# COHERENT FUNCTORS IN STABLE HOMOTOPY THEORY

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Let $\mathcal{S}$ be a compactly generated triangulated category, for example the stable homotopy category of CW-spectra. We call a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ into the category of abelian groups *coherent* if there exists an exact sequence

$$\mathrm{Hom}(D,-) \longrightarrow \mathrm{Hom}(C,-) \longrightarrow F \longrightarrow 0$$

such that $C$ and $D$ are compact objects in $\mathcal{S}$ (an object $X$ in $\mathcal{S}$ is *compact* if the representable functor $\mathrm{Hom}(X,-)$ preserves arbitrary coproducts).

The concept of a coherent functor has been introduced explicitly for abelian categories by Auslander [1], but it is also implicit in the work of Freyd on stable homotopy [6]. In this paper we characterize coherent functors in a number of ways and use them to study a wider class of functors $\mathcal{S} \to \mathrm{Ab}$ which share a weak exactness property. Another purpose of this paper is to investigate certain subcategories of $\mathcal{S}$ which are defined in terms of coherent functors.

In the category $\mathrm{Mod}\,\Lambda$ of modules over an associative ring $\Lambda$, the analogue of a compact object is a finitely presented module. This fact can be made precise (cf. the Appendix), and one has in this context the following classical result: a functor $F\colon \mathrm{Mod}\,\Lambda \to \mathrm{Ab}$ is coherent precisely if $F$ preserves products and filtered colimits. There is no obvious way to formulate such a characterization for compactly generated triangulated categories because filtered colimits rarely exist in triangulated categories. Nevertheless, we are able to characterize the coherent functors as follows.

**Theorem A.** *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is coherent.*
2. *$F$ preserves products and sends every homology colimit to a colimit.*
3. *$F$ preserves products and coproducts, and for every triangle $X \to Y \to Z \to \Sigma X$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact, provided that $0 \to \mathrm{Hom}(C,X) \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(C,Z) \to 0$ is exact for each compact $C$.*

*In the presence of Brown representability (for homology theories), there is a further equivalent condition:*

4. *$F$ preserves products and minimal weak filtered colimits of compact objects.*

Coherent functors satisfy a weak exactness property which seems to be interesting. To explain this, let us introduce the following notation. The full subcategory of compact objects in $\mathcal{S}$ is denoted by $\mathcal{F}$ and $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ denotes the category of additive functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups. For every object $X$ in $\mathcal{S}$ consider the functor

$$H_X = \mathrm{Hom}(-,X)|_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{op}} \longrightarrow \mathrm{Ab}.$$

This is an example of an exact functor. Recall that a functor from a triangulated category to the category of abelian groups is *exact* if it sends triangles to exact sequences. We call a triangle

$$(*) \qquad X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathcal{S}$ *pure* if the induced sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact, and therefore a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is said to be *short exact* if for every pure triangle $(*)$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.

For the stable homotopy category, exact and coproduct preserving functors have been characterized by Brown and Adams [1]; they are precisely the functors $F\colon \mathcal{S} \to \mathrm{Ab}$ that are 'represented' by an object $Y$ in $\mathcal{S}$ in the sense that

$$F(X) \cong \mathrm{Hom}(S, X \wedge Y)$$

for all $X$ in $\mathcal{S}$ (where $S$ denotes the sphere spectrum). One can also use the tensor-product

$$(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{F}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (H, G) \to H \otimes_{\mathcal{F}} G$$

to study functors $\mathcal{S} \to \mathrm{Ab}$. Recall that the tensor functor $- \otimes_{\mathcal{F}} G$ is determined by the fact that it preserves colimits and $H_X \otimes_{\mathcal{F}} G \cong G(X)$ for all $X$ in $\mathcal{F}$. It turns out that $F\colon \mathcal{S} \to \mathrm{Ab}$ is exact and preserves coproducts if and only if there is a functorial isomorphism

$$F(X) \cong H_X \otimes_{\mathcal{F}} G$$

for some exact functor $G\colon \mathcal{F} \to \mathrm{Ab}$. The following result characterizes the functors which are 'represented' by an arbitrary functor $G\colon \mathcal{F} \to \mathrm{Ab}$.

**Theorem B.** *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is short exact and preserves coproducts.*
2. *There exist an additive functor $G\colon \mathcal{F} \to \mathrm{Ab}$ and a functorial isomorphism $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all $X$ in $\mathcal{S}$.*
3. *There exist a filtered diagram $(F_i)_{i \in \mathcal{I}}$ of coherent functors and a functorial isomorphism $F(X) \cong \mathrm{colim}_i F_i(X)$ for all $X$ in $\mathcal{S}$.*

Let us consider the collection of all coherent functors $\mathcal{S} \to \mathrm{Ab}$ which we denote by $\mathrm{Coh}\,\mathcal{S}$. In fact, $\mathrm{Coh}\,\mathcal{S}$ is an abelian category if we take as maps the natural transformations. This category has been studied by Freyd in [6]. Here, we exhibit an interesting closure operation which is defined in terms of coherent functors. Given a class $\mathcal{C}$ of objects in $\mathcal{S}$, we define

$$\mathrm{Def}\,\mathcal{C} = \{X \in \mathcal{S} \mid F \in \mathrm{Coh}\,\mathcal{S} \text{ and } F(Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ implies } F(X) = 0\}.$$

For example, Freyd's Generating Hypothesis [6] for the stable homotopy category could be reformulated as follows.

**Generating Hypothesis (Freyd).** $\mathrm{Def}\{S^n \mid n \in \mathbb{Z}\} = \mathcal{S}$.

There is an explicit construction which produces all objects in $\mathrm{Def}\,\mathcal{C}$, at least if we assume Brown representability. We call an object $X$ the *reduced product* of a family of objects $(X_i)_{i \in I}$ in $\mathcal{S}$ with respect to a filter $\mathcal{U}$ on the set $I$, if

$$H_X \cong \mathrm{colim}_{J \in \mathcal{U}} \prod_{i \in J} H_{X_i}$$

where the filtered colimit is taken over the canonical projections $\prod_{i \in J_1} H_{X_i} \to \prod_{i \in J_2} H_{X_i}$ which are induced by the inclusions $J_2 \subseteq J_1$ of subsets $J_1, J_2 \in \mathcal{U}$. Note that a reduced product always exists; it is unique up to isomorphism and denoted by $\prod_{i \in I} X_i/\mathcal{U}$.

**Theorem C.** *Suppose that Brown representability holds for $\mathcal{S}$, and let $\mathcal{C}$ be a class of objects in $\mathcal{S}$. Then an object $X$ in $\mathcal{S}$ belongs to $\mathrm{Def}\,\mathcal{C}$ if and only if there is a pure triangle $X \to Y \to Z \to \Sigma X$ such that $Y$ is a reduced product of objects in $\mathcal{C}$.*

We say that a full subcategory $\mathcal{C}$ of $\mathcal{S}$ is *definable* if $\mathcal{C} = \operatorname{Def}\mathcal{C}$, equivalently if $\mathcal{C} = \{X \in \mathcal{S} \mid F_i(X) = 0 \text{ for all } i \in I\}$ for some family $(F_i)_{i\in I}$ of coherent functors. This concept has its origin in model theory of modules; in this context a definable subcategory corresponds to a complete theory of modules [18, 5]. There are three other concepts equivalent to definable subcategories:
• *Ziegler-closed subsets* of the set $\operatorname{Sp}\mathcal{S}$ of isomorphism classes of indecomposable pure-injective objects in $\mathcal{S}$. Recall that $X$ in $\mathcal{S}$ is *pure-injective* if for every pure triangle $X \to Y \to Z \to \Sigma X$ the first map is a section. A subset of $\operatorname{Sp}\mathcal{S}$ is Ziegler-closed if it is of the form $\mathcal{C} \cap \operatorname{Sp}\mathcal{S}$ for some definable subcategory $\mathcal{C}$ of $\mathcal{S}$.
• *Serre subcategories* of $\operatorname{Coh}\mathcal{S}$. These are full subcategories of $\operatorname{Coh}\mathcal{S}$ which are closed under forming subobjects, quotient objects, and extensions.
• *Cohomological ideals* in $\mathcal{F}$. These are ideals of maps in $\mathcal{F}$ which are of the form $\{\phi \in \mathcal{F} \mid F(\phi) = 0\}$ for some exact functor $F\colon \mathcal{F} \to \mathrm{Ab}$. For example, given $X$ in $\mathcal{S}$, the *annihilator*

$$\operatorname{Ann} X = \{\phi \in \mathcal{F} \mid \operatorname{Hom}(\phi, X) = 0\}$$

is cohomological.

