Modality for Parabolic Group Actions

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Introduction

Let G be a linear reductive algebraic group defined over an algebraically closed field k and let P be a parabolic subgroup of G, that is G/P is a complete variety. We consider the action of P on its unipotent radical P_u via conjugation and on $\operatorname{Lie} P_u = \mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint representation. The *modality of the action of* P *on* $\mathfrak{p}_u$, denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$, is the maximal number of parameters upon which a family of P–orbits on $\mathfrak{p}_u$ depends. Similarly for the action of P on P_u. See Section 1.2 for a precise definition of this notion. We write $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$ and call this the *modality of* P. Note that $\operatorname{mod} P = 0$ precisely when P acts on $\mathfrak{p}_u$ with a finite number of orbits.

The aim of these notes is to delineate new developments and results in the theory of modality of parabolic groups. In [**58**] the problem was posed to determine each parabolic subgroup P of G that has a finite number of orbits on P_u, as well as on $\mathfrak{p}_u$, see also [**33**, 5.4(8)]. We present the classifications from [**32**] and [**37**] of all these cases along with the essential ideas and methods of proofs, see the theorem below.

In 1974 R.W. Richardson proved that P admits an open dense orbit on P_u, similarly for the adjoint action of P on $\mathfrak{p}_u$ [**62**]. There results a natural dichotomy: the instances when P acts on $\mathfrak{p}_u$ with a finite number of orbits versus the cases when $\operatorname{mod} P$ is positive. Likewise for the action on P_u. In the instances when the number of orbits is infinite, the geometry of orbits is somewhat intricate, because then, by Richardson's Dense Orbit Theorem, infinitely many orbits must occur in a proper invariant subvariety of $\mathfrak{p}_u$, while the complement to this subvariety is the dense orbit in $\mathfrak{p}_u$. In Corollary 5.33 we show that in almost all of these instances we can find a subvariety admitting an infinite number of orbits which is a P–submodule of $\mathfrak{p}_u$.

Let $P = LP_u$ be a Levi decomposition of P. It follows from another result of R.W. Richardson that L only has a finite number of orbits on consecutive quotients of the descending central series of P_u [**64**, Thm. E]. In particular, this implies that the number of P–orbits on P_u is finite if P_u is abelian. Thus parabolic subgroups with an abelian unipotent radical provide a natural family with this finiteness property. A detailed analysis of this case can be found in [**65**], see also [**51**]. This finiteness property was extended to arbitrary closed abelian unipotent normal subgroups of any parabolic group in [**72**]. In Chapter 7 we present the a and elegant approach to this result from [**56**].

A. Borel and R. Steinberg are credited with posing the question whether the number of unipotent classes of a reductive group G is finite. Since any unipotent element of G lies in the unipotent radical of some Borel subgroup B of G and as all Borel subgroups are conjugate under the action of G, the answer is affirmative in those cases where B has a finite number of orbits on its unipotent radical. Consequently, this question is of interest only in those instances when the Borel subgroups of G are of positive modality. In this context this question was first pursued by A.E. Zalesskiĭ [**94**] in 1968. In his note Zalesskiĭ showed that B operates on B_u with an infinite number of orbits whenever G is of type A_r, and $r \geq 5$, B_r, C_r, D_r, and $r \geq 6$, and also for E_6, E_7, and E_8. We present Zalesskiĭ's argument in a slightly more general setting in Proposition 3.12 below. The question of finiteness for the number of unipotent classes of G was settled by R.W. Richardson at the time under some mild restrictions on $\operatorname{char} k$ [**61**] which were subsequently removed by G. Lusztig [**47**].

The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [**58**]. One goal of these notes is to generalize two basic "monotonicity" results on the modality of parabolic subgroup actions from [**58**]. Analogues of Corollary 3.4 and Theorem 3.10 were proved in [**58**, Thm. 2.13] under the assumption that $\operatorname{char} k$ is zero. The proofs in [**58**] do not generalize to positive characteristic as they make use of the separability of the orbit maps for the actions involved.

Theorems 2.3 and 2.4 give two general monotonicity results for the modality of algebraic group actions based on work of R.W. Richardson [**63**]. This is followed by a discussion of the basic properties of linearly reductive groups which are relevant to our purpose.

In Section 3.1 we apply the forgoing theorems in the context of parabolic group actions. The advantage of this approach is that we obtain simultaneously the desired monotonicity results for the modality of the conjugation action of P on P_u and for the adjoint action of P on $\mathfrak{p}_u$. This avoids any separability considerations of the orbit maps involved. These monotonicity theorems provide helpful inductive tools for comparing the modality of various parabolic group actions.

What follows in Section 3.2 is a brief description of some aspects of the GAP share package MOP developed jointly with U. Jürgens to address questions of modality of parabolic group actions algorithmically. The computational results of MOP were crucial in the classification of the finite orbit instances for exceptional groups.

Instances when a parabolic group controls fusion in its unipotent radical or its nilradical are given in Propositions 3.12 and 3.13.

In Section 3.4 we relate the modality of the action of P on $\mathfrak{p}_u$ and on P_u. Assuming that $\operatorname{char} k$ is a good prime we show $\operatorname{mod}(P : \mathfrak{p}_u) = \operatorname{mod}(P : P_u)$. The proof utilizes Springer's map $\varphi : \mathcal{U} \longrightarrow \mathcal{N}$, a G–equivariant bijective morphism between the unipotent variety $\mathcal{U}$ of G and the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$.

This is followed by a short discussion on the modality of the coadjoint action of P on the dual space $\mathfrak{p}_u^*$ in §3.5. Corollary 3.21 states the equality $\operatorname{mod}(P : \mathfrak{p}_u^*) = \operatorname{mod}(P : \mathfrak{p}_u)$.

We close this chapter with a glance at the action of G on the tangent and cotangent bundles of the flag manifold G/P and relate the modality of the G–action on these bundles to $\operatorname{mod} P$.

In Chapter 4 we discuss parabolic groups of positive modality, for classical groups, utilizing the inductive tools from Section 3. In Proposition 4.5 we indicate the results for exceptional groups from [**69**]. In Theorem 4.2 we present the classification of Borel subgroups of modality zero due to V. Kashin [**41**, Thm. 1]. We also include the classification of semisimple rank 1 parabolics with that finiteness property in Theorem 4.3. This was proved in [**58**, Cor. 1.4].

Combining the results from Chapters 4 and 5, we obtain in Theorem 5.22 the classification of modality zero parabolic groups for classical groups from [**32**]. In Theorem 5.30 we present the classification of modality zero parabolic subgroups for groups of exceptional type from [**37**]. The combined statement of the classifications is presented in the following theorem. By saying that P is of a particular type, we mean the Dynkin type of a Levi subgroup L_P of P. The class of nilpotency of P_u is denoted by $\ell(P_u)$.

THEOREM. *Let G be a simple algebraic group and $P \subseteq G$ parabolic. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Then P acts on $\mathfrak{p}_u$ with a finite number of orbits if and only if one of*

(i) $\ell(P_u) \leq 4$;
(ii) *G is of type D_r, $\ell(P_u) = 5$, $\tau P \neq P$, and the semisimple part of L_P consists of two simple components;*
(iii) *G is of type E_6, $\ell(P_u) = 5$, and P is of type $A_1^2 A_2$ or A_3;*
(iv) *G is of type E_7, $\ell(P_u) = 5$, and P is of type $A_1 A_4$.*

In Section 5.1 we outline the classification of modality zero parabolics for general linear groups in Theorem 5.6. This appears in [**31, 32**]. In Theorem 5.19 we furnish a complete combinatorial description of the closure relation on the set of P–orbits on $\mathfrak{p}_u$ for every finite orbit case from [**19**]. Let V be a finite dimensional k–space, $\mathrm{GL}(V)$ the general linear group of V and P a parabolic subgroup of $\mathrm{GL}(V)$. In Lemma 5.1 we show there is a canonical bijection between the set of P–orbits on $\mathfrak{p}_u$ and the set of isomorphism classes of Δ–filtered modules of a particular dimension vector $\mathbf{e}$ of a certain quasi-hereditary algebra $\mathcal{A}(t)$. These isomorphism classes in turn are given by the orbits of a reductive group $G(\mathbf{e})$ on a certain affine variety $\mathcal{R}(\Delta)(\mathbf{e})$ of $\mathcal{A}(t)$–modules with Δ–filtration and dimension vector $\mathbf{e}$. As it turns out, the *closure order* of the action of P on $\mathfrak{p}_u$ coincides with the closure order of the action of $G(\mathbf{e})$ on $\mathcal{R}(\Delta)(\mathbf{e})$ irrespective of the modality of P. This is presented in Theorem 5.20.

Let X and Y be in $\mathcal{F}(\Delta)(\mathbf{e})$, the subcategory of of $\mathcal{A}(t)$–mod of Δ–filtered modules of dimension vector $\mathbf{e}$. We write $X \geq_{\mathrm{hom}} Y$ provided $\dim \mathrm{Hom}(X, I) \geq \dim \mathrm{Hom}(Y, I)$ for every indecomposable module I in $\mathcal{F}(\Delta)$. As in the case of Artin algebras, it turns out that this induces a partial order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$. We refer to this as the hom–*order* on $\mathcal{F}(\Delta)$, see 5.1.4. The principal result concerning this combinatorial ordering is Theorem 5.18. Under the assumption that the subcategory $\mathcal{F}(\Delta)$ of $\mathcal{A}$–mod is of finite representation type, the closure order on the set of $G(\mathbf{e})$–orbits on $\mathcal{R}(\Delta)(\mathbf{e})$ coincides with the poset *opposite* to the hom–order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$.

In the Section 5.4 of Chapter 5 we discuss associated parabolics, i.e., parabolic subgroups with conjugate Levi subgroups. We present some evidence indicating that the modality is constant on classes of associated parabolic groups.

In Chapter 6 we study aspects of parabolic groups of higher modality. In Section 6.1 we give some lower bounds for the modality of parabolics of classical groups. In Corollary 6.6 we derive a finiteness result of V.L. Popov [**57**] asserting that there is only a finite number of simple algebraic groups admitting parabolic subgroups with prescribed semisimple rank and fixed modality.

This is followed in Section 6.2 where we present some small rank cases when the value of $\operatorname{mod} B$ is actually known, see Tables 6.3 and 6.4. Here the upper bounds for $\operatorname{mod} B$ were computed by MOP or its precursor, cf. [**35**]. In Proposition 6.8 we show some instances where the precise value of $\operatorname{mod} P$ can be determined using MOP.

In Chapter 7 we present the results from [**56**] where we study the relationship between spherical nilpotent orbits in the Lie algebra of a complex reductive group and abelian ideals $\mathfrak{a}$ of $\mathfrak{b} = \operatorname{Lie} B$. The principal result in this context is that, for an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$, any nilpotent orbit meeting $\mathfrak{a}$ is a spherical G–variety, see Theorem 7.3. As a consequence of this we obtain a short conceptual proof of a finiteness theorem from [**72**, Thm. 1.1]. Namely, for a parabolic subgroup P of G and an abelian ideal $\mathfrak{a}$ of $\mathfrak{p}$ in the nilpotent radical $\mathfrak{p}_u$, the group P operates on $\mathfrak{a}$ with finitely many orbits.

In our final chapter we present some examples of *Hasse diagrams* of the Bruhat-Chevalley order of the action of $P \subset \mathrm{GL}(V)$ on $\mathfrak{p}_u$ in some finite orbit cases.

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CHAPTER 1

Notation and Preliminaries

All algebraic varieties are taken over a fixed algebraically closed field k and all algebraic groups considered are affine. As general references for algebraic groups we cite the books by A. Borel [**6**] and T.A. Springer [**82**], as well as the notes from the Séminaire C. Chevalley [**23**]. In general, we follow the notation and terminology therein. For facts about conjugacy classes of algebraic groups, see R. Steinberg's lecture notes [**87**] and J.E. Humphreys' survey [**33**]. Concerning basic properties of subgroups of reductive groups normalized by a maximal torus, we refer to the work [**8**] by A. Borel and J. Tits. For information on root systems the reader should consult N. Bourbaki [**11**].

The general theory of representations of finite-dimensional algebras can be found in the books by M. Auslander, I. Reiten, and S. Smalø [**1**], P. Gabriel and A.V. Roiter [**28**], and C.M. Ringel's monograph [**66**].

1.1. General Notation

Throughout, the Lie algebra of an algebraic group G is denoted by $\operatorname{Lie} G$ or $\mathfrak{g}$, the identity component of G by G^0, and its unipotent radical by G_u. For the Lie algebra of G_u we write $\mathfrak{g}_u$. The identity element of G is labeled by e.

For $x \in G$ let $\operatorname{Int}(x)$ be the inner automorphism of G given by conjugation by x. For a subset S of G we denote the centralizer of S in G by $C_G(S) := \{g \in G \mid \operatorname{Int}(s)g = g \text{ for all } s \in S\}$, and likewise the centralizer of S in $\mathfrak{g}$ by $\mathfrak{c}_{\mathfrak{g}}(S) := \{Y \in \mathfrak{g} \mid \operatorname{Ad}(s)Y = Y \text{ for all } s \in S\}$.

