^2(d) = \left(\rho_{i,j}(k,d)\right)_{i,j=1}^n$. Since $|\sigma_k(d)| \leqslant v^{-1}$ and

$$\mathbf{E}_{k-1}\sigma_{k(d)}^{(1)} - \mathbf{E}_{k}\sigma_{k(d)}^{(1)} = 0,$$

we get

$$\mathbf{E}\,|\gamma_k(d)|^2 \leqslant \frac{2\mathbf{E}\,|\varepsilon_k(d)|^2}{v^2|z + s_{nk(d)}(z)|^2} + \frac{4M\sum_{i,j=1}^n \mathbf{E}\,|\rho_{ij}(k,d)|^2}{n^2|z + s_{nk(d)}(z)|^2}. \tag{5.32}$$

First note that

$$|z + s_n(z)| - |s_n(z) - s_{nk(d)}(z)| \leqslant |z + s_{nk(d)}(z)| \leqslant |z + s_n(z)| + |s_n(z) - s_{nk(d)}(z)|. \tag{5.33}$$

Using Lemma 3.3 we obtain that

$$|s_n(z) - s_{nk(d)}(z)| \leqslant \frac{2}{nv}. \tag{5.34}$$

The condition (5.1) and inequalities (5.32) and (5.34) together imply that

$$\frac{a_1}{2} \leqslant |z + s_{nk(d)}(z)| \leqslant \frac{3a_2}{2}. \tag{5.35}$$

Note also that by the relations (5.11)

$$\begin{aligned}\frac{1}{n}\sum_{i,j=1}^n \mathbf{E}\,|\rho_{ij}(k,d)|^2 &= \int_{-\infty}^{\infty}\frac{1}{|x-z|^4}dF_n^{d,k}(x)\\ &\leqslant v^{-2}\int_{-\infty}^{\infty}\frac{1}{|x-z|^2}dF_n^{d,k}(x) \leqslant v^{-3}|s_{nk(d)}(z) + z| \leqslant C(a_1,a_2)v^{-3}.\end{aligned} \tag{5.36}$$

The inequalities (5.32)–(5.34) together imply that

$$\mathbf{E}\,|\gamma_k(d)|^2 \leqslant \frac{C(a_1,a_2)\mathbf{E}\,|\varepsilon_k(d)|^2}{v^2} + \frac{C(a_1,a_2)M}{nv^3} \tag{5.37}$$

Using Lemma 5.3 we get

$$\mathbf{E}\,|\gamma_k(d)|^2 \leqslant \frac{C(a_1,a_2)\mathbf{E}\,|\varepsilon_k|^2}{v^2} + \frac{C(a_1,a_2)M}{nv^3} \tag{5.38}$$

The equality (5.25) and the inequalities (5.27), (5.28) and (5.38) and Lemma 4.1 together imply

$$\mathbf{E}\,|\varepsilon_d|^2 \leqslant \frac{C(a_1,a_2)M}{nv} + \frac{C(a_1,a_2)\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k|^2}{n^2v^2} + \frac{C(a_1,a_2)M}{n^2v^3} \tag{5.39}$$

If we choose $v \geqslant \sqrt{2C(a_1,a_2)}n^{-\frac{1}{2}}$ we get

$$\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k|^2 \leqslant \frac{2C(a_1,a_2)M}{nv^3} + \frac{2C(a_1,a_2)M}{v} \tag{5.40}$$

and

$$\mathbf{E}\,|\varepsilon_d|^2 \leqslant \frac{2C(a_1,a_2)M}{nv} + \frac{2C(a_1,a_2)M}{n^3v^5} + \frac{2C(a_1,a_2)M}{n^2v^3}.$$