**Fundamental Correspondence.** *There are bijections between*

- *the set of definable subcategories* $\mathcal{C}$ *of* $\mathcal{S}$*,*
- *the set of Ziegler-closed subsets* $\mathbf{U}$ *of* $\operatorname{Sp}\mathcal{S}$*,*
- *the set of Serre subcategories* $\mathcal{T}$ *of* $\operatorname{Coh}\mathcal{S}$*,*
- *the set of cohomological ideals* $\mathfrak{I}$ *in* $\mathcal{F}$*.*

*These bijections are defined as follows:*

$$\mathcal{C} \mapsto \begin{cases} \mathbf{U} = \mathcal{C} \cap \operatorname{Sp}\mathcal{S} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \textit{ for all } X \in \mathcal{C}\} \\ \mathfrak{I} = \bigcap_{X\in\mathcal{C}} \operatorname{Ann} X \end{cases}$$

$$\mathbf{U} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid \textit{there are } Y_i \in \mathbf{U} \textit{ and a pure triangle } X \to \prod_i Y_i \to Z \to \Sigma X\} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \textit{ for all } X \in \mathbf{U}\} \\ \mathfrak{I} = \bigcap_{X\in\mathbf{U}} \operatorname{Ann} X \end{cases}$$

$$\mathcal{T} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid F(X) = 0 \textit{ for all } F \in \mathcal{T}\} \\ \mathbf{U} = \{X \in \operatorname{Sp}\mathcal{S} \mid F(X) = 0 \textit{ for all } F \in \mathcal{T}\} \\ \mathfrak{I} = \{\phi \in \mathcal{F} \mid \operatorname{Im}\operatorname{Hom}(\phi, -) \in \mathcal{T}\} \end{cases}$$

$$\mathfrak{I} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid \mathfrak{I} \subseteq \operatorname{Ann} X\} \\ \mathbf{U} = \{X \in \operatorname{Sp}\mathcal{S} \mid \mathfrak{I} \subseteq \operatorname{Ann} X\} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F = \operatorname{Im}\operatorname{Hom}(\phi, -) \textit{ for some } \phi \in \mathfrak{I}\} \end{cases}$$

This correspondence is the analogue of a correspondence for module categories which is based on work of several mathematicians [18, 9, 5, 11]. For instance, Ziegler introduced the closed subsets of indecomposable pure-injective modules in model-theoretic terms [18]. In our setting, one obtains a topology on $\operatorname{Sp}\mathcal{S}$ by taking the Ziegler-closed subsets as closed subsets [11]. Examples of definable subcategories arise quite naturally. Take for instance a localization functor $L\colon \mathcal{S} \to \mathcal{S}$ which is smashing, i.e. $L$ preserves coproducts. Then the $L$-local objects form a definable subcategory [13]. Or take an endofinite object $X$ in $\mathcal{S}$ (in the sense of [14]). Then the direct factors of coproducts of copies of $X$ form a definable subcategory.

## 1. The functor category

**Purity.** We fix a triangulated category $\mathcal{S}$ and make the following additional assumptions:

- $\mathcal{S}$ has arbitrary coproducts;
- the isomorphism classes of compact objects in $\mathcal{S}$ form a set;
- $\operatorname{Hom}(C, X) = 0$ for all compact $C$ implies $X = 0$ for every object $X$ in $\mathcal{S}$.

A triangulated category satisfying these conditions is called *compactly generated*. The full subcategory of compact objects in $\mathcal{S}$ is always denoted by $\mathcal{F}$. Recall that $X$ in $\mathcal{S}$ is *compact* if the representable functor $\operatorname{Hom}(X, -)$ preserves arbitrary coproducts. Our basic tool is the category of additive functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ which we denote by $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. The *restricted Yoneda functor*

$$\mathcal{S} \longrightarrow (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{F}},$$

relates the triangulated structure of $\mathcal{S}$ to the abelian structure of $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. The functor identifies the full subcategory of pure-projective objects in $\mathcal{S}$ with the full subcategory of projective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$, and it identifies the full subcategory of pure-injective objects in $\mathcal{S}$ with the full subcategory of injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. We recall briefly the relevant definitions and refer to [13] for more details.

**Definition 1.1.** Let $\mathcal{S}$ be a compactly generated triangulated category.

(1) A map $X \to Y$ in $\mathcal{S}$ is a *pure monomorphism* if the map $H_X \to H_Y$ is a monomorphism. An object $X$ in $\mathcal{S}$ is *pure-injective* if every pure monomorphism $X \to Y$ is a split monomorphism.
(2) A map $Y \to Z$ in $\mathcal{S}$ is a *pure epimorphism* if the map $H_Y \to H_Z$ is an epimorphism. An object $Z$ in $\mathcal{S}$ is *pure-projective* if every pure epimorphism $Y \to Z$ is a split epimorphism.
(3) A triangle $X \to Y \to Z \to \Sigma X$ is *pure* if the sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact.

One can prove easily that an object in $\mathcal{S}$ is pure-projective if and only if it is a direct factor of a coproduct of compact objects. The following well-known lemma describes some essential properties of pure-projective objects.

**Lemma 1.2.** *Let $\mathcal{S}$ be a compactly generated triangulated category, and let $P$ be a projective object in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then there exists, up to isomorphism, a unique object $X$ in $\mathcal{S}$ such that $P \cong H_X$. Moreover, the map*

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(H_X, H_Y), \quad \alpha \mapsto H_\alpha,$$

*is an isomorphism for all $Y$ in $\mathcal{S}$.*

*Proof.* Every projective $P$ is a direct factor of some coproduct $\coprod_{i \in I} H_{C_i}$ of representable functors with $C_i \in \mathcal{F}$ for all $i \in I$. Assume first that $P = \coprod_{i \in I} H_{C_i}$. Then one takes $X = \coprod_{i \in I} C_i$ and the isomorphism $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(H_X, H_Y)$ is an immediate consequence of Yoneda's lemma. The general case reduces to the first. In fact, if $P$ is a proper direct factor of $\coprod_{i \in I} H_{C_i}$, then we get a corresponding idempotent in $\operatorname{End}(\coprod_{i \in I} C_i)$ which gives an object $X$ in $\mathcal{S}$ satisfying $P \cong H_X$ since idempotents in $\mathcal{S}$ split. $\square$

The next lemma describes some properties of pure-injective objects. The proof is essentially an application of Brown's representability theorem.

**Lemma 1.3.** *Let $\mathcal{S}$ be a compactly generated triangulated category, and let $I$ be an injective object in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then there exists, up to isomorphism, a unique object $Y$ in $\mathcal{S}$ such that $I \cong H_Y$. Moreover, the map*

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(H_X, H_Y), \quad \alpha \mapsto H_\alpha,$$

*is an isomorphism for all $X$ in $\mathcal{S}$.*

*Proof.* See Lemma 1.7 in [13]. □

**Injective envelopes.** We shall also need to use the fact that $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is a *Grothendieck category*, which as far as we are concerned means that it has injective envelopes [8]. The definition of an injective envelope can be reformulated as follows.

**Lemma 1.4.** *A monomorphism $\alpha\colon X \to Y$ is an injective envelope of $X$ if and only if $Y$ is an injective object and every endomorphism $\beta\colon Y \to Y$ satisfying $\beta \circ \alpha = \alpha$ is an isomorphism.*

**Brown representability.** Sometimes we shall use an additional assumption on the category $\mathcal{S}$. To this end recall that a functor from a triangulated category to the category of abelian groups is *exact* if it sends triangles to exact sequences. For example, every functor of the form $H_X$ is exact. In some cases also the converse is true. More precisely, one says that *Brown representability* holds for $\mathcal{S}$, if

- every exact functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to $H_X$ for some object $X$ in $\mathcal{S}$, and
- every natural transformation $H_X \to H_Y$ is of the form $H_\alpha$ for some map $\alpha\colon X \to Y$.

A classical theorem due to Brown and Adams states that Brown representability holds for the stable homotopy category [1].

**Flat functors.** Recall that there exists a *tensor product*

$$(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{F}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (F, G) \to F \otimes_{\mathcal{F}} G$$

where for any functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, the tensor functor $F \otimes_{\mathcal{F}} -$ is determined by the fact that it preserves colimits and $F \otimes_{\mathcal{F}} \mathrm{Hom}(X, -) \cong F(X)$ for all $X$ in $\mathcal{F}$. A functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is *flat* if the tensor functor $F \otimes_{\mathcal{F}} -$ is exact. The following well-known characterization will be needed.