Let X be a G–set. We write $g \cdot x$ for the image of $x \in X$ under the action of $g \in G$, and denote by $G \cdot x$ the *G–orbit of x in X*. For a subset Z of X, the *G–saturation* is the set of G–orbits $\cup_{z \in Z} G \cdot z$ and is denoted by $G \cdot Z$. For a subset $S \subseteq G$, we write X^S for the set of S–fixed points on X, that is $X^S := \{x \in X \mid s \cdot x = x \text{ for all } s \in S\}$. If $S = \{s\}$, then we write X^s instead of $X^{\{s\}}$. In particular, viewing G as a G–set, we have $G^S = C_G(S)$.

For an algebraic variety X and $x \in X$ we write $\mathrm{T}_x(X)$ for the *tangent space of X at x*.

If G acts morphically on the algebraic variety X, we also call X a *G–variety.* In that case we write $\phi_x : G \longrightarrow G \cdot x$ for the *orbit map at* $x \in X$ and say that G *acts separably on* $G \cdot x$ provided ϕ_x is a separable morphism, that is provided the differential $(d\phi_x)_e : \mathfrak{g} \longrightarrow \mathrm{T}_x(G \cdot x)$ of ϕ_x is onto [**6**, Prop. 6.7]. The *closure order*, or *Bruhat-Chevalley order*, on the set of G–orbits on X is given by the orbit closures, that is given two G–orbits, $\mathcal{O}_1$ and $\mathcal{O}_2$, we write

$$\mathcal{O}_1 \leq \mathcal{O}_2$$

whenever $\mathcal{O}_1$ is contained in the Zariski closure of $\mathcal{O}_2$, see [**6**, Prop. 1.8]. If S is an algebraic group acting on G by means of automorphisms of G, we say that G is an *S–group.*

Let Θ be an automorphism of G. Then $\theta = d\Theta_e$ is the corresponding automorphism of $\mathfrak{g}$. By G^Θ and $\mathfrak{g}^\theta$ we denote the Θ–fixed point subgroup of G and the θ–fixed point subalgebra of $\mathfrak{g}$ respectively. An automorphism of G is called *semisimple* if it can be achieved by conjugation by a semisimple element of some general linear group containing G [**85**] or [**82**, §4.4].

1.2. The Modality of Group Actions

Suppose that the algebraic group G acts morphically on an algebraic variety X. The *modality of the action of G on X* is defined as

$$\operatorname{mod}(G : X) := \max_{Z} \min_{z \in Z} \operatorname{codim}_Z G^0 \cdot z, \tag{1}$$

where Z runs through all irreducible G^0–invariant subvarieties of X. In case X is an irreducible variety let $k(X)^G$ denote the field of G–invariant rational functions on X. The following fundamental invariant-theoretic fact is due to M. Rosenlicht [**74**] (see also [**59**, 2.3])

$$\min_{x \in X} \operatorname{codim}_X G \cdot x = \operatorname{trdeg} k(X)^G. \tag{2}$$

Therefore, $\operatorname{mod}(G : X)$ measures the maximal number of parameters upon which a family of G–orbits on X depends. The modality of the action of G on X is zero precisely when G admits only a finite number of orbits on X. In turn, using (2), we can interpret the modality in terms of invariant theory. For instance, if $\operatorname{mod}(G : X) = 0$ then $k(Z)^G = k$ for every irreducible, G–invariant subvariety Z of X.

The notion of modality originates in the work of V.I. Arnold on the theory of singularities [**3**]. Our definition (1) here is due to E.B. Vinberg [**91**], see also [**59**, 5.2].

1.3. Reductive Groups

Suppose that G is reductive. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T, and let $r = \dim T = \operatorname{rank} G$ be the rank of G. Fix a Borel subgroup B of G containing T and let $\Pi = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Π in Ψ is $\Psi^+ = \Psi(B)$. Let W be the Weyl group of G. The highest (long) root in Ψ is denoted by ϱ. If all roots in Ψ are of the same length, they are all called long.

Suppose that G is simple (over its center). A prime is said to be *bad for G* if it divides the coefficient of a simple root in ϱ, else it is called *good for G* [**84**, §I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the *Chevalley commutator relations* for G; for these structural relations see for example [**85**, p. 30]. Thus, if $\operatorname{char} k = p$ is very bad for G, there are degeneracies in these relations. This only occurs if $p = 2$ and G is of type B_r, C_r, F_4, or G_2, or $p = 3$ and G is of type G_2. The same notions apply to reductive groups by means of simple components [**86**, 3.6].

Let N be a closed subgroup of G in B_u normalized by T. Then also $\mathfrak{n}$ is normalized by T, and the root spaces of $\mathfrak{n}$ relative to T are root spaces of $\mathfrak{g}$ (relative to T), that is $\mathfrak{n}$ is *T–regular* in the sense of [**27**, Ch. II]. As a consequence, $\mathfrak{n}$ is the direct sum of its root spaces, [**6**, Prop. 13.20]. Define the *set of roots of* $\mathfrak{n}$ *or* N (with respect to T) by

$$\Psi(\mathfrak{n}) := \Psi(N) := \{\beta \in \Psi \mid \mathfrak{g}_\beta \subseteq \mathfrak{n}\},$$

and likewise we define the *set of simple roots of* $\mathfrak{n}$ *or* N by

$$\Pi(\mathfrak{n}) := \Pi(N) := \Psi(\mathfrak{n}) \cap \Pi.$$

In particular, we have

$$\mathfrak{n} = \bigoplus_{\beta \in \Psi(\mathfrak{n})} \mathfrak{g}_\beta.$$

Thanks to [**6**, Prop. 14.4(2a)], N is connected and moreover $N = \prod U_\beta$, where the product is taken in some fixed order over $\Psi(N)$. In the situations we are going to study $\Psi(N)$ is usually closed under addition in Ψ. This is automatically satisfied whenever $\operatorname{char} k$ is not a very bad prime for G.

We may assume that each parabolic subgroup P of G considered contains B, i.e. that P is *standard*. Sometimes we denote a Levi subgroup of P by L_P, so that

$$P = L_P P_u$$

is a *Levi decomposition* of P. By the *semisimple rank of* P we mean the rank of the derived group of L_P, and denote it by $\operatorname{rank}_s P$. By saying that P is of a particular type, we mean the Dynkin type of a Levi subgroup of P.

Let $\beta = \sum n_\sigma(\beta)\sigma$ be a root, where σ is in Π. The P*–level of* β is the sub-sum of the coefficients $n_\sigma(\beta)$ over the elements of $\Pi(P_u)$, that is $\sum_{\Pi(P_u)} n_\sigma(\beta)$, see [**2**].

By a Levi subgroup of G we mean the Levi subgroup of some parabolic of G.

The descending central series of P_u is defined as usual by $\mathcal{C}^0 P_u := P_u$ and $\mathcal{C}^{i+1} P_u := (\mathcal{C}^i P_u, P_u)$ for $i \geq 0$. Since P_u is nilpotent, the smallest integer m such that $\mathcal{C}^m P_u = \{e\}$ is the class of nilpotency of P_u, that is the length of this series, and is also denoted by $\ell(P_u)$. If $\operatorname{char} k$ is not a very bad prime for G, then $\Psi(\mathcal{C}^i P_u)$ consists precisely of all roots whose P–level is at least $i+1$, see [**2**].

Suppose that $\operatorname{char} k$ is not a very bad prime for G. Owing to the Chevalley commutator relations, the class of nilpotency of P_u is readily determined to be the P–level of the highest root ϱ, that is

$$\ell(P_u) = \sum_{\sigma \in \Pi(P_u)} n_\sigma(\varrho), \tag{3}$$

see [**8**, Prop. 4.7(iii)].

Throughout, we use the labeling of the Dynkin diagram of G, that is of Π, in accordance with N. Bourbaki [**11**, Planches I - IX].

1.4. Richardson's Dense Orbit Theorem

Suppose that G is connected and reductive and P is a parabolic subgroup of G. R.W. Richardson proved that P acts on P_u with a dense orbit; likewise for the adjoint action of P on $\mathfrak{p}_u$ [**62**]. The proof relies on the fact that the number of unipotent classes of G is finite. This was first proved also by Richardson under some mild restrictions on the characteristic of the ground field [**61**]. Afterwards, these were removed by G. Lusztig [**47**]. An alternative proof of Richardson's Dense Orbit Theorem, due to R. Steinberg, can be found in [**88**, Cor. 4.2]; see also [**33**, 5.3]. For a detailed proof of the Lie algebra analogue, see [**22**, Thm. 5.2.3]. We refer to G. Lusztig and N. Spaltenstein [**48**] for an important generalization of this theorem, in the context of *induction of unipotent classes.*

The unique nilpotent class of $\mathfrak{g}$ which meets $\mathfrak{p}_u$ in the open P–orbit is called the *Richardson class of* P; likewise for the unique unipotent class of G meeting P_u densely.

The existence of a dense orbit is of course a necessary condition for P to have a finite number of orbits on $\mathfrak{p}_u$ or on P_u, as both, $\mathfrak{p}_u$ and P_u are irreducible varieties. It is, however, not sufficient. So, in this setting, the question of finiteness is a completely different one from that of density; unfortunately, the existence of a dense P–orbit is of no help in determining the instances when P operates on $\mathfrak{p}_u$ or P_u with finitely many orbits only.

1.5. Prehomogeneous Vector Spaces

Suppose the connected algebraic group H acts on the rational H–module V with a dense orbit. Then V is called a *prehomogeneous vector space for* H. Owing to Richardson's Dense Orbit Theorem, $\mathfrak{p}_u$ is a prehomogeneous vector space for the parabolic group P. If V is faithful and irreducible as an H–module, then H is reductive. In characteristic zero prehomogeneous vector spaces for reductive groups were classified by M. Sato and T. Kimura [**75**]. Call V a *finite orbit module* if H has only a finite number of orbits on V. Clearly, if V is a finite orbit module of H, then it is a prehomogeneous vector space for H. In [**39**, Thm. 2] V. Kac classified all the finite orbit modules for H reductive and $k = \mathbb{C}$. This classification was extended to positive characteristic by R. Guralnick, M. Liebeck, D. Macpherson, and G. Seitz in [**30**].

The classification results from Chapter 5 address the question when the prehomogeneous vector space $\mathfrak{p}_u$ is a finite orbit module for the non-reductive group P.

Frequently, the basis for determining parabolic subgroups with an infinite number of orbits on their nilradical is the existence of a proper P–invariant subspace $\mathfrak{n}$ of $\mathfrak{p}_u$, that is an ideal of $\mathfrak{p}$, which fails to be a prehomogeneous vector space for P in certain low rank cases, see Table 4.1. Accordingly, in default of a dense orbit, P acts on $\mathfrak{n}$ with an infinite number of orbits and likewise on all of $\mathfrak{p}_u$, see Lemma 4.1. Further examples of parabolic groups with an infinite number of orbits are then readily established by means of the inductive methods from Section 3.1.

There are examples, however, where P acts on $\mathfrak{p}_u$ with infinitely many orbits, and nevertheless, *every* linear P–submodule of $\mathfrak{p}_u$ is still prehomogeneous for P, see Remark 5.34 below.

CHAPTER 2

Monotonicity of Modality

2.1. Richardson's Lemma and Modality

In this section we present two monotonicity results for the modality of algebraic group actions. Throughout this section, G is an algebraic group acting morphically on an algebraic variety X and H is a closed subgroup of G.

LEMMA 2.1. *Suppose that X is irreducible. Let Z be an irreducible H–invariant subvariety of X. Assume that*

(i) $X = \overline{G \cdot Z}$, *and*
(ii) $G \cdot z \cap Z$ *is a finite union of H–orbits for $z \in Z$ in general position.*

Then

$$\operatorname{trdeg} k(X)^G = \operatorname{trdeg} k(Z)^H.$$

PROOF. Changing X and Z by appropriate invariant open subsets, we may assume, by Rosenlicht's Theorem [**74**, Thm. 2] (see also [**59**, Thm. 4.4]), that there exist geometric quotients

$$\pi_{G,X} : X \longrightarrow X/G \quad \text{and} \quad \pi_{H,Z} : Z \longrightarrow Z/H.$$

For any function $f \in k(X)^G$ the set of points in X where f is not regular is G–invariant and closed. Therefore, it follows from (i) that the restriction $f|_Z$ of f to Z is well-defined. The mapping $f \longrightarrow f|_Z$ is the embedding of the fields $k(X)^G \hookrightarrow k(Z)^H$. Since

$$\pi^*_{G,X}(k(X/G)) = k(X)^G \quad \text{and} \quad \pi^*_{H,Z}(k(Z/H)) = k(Z)^H, \tag{4}$$

this embedding defines a dominant rational mapping $\eta : Z/H \longrightarrow X/G$ such that the following diagram is commutative:

$$\begin{array}{ccc} Z & \xrightarrow{id} & X \\ {\scriptstyle \pi_{H,Z}}\downarrow & & \downarrow{\scriptstyle \pi_{G,X}} \\ Z/H & \xrightarrow{\eta} & X/G \end{array}$$

Since the fibers of $\pi_{G,X}$ are G–orbits, it follows from this diagram and (ii) that, for a point $a \in X/G$ in general position, the set

$$(\eta \circ \pi_{H,Z})^{-1}(a) = \pi_{G,X}^{-1}(a) \cap Z$$

is a union of finitely many H–orbits. Since the fibers of $\pi_{H,Z}$ are H–orbits, this shows that $\eta^{-1}(a)$ is a finite set. Further, as η is dominant, it follows from here and from the theorem on the dimension of the fibers of a morphism [**6**, Thm. 10.1] that $\dim X/G = \dim Z/H$. The claim now follows from (4) and the definition of dimension. □

Corollary 2.2. *Assume as in Lemma 2.1. Then*

$$\min_{x \in X} \operatorname{codim}_X G \cdot x = \min_{z \in Z} \operatorname{codim}_Z H \cdot z.$$

Proof. This follows from Lemma 2.1, the equation (2) on page 2, and the fact that the minimum of the codimensions of orbits is attained on the orbits of points in general position [**59**, §1.4]. □

Lemma 2.1 and Corollary 2.2 appear in [**58**, Thm. 2.11].