From the last inequality it follows that for $v \geqslant n^{-\frac{1}{2}}$

$$\mathbf{E}\,|\varepsilon_d|^2 \leqslant \frac{C(a_1,a_2)M}{nv}. \tag{5.41}$$

This proves Lemma 5.4. □

Lemma 5.5. *Assume that condition* (5.1) *holds. Then there exist some positive constants $C_1(a_1,a_2)$ and $C_2(a_1,a_2)$ depending on a_1 and a_2 such that for any $1 \geqslant v \geqslant C_1(a_1,a_2)\sqrt{M}n^{-1/2}$ the following inequality holds*

$$\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|R(k,k)|^2 \leqslant C_2(a_1,a_2).$$

Proof. We represent ε_k for $k=1,\dots,n$ in the form

$$\varepsilon_k = \sum_{\nu=1}^5 \varepsilon_k^{(\nu)}, \tag{5.42}$$

where

$$\varepsilon_k^{(1)} = \frac{1}{\sqrt{n}}X_{kk}, \quad \varepsilon_k^{(2)} = \frac{1}{n}\sum_{j=1,j\neq k}^n (X_{kj}^2-1)R_k(j,j),$$

$$\varepsilon_k^{(3)} = \frac{1}{n}\sum_{\substack{j=1,\\ j\neq k}}^n \sum_{\substack{l=1,\\ l\neq k,l\neq j}}^n X_{kj}X_{kl}R_k(j,l)$$

$$\varepsilon_k^{(4)} = \frac{1}{n}(\operatorname{Tr}R_k - \operatorname{Tr}R), \quad \varepsilon_k^{(5)} = \frac{1}{n}(\operatorname{Tr}R - E\operatorname{Tr}R)$$

The relations (4.5), (5.42), and (5.1) together imply

$$\mathbf{E}\,|R(k,k)|^2 \leqslant 2a_1^{-2} + 2\sum_{\nu=1}^5 \mathbf{E}\,|\varepsilon_k^{(\nu)}|^2|R(k,k)|^2. \tag{5.43}$$

By equality (4.5) and inequality (4.17) we have

$$|R(k,k)| \leqslant v^{-1}.$$

From here it follows that

$$\mathbf{E}\,|\varepsilon_k^{(1)}|^2|R(k,k)|^2 \leqslant \frac{1}{nv^2}. \tag{5.44}$$

By Lemma 3.3 we have

$$\mathbf{E}\,|\varepsilon_k^{(4)}|^2|R(k,k)|^2 \leqslant \frac{1}{n^2v^4}. \tag{5.45}$$

Using the Rosenthal's inequality for quadratic forms or direct calculation we get

$$\mathbf{E}\left|\varepsilon_k^{(3)}\right|^4 \leqslant \frac{CM^2}{n^4}\mathbf{E}\Big(\sum_{\substack{l,m=1\\ l\neq k, m\neq k}}^n |R_k(l,m)|^2\Big)^2. \tag{5.46}$$

We can write

$$\mathbf{E}\Big(\sum_{\substack{l,m=1\\ l\neq k, m\neq k}}^n |R_k(l,m)|^2\Big)^2 \leqslant |\mathbf{E}\,\operatorname{Tr}|R_k|^2|^2 + \mathbf{E}\,|\operatorname{Tr}|R_k|^2 - \mathbf{E}\,\operatorname{Tr}|R_k|^2|^2.$$

By inequalities (5.42) and (5.421) we have

$$\mathbf{E}\,\|R_k\|^2 \leqslant \frac{C(a_1,a_2)n}{v}. \tag{5.47}$$

Similar as bounds for $\mathbf{E}\left|\operatorname{Tr}R_d - \mathbf{E}\,\operatorname{Tr}R_d\right|^2$ we introduce the random variables

$$\widetilde{\gamma}_k(d) = \mathbf{E}_{\,k-1}\operatorname{Tr}|R_d|^2 - \mathbf{E}_{\,k}\operatorname{Tr}|R_d|^2 = \mathbf{E}_{\,k-1}\widetilde{\sigma}_k(d) - \mathbf{E}_{\,k}\widetilde{\sigma}_k(d)$$

with

$$\widetilde{\sigma}_k(d) = \operatorname{Tr}|R_d|^2 - \operatorname{Tr}|R_d(k)|^2$$