**Lemma 1.5.** *For a functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ the following are equivalent:*

1. *$F$ is flat.*
2. *$F$ is an exact functor.*
3. *$F$ is a filtered colimit of representable functors.*

*Proof.* For the characterization of flatness via condition (2), see Lemma 2.7 in [13], for (3), see Theorem 3.2 in [16]. □

**Finitely presented functors.** Some of our constructions involve finitely presented functors. Let us recall that a functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is *finitely presented* if there exists an exact sequence

$$\mathrm{Hom}(-, C) \longrightarrow \mathrm{Hom}(-, D) \longrightarrow F \longrightarrow 0$$

with $C$ and $D$ in $\mathcal{F}$. This generalizes the concept of a finitely presented module, and we shall use a few basic facts about finitely presented functors which are well-known in the context of modules over a ring. For instance, every additive functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is a filtered colimit of finitely presented functors. The following characterization is another example.

**Lemma 1.6.** *For an additive functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ the following are equivalent:*

(1) *$F$ is finitely presented.*
(2) *The representable functor $\mathrm{Hom}(F,-)$ preserves filtered colimits.*
(3) *The tensor functor $F \otimes_{\mathcal{F}} -$ preserves products.*
(4) *The map $F \otimes_{\mathcal{F}} (\prod_i \mathrm{Hom}(C_i,-)) \to \prod_i (F \otimes_{\mathcal{F}} \mathrm{Hom}(C_i,-))$ is an isomorphism for every family $(C_i)_{i \in I}$ in $\mathcal{F}$.*

*Proof.* Adapt the proof for modules over a ring (cf. [17]). ☐

## 2. Weak colimits

A *diagram* in a category $\mathcal{C}$ is a functor $\mathcal{I} \to \mathcal{C}$, $i \to X_i$, from a small category $\mathcal{I}$ to $\mathcal{C}$. We denote such a diagram by $(X_i)_{i \in \mathcal{I}}$ and call a family of maps $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ a *cone* if $\mu_j \circ X_\lambda = \mu_i$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$.

**Definition 2.1.** Let $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ be a cone of a diagram $(X_i)_{i \in \mathcal{I}}$.

(1) The cone is a *weak colimit* of the diagram $(X_i)_{i \in \mathcal{I}}$ if for every cone $\nu_i\colon X_i \to Y$ $(i \in \mathcal{I})$ there exists a map $\alpha\colon X \to Y$ such that $\alpha \circ \mu_i = \nu_i$ for each $i \in \mathcal{I}$.
(2) The cone is *minimal* if every endomorphism $\alpha\colon X \to X$ satisfying $\alpha \circ \mu_i = \mu_i$ for each $i \in \mathcal{I}$ is an isomorphism.

If we require the factorization $\alpha\colon X \to Y$ in the definition of a weak colimit to be unique, this is the definition of a *colimit* which we denote by $\mathrm{colim}_{i \in \mathcal{I}} X_i$. Note that every colimit is a minimal weak colimit. A minimal weak colimit of a diagram $(X_i)_{i \in \mathcal{I}}$ is unique up to a (non-unique) isomorphism. Our terminology is borrowed from Auslander [3]. He calls a map $\alpha\colon X \to Y$ *left minimal* if every endomorphism $\beta\colon Y \to Y$ satisfying $\beta \circ \alpha = \alpha$ is an isomorphism. Viewing a cone of a diagram $\mathcal{I} \to \mathcal{C}$ as a map in the category of all functors $\mathcal{I} \to \mathcal{C}$, it is clear that this map is left minimal if and only if the cone is minimal.

**Definition 2.2.** Let $(X_i)_{i \in \mathcal{I}}$ be a diagram in a compactly generated triangulated category $\mathcal{S}$. A cone $X_i \to X$ $(i \in \mathcal{I})$ is called a *homology colimit* of the diagram $(X_i)_{i \in \mathcal{I}}$ if the induced map

$$\mathrm{colim}_{i \in \mathcal{I}} \mathrm{Hom}(C, X_i) \longrightarrow \mathrm{Hom}(C, X)$$

is an isomorphism for every compact object $C$ in $\mathcal{S}$.

Note that a homology colimit of a diagram $(X_i)_{i \in \mathcal{I}}$ is minimal and therefore unique up to a (non-unique) isomorphism; it is denoted by $\mathrm{hocolim}_{i \in \mathcal{I}} X_i$. Our terminology is justified by the following observation.

**Proposition 2.3.** *A cone $X_i \to X$ $(i \in \mathcal{I})$ is a homology colimit if and only if for every exact and coproduct preserving functor $H\colon \mathcal{S} \to \mathrm{Ab}$ the induced map $\mathrm{colim}_i H(X_i) \to H(X)$ is an isomorphism.*

*Proof.* One direction is clear. Therefore suppose that the cone $X_i \to X$ $(i \in \mathcal{I})$ is a homology colimit and fix an exact and coproduct preserving functor $H\colon \mathcal{S} \to \mathrm{Ab}$. The restriction $H|_{\mathcal{F}}$ is exact and therefore a filtered colimit of representable functors by Lemma 1.5, that is

$$H|_{\mathcal{F}} \cong \mathrm{colim}_{j \in \mathcal{J}} \mathrm{Hom}(C_j, -)|_{\mathcal{F}}.$$

This isomorphism induces a functorial isomorphism $H(X) \cong \operatorname{colim}_{j\in\mathcal{J}} \operatorname{Hom}(C_j, X)$ for all $X$ in $\mathcal{S}$ because both sides are exact and preserve coproducts (cf. [13, Proposition 3.2]). We obtain the following commutative diagram:

$$\begin{tikzcd}
\operatorname{colim}_i H(X_i) \arrow[r, phantom, "\cong"] \arrow[d, "\alpha"] & \operatorname{colim}_i \operatorname{colim}_j \operatorname{Hom}(C_j, X_i) \arrow[r, phantom, "\cong"] \arrow[d, "\beta"] & \operatorname{colim}_j \operatorname{colim}_i \operatorname{Hom}(C_j, X_i) \arrow[d, "\gamma"] \\
H(X) \arrow[r, phantom, "\cong"] & \operatorname{colim}_j \operatorname{Hom}(C_j, X) \arrow[r, phantom, "="] & \operatorname{colim}_j \operatorname{Hom}(C_j, X)
\end{tikzcd}$$

The map $\gamma$ is the colimit of isomorphisms by our assumption on the cone, and we conclude that $\alpha$ is an isomorphism. This completes the proof. $\square$

In [15], Margolis discusses weak colimits for the stable homotopy category. However, his definition of a minimal weak colimit is more restrictive than the one given here. Given a diagram $(X_i)_{i\in\mathcal{I}}$ in a compactly generated triangulated category, a weak colimit always exists. In fact, a weak colimit can be computed by taking the cofiber of an appropriate map

$$\coprod_{\lambda\colon i\to j} X_i \longrightarrow \coprod_k X_k$$

where $\lambda\colon i \to j$ runs through all maps and $k$ runs through all objects in $\mathcal{I}$. The following result is essentially due to Margolis [15].

**Proposition 2.4.** *Let $\mathcal{S}$ be a compactly generated triangulated category and suppose that Brown representability holds. Then every filtered diagram of pure-projective objects in $\mathcal{S}$ has a homology colimit which is also a minimal weak colimit.*

*Proof.* Let $(X_i)_{i\in\mathcal{I}}$ be a filtered diagram of pure-projective objects. The functor $\operatorname{colim}_i H_{X_i}$ is exact and therefore isomorphic to $H_X$ for some $X$ in $\mathcal{S}$ since we assume Brown representability. Using Lemma 1.2 and the fact that each $X_i$ is pure-projective, we get a family of maps $X_i \to X$ ($i \in \mathcal{I}$) which is a cone for $(X_i)_{i\in\mathcal{I}}$. Moreover, this cone is a homology colimit by construction. In order to show that the cone is a weak colimit, let $X_i \to Y$ ($i \in \mathcal{I}$) be another cone. Using Brown representability again, the induced map $H_X \cong \operatorname{colim}_i H_{X_i} \to H_Y$ is of the form $H_\alpha$ for some $\alpha\colon X \to Y$. The map $\alpha$ is compatible with the structural maps $X_i \to X$ and $X_i \to Y$ by Lemma 1.2, and therefore $X_i \to X$ ($i \in \mathcal{I}$) is a weak colimit. $\square$

Next we collect a few basic facts about the existence of minimal weak colimits for arbitrary diagrams.