Theorem 2.3. *Let G be an algebraic group acting on an algebraic variety X. Let H be a closed subgroup of G and Y a locally closed H–invariant subset of X. Let $\phi_y : G \longrightarrow G \cdot y$ be the orbit map in X at $y \in Y$. Assume that*

$$\mathrm{T}_y(G \cdot y) \cap \mathrm{T}_y(Y) \subseteq (d\phi_y)_e(\mathfrak{h}) \quad \textit{for each point } \; y \in Y. \tag{5}$$

Choose $y \in Y$ and let $\mathcal{O} := G \cdot y$. Then:

- (i) *Each irreducible component of $\mathcal{O} \cap Y$ is a single H^0–orbit. In particular, $\mathcal{O} \cap Y$ is a union of finitely many H–orbits each of which is open and closed in $\mathcal{O} \cap Y$.*
- (ii) *H acts separably on each orbit $H \cdot y$ for any $y \in \mathcal{O} \cap Y$.*
- (iii) *We have the inequality*

$$\operatorname{mod}(G : X) \geq \operatorname{mod}(H : Y).$$

Proof. Without loss we may assume that $H = H^0$. Since $H \cdot y \subseteq G \cdot y \cap Y$, we have $\mathrm{T}_y(H \cdot y) \subseteq \mathrm{T}_y(G \cdot y \cap Y)$, and since $\mathrm{T}_y(G \cdot y \cap Y) \subseteq \mathrm{T}_y(G \cdot y) \cap \mathrm{T}_y(Y)$, it follows from (5) that

$$\mathrm{T}_y(H \cdot y) \subseteq \mathrm{T}_y(G \cdot y) \cap \mathrm{T}_y(Y) \subseteq (d\phi_y)_e(\mathfrak{h}) \subseteq \mathrm{T}_y(H \cdot y). \tag{6}$$

Consequently, we have

$$\mathrm{T}_y(H \cdot y) = \mathrm{T}_y(G \cdot y \cap Y).$$

Let Z be an irreducible component of $G \cdot y \cap Y$ containing $H \cdot y$. Then the containments $H \cdot y \subseteq Z \subseteq G \cdot y \cap Y$ imply

$$\dim H \cdot y \leq \dim Z \leq \dim \mathrm{T}_y(G \cdot y \cap Y).$$

Because $H \cdot y$ is smooth, we have $\dim H \cdot y = \dim \mathrm{T}_y(H \cdot y)$. It then follows that $\dim H \cdot y = \dim Z$. Whence the closure of $H \cdot y$ in $G \cdot y \cap Y$ coincides with Z. Thus, as $H \cdot y$ is open in its closure, $H \cdot y$ is open in Z. Further, $Z \setminus H \cdot y$ is a union of H–orbits each of which is also open in Z by the same argument. Therefore, $H \cdot y$ is also closed in Z and we conclude that $H \cdot y = Z$. Thus (i) now follows.

By (6) we have $(d\phi_y)_e(\mathfrak{h}) = \mathrm{T}_y(H \cdot y)$, and thus $\phi_y|_H : H \longrightarrow H \cdot y$ is separable [**6**, Prop. 6.7], whence (ii) holds.

For the modality statement (iii) we argue as follows, see [**58**, Thm. 2.13]. Let Z be an irreducible H–invariant subvariety of Y such that

$$\operatorname{mod}(H : Y) = \min_{z \in Z} \operatorname{codim}_Z H \cdot z.$$

By the first part of Theorem 2.3 conditions (i) and (ii) of Lemma 2.1 are satisfied for Z and $\overline{G^0 \cdot Z}$. Then it follows from Corollary 2.2 that

$$\min_{z \in Z} \operatorname{codim}_Z H \cdot z = \min_{x \in \overline{G^0 \cdot Z}} \operatorname{codim}_{\overline{G^0 \cdot Z}} G^0 \cdot x.$$

From the definition of modality (1) on page 2, we have

$$\operatorname{mod}(G : X) \geq \min_{x \in \overline{G^0 \cdot Z}} \operatorname{codim}_{\overline{G^0 \cdot Z}} G^0 \cdot x.$$

Finally, the desired inequality in (iii) follows. □

Parts (i) and (ii) of Theorem 2.3 are proved in [**63**, Lem. 3.1]; see also [**90**, p. 469], [**59**, Lem. 1.10], or [**58**, Thm. 2.13].

For our purposes we require the following

Theorem 2.4. *Let A be an affine algebraic group, G a closed connected normal subgroup of A, and S a closed subgroup of A. Suppose that A acts morphically on an algebraic variety X. Set $H := (G^S)^0$ and $Y := X^S$. For $y \in Y$ let $\mathcal{O} := G \cdot y$. Assume that*

(i) *the adjoint action of S on $\mathfrak{g}$ is semisimple, and*
(ii) $\operatorname{Lie} H = \mathfrak{g}^S$.

Then H acts transitively on each connected component of $\mathcal{O}^S = \mathcal{O} \cap Y$. In particular, H has only a finite number of orbits on $\mathcal{O} \cap Y$ and each such H–orbit is open and closed in $\mathcal{O} \cap Y$.

As a consequence, we have the inequality

$$\operatorname{mod}(G : X) \geq \operatorname{mod}(H : Y).$$

Proof. The first part of the theorem is the principal result from [**63**, Thm. A]. The proof of the inequality for the modality of both actions is identical to that of Theorem 2.3. □

Suppose as in Theorem 2.4; if G acts separably on $\mathcal{O}$, then so does H on each orbit $H \cdot y$, where $y \in \mathcal{O} \cap Y$ [**63**, Cor. 3.2].

REMARK 2.5. The H–orbits in $\mathcal{O} \cap Y$ in Theorem 2.3 necessarily all have the same dimension. For, in its proof we see that for any $y \in \mathcal{O} \cap Y$ we have

$$\dim H \cdot y \leq \dim \mathcal{O} \cap Y \leq \dim \mathrm{T}_y(\mathcal{O} \cap Y) = \dim \mathrm{T}_y(H \cdot y) = \dim H \cdot y.$$

So that $\dim H \cdot y = \dim \mathcal{O} \cap Y$ for any $y \in \mathcal{O} \cap Y$.

In contrast, this is not the case in the situation of Theorem 2.4 in general, for instance, see [**63**, Ex. 3.3].

REMARK 2.6. We emphasize that the connectedness assumptions on G and H in Theorem 2.4 are not required for the finiteness and modality statements, as G^0 has finite index in G and $\mathrm{mod}(G^0 : X) = \mathrm{mod}(G : X)$ by definition of modality (1); likewise for the action of H on Y (see [**77**, Rem. p. 334]).

Finite decomposition results like Theorems 2.3, 2.4, or variations thereof, such as [**77**, I.2], are often referred to in the literature as *Richardson's Lemma*. In connection with so called *reductive pairs* such a result appeared for the first time in Richardson's fundamental paper [**61**, Thm. 3.1]; see [**77**, I.3] for examples of such reductive pairs.

2.2. Linearly Reductive Groups

An algebraic group S defined over k is called *linearly reductive* provided every rational representation of S is semisimple. Concerning basic properties of these groups, see T.A. Springer's review [**83**] or H. Kraft's book [**45**, II.3.5, AII]. In characteristic zero, S is linearly reductive if and only if S is reductive, see [**83**, V. §1.1], or [**45**, II.3.5 p. 109]. If $\mathrm{char}\, k = p$, then S is linearly reductive if and only if S^0 is a torus and the order of S/S^0 is relatively prime to p, by work of M. Nagata [**52**]. In particular, any finite group (viewed as an algebraic group over k) whose order is relatively prime to p is linearly reductive (see Maschke's Theorem).

Our aim is to apply Theorem 2.4 for linearly reductive groups. The following fact is [**63**, Lem. 4.1].

LEMMA 2.7. *Let S be linearly reductive and G an S–group. Then we have* $\mathrm{Lie}(G^S) = \mathfrak{g}^S$. *In particular, conditions (i) and (ii) of Theorem 2.4 are satisfied in this instance.*

LEMMA 2.8. *Let G and S be closed subgroups of the affine algebraic group A. Suppose that G is normal in A and S is linearly reductive. Then we have*

$$(G_u^S)^0 = (G^S)_u.$$

PROOF. As G_u is a characteristic subgroup of G, it is also an S–group and so is G/G_u for the induced action. By definition G/G_u is reductive and therefore, so is $(G/G_u)^S$, by [**63**, Prop. 10.1.5].

Since S is linearly reductive, we have $\mathrm{Lie}(G^S) = \mathfrak{g}^S$; and likewise for G_u and G/G_u, by Lemma 2.7. As $\mathfrak{g}$ is a semisimple S–module and $\mathfrak{g}_u$ is an S–submodule, there exists an S–submodule $\mathfrak{r}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}_u \oplus \mathfrak{r}$. Therefore, $\mathfrak{g}/\mathfrak{g}_u \cong \mathfrak{r}$ as S–modules, and so $(\mathfrak{g}/\mathfrak{g}_u)^S \cong \mathfrak{r}^S$. Furthermore, since $\mathfrak{g}_u$ and $\mathfrak{r}$ are S–submodules of $\mathfrak{g}$, we also have $\mathfrak{g}^S = \mathfrak{g}_u^S \oplus \mathfrak{r}^S$, and so $\mathfrak{g}^S/\mathfrak{g}_u^S \cong \mathfrak{r}^S$ as S–modules. In particular, $\dim \mathfrak{g}^S/\mathfrak{g}_u^S = \dim(\mathfrak{g}/\mathfrak{g}_u)^S$.

Because $\mathrm{Lie}(G/G_u)^S \cong (\mathfrak{g}/\mathfrak{g}_u)^S$ and $\mathrm{Lie}\, G^S/G_u^S \cong \mathfrak{g}^S/\mathfrak{g}_u^S$, we conclude that $\dim G^S/G_u^S = \dim(G/G_u)^S$.

Note that the image of G^S under the canonical epimorphism $G \longrightarrow G/G_u$ is a subgroup of $(G/G_u)^S$ isomorphic to G^S/G_u^S. Then, since these two groups have the same dimension, their identity components are isomorphic. A linear algebraic group is reductive precisely when its identity component has this property. Thus G^S/G_u^S is reductive. As $G^S/(G^S)_u$ is the largest reductive quotient of G^S, we infer that $\dim(G^S)_u \leq \dim G_u^S$. Clearly, G_u^S is unipotent. Therefore, $(G_u^S)^0 \subseteq (G^S)_u$. Finally, comparing dimensions, we conclude that $(G^S)_u = (G_u^S)^0$, as desired. □

We emphasize that if S is diagonalizable and specifically if $S = \langle \Theta \rangle$, where Θ is a semisimple automorphism of G, then S is linearly reductive, and thus conditions (i) and (ii) of Theorem 2.4 are satisfied. In case S is diagonalizable Lemmas 2.7 and 2.8 are well known, see [**6**, Prop. 9.4(1)] and [**6**, Cor. 13.17]. Also, in that case G_u^S is known to be connected [**6**, Prop. 9.4(1)], or [**6**, Thm. 10.6(5)].

REMARK 2.9. Suppose G and S are closed subgroups of the affine algebraic group A, that G is connected and normal in A and S is linearly reductive. Let P be a parabolic subgroup of G which is normalized by S, that is P is an S–group. Then P^S is a parabolic subgroup of $(G^S)^0$ [**63**, Prop. 10.2.1]. Note that, as a characteristic subgroup of P, its unipotent radial P_u is also an S–group. Now suppose in addition that G is reductive. Then G^S is again reductive, by [**63**, Prop. 10.1.5]. Moreover, by [**63**, Prop. 6.1], there exists a Levi subgroup L of P which is also normalized by S. Consequently, using Lemma 2.8,

$$P^S = L^S P_u^S$$

is a Levi decomposition of P^S.