Since $\widetilde{\gamma}_k(d)$ are orthogonal $k=1,\dots,n$, then

$$\frac{1}{n^4}\mathbf{E}\,|\operatorname{Tr}|R_d^2| - \mathbf{E}\,\operatorname{Tr}|R_d|^2|^2 \leqslant \frac{1}{n^4}\sum_{k=1}^n \mathbf{E}\,|\widetilde{\gamma}_k(d)|^2.$$

Note that

$$|\operatorname{Tr}|R_d|^2 - \operatorname{Tr}|R_d(k)|^2| = \frac{1}{v}|\Im\Big(\operatorname{Tr}R_d - \operatorname{Tr}R_d(k)\Big)| \leqslant \frac{1}{v^2}.$$

This implies that $|\widetilde{\gamma}_k(d)| \leqslant \frac{C}{v^2}$ and

$$\frac{1}{n^4}\mathbf{E}\,|\operatorname{Tr}|R_d^2| - \mathbf{E}\,\operatorname{Tr}|R_d|^2|^2 \leqslant \frac{C}{n^3v^4}. \tag{5.48}$$

The inequalities (5.46)–(5.48) together imply that

$$\mathbf{E}\,|\varepsilon_k^{(3)}|^4 \leqslant \frac{CM^2}{n^2v^2}. \tag{5.49}$$

Using Cauchy's inequality we obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k^{(3)}|^2|R(k,k)|^2 &\leqslant v^{-1}\Big(\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k^{(3)}|^4\Big)^{1/2}\Big(\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|R(k,k)|^2\Big)^{1/2}\\ &\leqslant \frac{CM}{nv}\Big(\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|R(k,k)|^2\Big)^{1/2}. \end{aligned}\tag{5.50}$$

Notice that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k^{(5)}|^2|R(k,k)|^2 &= \mathbf{E}\,|\varepsilon_1^{(5)}|^2\Big(\frac{1}{n}\sum_{k=1}^n |R(k,k)|^2\Big)\\ \leqslant \mathbf{E}\,|\varepsilon_1^{(5)}|^2\Big(\frac{1}{n}\sum_{k,j=1}^n |R(k,j)|^2\Big) &= v^{-1}\mathbf{E}\,|\varepsilon_1^{(5)}|^2\Im\Big(\frac{1}{n}\operatorname{Tr}R\Big)\\ &\leqslant v^{-1}\mathbf{E}\,|\varepsilon_1^{(5)}|^3 + \frac{|s_n(z)|}{v}\mathbf{E}\,|\varepsilon_1^{(5)}|^2. \end{aligned}\tag{5.51}$$

By Rosenthal's inequality for martingale (see Hall and Heyde (1980), p24) we obtain

$$\mathbf{E}\,|\operatorname{Tr}R - \mathbf{E}\,\operatorname{Tr}R|^3 \leqslant C\sqrt{n}\sum_{k=1}^n \mathbf{E}\,|\gamma_k|^3. \tag{5.52}$$

Inequalities (5.39) and (5.42) together imply that for $1 \geqslant v \geqslant C_1(a_1,a_2)n^{-1/2}$

$$\mathbf{E}\,|\gamma_k|^2 \leqslant \frac{C(a_1,a_2)}{nv^3}.$$

Since $|\gamma_k| \leqslant 2v^{-1}$, we get

$$\mathbf{E}\,|\gamma_k|^3 \leqslant \frac{C(a_1,a_2)}{nv^4}. \tag{5.53}$$

From inequalities (5.52) and (5.53) we obtain

$$\frac{1}{n^3}\mathbf{E}\,|\operatorname{Tr}R - \mathbf{E}\,\operatorname{Tr}R|^3 \leqslant \frac{C(a_1,a_2)}{n^{\frac{5}{2}}v^4}. \tag{5.54}$$