**Lemma 2.5.** *Let $(X_i)_{i\in\mathcal{I}}$ be a diagram in an additive category $\mathcal{C}$ and suppose that idempotents in $\mathcal{C}$ split. Let $X_i \to X$ ($i \in \mathcal{I}$) be a weak colimit and denote by $M$ the image of the induced map $\operatorname{Hom}(X, X) \to \prod_i \operatorname{Hom}(X_i, X)$. Then the following conditions are equivalent:*

1. *The $\operatorname{End}(X)$-module $M$ has a projective cover.*
2. *The diagram $(X_i)_{i\in\mathcal{I}}$ has a minimal weak colimit.*

*Moreover, in this case $X_i \to X$ ($i \in \mathcal{I}$) is minimal if and only if the canonical map $\operatorname{Hom}(X, X) \to M$ is a projective cover.*

*Proof.* The proof is straightforward if one observes that an epimorphism $\pi\colon P \to M$ with $P$ projective is a projective cover of $M$ if and only if every endomorphism $\varepsilon\colon P \to P$ satisfying $\pi \circ \varepsilon = \pi$ is an isomorphism. $\square$

Let $X$ be an object in an additive category and suppose that idempotents split. Then every finitely generated $\mathrm{End}(X)$-module has a projective cover if and only if $X$ decomposes into finitely many indecomposable objects with local endomorphism rings. Using this elementary fact, one can prove the following.

**Proposition 2.6.** *Every finite diagram of compact objects in the category of $p$-local spectra has a minimal weak colimit.*

*Proof.* Every compact $p$-local spectrum $X$ decomposes into finitely many indecomposable objects with local endomorphism rings (cf. [7]). The assertions is therefore a consequence of Lemma 2.5 because every finitely generated $\mathrm{End}(X)$-module has a projective cover. □

Another method to produce minimal weak colimits is to construct appropriate injective envelopes.

**Proposition 2.7.** *A diagram in a compactly generated triangulated category has a minimal weak colimit provided there exist a weak colimit which is pure-injective.*

*Proof.* Let $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ be a cone of some diagram and suppose that $X$ is pure-injective. It follows from Lemma 1.3 and the characterization of injective envelopes in Lemma 1.4 that the cone is minimal if and only if for $\mu\colon \coprod_i X_i \to X$ the induced map $\operatorname{Im} H_\mu \to H_X$ is an injective envelope. Now suppose that the above cone is a weak colimit. Taking an injective envelope $\operatorname{Im} H_\mu \to H_Y$ produces a new cone $\mu_i\colon X_i \to Y$ $(i \in \mathcal{I})$ which is a minimal weak colimit. □

We end this section with a characterization of pure triangles. Let us call a triangle $X \to Y \to Z \to \Sigma X$ *split* if the sequence

$$0 \longrightarrow \mathrm{Hom}(-, X) \longrightarrow \mathrm{Hom}(-, Y) \longrightarrow \mathrm{Hom}(-, Z) \longrightarrow 0$$

is split exact.

**Lemma 2.8.** *The following are equivalent for a triangle $\delta\colon X \to Y \to Z \to \Sigma X$ in a compactly generated triangulated category:*

(1) *$\delta$ is pure.*
(2) *$\delta$ is a homology colimit of a filtered diagram of split triangles of compact objects.*
(3) *$\delta$ is a homology colimit of a filtered diagram of split triangles.*

*Proof.* (1) ⇒ (2) Given an object $X$ in $\mathcal{S}$, the functor $H_X\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is exact and therefore flat by Lemma 1.5. A well-known consequence of this is the fact that the category $\mathcal{I}_X$ whose objects are the maps $X_i \to X$ with $X_i$ compact and whose maps are the obvious commuting triangles forms a small filtered category with $\operatorname{hocolim}_{i \in \mathcal{I}_X} X_i = X$ (cf. [16, Theorem 3.2]). Analogously, one shows that for any map $\beta\colon Y \to Z$ in $\mathcal{S}$ the category $\mathcal{I}_\beta$ whose objects are the commuting squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\beta_i} & Z_i \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\beta} & Z \end{array}$$

with $Y_i$ and $Z_i$ compact and whose maps $i \to j$ are the obvious commuting squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\beta_i} & Z_i \\ \downarrow & & \downarrow \\ Y_j & \xrightarrow{\beta_j} & Z_j \end{array}$$

form a small filtered category with $\operatorname{hocolim}_{i\in\mathcal{I}_\beta}\beta_i = \beta$.

Now suppose that the triangle $\delta\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is pure. It follows that the commuting squares $i \in \mathcal{I}_\beta$ with $\beta_i$ a split epimorphism form a cofinal subcategory of $\mathcal{I}_\beta$ which we denote by $\mathcal{I}$. In fact, every commuting square $i \in \mathcal{I}_\beta$ fits into a commutative diagram of the form

$$\begin{array}{ccc}
Y_i & \xrightarrow{\beta_i} & Z_i \\
\downarrow & & \| \\
Y_i \coprod Z_i & \xrightarrow{\beta_i'} & Z_i \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\beta} & Z
\end{array}$$

where the second component of $\beta_i'$ is the identity since the map $Z_i \to Z$ factors through $\beta$. We obtain a filtered diagram $(\delta_i)_{i\in\mathcal{I}}$ of split triangles $\delta_i\colon X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\beta_i} Z_i \xrightarrow{\gamma_i} \Sigma X_i$ and one checks easily that the commuting squares corresponding to the $\alpha_i$ and $\gamma_i$ with $i \in \mathcal{I}$ form cofinal subcategories of $\mathcal{I}_\alpha$ and $\mathcal{I}_\gamma$, respectively. We conclude that $\operatorname{hocolim}_{i\in\mathcal{I}}\alpha_i = \alpha$, $\operatorname{hocolim}_{i\in\mathcal{I}}\beta_i = \beta$, and $\operatorname{hocolim}_{i\in\mathcal{I}}\gamma_i = \gamma$. Thus $\delta = \operatorname{hocolim}_{i\in\mathcal{I}}\delta_i$ .

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Suppose that $\delta = \operatorname{hocolim}_i \delta_i$ and that each $\delta_i$ is split. A split triangle $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$ induces an exact sequence $0 \to H_{X_i} \to H_{Y_i} \to H_{Z_i} \to 0$. Taking filtered homology colimits preserves exactness and we get therefore an exact sequence $0 \to H_X \to H_Y \to H_Z \to 0$. Thus $\delta$ is pure. □

## 3. Weak limits

The concept of a (minimal) weak limit is the obvious analogue of a (minimal) weak colimit which one obtains by reversing all the arrows in Definition 2.1. In this section we investigate the existence of minimal weak limits. We need the following lemma.

**Lemma 3.1.** *Let $X \xrightarrow{\alpha} I \xrightarrow{\beta} J \xrightarrow{\gamma} \Sigma X$ be a triangle in $\mathcal{S}$ and suppose that the induced map $\operatorname{Im} H_\beta \to H_J$ is an injective envelope. Then every endomorphism $\varepsilon\colon X \to X$ satisfying $\alpha \circ \varepsilon = \alpha$ is an isomorphism.*

*Proof.* Choose a map $\phi\colon J \to J$ which completes the following commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & J & \xrightarrow{\gamma} & \Sigma X \\
\downarrow{\scriptstyle\varepsilon} & & \| & & & & \downarrow{\scriptstyle\Sigma\varepsilon} \\
X & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & J & \xrightarrow{\gamma} & \Sigma X
\end{array}$$

The assumption on $\varepsilon$ implies that $H_\phi$ keeps $\operatorname{Im} H_\beta$ fixed, and therefore $H_\phi$ is an isomorphism by Lemma 1.4. Thus $\phi$ is an isomorphism, and we conclude that $\varepsilon$ is an isomorphism. □

We are now in a position to prove an existence criterion for minimal weak limits.

**Theorem 3.2.** *Let $\mathcal{S}$ be a compactly generated triangulated category. Then every diagram of pure-injective objects in $\mathcal{S}$ has a minimal weak limit.*

*Proof.* The proof uses the fact that the category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ has injective envelopes. Let $(X_i)_{i\in\mathcal{I}}$ be a diagram of pure-injective objects in $\mathcal{S}$ and let $F = \lim_i H_{X_i}$ be the corresponding limit in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. There exists a minimal injective copresentation of $F$ which is of the form

$$0 \longrightarrow F \longrightarrow H_I \xrightarrow{H_\beta} H_J$$

by Lemma 1.3. We complete $\beta$ to a triangle $X \xrightarrow{\alpha} I \xrightarrow{\beta} J \xrightarrow{\gamma} \Sigma X$. The map $H_\alpha$ induces a map $H_X \to F$ which we compose with the structural maps $F \to H_{X_i}$ to obtain a family of maps $\mu_i\colon X \to X_i$, using Lemma 1.3 and the fact that each $X_i$ is pure-injective. We claim that $\mu_i\colon X \to X_i$ $(i \in \mathcal{I})$ is a minimal weak limit of the diagram $(X_i)_{i\in\mathcal{I}}$. Observe first that $X_\lambda \circ \mu_i = \mu_j$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$ since $H_{X_\lambda} \circ H_{\mu_i} = H_{\mu_j}$, again by Lemma 1.3. Now suppose there is another family $\nu_i\colon Y \to X_i$ $(i \in \mathcal{I})$ of maps satisfying $X_\lambda \circ \nu_i = \nu_j$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$. The family $H_Y \to H_{X_i}$ $(i \in \mathcal{I})$ induces a map $H_Y \to F$ which we compose with $F \to H_I$ to get a map $Y \to I$. The composition of this map with $\beta$ is zero and therefore $Y \to I$ factors through $\alpha$ via some map $Y \to X$. Using again the pure-injectivity of the $X_i$, it is easy to check that $Y \to X$ is compatible with the strucural maps $\mu_i$ and $\nu_i$.