2.3. Controlling Fusion

Let G be an algebraic group and H a closed subgroup of G. Suppose that X is a G–variety and Y is an H–invariant subvariety of X. We say that H *controls fusion in* Y *with respect to the action of* G provided

$$G \cdot y \cap Y = H \cdot y \quad \text{for any } y \in Y.$$

In certain instances of the setting of Theorem 2.4 it turns out that H controls fusion in Y for the action of G. Naturally, this is of interest if one is concerned with comparing the numbers of orbits of the actions of G on X versus that of H on Y in case $\operatorname{mod}(G : X) = 0$. As an example, we refer to [**49**, Prop. 2.1], where the group S in Theorem 2.4 is a cyclic group of order two. We describe several instances of fusion for parabolic group actions in Section 3.3 below; see Propositions 3.13 and 3.12, as well as Remarks 3.15 and 3.16.

CHAPTER 3

The Modality of Parabolic Group Actions

The results in this chapter generalize work from [**58**] and [**73**]. More specifically, Corollaries 2.6 and 2.8 from [**58**] (in case $\operatorname{char} k = 0$) and Theorems 1.1 and 1.2 in [**73**] (in positive characteristic). In Sections 3.2 through 3.6 we assume that G is reductive.

3.1. Monotonicity for the Modality of P–Actions

Throughout this section, G denotes a linear algebraic group defined over an algebraically closed field k and P is a parabolic subgroup of G with unipotent radical P_u. We consider the modality of the action of P on closed connected normal subgroups $N \subseteq P_u$ by conjugation, as well as the adjoint action of P on ideals $\mathfrak{n} \subseteq \mathfrak{p}_u$ of $\mathfrak{p}$. With the aid of Theorem 2.4 we first show some monotonicity results for the modality of the action of P on N and $\operatorname{Lie} N = \mathfrak{n}$. We call

$$\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$$

the *modality of* P.

The advantage of Theorem 2.4 for our purpose is that it allows us to obtain the desired monotonicity results for the modality of the conjugation action of P on P_u as well as for the adjoint action of P on $\mathfrak{p}_u$ simultaneously, avoiding any separability considerations of the orbit maps involved.

On the other hand, Theorem 2.3 is particularly useful if the base field is of characteristic zero, e.g., see [**58**], when separability of the orbit maps is guaranteed.

LEMMA 3.1. *Let $Q \subseteq P$ be parabolic subgroups of G. Let $N \subseteq P_u$ be a closed normal subgroup of P. Then* $\operatorname{mod}(P : N) \leq \operatorname{mod}(Q : N)$. *In particular, we deduce*

(i) $\operatorname{mod}(P : P_u) \leq \operatorname{mod}(Q : Q_u)$, *and*
(ii) $\operatorname{mod} P \leq \operatorname{mod} Q$.

PROOF. Since $Q \subseteq P$, we have $N \subseteq P_u \subseteq Q_u$. Any irreducible P–invariant subvariety Z of N is also Q–invariant and $\operatorname{codim}_Z P \cdot z \leq \operatorname{codim}_Z Q \cdot z$ for any z in Z. Consequently, we get $\operatorname{mod}(P : N) \leq$

$\mathrm{mod}(Q : N)$, by the definition of modality (1) above. In particular, setting $N = P_u$, we infer

$$\mathrm{mod}(P : P_u) \leq \mathrm{mod}(Q : P_u) \leq \mathrm{mod}(Q : Q_u),$$

and (i) follows.

The proof of (ii) is analogous to that of (i) replacing N by $\mathfrak{n}$, etc. □

Next we address our principal application of Theorem 2.4.

COROLLARY 3.2. *Suppose that $G, S \subseteq A$ are closed subgroups of the affine algebraic group A, that G is normal in A and S is linearly reductive. Let P be a parabolic subgroup of G and let $N \subseteq P_u$ be a closed normal subgroup of P. Suppose that both, P and N are normalized by S, that is both are S–groups. Then we have the inequalities*

(i) $\mathrm{mod}(P^S : N^S) \leq \mathrm{mod}(P : N)$, *and*
(ii) $\mathrm{mod}(P^S : \mathfrak{n}^S) \leq \mathrm{mod}(P : \mathfrak{n})$.

PROOF. For $A := P \rtimes S$, P^0, and $X := N$ the hypotheses of Theorem 2.4 are satisfied. Note that P^0 is a normal subgroup of $P \rtimes S$. Hence for each x in N^S the intersection $P^0 \cdot x \cap N^S$ is a finite union of $(P^S)^0$–orbits and $\mathrm{mod}((P^S)^0 : N^S) \leq \mathrm{mod}(P^0 : N)$ by Theorem 2.4. Finally, by Remark 2.6 we infer that $\mathrm{mod}(P^S : N^S) \leq \mathrm{mod}(P : N)$, as claimed.

Part (ii) follows *mutatis mutandis*. □

As a special case of Corollary 3.2 we obtain

THEOREM 3.3. *Suppose that $G, S \subseteq A$ are closed subgroups of the affine algebraic group A, that G is normal in A and S is linearly reductive. Let P be a parabolic subgroup of G which is normalized by S. Then we have*

(i) $\mathrm{mod}(P^S : (P^S)_u) \leq \mathrm{mod}(P : P_u)$, *and*
(ii) $\mathrm{mod}\, P^S \leq \mathrm{mod}\, P$.

PROOF. Since P_u is a characteristic subgroup of P, it is also S–stable, as S operates on P by automorphisms. It follows from Lemma 2.8 above that $(P^S)_u = (P_u^S)^0$. The statement (i) now follows from Corollary 3.2(i).

For part (ii) observe that $\mathrm{Lie}(P^S)_u = \mathrm{Lie}\, P_u^S$ by Lemma 2.8 and $\mathrm{Lie}\, P_u^S = (\mathrm{Lie}\, P_u)^S$ by Lemma 2.7. The result then follows from Corollary 3.2(ii):

$$\begin{aligned} \mathrm{mod}\, P^S &= \mathrm{mod}(P^S : \mathrm{Lie}(P^S)_u) = \mathrm{mod}(P^S : \mathrm{Lie}\, P_u^S) \\ &= \mathrm{mod}(P^S : (\mathrm{Lie}\, P_u)^S) \leq \mathrm{mod}(P : \mathfrak{p}_u), \end{aligned}$$

as desired. □

We proceed with some particular instances of Theorem 3.3.

COROLLARY 3.4. *Suppose that Θ is a semisimple automorphism of G and that P is Θ–stable. Set $H := G^\Theta$ and $Q := P^\Theta = P \cap H$. Then we get*

(i) $\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u)$, *and*
(ii) $\operatorname{mod} Q \leq \operatorname{mod} P$.

PROOF. This is the special case $S = \langle\Theta\rangle$ of Theorem 3.3. □

COROLLARY 3.5. *Let $s \in P$ be semisimple. Set $H := C_G(s)$ and $Q := P \cap H$. Then we have the inequalities*

(i) $\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u)$, *and*
(ii) $\operatorname{mod} Q \leq \operatorname{mod} P$.

PROOF. Observe that $\Theta = \operatorname{Int}(s)$ is a semisimple automorphism of G, and $H = G^\Theta$. The result is a special case of Corollary 3.4. □

COROLLARY 3.6. *Let T be a maximal torus of G in P and S a subtorus of T. Set $H := C_G(S)$ and $Q := P \cap H$. Then we have the inequalities*

(i) $\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u)$, *and*
(ii) $\operatorname{mod} Q \leq \operatorname{mod} P$.

PROOF. By [**6**, Prop. 8.18] there exists an element $s \in S$ such that $C_G(s) = C_G(S)$. Thus Corollary 3.5 applies and the desired result follows. Note that H is connected whenever G is, by [**6**, Cor. 11.12]. □

For the remainder of this section we assume that G is reductive.

REMARK 3.7. For our main concern of studying $\operatorname{mod}(P : P_u)$ and $\operatorname{mod} P$ we may assume without loss that G is connected, simply connected, and semisimple. For, as $P \cap G^0$ has finite index in P, we have $\operatorname{mod} P \cap G^0 = \operatorname{mod} P$. So, we may suppose that G is connected. Then $G = G'Z(G)^0$, where G' is semisimple. Since P and $P \cap G'$ only differ by central elements, we have $\operatorname{mod} P \cap G' = \operatorname{mod} P$. So we may further assume that G is semisimple. If G is simply connected and $\overline{G}$ is in the same isogeny class as G, then $\operatorname{mod} P \cap \overline{G} = \operatorname{mod} P$, as $P \longrightarrow P \cap \overline{G}$ has finite central kernel. Hence we may assume that G is a connected simply connected semisimple algebraic group.

The same reductions apply if we consider the related expressions $\operatorname{mod}(P : N)$, $\operatorname{mod}(P : N/M)$, or $\operatorname{mod}(P : \mathfrak{n}/\mathfrak{m})$, where $M \subseteq N \subseteq P_u$ are closed normal subgroups of P.

If G is a connected simply connected semisimple algebraic group and Θ is a semisimple automorphism of G, then G^Θ is reductive and again connected by [**86**, 8.1]. In particular, this applies to $C_G(s)$, where s is a semisimple element of G [**86**, 8.5], whence to H in Corollaries 3.5 and 3.6. Thus by Remark 3.7 we may replace H by H' in these statements. In Corollary 3.6, the centralizers of sub-tori of maximal tori of G are precisely the Levi subgroups of G [**8**, 4.15]. Thus we may reformulate Corollary 3.6 in this case as follows.

Theorem 3.8. *Let G be reductive and $P \subseteq G$ a parabolic subgroup of G containing T. Let H be a Levi subgroup of G normalized by T, or the derived subgroup thereof. Set $Q := P \cap H$. Then we have*

(i) $\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u)$, *and*
(ii) $\operatorname{mod} Q \leq \operatorname{mod} P$.

Let G be a connected, simply connected simple algebraic group. The following criterion of D.I. Deriziotis [**25**, Prop. 2.3] gives precise information as to when a reductive subgroup H of G is the centralizer of a semisimple element of G. Let $\widetilde{\Pi} := \Pi \cup \{-\varrho\}$. The set $\widetilde{\Pi}$ is associated with the *extended Dynkin diagram*, see [**11**], [**7**]. In [**21**, Prop. 11] R.W. Carter proves a criterion which is equivalent to the one by Deriziotis, but also applies in characteristic zero. Recall, W denotes the Weyl group of G.

Proposition 3.9. *Let G be a connected, simply connected simple algebraic group. Assume that* $\operatorname{char} k$ *is a good prime for G. Let H be a connected reductive subgroup of G of maximal rank. Fix a maximal torus T of G in H. Then H is the centralizer of some semisimple element of G if and only if $\Psi(H)$ has a basis which is W–conjugate to a proper subset of $\widetilde{\Pi}$.*

Ultimately, we consider the setting of a closed reductive subgroup of G normalized by a maximal torus T of G.

Theorem 3.10. *Let G be a reductive algebraic group and suppose that* $\operatorname{char} k$ *is zero or a good prime for G. Let H be a closed reductive subgroup of G normalized by a maximal torus T of G and P a parabolic subgroup of G containing T. Set $Q := P \cap H$. Then we have the inequalities*

(i) $\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u)$, *and*
(ii) $\operatorname{mod} Q \leq \operatorname{mod} P$.

Proof. Thanks to Remark 3.7, we may suppose that G is connected, simply connected, and semisimple, and likewise that H is connected and semisimple. Let $\Psi(H)$ be the set of roots of H relative to T which is a closed symmetric subsystem of the root system Ψ of G

relative to T. These can be determined by means of the algorithm of Borel-de Siebenthal [**7**] applied to each of the simple components of G, see also [**11**, Exc. Ch. VI §4.4]. According to that, H is the derived subgroup of a Levi subgroup of some connected semisimple subgroup of G of maximal rank. By Theorem 3.8 we may assume that H is a connected semisimple subgroup of G of maximal rank. Such an H can be obtained by successive applications of the Borel-de Siebenthal algorithm applied first to the simple components of G and then successively to further irreducible components. If H is maximal among such groups, it is then the centralizer of some semisimple element of G according to Proposition 3.9 applied to each of the simple components of G. Inductively we obtain a chain of connected semisimple subgroups of G of maximal rank each of which is the centralizer of some semisimple element of the next group with H at the beginning and G at the end of the chain. If $\operatorname{char} k$ is a good prime for G, it is also a good prime for each connected semisimple subgroup of G of maximal rank [**84**, 4.7]. Since each of these simple components is also simply connected, the result then follows from Corollary 3.5 and Proposition 3.9. In case $\operatorname{char} k = 0$ we argue analogously. □

REMARK 3.11. Observe that the proofs of Corollaries 3.5 and 3.6, and of Theorems 3.8 and 3.10 equally apply if we consider the P–action on a closed normal subgroup N of P in P_u instead of P_u itself and the Q–action on $Q_u \cap N$ instead of Q_u; likewise for the adjoint action of P on $\mathfrak{n}$ and the one of Q on $\mathfrak{q}_u \cap \mathfrak{n}$.