Inequalities (5.42), (5.25), (5.51) and (5.53) together imply

$$\frac{1}{n}\sum_{k=1}^n \mathbf{E}\,|\varepsilon_k^{(5)}|^2|R(k,k)|^2 \leqslant \frac{C(a_1,a_2)}{\sqrt{n^5v^{10}}} + \frac{C(a_1,a_2)}{nv^2}. \tag{5.55}$$

Finally, note that

$$\mathbf{E}\,|\varepsilon_k^{(2)}|^2|R(k,k)|^2 \leqslant \frac{1}{v^2}\mathbf{E}\,|\varepsilon_k^{(2)}|^2 \leqslant \frac{C(a_1,a_2)M}{nv^2}\Big(\frac{1}{n}\sum_{j=1,j\neq k}^n \mathbf{E}\,|R_k(j,j)|^2\Big). \tag{5.56}$$

The inequalities (5.43), (5.44), (5.45), (5.50), (5.55) and (5.56) together imply that for $1 \geqslant v \geqslant C_1 n^{-1/2}$

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2 \leqslant \frac{C(a_1,a_2)M}{nv^2}\Big(\frac{1}{n}\sum_{k=1}^{n}\Big(\frac{1}{n}\sum_{j=1,j\neq k}^{n}\mathbf{E}\,|R_k(j,j)|^2\Big)\Big)+C(a_1,a_2). \tag{5.57}$$

Introduce now some integer number $m = m(n)$ depending on n such that $m((n-m)v)^{-1} \leqslant a_1/4$. Without loss of generality we can assume that $m \leqslant n/2$. Since
$|s_{n-l}(z) - s_{n-l-1}(z)| \leqslant 2/(n-l)v$ we get

$$\frac{a_1}{2} \leqslant \min_{1\leqslant l\leqslant m}|s_{n-l}(z)+z| \leqslant \max_{1\leqslant l\leqslant m}|s_{n-l}(z)+z| \leqslant \frac{3a_2}{2}.$$

Let $\mathbf{j}^{(r)} = (j_1,\dots,j_r)$ with $1 \leqslant j_1 \neq j_2 \ldots \neq j_r \leqslant n$, $r = 1,\dots,m$. Denote by $W(\mathbf{j}^{(r)})$ the matrix which is obtained from W by deleting the j_1-th, …, j_r-th rows and columns, and let

$$R_{\mathbf{j}^{(r)}} = \Big(\frac{\sqrt{n}}{\sqrt{n-r}}W(\mathbf{j}^{(r)}) - z\mathbf{I}_{n-r}\Big)^{-1}.$$

Arguing similar as in inequality (5.56) we get that uniformly for $r = 1,\dots,m-1$,

$$\frac{1}{n}\sum_{\substack{k=1,\\ k\notin\mathbf{j}^{(r)}}}^{n}\mathbf{E}\,|R_{\mathbf{j}^{(r)}}(k,k)|^2 \leqslant \frac{C(a_1,a_2)M}{nv^2}\left(\frac{1}{n}\sum_{\substack{k=1,\\ k\notin\mathbf{j}^{(r)}}}^{n}\Big(\frac{1}{n}\sum_{j=1,j\notin\mathbf{j}^{(r+1)}}^{n}\mathbf{E}\,|R_{\mathbf{j}^{(r+1)}}(j,j)|^2\Big)\right) + C(a_1,a_2). \tag{5.58}$$

Applying inequality (5.58) recursively we get for $1 \geqslant v \geqslant C_1 n^{-1/2}$,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2 &\leqslant \frac{C(a_1,a_2)M}{nv^2}\sum_{r=1}^{m-1}\Big(\frac{C(a_1,a_2)M}{nv^2}\Big)^r\\ &+\Big(\frac{C(a_1,a_2)M}{nv^2}\Big)^m\left(\frac{1}{n}\sum_{\substack{k=1,\\ k\notin\mathbf{j}^{(m-1)}}}^{n}\Big(\frac{1}{n}\sum_{\substack{j=1,\\ j\notin\mathbf{j}^{(m)}}}^{n}\mathbf{E}\,|R_{\mathbf{j}^{(m)}}(j,j)|^2\Big)\right)+C(a_1,a_2).\end{aligned} \tag{5.59}$$