It remains to show that the family $X \to X_i$ $(i \in \mathcal{I})$ is minimal. Every endomorphism $\varepsilon\colon X \to X$ which is compatible with the $\mu_i$ induces a map $H_\varepsilon$ which is compatibel with the map $H_X \to F$. Therefore $\alpha \circ \varepsilon = \alpha$, and Lemma 3.1 implies that $\varepsilon$ is an isomorphism. This shows that the weak limit is minimal and the proof is complete. ☐

## 4. Extending functors

It is often useful to extend a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ to a functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ such that $\hat{F}(H_X) = F(X)$ for all $X$ in $\mathcal{S}$. We consider a number of conditions on $F$ which translate into properties of the functor $\hat{F}$.

- (E) For every pure triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{S}$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.
- (Π) $F(\prod_i X_i) \cong \prod_i F(X_i)$ for every family $(X_i)_{i\in I}$ of pure-injective objects in $\mathcal{S}$.
- (Σ) $\coprod_i F(X_i) \cong F(\coprod_i X_i)$ for every family $(X_i)_{i\in I}$ of pure-projective objects in $\mathcal{S}$.

It is sometimes convenient to work with the following variant of (Π) and (Σ), respectively.

- (Π′) $F(\prod_i X_i) \cong \prod_i F(X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{S}$.
- (Σ′) $\coprod_i F(X_i) \cong F(\coprod_i X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{S}$.

Note that (Σ′) and (Σ) are equivalent since every pure-projective object is a direct factor of a coproduct of compact objects. It turns out that condition (E) is sufficient to construct a functor $\hat{F}$ which extends $F$.

**Proposition 4.1.** *Let $\mathcal{S}$ be a compactly generated triangulated category and let $F\colon \mathcal{S} \to \mathrm{Ab}$ be a functor satisfying* (E). *Then there exists, up to isomorphism, a unique functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is left exact and extends $F$, that is, $\hat{F}(H_X) = F(X)$ for all $X$ in $\mathcal{S}$. Moreover,*

1. *if $F$ satifies* (Π)*, then $\hat{F}$ preserves products, and*
2. *if $F$ satifies* (Σ)*, then $\hat{F}$ preserves filtered colimits.*

*Proof.* Choose for every object $X$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ an injective copresentation

$$0 \longrightarrow X \longrightarrow H_I \xrightarrow{H_\alpha} H_J.$$

This is possible by Lemma 1.3. Now one defines $\hat{F}(X) = \operatorname{Ker} F(\alpha)$ and checks easily that this can be extended to maps in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and that it is well-defined. Condition

(E) implies that $\hat{F}(H_X) = F(X)$ for all $X$ in $\mathcal{S}$. In fact, we can choose for $X$ pure triangles

$$X \longrightarrow I_0 \longrightarrow X_1 \longrightarrow \Sigma X \quad \text{and} \quad X_1 \longrightarrow I_1 \longrightarrow X_2 \longrightarrow \Sigma X_1$$

with $I_i$ pure-injective. This gives an injective copresentation $0 \to H_X \to H_{I_0} \to H_{I_1}$ and $F(X) = \hat{F}(H_X)$ follows since $0 \to F(X) \to F(I_0) \to F(I_1)$ is exact. Clearly, $\hat{F}$ is left exact by construction. Moreover, any left exact functor $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ is uniquely determined by by its restriction to the full subcategory of injective objects.

Suppose now that ($\Pi$) holds. This condition says that the restriction of $\hat{F}$ to the full subcategory of injectives in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ preserves products. Let $(X_i)_{i\in I}$ be a family of arbitrary objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and choose injective copresentations $0 \to X_i \to I_i \to J_i$. We get the following commutative diagram with exact rows since $\hat{F}$ is left exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{F}(\prod_i X_i) & \longrightarrow & \hat{F}(\prod_i I_i) & \longrightarrow & \hat{F}(\prod_i J_i) \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & \prod_i \hat{F}(X_i) & \longrightarrow & \prod_i \hat{F}(I_i) & \longrightarrow & \prod_i \hat{F}(J_i) \end{array}$$

The maps $\beta$ and $\gamma$ are isomorphisms and it follows that $\alpha$ is an isomorphism. Thus $\hat{F}$ preserve products.

Finally suppose that ($\Sigma$) holds. We construct a new functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ as follows. For a finitely presented functor $X$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ choose a presentation

$$\mathrm{Hom}(-, A) \xrightarrow{\mathrm{Hom}(-,\alpha)} \mathrm{Hom}(-, B) \longrightarrow X \longrightarrow 0$$

and complete $\alpha$ to a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$. We get an exact sequence

$$0 \longrightarrow X \longrightarrow \mathrm{Hom}(-, C) \xrightarrow{\mathrm{Hom}(-,\gamma)} \mathrm{Hom}(-, \Sigma A).$$

Now define $\check{F}(X) = \operatorname{Ker} F(\gamma)$. Every object $X$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ can be written as a filtered colimit of finitely presented functors. More precisely, the category $\mathcal{I}_X$ whose objects are the maps $X_i \to X$ with $X_i$ finitely presented and whose maps are the obvious commuting triangles forms a small filtered category with $\operatorname{colim}_{i\in\mathcal{I}_X} X_i = X$. One defines $\check{F}(X) = \operatorname{colim}_i \check{F}(X_i)$ and checks easily that this definition can be extended to maps in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Clearly, $\check{F}$ preserves filtered colimits. We claim that $\check{F}$ is left exact. To see this, fix an exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and write $\beta = \operatorname{colim}_i \beta_i$ as a filtered colimit of maps $\beta_i\colon Y_i \to Z_i$ between finitely presented objects. We get a filtered diagram of exact sequences $0 \to X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\beta_i} Z_i$ and $0 \to \check{F}(X_i) \xrightarrow{\check{F}(\alpha_i)} \check{F}(Y_i) \xrightarrow{\check{F}(\beta_i)} \check{F}(Z_i)$ is exact by construction. Taking filtered colimits preserves exactness and it follows that $0 \to \check{F}(X) \xrightarrow{\check{F}(\alpha)} \check{F}(Y) \xrightarrow{\check{F}(\beta)} \check{F}(Z)$ is exact.

Next we use (E) and ($\Sigma$) to show that $\check{F}(H_X) = F(X)$ for every $X$ in $\mathcal{S}$. Condition ($\Sigma$) implies that this holds if $X$ is pure-projective. Otherwise choose for $X = X_0$ pure triangles

$$X_2 \longrightarrow P_1 \longrightarrow X_1 \longrightarrow \Sigma X_2 \quad \text{and} \quad X_1 \longrightarrow P_0 \longrightarrow X_0 \longrightarrow \Sigma X_1$$

with $P_i$ pure-projective. Each sequence $0 \to H_{X_{i+1}} \to H_{P_i} \to H_{X_i} \to 0$ is a filtered colimit of split exact sequences of the form $0 \to H_A \to H_B \to H_C \to 0$ with $A, B, C$ compact, by Lemma 2.8. Thus $0 \to \check{F}(H_{X_{i+1}}) \to \check{F}(H_{P_i}) \to \check{F}(H_{X_i}) \to 0$ is exact and therefore $\check{F}(H_{P_1}) \to \check{F}(H_{P_0}) \to \check{F}(H_X) \to 0$ is exact. On the other hand, $F(P_1) \to$

$F(P_0) \to F(X) \to 0$ is exact by (E), and therefore $\check{F}(H_X) = F(X)$. It follows that $\check{F}$ and $\hat{F}$ are isomorphic, and therefore $\hat{F}$ preserves filtered colimits. $\square$

The preceding proposition has an analogue for right exact functors $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ extending a functor $F\colon \mathcal{S} \to \mathrm{Ab}$. The construction uses projective presentations instead of injective copresentations.