In characteristic zero Corollary 3.4(ii) and Theorem 3.10(ii) are proved in [**58**, Cor. 2.6, 2.8] and in positive characteristic in [**73**, Thm. 1.1, 1.2].

REMARK. If Θ is a finite automorphism of G, then it is semisimple provided $\operatorname{char} k$ does not divide its order. If G is semisimple, then the group of outer automorphisms of G is finite [**86**]. In particular, if Θ is a graph automorphism of G, Theorem 3.3 applies provided $\operatorname{char} k$ satisfies this condition. No characteristic restrictions are required for applications of Corollaries 3.5, 3.6 and Theorem 3.8.

3.2. Algorithmic Modality Analysis

The results of Sections 4.1 and 6.1 show that we have good tools in order to construct lower bounds for $\operatorname{mod} P$ (e.g. see Lemma 4.1). The situation is different, once we turn to upper bounds. An algorithm, developed with U. Jürgens, is an effective method to establish such bounds for $\operatorname{mod} P$. This program, referred to as MOP (Modality Of Parabolics), is available as a GAP share package, cf. [**29**]. MOP's application is limited to the instances when the Dynkin diagram of G is simply laced. This program is specifically designed to address modality questions for groups of exceptional type, and in particular, to determine parabolic subgroups of the latter of modality zero, in default of an adequate reduction technique, as available for classical groups. (see Lemma 5.24).

The use of this algorithmic procedure was crucial in establishing the classification of the modality zero parabolic subgroups in exceptional algebraic groups from [**37**], see Theorem 5.30 below. The results obtained in [**35**] are based on a precursor of MOP.

The program generalizes an algorithm due to H. Bürgstein and W.H. Hesselink [**20**], which was designed to analyze the orbit structure of a Borel subgroup B for the adjoint and coadjoint actions on $\mathfrak{b}_u$ and on $\mathfrak{b}_u^*$, see also [**58**]. In contrast to the methods from [**20**] and [**58**], this algorithm is inductive in the following sense. Suppose G is reductive, $P \subseteq G$ is parabolic, and we aim to show that $\operatorname{mod} P \leq m$, for $m \in \mathbb{N}$. Let H be a proper semisimple regular subgroup of G and let $Q = P \cap H$. Inductively, $\operatorname{mod} Q$ is known and we may assume that $\operatorname{mod} Q$ is at most m, else $\operatorname{mod} P > m$ by Theorem 3.10. It follows from the proof of this theorem and Remark 3.11 that $\operatorname{mod}(P : P \cdot \mathfrak{q}_u) = \operatorname{mod} Q$. Therefore, we only need to consider the P–orbits in $\mathfrak{p}_u \setminus P \cdot \mathfrak{q}_u$. This applies to any such Q. Here it obviously suffices to only take those H which are maximal among such subgroups leading to maximal candidates for Q. Hence, we only take maximal rank subgroups or regular semisimple subgroups H of corank 1 in G. We form the list of all subsystems $\Psi(H)^+$, where H runs through this fixed set of regular semisimple subgroups of G of large rank such that $\Psi(H)^+ \subset \Psi^+$. The symmetric subsystems corresponding to such semisimple subgroups H of G can be determined by means of the algorithm of Borel-de Siebenthal [**7**], see also [**11**, Exc. Ch. VI §4.4]. In [**4**] all conjugacy classes of such subsystems of Ψ under the action of the Weyl group of G are classified.

This inductive feature allows one to effectively compute the upper bound of $\operatorname{mod} P$ in several instances. This is demonstrated for instance in the classification of the modality zero parabolic subgroups in exceptional groups in Theorem 5.30. In Section 6.2 we present several instances where the modality of parabolic groups of positive modality can be calculated explicitly with the aid of MOP.

For a detailed description of the MOP program, its mathematical background, and further applications, we refer to [**36**] and the MOP manual [**38**].

3.3. Controlling Fusion for P–Actions

In this section we delineate two fusion results for the P–actions studied above. The first one, generalizes an argument by A.E. Zalesskiĭ [**94**], while the second one is the Lie algebra counterpart of the first, extending V.V. Kashin's argument [**41**, Lem. 1].

Proposition 3.12. *Let G be reductive and $P \subseteq R \subseteq G$ parabolic subgroups of G containing T. Let H be the standard Levi subgroup of R or its derived subgroup. Set $Q := P \cap H$. Then Q controls fusion in Q_u for the action of P, that is*

$$P \cdot x \cap Q_u = Q \cdot x \quad \textit{for every} \;\; x \in Q_u.$$

As a consequence, we obtain

$$\operatorname{mod}(Q : Q_u) \leq \operatorname{mod}(P : P_u).$$

Proof. By Théorème 1 in [**23**, exp. 17] Q is a parabolic subgroup of H, see Remark 2.9. First we suppose that $H = L_R$. Since $P \subseteq R$, we have $L_P \subseteq H$ and thus $L_P = L_Q$. Thus it suffices to consider P_u–conjugate elements of Q_u. Moreover, by construction, $P_u = Q_u R_u$. Now let x, x' be two elements of Q_u and $y \in P_u$ such that $x' = yxy^{-1}$. Write $y = qu$, where $q \in Q_u$ and $u \in R_u$. Thus we have $Q_u \ni q^{-1}x'q = uxu^{-1}$. Now, as this element and x lie in Q_u and since R_u is normal in P_u, we get that $x^{-1}uxu^{-1} \in Q_u \cap R_u = \{e\}$. Consequently, $x' = qxq^{-1}$, that is x' and x are Q–conjugate, as desired.

In case $H = L'_R$ the same result also holds, since $T = T_H C_T(H)$ ([**23**, exp. 17, Lem. 2]), where $T_H := H \cap T$ is a maximal torus of H.

Now since any two P–conjugate elements of Q_u are already Q–conjugate, the map $Q \cdot x \mapsto P \cdot x$ defines a bijection between the set of Q–orbits in Q_u and the set of P–orbits in $P \cdot Q_u$. Accordingly, we conclude

$$\operatorname{mod}(Q : Q_u) = \operatorname{mod}(P : P \cdot Q_u) \leq \operatorname{mod}(P : P_u),$$

as claimed. □

Proposition 3.13. *Assume that G, $P \subseteq R$, H, and Q are as in Proposition 3.12. Then Q controls fusion in $\mathfrak{q}_u$ for the action of P, that is*

$$P \cdot x \cap \mathfrak{q}_u = Q \cdot x \quad \textit{for every} \;\; x \in \mathfrak{q}_u.$$

As a consequence, we obtain

$$\operatorname{mod} Q \leq \operatorname{mod} P.$$

PROOF. The argument is similar to that in the proof of Proposition 3.12. Suppose first again that $H = L_R$. Since $P \subseteq R$, we have $L_P \subseteq H$ and thus $L_P = L_Q$. Thus it suffices to consider P_u–conjugate elements of $\mathfrak{q}_u$. As before, $P_u = Q_u R_u$, and thus $\mathfrak{p}_u = \mathfrak{q}_u \oplus \mathfrak{r}_u$. Let v, v' be two elements of $\mathfrak{q}_u$ and $y \in P_u$ such that $v' = y \cdot v$. Write $y = xu$, where $x \in Q_u$ and $u \in R_u$. Thus we have $\mathfrak{q}_u \ni x^{-1} \cdot v' = u \cdot v$. Note that u and v are of the form

$$u = \prod_{\alpha \in \Psi(R_u)} U_\alpha(\xi_\alpha) \quad \text{for some } \xi_\alpha \in k, \quad \text{and}$$

$$v = \sum_{\beta \in \Psi(\mathfrak{q}_u)} \zeta_\beta e_\beta \quad \text{for some } \zeta_\beta \in k.$$

As the adjoint action of a root element on a root vector satisfies

$$U_\alpha(\xi_\alpha) \cdot e_\beta \in e_\beta + \sum_{i \geq 1} k e_{\beta + i\alpha}$$

(see [**85**]), we see that $u \cdot v = v + v''$, where $v'' \in \mathfrak{r}_u$, since $\Psi(\mathfrak{r}_u)$ is an *ideal* in Ψ^+ (see [**85**, p. 24]). Consequently, we have $v'' = u \cdot v - v \in \mathfrak{q}_u \cap \mathfrak{r}_u = \{0\}$. Thus, $u \cdot v = v$ and so $v' = y \cdot v = q \cdot v$, as claimed.

In case $H = L'_R$ we argue as in the proof of the previous proposition. The same result also holds, since $T = T_H C_T(H)$ (see [**23**, exp. 17, Lem. 2]), where $T_H := H \cap T$ is a maximal torus of H, and $C_T(H)$ acts trivially on $\mathfrak{h}$.

The argument for the conclusion on the modality of the two actions is analogous to that of Proposition 3.12. $\square$

REMARKS 3.14. The case $P = B$ in Propositions 3.12 and 3.13 is the one treated in [**94**] and [**41**], respectively. Note that there are no characteristic restrictions involved.

In [**94**, p. 130] A.E. Zalesskiĭ shows that $\mathrm{mod}(B : B_u) > 0$ provided G is of type A_r and $r \geq 5$, or B_r, C_r, D_r, and $r \geq 6$, E_6, E_7, or E_8. The modality statement follows from the result for type A_r for $r \geq 5$ (see [**94**, Thm. 1]), the observation that each of the simple groups in this list admits a Levi subgroup of type A_5, and the case $P = B$ from Proposition 3.12.

For our purpose of comparing the modality of different group actions the application of both propositions is limited to the case when the semisimple parts of the Levi subgroups of P and Q are isomorphic. The monotonicity statements for the modality of the actions involved in both propositions equally follow from Theorem 3.8.

REMARK 3.15. Suppose G, P, H and, $Q = H \cap P$ are as in the setting of Theorem 3.8 or Theorem 3.10. In addition suppose that

$\operatorname{char} k = 0$ and Q_u is abelian. Then Q controls fusion in Q_u (resp. $\mathfrak{q}_u$) for the action of P. For, since $\operatorname{char} k = 0$, condition (5) of Theorem 2.3 is satisfied for the actions of P and Q on $P \cdot Q_u$ (resp. $P \cdot \mathfrak{q}_u$), [**58**, Prop. 2.5]. Consequently, $P \cdot x \cap Q_u$ is a finite union of Q–orbits for $x \in Q_u$, each of which is of the same dimension, by Remark 2.5. By [**65**, Prop. 2.16], no two Q–orbits in Q_u have the same dimension. Thus $P \cdot x \cap Q_u = Q \cdot x$, as claimed. Likewise, for the adjoint action we derive $P \cdot x \cap \mathfrak{q}_u = Q \cdot x$, for any $x \in \mathfrak{q}_u$.

We close this section by pointing to a fusion result of a different nature.

REMARK 3.16. Suppose G is reductive and $P \subset G$ parabolic with P_u abelian. Then, owing to [**65**, Cor. 2.18], L_P controls fusion in P_u for the action of G. More generally, according to a result of G. Seitz, L_P controls fusion in $Z(P_u)$, the center of P_u, for the action of G for any parabolic P, [**68**, Prop. 2.12].

3.4. Global and Infinitesimal Modality

In characteristic zero the exponential mapping is a P–equivariant isomorphism between the affine varieties $\mathfrak{p}_u$ and P_u, and therefore $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$. In positive characteristic a similar result can be proved using Springer's map between the variety $\mathcal{U}$ of unipotent elements of G and the variety $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}$, [**80**].

For the remainder of this section, suppose that $\operatorname{char} k = p$.

PROPOSITION 3.17. *Let G be a connected, simply connected, simple algebraic group, $P \subset G$ parabolic, and $N \subseteq P_u$ a closed normal subgroup of P. Suppose that* $\operatorname{char} k$ *is a good prime for G. Then there is a P–equivariant bijective morphism of P–varieties $\varphi : N \longrightarrow \mathfrak{n}$ affording a bijection between the sets of P–orbits on N and on $\mathfrak{n}$.*

PROOF. Let $\mathcal{U}$ be the unipotent variety of G and $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$ with the usual G–actions. In [**80**] T.A. Springer showed that, under the hypotheses of the theorem, there is a G–equivariant, bijective morphism

$$\varphi : \mathcal{U} \longrightarrow \mathcal{N} \tag{7}$$

which is a homeomorphism of topological spaces; see also [**84**, Thm. 3.12]. If $\operatorname{char} k = p$ is a *very good* prime for G (that is p is a good prime for G and it does not divide $r + 1$ in case G is of type A_r), then there is such a map which is even an isomorphism of varieties, since $\mathcal{N}$ is known to be normal in this instance [**89**, 6.9], see also [**5**, 9.3.6(c)].

For an alternative approach to Springer's map (7) by means of Luna's slice theorem in positive characteristic, see [**5**, 9.3.4].

Let $x \in B_u$ be a regular unipotent element of G. Since φ is G–equivariant, $\varphi(x)$ is regular nilpotent, and thus is contained in the Lie algebra of a unique Borel subgroup $\widetilde{B}$ of G [**79**, Lem. 5.3]. Likewise, the centralizer of a regular unipotent element is contained in a unique Borel subgroup [**79**, Lem. 4.3]. Thus we have $C_G(x) \subset B$ and $C_G(\varphi(x)) \subset \widetilde{B}$. By the G–equivariance of φ, we have $C_G(x) = C_G(\varphi(x))$, and thus $B = \widetilde{B}$. Since φ is a homeomorphism and $B_u = \overline{B \cdot x}$, it follows that $\varphi(B_u) = \overline{B \cdot \varphi(x)} = \mathfrak{b}_u$.