Without loss of generality we can assume that

$$\frac{C(a_1,a_2)M}{nv^2} \leqslant \frac{1}{2}.$$

Similar to inequality (5.12) we get that

$$\frac{1}{n}\sum_{\substack{j=1,\\ j\notin\mathbf{j}^{(m)}}}^{n}\mathbf{E}\,|R_{\mathbf{j}^{(m)}}(j,j)|^2 \leqslant \mathbf{E}\,\|R_{\mathbf{j}^{(m)}}\|^2 \leqslant \frac{C}{v}. \tag{5.60}$$

The inequalities (5.59) and (5.60) together imply that

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2 \leqslant \frac{C(a_1,a_2)M}{nv^2}+\frac{1}{2^m}\frac{C}{v}. \tag{5.61}$$

Choosing $m=[C\log n]$ such that $2^{-m}\leqslant Cv$ concludes the proof. $\square$

Proof of Lemma 5.1. The equalities (4.4) and (4.5) imply that

$$|\delta_n(z)| \leqslant \frac{C}{|z+s_n(z)|^2}\Big(\frac{1}{n}\sum_{k=1}^{n}|\mathbf{E}\,\varepsilon_k|+\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k|^2|R(j,j)|\Big). \tag{5.62}$$

According to Lemmas 4.1 and inequality (5.1) we get

$$\frac{C}{|z+s_n(z)|^2}\Big(\frac{1}{n}\sum_{k=1}^{n}|\mathbf{E}\,\varepsilon_k|\Big) \leqslant \frac{C(a_1,a_2)}{nv}. \tag{5.63}$$

Using the representation (5.42) and inequality (5.1) we obtain

$$\frac{C}{|z+s_n(z)|^2}\Big(\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k|^2|R(j,j)|\Big) \leqslant C(a_1,a_2)\sum_{\nu=1}^{5}\Big(\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(\nu)}|^2|R(j,j)|\Big). \tag{5.64}$$

Similar to inequality (5.44) and by Lemma 5.4 we arrive at

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{((1)}|^2|R(k,k)| &\leqslant \Big(\frac{1}{n}\sum_{k=1}\mathbf{E}\,|\varepsilon_k^{(1)}|^4\Big)^{1/2}\Big(\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2\Big)^{1/2}\\ &\leqslant \frac{C(a_1,a_2)M}{n}.\end{aligned} \tag{5.65}$$

By Lemma 3.3 $|\varepsilon_k^{(4)}|\leqslant (nv)^{-1}$ and we have

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(4)}|^2|R(k,k)| \leqslant \frac{C(a_1,a_2)}{n^2v^2}\Big(\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2\Big)^{1/2} \leqslant \frac{C(a_1,a_2)}{nv}. \tag{5.66}$$

Similar to inequality (5.65) we get

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{((3)}|^2|R(k,k)| \leqslant \left(\frac{1}{n}\sum_{k=1}\mathbf{E}\,|\varepsilon_k^{(3)}|^4\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|R(k,k)|^2\right)^{1/2}$$

Applying inequality (5.48) and Lemma 5.4 we have

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(3)}|^2|R(k,k)| \leqslant \frac{C(a_1,a_2)}{nv}. \tag{5.67}$$

Finally note that

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(2)}|^2|R(k,k)| \leqslant \frac{1}{nv}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(2)}|^2 \leqslant \frac{C(a_1,a_2)M}{nv}\Big(\frac{1}{n}\sum_{j=1,j\neq k}\mathbf{E}\,|R_k(j,j)|^2\Big).$$

Applying Lemma 5.5 to the matrix $W(k)$ we get

$$\frac{1}{n}\sum_{k=1}^{n}\mathbf{E}\,|\varepsilon_k^{(2)}|^2|R(k,k)| \leqslant \frac{C(a_1,a_2)M}{nv}. \tag{5.68}$$