**Proposition 4.2.** *Let $\mathcal{S}$ be a compactly generated triangulated category and let $F\colon \mathcal{S} \to \mathrm{Ab}$ be a functor satisfying* (E)*. Then there exists, up to isomorphism, a unique functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is right exact and extends $F$, that is, $\check{F}(H_X) = F(X)$ for all $X$ in $\mathcal{S}$. Moreover, if $F$ satifies* $(\Sigma)$*, then $\check{F}$ preserves coproducts and is isomorphic to $- \otimes_{\mathcal{F}} G$ where $G = F|_{\mathcal{F}}$.*

*Proof.* The proof is analogous to that of Proposition 4.1 and we leave the details to the reader. The last assertion about $- \otimes_{\mathcal{F}} G$ follows from the fact that $\check{F}(H_C) = F(C) = H_C \otimes_{\mathcal{F}} G$ for each $C \in \mathcal{F}$ and every $X \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ has a presentation

$$\coprod_j H_{D_j} \longrightarrow \coprod_i H_{C_i} \longrightarrow X \longrightarrow 0$$

with $C_i, D_j \in \mathcal{F}$ for all $i, j$. $\square$

Recall that a map $X \to Y$ in a compactly generated triangulated category is a *phantom map* if the induced map $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$ iz zero for all compact $C$.

**Corollary 4.3.** *Let $F\colon \mathcal{S} \to \mathrm{Ab}$ be a functor and suppose that for every pure triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{S}$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact. Then $F(\alpha) = 0$ for every phantom map $\alpha$.*

In general, the converse is not true. Take for instance an object $Z$ in $\mathcal{S}$ which is not pure-projective, and let $F(X) = \mathrm{Hom}(H_Z, H_X)$ for $X$ in $\mathcal{S}$. Clearly, $F(\alpha) = 0$ for every phantom map $\alpha$. However, if $X \to Y \to Z \to \Sigma X$ is a pure triangle with $Y$ pure-projective, the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ cannot be exact.

## 5. Coherent functors

We are now in a position to prove the first portion of our characterization of coherent functors.

**Proposition 5.1.** *Let $\mathcal{S}$ be a compactly generated triangulated category. For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is coherent.*
2. *$F$ preserves products and sends every homology colimit to a colimit.*
3. *$F$ preserves products and coproducts, and for every pure triangle $X \to Y \to Z \to \Sigma X$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.*
4. *$F$ satisfies* (E), $(\Pi)$, *and* $(\Sigma)$.

*Proof.* (1) $\Rightarrow$ (2) Each representable functor $\mathrm{Hom}(C, -)$ with $C$ compact preserves products and sends every homology colimit to a colimit by the definition of a homology colimit. Clearly, this property is preserved if we pass to the cokernel of a map $\mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -)$. Thus (2) holds for every coherent functor $F$.

(2) $\Rightarrow$ (3) Suppose that $F$ preserves homology colimits. It follows that $F$ preserves coproducts because every coproduct in $\mathcal{S}$ is a homology colimit. Now suppose that $\delta\colon X \to Y \to Z \to \Sigma X$ is a pure triangle. It has been shown in Lemma 2.8 that $\delta$ is a homology colimit of split triangles $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$. Clearly, each sequence

$0 \to F(X_i) \to F(Y_i) \to F(Z_i) \to 0$ is exact since $F$ is additive, and therefore the colimit $0 \to \operatorname{colim}_i F(X_i) \to \operatorname{colim}_i F(Y_i) \to \operatorname{colim}_i F(Z_i) \to 0$ is exact. However, this sequence is isomorphic to $0 \to F(X) \to F(Y) \to F(Z) \to 0$ by our assumption on $F$. This proves (3).

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) We apply Proposition 4.1 to get a functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is left exact and extends $F$. Moreover, ($\Pi$) and ($\Sigma$) imply that $\hat{F}$ preserves products and filtered colimits. It follows that $\hat{F}$ preserves limits since every limit can be computed by taking kernels and products. Therefore the Adjoint Functor Theorem implies the existence of a left adjoint $G\colon \mathrm{Ab} \to (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ for $\hat{F}$. This gives for $X$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ a functorial isomorphism

$$\hat{F}(X) \cong \mathrm{Hom}(\mathbb{Z}, \hat{F}(X)) \cong \mathrm{Hom}(G(\mathbb{Z}), X).$$

The criterion of Lemma 1.6 implies that $G(\mathbb{Z})$ is a finitely presented functor since $\hat{F}$ preserves filtered colimits. Choose a presentation

$$\mathrm{Hom}(-, A) \xrightarrow{\mathrm{Hom}(-,\alpha)} \mathrm{Hom}(-, B) \longrightarrow G(\mathbb{Z}) \longrightarrow 0.$$

Applying $\mathrm{Hom}(-, H_X)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(G(\mathbb{Z}), H_X) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(-, B), H_X) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(-, A), H_X)$$

which is isomorphic to

$$0 \longrightarrow F(X) \longrightarrow \mathrm{Hom}(B, X) \xrightarrow{\mathrm{Hom}(\alpha,X)} \mathrm{Hom}(A, X)$$

thanks to Yoneda's lemma and the isomorphism $F(X) \cong \mathrm{Hom}(G(\mathbb{Z}), H_X)$. This sequence is functorial in $X$, and if we complete $\alpha\colon A \to B$ to a triangle $A \to B \to C \to \Sigma A$, we get the desired presentation

$$\mathrm{Hom}(\Sigma A, -) \longrightarrow \mathrm{Hom}(C, -) \longrightarrow F \longrightarrow 0$$

which shows that $F$ is coherent. ∎

**Proposition 5.2.** *Let $\mathcal{S}$ be a compactly generated triangulated category. For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is coherent.*
2. *$F$ satisfies* (E), ($\Pi'$), *and* ($\Sigma'$).

*Proof.* (1) $\Rightarrow$ (2) is shown in Proposition 5.1. Therefore suppose that $F$ satisfies (E), ($\Pi'$), and ($\Sigma'$). Using conditions (E) and ($\Sigma'$), we can apply Proposition 4.2 and extend $F$ to a functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is isomorphic to $- \otimes_{\mathcal{F}} G$ for $G = F|_{\mathcal{F}}$. In particular, $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all $X \in \mathcal{S}$. We claim that $G$ is finitely presented. In fact, this follows from Lemma 1.6 and condition ($\Pi'$) since we have for every family $(C_i)_{i \in I}$ in $\mathcal{F}$

$$(\prod_i H_{C_i}) \otimes_{\mathcal{F}} G \cong H_{\prod_i C_i} \otimes_{\mathcal{F}} G \cong F(\prod_i C_i) \cong \prod_i F(C_i) \cong \prod_i (H_{C_i} \otimes_{\mathcal{F}} G).$$

Tensoring a presentation $\mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -) \to G \to 0$ with $H_X$ for $X \in \mathcal{S}$ gives an exact sequence

$$H_X \otimes_{\mathcal{F}} \mathrm{Hom}(D, -) \longrightarrow H_X \otimes_{\mathcal{F}} \mathrm{Hom}(C, -) \longrightarrow H_X \otimes_{\mathcal{F}} G \longrightarrow 0$$

which is isomorphic to

$$\mathrm{Hom}(D, X) \longrightarrow \mathrm{Hom}(C, X) \longrightarrow F(X) \longrightarrow 0.$$

This sequence is functorial in $X$ and therefore $F$ is coherent. This completes the proof. $\square$

The next proposition completes our characterization of coherent functors.

**Proposition 5.3.** *Let $\mathcal{S}$ be a compactly generated triangulated category and suppose that Brown representability holds. For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is coherent.*
2. *$F$ preserves products of families $(X_i)_{i\in I}$ and minimal weak colimits of filtered diagrams $(X_j)_{j\in\mathcal{J}}$ provided that each $X_i$ and each $X_j$ is a direct factor of a coproduct of compact objects.*
3. *$F$ preserves products of families $(X_i)_{i\in I}$ and minimal weak colimits of filtered diagrams $(X_j)_{j\in\mathcal{J}}$ provided that each $X_i$ and each $X_j$ is a compact object.*
4. *$F$ satisfies (E), $(\Pi')$, and $(\Sigma')$.*

*Proof.* (1) $\Rightarrow$ (2) A coherent functor preserves products and sends homology colimits to colimits by Proposition 5.1. Every minimal weak colimit of a filtered diagram of pure-projective objects in $\mathcal{S}$ is also a homology colimit by Proposition 2.4. Therefore (1) implies (2).

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) We check that $F$ satisfies the conditions (E) and $(\Sigma')$.