Observe that N is connected and $N = \prod U_\beta$, where the product is taken over $\Psi(N)$ in some fixed order, and $\mathfrak{n} = \oplus \mathfrak{g}_\beta$, where $\beta \in \Psi(N)$ [**6**, Prop. 14.4(2a)]. Let $\alpha \in \Psi^+$. Since $\dim \varphi(U_\alpha) = 1$ and $\varphi(U_\alpha)$ is a T–invariant subvariety of $\mathfrak{b}_u$, there is some root $\beta \in \Psi^+$ such that $\varphi(U_\alpha) \subseteq \mathfrak{g}_\beta$. For, if $\varphi(U_\alpha)$ meets at least two root spaces non-trivially, then, using the action of T, we see that $\dim \varphi(U_\alpha) \geq 2$ (assuming that $\operatorname{rank} G > 1$, else $\varphi(U_\alpha) = \mathfrak{g}_\alpha$ for the single root α in Ψ^+ by the previous paragraph). But then the T–invariance forces the equality $\varphi(U_\alpha) = \mathfrak{g}_\beta$. Since φ is T–equivariant and k is infinite, $\beta = \alpha$. In particular, this shows that any connected T–regular subgroup of B_u corresponds to its Lie algebra under φ. The desired result thus follows. □

COROLLARY 3.18. *Let G be reductive, P a parabolic subgroup of G, and N a closed normal subgroup of P contained in P_u. Suppose that* $\operatorname{char} k$ *is a good prime for G. Then we have the equality*

$$\operatorname{mod}(P : N) = \operatorname{mod}(P : \mathfrak{n}).$$

PROOF. Thanks to Remark 3.7 we may suppose that G is connected, simply connected, and semisimple. Then, as each simple component of G is also simply connected, and $\operatorname{char} k$ is a good prime for each of these components, and, since P is the direct product of its irreducible components and likewise for N, there results a P–equivariant bijective morphism of P–varieties $\varphi : N \longrightarrow \mathfrak{n}$ affording a bijection between the sets of P–orbits on N and $\mathfrak{n}$ by Proposition 3.17. □

The case $N = P_u$ of Corollary 3.18 then yields

COROLLARY 3.19. *Let G be reductive and $P \subseteq G$ parabolic. Suppose that* $\operatorname{char} k$ *is a good prime for G. Then we have the equality*

$$\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u).$$

A bijection between the sets of P–orbits on $\mathfrak{p}_u$ and on P_u, as asserted by Proposition 3.17, is considerably stronger than the equality statement of Corollary 3.19 which merely requires that the maximal number of parameters of both actions are the same. It would be interesting to know whether this equality also holds even when $\operatorname{char} k$ is a bad prime for G.

3.5. Modality and Coadjoint Action

Let G be a linear algebraic group. In this section we are concerned with the modality of the coadjoint action of a parabolic subgroup P of G on the dual space of $\mathfrak{p}_u$.

Modifying an argument from the proof of [**60**, Thm. 1] we show

PROPOSITION 3.20. *Let G be an algebraic group and let V be a rational G–module. Then we have the equality*

$$\operatorname{mod}(G : V^*) = \operatorname{mod}(G : V).$$

Moreover, if $\operatorname{mod}(G : V) = 0$, then there is a bijection between the sets of G–orbits on V and V^*.

PROOF. Let $\dim V = n$. For each $j \in \mathbb{N}$ we define $V^j := \{v \in V \mid \dim G \cdot v = j\}$; then $V = \bigcup_{j \geq 0} V^j$. Consider the subvariety

$$M := \{(v, f) \in V \times V^* \mid f(gv) = 0 \text{ for all } g \in G\} \subseteq V \times V^*.$$

Let $\pi : V \times V^* \longrightarrow V$ be the projection onto the first factor. We define $M^j := \pi^{-1}(V^j) \cap M$. Then $M = \bigcup_{j \geq 0} M^j$ and $\dim M = \max_{j \geq 0} \dim M^j$. By duality, each fiber of the map $\pi : M^j \longrightarrow V^j$ has dimension $n - j$. Therefore, $\dim M^j = \dim V^j + n - j$. As $\operatorname{mod}(G : V) = \max_{j \geq 0}(\dim V^j - j)$, see [**40**, 1.9], we have

$$\dim M = \max_{j \geq 0} \dim M^j = \max_{j \geq 0}(\dim V^j - j) + n = \operatorname{mod}(G : V) + n.$$

By dual considerations we also have $\dim M = \operatorname{mod}(G : V^*) + n$, and thus the desired equality follows.

The second statement of Proposition 3.20 follows from the proof of [**60**, Cor. 2], which is also valid in positive characteristic. $\square$

Proposition 3.20 is also proved in [**54**, §2].

COROLLARY 3.21. *Let G be a linear algebraic group and P a parabolic subgroup of G. Then we have the equality*

$$\operatorname{mod}(P : \mathfrak{p}_u^*) = \operatorname{mod}(P : \mathfrak{p}_u).$$

Moreover, if $\operatorname{mod} P = 0$, then there is a bijection between the sets of P–orbits on $\mathfrak{p}_u$ and those on $\mathfrak{p}_u^*$.

3.6. Bundles on Flag Manifolds

Now consider the principal fiber bundle $G \longrightarrow G/P$; since P is parabolic this bundle is locally trivial [**8**, 3.24, 3.25]. For Y a P–variety, we may consider the associated fiber bundle $G \times^P Y$ which is the quotient of $G \times Y$ by P, see [**76**, II.3.7]. The class of (g, y) in $G \times^P Y$ is denoted by $g * y$. Clearly, G acts on $G \times^P Y$ via left translation. This construction plays an important role in the resolution of singularities, see [**76**], [**10**]. Observe that $G \times^P \mathfrak{p}_u$ is the *cotangent bundle* and $G \times^P \mathfrak{p}_u^*$ the *tangent bundle* of the flag manifold G/P. In the special case when $P = B$, the collapsing of the bundle

$$G \times^B B_u \longrightarrow \mathcal{U},$$

given by $g * x \mapsto gxg^{-1}$, is the desingularization of the unipotent variety $\mathcal{U}$ [**80**, 1.4], [**88**, 1.1]. Analogously, the *moment map*

$$G \times^B \mathfrak{b}_u \longrightarrow \mathcal{N},$$

given by $g * x \mapsto \mathrm{Ad}(g)x$, is a resolution of singularities of the nilpotent variety $\mathcal{N}$ [**80**], [**81**]. There is a close relation between these two resolutions by means of Springer's map (7) from page 21, see [**76**, II.4.7].

By construction, the G–orbit of a point in $G \times^P Y$ meets the fiber over some point in G/P (which is isomorphic to Y) in a single P–orbit on Y. So, there is a canonical bijection between the G–orbits on $G \times^P Y$ and the P–orbits on Y. In particular, considering the case when $Y = \mathfrak{p}_u$, we get $\operatorname{mod}(G : G \times^P \mathfrak{p}_u) = \operatorname{mod} P$. This fact and Corollary 3.21 imply

Corollary 3.22. *Let G be a linear algebraic group and $P \subseteq G$ a parabolic subgroup of G. Then we have*

$$\operatorname{mod}(G : G \times^P \mathfrak{p}_u) = \operatorname{mod} P = \operatorname{mod}(G : G \times^P \mathfrak{p}_u^*).$$

Moreover, if $\operatorname{mod} P = 0$, then there is a bijection between the sets of G–orbits on these two G–bundles.

Remark. By Corollary 3.22 and Theorems 5.22 and 5.30 below we get a complete description of all instances when G admits only a finite number of orbits on the tangent and cotangent bundles of G/P.

CHAPTER 4

Parabolic Groups of Positive Modality

Throughout this chapter G is a (connected) simple algebraic group, $T \subset B$ are a fixed maximal torus of G and a fixed Borel subgroup and $P \supseteq B$ is a parabolic subgroup of G.

4.1. Preliminary Results

If N is a closed normal subgroup of P contained in P_u, then N is connected and $N = \prod U_\beta$, where the product is taken over $\Psi(N)$ in some fixed order [**6**, Prop. 14.4(2a)]. Moreover, $\mathfrak{n} = \oplus \mathfrak{g}_\beta$, where $\beta \in \Psi(N)$. Let $r = \operatorname{rank} G = \dim T$ denote the rank of G. Define

$$\mu(N) := \mu(\mathfrak{n}) := 2\dim \mathfrak{n} - \dim \mathfrak{p} - \dim[\mathfrak{n}, \mathfrak{n}].$$

The following bound appears in [**57**, Prop. 1].

Lemma 4.1. *Let P and N be as above. Then*

$$\operatorname{mod} P \geq \mu(\mathfrak{n}).$$

Proof. The action of P on $\mathfrak{n}$ induces an action of P on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ and this action factors through P/N, as N acts trivially on this coset space. Whence we have

$$\begin{aligned} \operatorname{mod} P &\geq \operatorname{mod}(P : \mathfrak{n}) \geq \operatorname{mod}(P : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) \\ &= \operatorname{mod}(P/N : \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) \geq \dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] - \dim P/N \\ &= \mu(\mathfrak{n}), \end{aligned}$$

as desired. □

Remark. The proof above equally shows that $\mu(N)$ is a lower bound for $\operatorname{mod}(P : P_u)$. When $\operatorname{mod} P$ is positive, there need not be an ideal $\mathfrak{n} \subseteq \mathfrak{p}_u$ of $\mathfrak{p}$ with $\mu(\mathfrak{n}) > 0$. For instance, see [**58**, Rem. 3.5].

The following classification of Borel subgroups of modality zero is due to V.V. Kashin [**41**, Thm. 1] (in case $\operatorname{char} k = 0$).

Theorem 4.2. *We have* $\operatorname{mod} B = 0$ *if and only if one of*

(i) *G is of type A_r for $r \leq 4$;*
(ii) *G is of type B_2.*

Proof. It follows from Lemma 4.1 and Table 4.1 below that $\operatorname{mod} B$ is positive in each of the cases listed. Each of the simple groups A_r for $r \geq 6$, B_r and C_r for $r \geq 4$, D_r for $r \geq 5$, E_6, E_7, E_8, and F_4 admits a standard Levi subgroup whose type appears in Table 4.1. By Proposition 3.13, we see that $\operatorname{mod} B > 0$ in each of these instances as well. For let R be the parabolic subgroup of G whose Levi subgroup is of the appropriate type from Table 4.1 and $P = B$ is the Borel subgroup of G. According to Proposition 3.13 we infer that

$$0 < \operatorname{mod} B_1 \leq \operatorname{mod} B,$$

where $B = B_1 \cap L'_R$ is the Borel subgroup of the corresponding group from Table 4.1. Consequently, $\operatorname{mod} B$ is positive whenever G is not as in (i) or (ii).

The fact that $\operatorname{mod} B = 0$ in all instances when G is as in (i) or (ii) follows from the results in [**20**]. ☐

In Table 4.1 we record some ideals $\mathfrak{n}$ of $\mathfrak{b}$ with the property that $\mu(\mathfrak{n}) = 1$ (relative B). In the second column we list the roots α such that $\mathfrak{n}$ is the minimal B–submodule of $\mathfrak{b}_u$ containing the root spaces $\mathfrak{g}_\alpha$. By Lemma 4.1 we have $\mu(\mathfrak{n}) \leq \operatorname{mod} B$. In fact $\operatorname{mod} B = 1$ holds in each of the cases in Table 4.1, see Section 6.2 below.

Type of G	$\mathfrak{n}$	$\dim \mathfrak{n}$	$\mu(\mathfrak{n})$
A_5	$\alpha_1, \alpha_3, \alpha_5$	13	1
B_3	α_2	7	1
C_3	α_1, α_3	8	1
D_4	α_2	9	1
G_2	α_1	5	1

Table 4.1. The critical Borel cases

In [**41**] Kashin uses a different ideal for D_4, namely the one generated by the root spaces relative to the simple roots corresponding to the end nodes of the D_4 diagram.

In [**94**] A.E. Zalesskiĭ shows that $\operatorname{mod}(B : B_u) > 0$ whenever G is of type A_r for $r \geq 5$, B_r, C_r, D_r, for $r \geq 6$, E_6, E_7, and E_8 using the A_5 entry from Table 4.1 in connection with Proposition 3.12.

The following classification of semisimple rank 1 parabolics of modality zero is proved in [**58**, Cor. 1.4] in case $\operatorname{char} k = 0$; like Theorem 4.2 it is valid in any characteristic.