The inequalities (5.62)–(5.68) together imply that for $1 \geqslant v \geqslant C_1(a_1,a_2)n^{-1/2}$

$$|\delta_n(z)| \leqslant \frac{C(a_1,a_2)M}{nv},$$

which proves Lemma 5.2. □

Proof of Lemma 5.1. Assume that the imaginary part of $z+\delta_n(z)$ satisfies

$$\Im(z+\delta_n(z)) = 0. \tag{5.69}$$

Since

$$s_n(z)+z = -\frac{1}{z+s_n(z)} + \delta_n(z) + z$$

this immediately implies that

$$\Im(z+s_n(z)) = -\Im\Big\{\frac{1}{z+s_n(z)}\Big\}.$$

Note that if a a complex number such that $\Im a \neq 0$ and $\Im a = -\Im\big\{\frac{1}{a}\big\}$, then $|a| = 1$. Indeed,

$$\Im a = -\Im\Big\{\frac{1}{a}\Big\} = \frac{\Im a}{|a|}$$

Since $\Im a \neq 0$ this implies that

$$|a| = 1.$$

Let $a = z+s_n(z)$. Since $\Im(z+s_n(z)) \geqslant \Im z = v > 0$ this implies that

$$|z+s_n(z)| = 1.$$

Hence the condition (5.1) holds with $a_1 = a_2 = 1$ and we have

$$|\delta_n(z)| \leqslant \frac{CM}{nv}.$$

Then for any $v \geqslant 2n^{-\frac{1}{2}}\sqrt{CM}$,

$$|\delta_n(z)| \leqslant \frac{1}{4}v < v,$$

holds. But condition (5.69) implies that

$$|\delta_n(z)| \geqslant v,$$

which is a contradiction. Hence we conclude that $\Im\{z+\delta_n(z)\} \neq 0$ in the region $v \geqslant 2n^{-\frac{1}{2}}\sqrt{CM}$. From inequality (4.18) it follows for example that for $v = 1$

$\Im\{z+\delta_n(z)\} > 0$. Since the function $\Im\{z+\delta_n(z)\}$ is continuous in the region $v \geqslant C_1 n^{-\frac{1}{2}}$ we get that $\Im\{z+\delta_n(z)\} > 0$ for $v \geqslant C_1 n^{-\frac{1}{2}}$. This proves Lemma 5.1.□

Proof of Theorem 1.1. Let $v_0 = \max\{\gamma_0\Delta_n, n^{-\frac{1}{2}}C_1\sqrt{M}\}$ with a γ_0 such that $1 > \gamma_0 > 0$ to be choose later. The constant C_1 we choose such that the conclusion of Lemma 5.1 holds, that is for any $1 \geqslant v \geqslant v_0$ we have

$$\Im\{z+\delta_n(z)\} > 0.$$

Note that the constant C_1 does not depend on γ_0. In addition we have

$$\begin{aligned}|s_n(z)-s(z)| &= \Big|\int_{-\infty}^{\infty}\frac{1}{x-z}d\Big(\mathbf{E}\,F_n(x)-F(x)\Big)\Big| \\ &= \Big|\int_{-\infty}^{\infty}\frac{\mathbf{E}\,F_n(x)-F(x)}{(x-z)^2}dx\Big| \leqslant \frac{\Delta_n}{v} \leqslant \frac{1}{\gamma_0}.\end{aligned}$$

This implies that for $z = u+iv$ such that $u \in [-2,2]$ and $1 \geqslant v \geqslant v_0$,

$$|s_n(z)+z| \leqslant \frac{1}{\gamma_0} + 5. \tag{5.70}$$

We set $\varepsilon = v_0^{\frac{2}{3}}$. From the equality (4.6) it follows that

$$\Im\Big(z+s_n(z)\Big) = -\Im\Big(\frac{1}{z+s_n(z)}\Big) + \Im(z+\delta_n(z)) = \frac{\Im(z+s_n(z))}{|z+s_n(z)|^2} + \Im\Big(z+\delta(z)\Big).$$