(E) It has been shown in Lemma 2.8 that a pure triangle $\delta\colon X \to Y \to Z \to \Sigma X$ can be expressed as a homology colimit of a diagram $(\delta_i)_{i\in\mathcal{I}}$ of split triangles $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$ of compact objects. Note that $\delta$ is also a minimal weak colimit of the diagram $(\delta_i)_{i\in\mathcal{I}}$ by Proposition 2.4. Applying $F$ gives a filtered diagram of exact sequences $0 \to F(X_i) \to F(Y_i) \to F(Z_i) \to 0$. The colimit of these exact sequences is again exact and isomorphic to the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$, by our assumptions on the functor $F$.

$(\Sigma')$ The coproduct of a family $(X_i)_{i\in I}$ is the filtered colimit of the finite coproducts $\coprod_{i\in J} X_i$ where $J$ runs through all finite subsets of $I$. Note that $\coprod_{i\in J} X_i = \prod_{i\in J} X_i$ if $J$ is finite. Thus (3) implies that $F$ preserves coproducts of compact objects in $\mathcal{S}$.

(4) $\Rightarrow$ (1) See Proposition 5.2. $\square$

## 6. Short exact functors

In this section we characterize short exact functors. Recall that a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is *short exact* if for every triangle $X \to Y \to Z \to \Sigma X$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact whenever $0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y) \to \mathrm{Hom}(C, Z) \to 0$ is exact for every compact $C$.

**Theorem 6.1.** *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *$F$ is short exact and preserves coproducts.*
2. *There exists an additive functor $G\colon \mathcal{F} \to \mathrm{Ab}$ and a functorial isomorphism $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all $X$ in $\mathcal{S}$.*
3. *There exists a filtered diagram $(F_i)_{i\in\mathcal{I}}$ of coherent functors and a functorial isomorphism $F(X) \cong \mathrm{colim}_i F_i(X)$ for all $X$ in $\mathcal{S}$.*

*Proof.* (1) $\Rightarrow$ (2) Let $G = F|_{\mathcal{F}}$. Condition (1) implies that $F$ extends to a functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is isomorphic to $-\otimes_{\mathcal{F}} G$ by Proposition 4.2. Therefore

$$F(X) \cong \check{F}(H_X) \cong H_X \otimes_{\mathcal{F}} G$$

for all $X$ in $\mathcal{S}$.

(2) $\Rightarrow$ (3) Suppose that $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for some functor $G \colon \mathcal{F} \to \mathrm{Ab}$. Writing $G = \mathrm{colim}_i\, G_i$ as a filtered colimit of finitely presented functors, we get a filtered diagram of coherent functors $F_i \colon \mathcal{S} \to \mathrm{Ab}$ if we define $F_i(X) = H_X \otimes_{\mathcal{F}} G_i$ for each $i$. This gives an isomorphism

$$F(X) \cong H_X \otimes_{\mathcal{F}} (\mathrm{colim}_i\, G_i) \cong \mathrm{colim}_i (H_X \otimes_{\mathcal{F}} G_i) = \mathrm{colim}_i\, F_i(X)$$

since $H_X \otimes_{\mathcal{F}} -$ preserves colimits.

(3) $\Rightarrow$ (1) A coherent functor is short exact and preserves coproducts by Proposition 5.1. Taking filtered colimits preserves exactness and coproducts, and therefore a filtered colimit of coherent functors is short exact and preserves coproducts. □

## 7. Definable subcategories

In this section we use coherent functors to study certain subcatgories of $\mathcal{S}$.

**Definition 7.1.** We call a full subctegory $\mathcal{C}$ of $\mathcal{S}$ *definable* if it is of the form

$$\mathcal{C} = \{X \in \mathcal{S} \mid F_i(X) = 0 \text{ for all } i \in I\}$$

for some family $(F_i)_{i \in I}$ of coherent functors.

There are three other concepts equivalent to definable subcategories:

- Ziegler-closed subsets of the set $\mathrm{Sp}\,\mathcal{S}$ of indecomposable pure-injectives in $\mathcal{S}$,
- Serre subcategories of $\mathrm{Coh}\,\mathcal{S}$, and
- cohomological ideals in $\mathcal{F}$.

We refer to the introduction for precise definitions and the statement of the 'fundamental correspondence' which relates these concept to each other. Here, we use the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ to prove this correspondence. We start with some preparations.

Let $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ be the full subcategory formed by the finitely presented functors in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Note that $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is abelian since $\mathcal{F}$ has weak kernels. Given a functor $F \colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, we define a functor $F^\vee \colon \mathcal{S} \to \mathrm{Ab}$ by

$$F^\vee(X) = \mathrm{Hom}(F, H_X).$$

**Lemma 7.2.** *The assignment $F \mapsto F^\vee$ induces an equivalence*

$$(\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}))^{\mathrm{op}} \longrightarrow \mathrm{Coh}\,\mathcal{S}.$$

*Proof.* Let $F \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and fix a presentation

$$\mathrm{Hom}(-, A) \longrightarrow \mathrm{Hom}(-, B) \longrightarrow F \longrightarrow 0.$$

Completing the map $A \to B$ to a triangle $A \to B \to C \to \Sigma A$, it follows from Yoneda's lemma that we get a presentation

$$\mathrm{Hom}(\Sigma A, -) \longrightarrow \mathrm{Hom}(C, -) \longrightarrow F^\vee \longrightarrow 0.$$

Thus $F^\vee$ is coherent. It is now straightforward to write down an inverse for $F \mapsto F^\vee$. □

We denote by $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ the set of isomorphism classes of indecomposable injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. A subset of $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is *closed* if it is of the form

$$\{X \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F_i, X) = 0 \text{ for all } i \in I\}$$

for some family $(F_i)_{i \in I}$ of finitely presented functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$.

**Proposition 7.3.** *The assignments*

$$\mathbf{U} \mapsto \{F \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F, X) = 0 \textit{ for all } X \in \mathbf{U}\} \textit{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F, X) = 0 \textit{ for all } F \in \mathcal{T}\}$$

*induce mutually inverse bijections between the set of closed subsets of* $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ *and the set of Serre subcategories of* $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$.

*Proof.* See Theorem 4.2 in [11]. □

Given an object $X$ in $\mathcal{S}$, we consider the *annihilator*

$$\mathrm{Ann}\, X = \{\phi \in \mathcal{F} \mid \mathrm{Hom}(\phi, X) = 0\}.$$

Clearly, $\mathrm{Ann}\, X$ is a cohomological ideal in $\mathcal{F}$, and the converse is also true.

**Proposition 7.4.** *Every cohomological ideal in* $\mathcal{F}$ *is of the form* $\mathrm{Ann}\, X$ *for some pure-injective object* $X$ *in* $\mathcal{S}$.

*Proof.* We fix a cohomological ideal $\mathfrak{I}$. By definition, there exists an exact functor $F\colon \mathcal{F} \to \mathrm{Ab}$ such that $\mathfrak{I} = \{\phi \in \mathcal{F} \mid F(\phi) = 0\}$. The functor $- \otimes_{\mathcal{F}} F\colon (\mathcal{F}^{\mathrm{op}}, Ab) \to \mathrm{Ab}$ is exact by Lemma 1.5, and we obtain therefore a Serre subcategory of $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ by taking

$$\mathcal{T} = \{G \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid G \otimes_{\mathcal{F}} F = 0\}.$$

Now let $I$ be the product of all $Y \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ such that $\mathrm{Hom}(G, Y) = 0$ for all $G \in \mathcal{T}$. The correspondence in Proposition 7.3 implies

$$\mathcal{T} = \{G \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(G, I) = 0\},$$

and we find $X \in \mathcal{S}$ with $H_X \cong I$ by Lemma 1.3. Now let $\phi$ be an arbitrary map in $\mathcal{F}$ and put $G = \mathrm{Im}\, H_\phi$. We get

$$F(\phi) = 0 \;\Leftrightarrow\; G \otimes_{\mathcal{F}} F = 0 \;\Leftrightarrow\; \mathrm{Hom}(G, H_X) = 0 \;\Leftrightarrow\; \mathrm{Hom}(\phi, X) = 0 \;\Leftrightarrow\; \phi \in \mathrm{Ann}\, X.$$

Thus $\mathfrak{I}$ is of the form $\mathrm{Ann}\, X$. □

*Proof of the Fundamental Correspondence.* The assignment $X \mapsto H_X$ identifies the pure-injective objects in $\mathcal{S}$ with the injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ (cf. [13, Corollary 1.9]) and induces therefore a bijection $\mathrm{Sp}\, \mathcal{S} \to \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ which identifies the Ziegler closed subsets of $\mathrm{Sp}\, \mathcal{S}$ with the closed subsets of $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ by Lemma 7.2. We conclude from Proposition 7.3 that

$$\mathbf{U} \mapsto \{F \in \mathrm{Coh}\, \mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathbf{U}\} \text{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathrm{Sp}\, \mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$$

induce mutually inverse bijections between Ziegler-closed subsets of $\mathrm{Sp}\, \mathcal{S}$ and Serre subcategories of $\mathrm{Coh}\, \mathcal{S}$. It is an immediate consequence that

$$\mathcal{C} \mapsto \{F \in \mathrm{Coh}\, \mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathcal{C}\} \text{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$$

induce mutually inverse bijections between definable subcategories of $\mathcal{S}$ and Serre subcategories of $\mathrm{Coh}\, \mathcal{S}$. In other words: a definable subcategory $\mathcal{C}$ is already determined by $\mathcal{C} \cap \mathrm{Sp}\, \mathcal{S}$. In fact, each definable subcategory $\mathcal{C}$ can be reconstructed explicitly from the corresponding Ziegler-closed subset $\mathbf{U} = \mathcal{C} \cap \mathrm{Sp}\, \mathcal{S}$ since

$$\mathcal{C} = \{X \in \mathcal{S} \mid \text{there are } Y_i \in \mathbf{U} \text{ and a pure triangle } X \to \textstyle\prod_i Y_i \to Z \to \Sigma X\}.$$

This follows from Proposition 3.2 in [11].