THEOREM 4.3. *Suppose* $\operatorname{rank}_s P = 1$. *Then* $\operatorname{mod} P = 0$ *if and only if one of the following holds:*

(i) *G is of type A_r for $r \leq 5$;*
(ii) *G is of type B_r or C_r for $r \leq 3$;*
(iii) *G is of type D_4, or G_2.*

4.2. The Classical Groups

It was observed in [**69**] that there is a close connection between $\ell(P_u)$, the length of the descending central series of P_u, and the question whether $\operatorname{mod} P$ is positive. Earlier results showed that the number of P–orbits on $\mathfrak{p}_u$ is infinite given that $\ell(P_u)$ is sufficiently large, e.g., see [**94**], [**41**]. On the other hand there is a finite number of P–orbits on $\mathfrak{p}_u$ provided $\ell(P_u)$ is small. For instance, if $\ell(P_u) = 1$, that is when P_u is abelian, then $\operatorname{mod} P = 0$, see [**64**, Thm. E]. Also, in [**71**, Thm. 1.4] it was shown that $\operatorname{mod} P = 0$ provided G is classical and $\ell(P_u) \leq 2$.

In case G is of type D_r let τ be the graph automorphism of G of order 2 (stemming from the interchange of the simple roots σ_{r-1} and σ_r). Our next result combines [**69**, Thm. 3.1] for classical groups and [**32**, Lem. 3.2].

PROPOSITION 4.4. *Let G be classical and $P \subseteq G$ is parabolic. Then* $\operatorname{mod} P$ *is positive provided*

(a) *G is of type A_r, B_r, or C_r and $\ell(P_u) \geq 5$; or*
(b) *G is of type D_r, and one of the following holds:*
   (i) *$\ell(P_u) \geq 6$; or*
   (ii) *$\ell(P_u) = 5$ and $\tau P = P$; or*
   (iii) *$\ell(P_u) = 5$, $\tau P \neq P$, and L_P' consists of three simple components.*

PROOF. First, we combine the proofs for (a) and (b)(i - ii). In all these cases the idea is to induct on the rank of G by reducing to a suitable simple regular subgroup H of G of the same classical type as that of G, but of smaller rank, and then to invoke Theorem 3.8.

Let G and P be as in (a) or (b)(i - ii). We may assume that P is standard, that is $P = P_J$, where $J \subseteq \Pi$ and in view of Lemma 3.1, it suffices to consider only those P which are *maximal* subject to the conditions in our proposition, that is

(A) If G is of type A_r, then we may assume that $\ell(P_u) = 5$.
(B) If G is of type B_r, C_r, or D_r then we may assume that either $\ell(P_u) = 5$, or $\ell(P_u) = 6$ and each parabolic subgroup R of G which contains P properly satisfies $\ell(R_u) \leq 4$.

In case G is of type D_r this implies that $\tau P = P$. For, this is clear if $\ell(P_u) = 5$ by hypothesis (b)(ii). Suppose that P satisfies the maximality condition (B), and $\ell(P_u) = 6$, but $\tau P \neq P$. Then $|\{\sigma_{r-1}, \sigma_r\} \cap J| = 1$. Let R be the standard parabolic subgroup of G corresponding to $J \cup \{\sigma_{r-1}, \sigma_r\}$. Then R contains P properly and $\ell(R_u) = 5$, and $\tau R = R$. This contradicts the choice of P. Thus $\tau P = P$, as claimed.

We argue by induction on $\operatorname{rank} G = r$. In case G is of type A_r and $r \leq 4$, or in case of B_r or C_r with $r \leq 2$, it follows by (3) on page 4 that the statement of the proposition holds trivially.

If G is as in Table 4.1, then J is empty, that is $P = B$ and $\ell(B_u) = 5$. The result follows in each of these cases from Theorem 4.2.

So we may assume that J is non-empty. Let G and P be as in (a), (b)(i - ii) with P satisfying (A) or (B), and suppose that these statements hold for all simple groups of rank less than r.

We are going to construct a simple regular subgroup H of G so that H is the derived subgroup of a Levi subgroup of G and $\operatorname{rank} H = \operatorname{rank} G - 1$. Since J is non-empty, there exists a pair of simple roots which are adjacent in the Dynkin diagram of G where precisely one of them is in J. We fix such a pair which is either of the form $\{\sigma_i, \sigma_{i+1}\}$ for some $i < r$, or $\{\sigma_{r-2}, \sigma_r\}$ in case G is of type D_r. For the definition of H we distinguish three cases:

(I) For G of type A_r, B_r, or C_r and $i < r-1$, or D_r and $i < r-2$, let H be the connected simple regular subgroup of G defined by the set of simple roots

$$\Pi(H) := \{\sigma_1, \dots, \sigma_{i-1}, \sigma_i + \sigma_{i+1}, \sigma_{i+2}, \dots, \sigma_r\}.$$

(II) If G is of type C_r and $i = r-1$, then let H be the connected simple regular subgroup of G with simple roots

$$\Pi(H) := \{\sigma_1, \dots, \sigma_{r-2}, 2\sigma_{r-1} + \sigma_r\}.$$

(III) In case G is of type D_r and the chosen pair of consecutive simple roots is either $\{\sigma_{r-2}, \sigma_{r-1}\}$ or $\{\sigma_{r-2}, \sigma_r\}$ we let H be the connected simple regular subgroup of G given by

$$\Pi(H) := \{\sigma_1, \dots, \sigma_{r-2} + \sigma_{r-1}, \sigma_{r-2} + \sigma_r\}.$$

In each case the subgroup H is of the same classical type as that of G and

$$\operatorname{rank} H = \operatorname{rank} G - 1.$$

Define $Q := P \cap H$. As, by construction, only one of the two chosen simple roots is in J, we see that either

$$
\begin{aligned}
(8) \qquad \operatorname{rank} H - \operatorname{rank}_{\mathrm{s}} Q &= \operatorname{rank} G - \operatorname{rank}_{\mathrm{s}} P, \quad \text{or} \\
\operatorname{rank} H - \operatorname{rank}_{\mathrm{s}} Q &= \operatorname{rank} G - \operatorname{rank}_{\mathrm{s}} P + 1.
\end{aligned}
$$

The latter equality only occurs in case (III) above when $\{\sigma_{r-1}, \sigma_r\} \subseteq J$.

We wish to compare $\ell(Q_u)$ with $\ell(P_u)$. From (8) we get

(a) $\ell(Q_u) = \ell(P_u) \geq 5$, or
(b) G is of type B_r or D_r, $i = 1$, σ_1 is in J, and $\ell(Q_u) = \ell(P_u) - 1$, or
(c) G is of type C_r, $i = r - 1$, σ_r is in J, and $\ell(Q_u) = \ell(P_u) - 1$.

Further, in the latter two cases it follows from properties of the respective root systems and (B) above that $\ell(P_u)$ must be even, whence, $\ell(P_u) = 6$. Therefore, $\ell(Q_u) = 5$ in (b) and (c). Thus, in all cases, H and Q satisfy the hypotheses of the proposition. By applying the induction hypothesis to H and Q, we have $\operatorname{mod} Q > 0$. Owing to Theorem 3.8 we conclude $\operatorname{mod} P > 0$. This completes the proof of parts (a) and (b)(i - ii) of the proposition.

Finally, we consider the case (b)(iii). Combined, the conditions there imply that r is at least 6. For $r = 6$ there is just one case up to conjugacy by the graph automorphism. Namely, for G of type D_6 the standard parabolic subgroup P of G with $\Psi(P) = \{-\sigma_1, -\sigma_3, -\sigma_5\} \cup \Psi^+$ satisfies the conditions of (b)(iii).

In this case let H be the regular simple subgroup of G relative to the subsystem of Ψ which is spanned by

$$\Pi(H) := \{\sigma_1 + \sigma_2 + \sigma_3, \sigma_4, \sigma_6, \sigma_3 + \sigma_4 + \sigma_5, \sigma_2\}.$$

Then H is the derived subgroup of a Levi subgroup of G of type A_5 and $Q = H \cap P$ is the standard Borel subgroup of H relative to $\Pi(H)$. It follows from Theorem 4.2 that $\operatorname{mod} Q > 0$ thus by Theorem 3.8 $\operatorname{mod} P$ is positive as well. We illustrate this critical D_6 case in Figure 4.1 below.

The general case where G is of type D_r for $r > 6$ and P is as in (b)(iii) reduces inductively to this particular D_6 configuration just discussed by applying Theorem 3.8. We can argue in the induction as in the previous cases. Again, by Lemma 3.1, we may suppose that P is maximal with respect to the conditions in (b)(iii). Since $r > 6$, we can construct a simple regular subgroup H of G of type D_{r-1} by means of (I).

This completes the proof of our proposition. $\square$

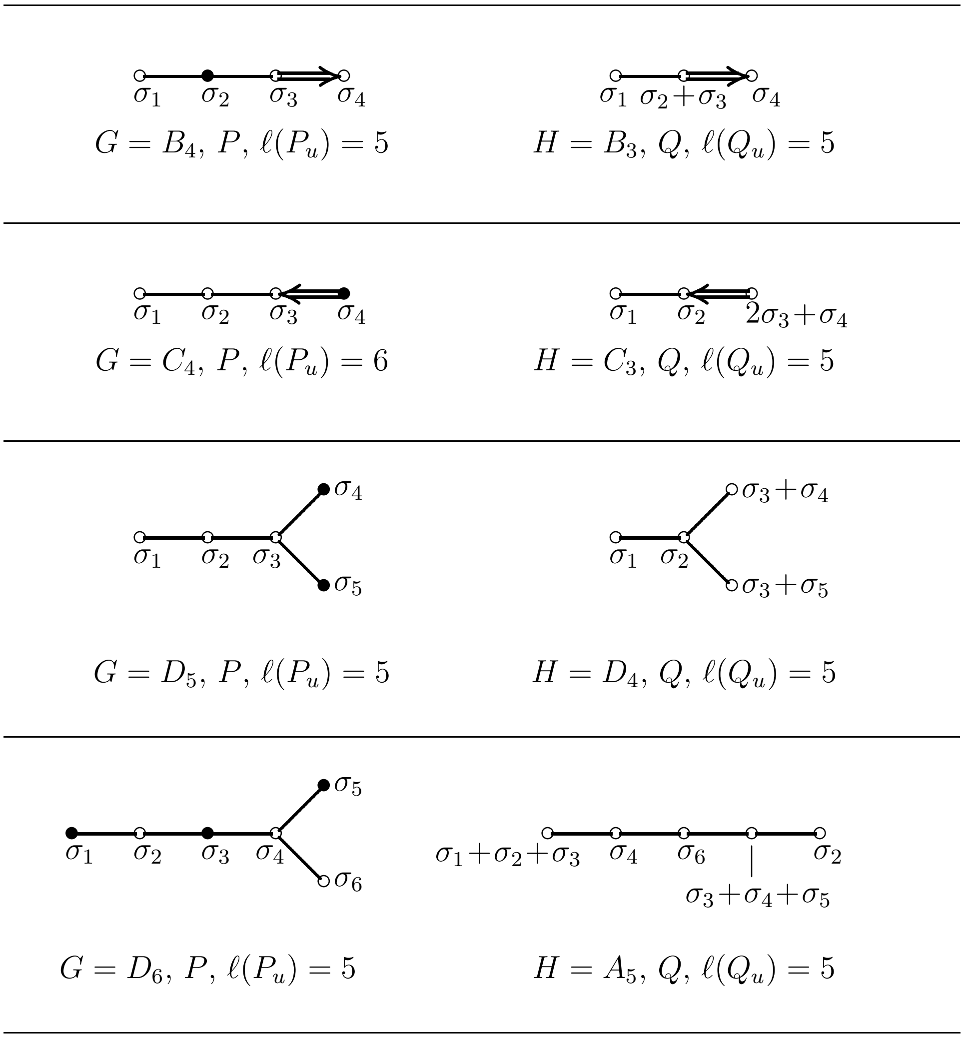


FIGURE 4.1. Some classical examples

EXAMPLES. We illustrate the construction in some cases from the proof of Proposition 4.4 in terms of Dynkin diagrams in Figure 4.1 below.

Our first example is, in a sense, the generic situation (I) from the proof above; namely, when σ_i and σ_{i+1} are both of the same length and $\ell(Q_u) = \ell(P_u)$. In the second one we have $\ell(Q_u) = \ell(P_u) - 1$, and σ_i and σ_{i+1} are of different lengths. This is an instance from (II) above. The next case is an occurrence of the construction from (III). Finally, we present the significant D_6 configuration from Proposition 4.4(b)(iii).

In the first column of Figure 4.1 the parabolic subgroup $P = P_J$ of G is indicated by coloring the nodes corresponding to the simple roots $J \subseteq \Pi$ in the Dynkin diagram of G. Here we follow the labeling of Π as in [**11**]. In the second column we give the diagram for H, labeled by the simple roots $\Pi(H)$ of H. Also, $Q = H \cap P$ is shown in a similar fashion to P. The absence of any colored nodes in this diagram indicates that Q is the corresponding Borel subgroup for H. This is the case in each of these instances.

4.3. The Exceptional Groups

Combining the results of [**73**, Thm. 3.1] for exceptional groups and [**35**, Lem. 3.13] we get the following counterpart to Proposition 4.4.