The last equality and the equality (4.1) together imply that for $v \geqslant v_0$

$$\Im\Big(z+s_n(z)\Big)\Big(1-\frac{1}{|z+s_n(z)|^2}\Big) = \Im\Big(z+\delta(z)\Big) > 0. \tag{5.71}$$

From (5.71) we immediately obtain that

$$|z+s_n(z)| \geqslant 1. \tag{5.72}$$

From the inequalities (5.70) and (5.72) it follows that condition (5.1) holds with $a_1 = 1$, and $a_2 = \frac{1}{\gamma_0} + 5$. By Lemma 5.2 there exist constants $C_1(\gamma_0)$ and $C_2(\gamma_0)$ such that for any $v \geqslant C_1(\gamma_0)\sqrt{M}$ we have

$$|\delta_n(z)| \leqslant \frac{C_2(\gamma_0)M}{nv}. \tag{5.73}$$

Now we redefine v_0 as follows

$$v_0 := max\{v_0, C_1(\gamma_0)\sqrt{M}\}.$$

For $|u| \leqslant 2$ and $1 \geqslant v \geqslant v_0$ we rewrite the relation (4.6) in the form

$$s_n^2(z) + zs_n(z) + 1 = (s_n(z)+z)\delta_n(z).$$

use the equality (4.19) we arrive at

$$(s_n(z) - s(z))(s_n(z) + s(z) + z) = (s_n(z) + z)\delta_n(z). \tag{5.74}$$

Inequality (5.74) implies that

$$|s_n(z) - s(z)| \leqslant \frac{|z + s_n(z)|}{|s_n(z) + s(z) + z|}|\delta_n(z)|$$

By inequality (5.70) we have

$$|s_n(z) - s(z)| \leqslant Cv^{-1}|\delta_n(z)|. \tag{5.75}$$

This implies that for $z = u + iv$ such that $u \in I'_\varepsilon$ and $1 \geqslant v \geqslant v_0$

$$|s_n(z) - s(z)| \leqslant Cv^{-1}|\delta_n(z)|. \tag{5.76}$$

From (5.76) and (5.73) it follows that

$$|s_n(z) - s(z)| \leqslant \frac{C(\gamma_0)M}{nv^2}.$$

Choosing in Corollary 2.3 $V = 1$ and using the inequality (4.28) we get after integrating in u and v

$$\Delta_n \leqslant C_1 n^{-1} + C_2 v_0 + C_3(\gamma_0) M n^{-1} v_0^{-1}.$$

Since $v_0 \geqslant n^{-\frac{1}{2}} C_1 \sqrt{M}$ we get

$$\Delta_n \leqslant C(\gamma_0)\sqrt{M} n^{-\frac{1}{2}} + C_2 v_0$$

Recall that C_2 does not depend on γ_0. We choose $\gamma_0 = \frac{1}{2C_2}$.
If $v_0 = \max\{C_1, C_1(\gamma_0)\}\sqrt{M} n^{-\frac{1}{2}}$ then

$$\Delta_n \leqslant \max\{C_1, C_1(\gamma_0)\}\sqrt{M} n^{-\frac{1}{2}}.$$

If $v_0 = \gamma_0 \Delta_n$ then

$$\Delta_n \leqslant C(\gamma_0)\sqrt{M}(1 - C_2\gamma_0)^{-1} n^{-\frac{1}{2}} \leqslant 2C(\gamma_0)\sqrt{M} n^{-\frac{1}{2}}.$$

This completes the proof of the Theorem 1.1. □

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Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld 1
Germany
E-mail address: goetze@mathematik.uni-bielefeld.de

Aleksander Tikhomirov
Fakulty of Mathematics
Oktjabrskyi prospekt 55
167001, Syktyvkar
Russia
E-mail address: tikhomir@ssu.komi.com