Next we consider the cohomological ideals. Observe that a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is coherent precisely if $F = \operatorname{Im}\operatorname{Hom}(\phi, -)$ for some map $\phi\colon C \to D$ in $\mathcal{F}$. Clearly, $F(X) = 0$ for some $X$ in $\mathcal{S}$ if and only if $\phi \in \operatorname{Ann} X$. Using the correspondence between definable subcategories and Serre subcategories of coherent functors, it follows that

$$\mathcal{C} \mapsto \bigcap_{X\in\mathcal{C}} \operatorname{Ann} X$$

induces an injective map from the set of definable subcategories of $\mathcal{S}$ into the set of cohomological ideals in $\mathcal{F}$. It remains to show that this map is surjective. To this end fix a cohomological ideal $\mathfrak{I}$ in $\mathcal{F}$. We have $\mathfrak{I} = \operatorname{Ann} Y$ for some $Y \in \mathcal{S}$ by Proposition 7.4. Thus $\mathcal{C} = \{X \in \mathcal{S} \mid \mathfrak{I} \subseteq \operatorname{Ann} X\}$ is a definable subcategory satisfying

$$\mathfrak{I} = \bigcap_{X\in\mathcal{C}} \operatorname{Ann} X.$$

This completes the proof of the correspondence between definable subcategories, Ziegler closed subsets, Serre subcategories and cohomological ideals. □

Given a class $\mathcal{C}$ of objects in $\mathcal{S}$, the definable subcategory generated by $\mathcal{C}$ is

$$\operatorname{Def}\mathcal{C} = \{X \in \mathcal{S} \mid F \in \operatorname{Coh}\mathcal{C} \text{ and } F(Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ implies } F(X) = 0\}.$$

It remains to prove the following description of $\operatorname{Def}\mathcal{C}$ via reduced products which is formulated in Theorem C.

**Theorem 7.5.** *Suppose that Brown representability holds for $\mathcal{S}$, and let $\mathcal{C}$ be a class of objects in $\mathcal{S}$. Then an object $X$ in $\mathcal{S}$ belongs to* $\operatorname{Def}\mathcal{C}$ *if and only if there is a pure triangle $X \to Y \to Z \to \Sigma X$ such that $Y = \prod_{i\in I} X_i/\mathcal{U}$ for some family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ and some filter $\mathcal{U}$ on $I$.*

*Proof.* We fix a class $\mathcal{C}$ of objects in $\mathcal{S}$ and put

$$\mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathcal{C}\}.$$

We use again the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Recall that $F \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is *fp-injective* if $\operatorname{Ext}^1(G, F) = 0$ for all $G \in \operatorname{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. A functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is fp-injective if and only if it is exact (cf. Lemma 2.7 in [13]) and therefore the restricted Yoneda functor

$$\mathcal{S} \longrightarrow (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{F}},$$

identifies the objects in $\mathcal{S}$ with the fp-injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ since we assume Brown representability. Now let $\mathcal{C}' = \{H_X \mid X \in \mathcal{C}\}$ and put

$$\mathcal{T}' = \{F \in \operatorname{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \operatorname{Hom}(F, X) = 0 \text{ for all } X \in \mathcal{C}'\}.$$

Note that $\mathcal{T} = \{F^\vee \mid F \in \mathcal{T}'\}$ since $F^\vee(X) = \operatorname{Hom}(F, H_X)$ for $X \in \mathcal{S}$. A reduced product of a family $(X_i)_{i\in I}$ of objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ with respect to some filter $\mathcal{U}$ on $I$ is by definition the filtered colimit $\operatorname{colim}_{J\in\mathcal{U}} \prod_{i\in J} X_i$ so that the restricted Yoneda functor preserves reduced products. It follows from Proposition 4.5 in [12] that an fp-injective object $X$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is a subobject of some reduced product of objects in $\mathcal{C}'$ if and only if $\operatorname{Hom}(F, X) = 0$ for all $F \in \mathcal{T}'$. Using again the restricted Yoneda functor, it follows that $X \in \mathcal{S}$ fits into a triangle $X \to Y \to Z \to \Sigma X$ such that $Y$ is a reduced product of objects in $\mathcal{C}$ if and only if $F(X)$ for all $F \in \mathcal{T}$. This completes the proof since $\operatorname{Def}\mathcal{C} = \{X \in \mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$. □

## Appendix: Finitely presented modules versus compact objects

In this appendix we explain the analogy between compact objects in a compactly generated triangulated category, and finitely presented modules in the category of modules over an associative ring.

Let $\mathcal{A}$ be an additive category and suppose that it has arbitrary products and coproducts. We make the following definitions:

- An object $Q$ is *p-injective* if for every set $I$ the summation map $\coprod_I Q \to Q$ factors through the canonical map $\coprod_I Q \to \prod_I Q$.
- A sequence of maps $X \to Y \to Z$ is *p-exact* if for every p-injective object $Q$ in $\mathcal{A}$ the sequence $0 \to \mathrm{Hom}(Z,Q) \to \mathrm{Hom}(Y,Q) \to \mathrm{Hom}(X,Q) \to 0$ is exact.
- An object $P$ is *p-projective* if for every p-exact sequence $X \to Y \to Z$ the sequence $0 \to \mathrm{Hom}(P,X) \to \mathrm{Hom}(P,Y) \to \mathrm{Hom}(P,Z) \to 0$ is exact.
- An object $X$ is *compact* if the functor $\mathrm{Hom}(X,-)$ preserves coproducts.

If $\mathcal{A}$ is the category $\mathrm{Mod}\,\Lambda$ of modules over an associative ring $\Lambda$, then the above concept of p-exactness coincides with the concept of pure-exactness introduced by Cohn [4]. This follows essentially from the characterization of pure-injective modules via the summation map which is due to Jensen and Lenzing (cf. [10, Proposition 7.32]). In this context the compact p-projective objects are characterized as follows.

**Proposition.** *Let $\mathcal{A}$ be the category of modules over an associative ring. Then an object in $\mathcal{A}$ is compact and p-projective if and only if it is a finitely presented module.*

*Proof.* The assertion is an immediate consequence of the well-known fact that a module is pure-projective if and only if it is a direct factor of a coproduct of finitely presented modules. □

Now suppose that $\mathcal{A}$ is a compactly generated triangulated category and denote by $\mathcal{F}$ the full subcategory of compact objects. Then the p-injective objects are precisely the objects which are pure-injective in the sense of Definition 1.1 (cf. [13, Theorem 1.8]). Therefore the functor $\mathcal{A} \to (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$, $X \mapsto H_X$, identifies the p-injective objects in $\mathcal{A}$ with the injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ (cf. [13, Corollary 1.9]). Note that $\mathrm{Hom}(X,Q) \cong \mathrm{Hom}(H_X, H_Q)$ for all $X$ in $\mathcal{A}$ and every p-injective object $Q$ by Lemma 1.3. Therefore $X \mapsto H_X$ identifies the p-exact sequences $X \to Y \to Z$ with the exact sequences $0 \to H_X \to H_Y \to H_Z \to 0$ since the injective objects cogenerate $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$.

**Proposition.** *Let $\mathcal{A}$ be a compactly generated triangulated category. Then an object in $\mathcal{A}$ is compact and p-projective if and only if it is compact.*

*Proof.* We need to show that every compact object is p-projective. However, this is just a reformulation of the fact that for each p-exact sequence $X \to Y \to Z$ the sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact. □

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