PROPOSITION 4.5. *Suppose G is of exceptional type and $P \subseteq G$ is parabolic. Then* $\operatorname{mod} P > 0$ *provided one of the following holds:*

(i) *G is of type E_8, F_4, or G_2 and $\ell(P_u) \geq 5$;*
(ii) *G is of type E_6 or E_7 and $\ell(P_u) \geq 6$;*
(iii) *G is of type E_6, $\ell(P_u) = 5$, and P is not of type $A_1^2 A_2$ or A_3;*
(iv) *G is of type E_7, $\ell(P_u) = 5$, and P is not of type $A_1 A_4$.*

PROOF. The result for G_2 follows directly from Theorem 4.2.

We illustrate the crucial cases of the argument in case of F_4 and E_6 in terms of Dynkin diagrams in Figures 4.2 and 4.3 below. In each instance we construct a simple regular subgroup H of G, so that $Q = H \cap P$ satisfies $\operatorname{mod} Q > 0$ according to Proposition 4.4 and thus, by Theorem 3.8, we obtain $\operatorname{mod} P > 0$, as claimed.

Thanks to Lemma 3.1 we only need to consider those P which are maximal with respect to satisfying $\ell(P_u) \geq 5$ in case of F_4. The simple regular subgroup H of G used in each instance is shown in Figure 4.2.

In case G is of type E_6 we argue similarly. Again, by Lemma 3.1, we only have to study those parabolics maximal with respect so certain conditions. There are two different kinds of parabolic subgroups P we have to study.

The first kind are those P which are maximal with respect to satisfying $\ell(P_u) \geq 5$ and are not of type $A_1^2 A_2$ or A_3. These are the first five cases illustrated in Figure 4.3 up to symmetry.

The ones of the second kind are those P which are maximal with respect to being properly contained in a parabolic subgroup of type $A_1^2 A_2$ or A_3; note that then $\ell(P_u) \geq 6$. Cases 7 through 8 of Figure 4.3 are of this nature. In each of the remaining instances it turns out that we can embed P properly in one of the parabolic subgroups of the first kind and then apply Lemma 3.1 to derive that $\operatorname{mod} P > 0$. We leave the details to the reader (see also Table V.2 in [**69**]).

For the cases shown in Figure 4.3, it follows from the construction, Theorem 3.8, and Proposition 4.4 that $\operatorname{mod} P > 0$.

For proofs of the relevant configurations in the remaining instances in E_7 and E_8 we refer to [**69**, §5] and [**35**, Lem. 3.13]. $\square$

EXAMPLES 4.6. In Figures 4.2 and 4.3 the significant F_4 and E_6 cases from Proposition 4.5 are listed respectively. For convenience we record $\ell(P_u)$ and $\ell(Q_u)$, see (3) on page 4.

In Figure 4.2 let $\beta = \sigma_2 + 2\sigma_3 + 2\sigma_4$ and $\gamma = \sigma_1 + 2\sigma_2 + 2\sigma_3$ be the highest roots of the standard subsystems of type C_3 and B_3 of F_4, respectively.

In order to avoid bulky labels in Figure 4.3, we abbreviate $\sigma_3 + \sigma_4$ by σ_{34}, and $\sigma_4 + \sigma_5 + \sigma_6$ by σ_{456}, etc.

G, P, $\ell(P_u)$ $\qquad$ H, $Q = H \cap P$, $\ell(Q_u)$

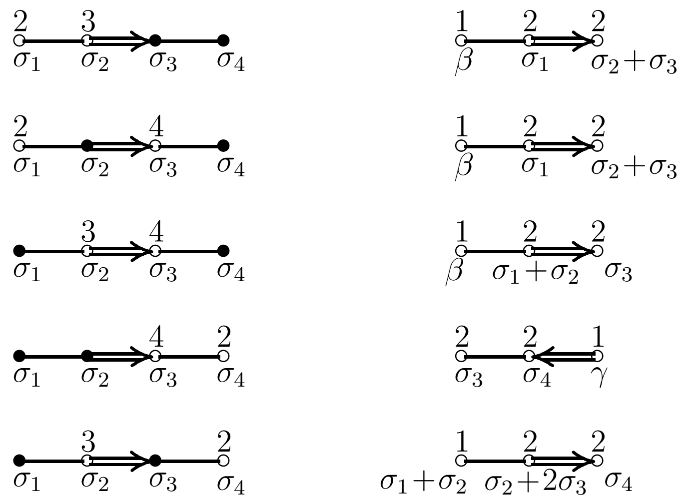


FIGURE 4.2. The critical F_4 cases

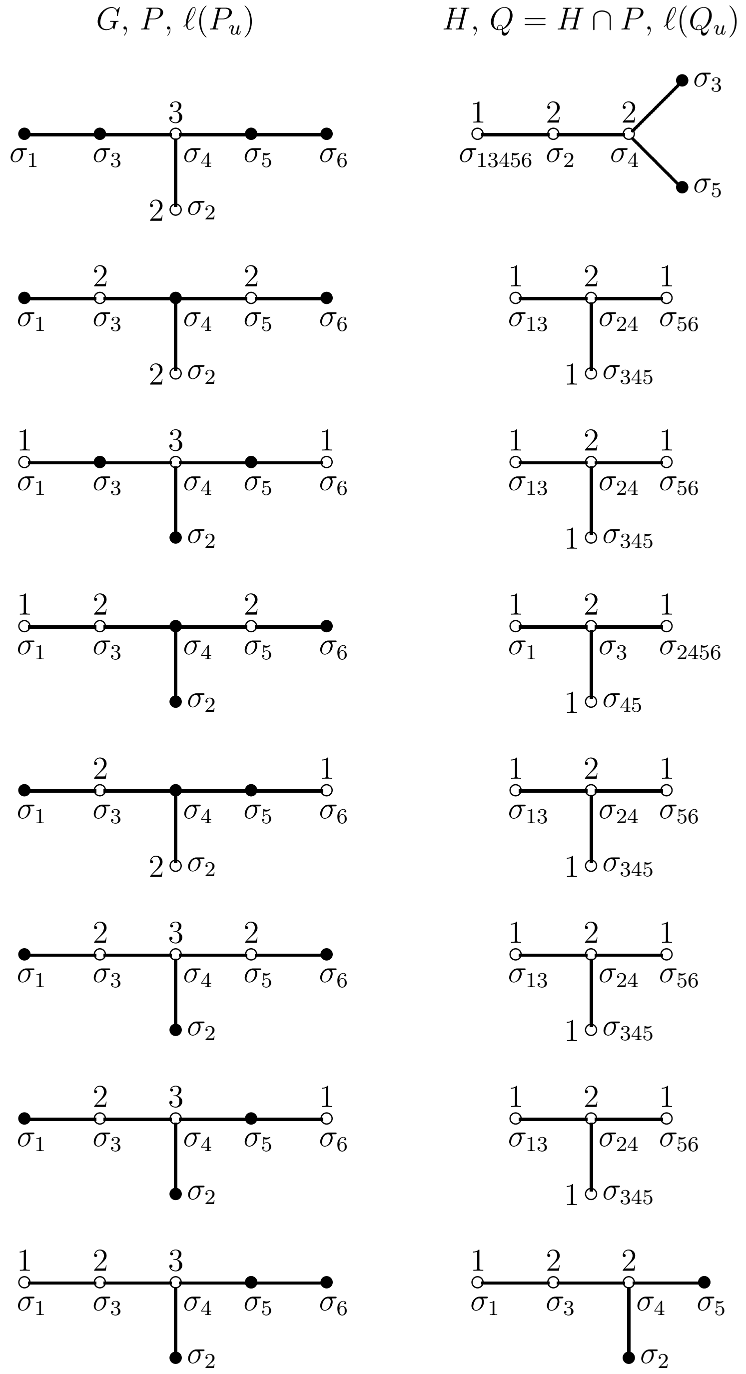


FIGURE 4.3. The critical E_6 cases

CHAPTER 5

Parabolic Groups of Modality zero

In this chapter, we delineate the classifications for classical groups from [**31, 32**] as well as those for groups of exceptional type from [**37**]. Moreover, for the general linear groups we render a complete combinatorial description from [**19**] of the closure relation on the set of P–orbits on $\mathfrak{p}_u$, that is the *Bruhat-Chevalley order*, in all the finite orbit events. Finally, we present generalizations of these classifications from [**14**] and [**18**].

5.1. The General Linear Groups

In the present section, we outline the classification from [**31, 32**] for general linear groups and furnish a complete combinatorial description of the closure relation on the set of P–orbits on $\mathfrak{p}_u$ for every finite orbit case from [**19**]. Our standard reference on categories of Δ–filtered modules over quasi-hereditary algebras is [**26**].

More generally, in [**14**] and [**19**] the action of P on the l-th member of the descending central series of $\mathfrak{p}_u$ is studied, where this series of $\mathfrak{p}_u$ is defined by $\mathfrak{p}_u^{(0)} := \mathfrak{p}_u$ and $\mathfrak{p}_u^{(l)} := [\mathfrak{p}_u, \mathfrak{p}_u^{(l-1)}]$, for $l \geq 1$, see Theorem 5.35 below. Though all the results from [**19**] apply in this more general situation, we only present them here for the action of P on $\mathfrak{p}_u$.

For further generalizations in the setting of general linear groups, we refer to more recent work by T. Brüstle and L. Hille, [**15**] and [**16**]. In [**15**] they show that the action of any standard parabolic subgroup P of $\mathrm{GL}(V)$ on a normal unipotent subgroup $U \subseteq P_u$ of P can be interpreted in terms of the Δ–filtered modules of a certain quasi-hereditary algebra. While [**16**] explains the occurrence of quasi-hereditary algebras in connection with classification results for parabolic group actions in general linear groups. In this context see also [**17**].

5.1.1. Parabolic Groups and Δ–Filtered Modules. We maintain the notation from above. A Levi subgroup L of P is (isomorphic to) a product of general linear groups, say $\mathrm{GL}(d_i)$, for $1 \leq i \leq t$ for some $t \in \mathbb{N}$, with $\dim V = \sum d_i$. The ordered tuple $\mathbf{d} = (d_1, \dots, d_t) \in \mathbb{N}^t$ determines the conjugacy class of P in $\mathrm{GL}(V)$. To indicate this, we often write $P = P(\mathbf{d})$. Fix $t \in \mathbb{N}$ and define a category $\mathcal{F}(t)$ as follows. The objects are pairs (F, f), where F is a flag $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = V$ of length t of subspaces of some finite-dimensional k–vector space V, and f is an endomorphism of V which lies in $\mathfrak{p}_u$, where P is the stabilizer of F in $\mathrm{GL}(V)$. Consequently, $f(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq t$. Let (F, f) and (F', f') be in $\mathcal{F}(t)$. A morphism $\varphi : (F, f) \longrightarrow (F', f')$ is a linear map $\varphi : V \longrightarrow V'$ such that $\varphi(V_i) \subseteq V_i'$ for $1 \leq i \leq t$ and $\varphi f = f'\varphi$. For (F, f) in $\mathcal{F}(t)$ we set $d_i := \dim V_i - \dim V_{i-1}$ for $1 \leq i \leq t$ and call $\mathbf{d} = (d_1, \ldots, d_t)$ the *dimension vector* of (F, f) and write

$$\underline{\dim}\, F := \mathbf{d}.$$

For $\mathbf{d} \in \mathbb{N}^t$ we denote by $\mathcal{F}(t)(\mathbf{d})$ the subcategory of $\mathcal{F}(t)$ of all objects of dimension vector $\mathbf{d}$.

If $\varphi : (F, f) \xrightarrow{\sim} (F', f')$ is an isomorphism in $\mathcal{F}(t)$, then $\mathbf{d} = \mathbf{d}'$ and so $\dim V = \dim V'$. After identifying V with V' and F with F' we see that φ lies in $P = P(\mathbf{d})$ and $f' = \varphi f \varphi^{-1}$, that is f and f' are endomorphisms in $\mathfrak{p}_u$ conjugate under the action of P.

Lemma 5.1. *For any $t \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{N}^t$ the isomorphism classes of objects in $\mathcal{F}(t)(\mathbf{d})$ correspond bijectively to the P–orbits on $\mathfrak{p}_u$ for $P = P(\mathbf{d})$. This correspondence is induced by the map $(F, f) \mapsto P \cdot f$, where P is the stabilizer of the flag F and f lies in $\mathfrak{p}_u$.*

Proof. It follows from the remarks above that there is a well-defined map from the set of isomorphism classes of objects in $\mathcal{F}(t)(\mathbf{d})$ to the set of P–orbits on $\mathfrak{p}_u$ for $P = P(\mathbf{d})$. Clearly, it is onto, as for any f in $\mathfrak{p}_u$, the pair (F, f) lies in $\mathcal{F}(t)(\mathbf{d})$ and its class maps to the P–orbit through f. Moreover, since for any two pairs $(F, f), (F, f')$ in $\mathcal{F}(t)(\mathbf{d})$ and $\varphi \in P = P(\mathbf{d})$ satisfying $f' = \varphi f \varphi^{-1}$ the linear map φ defines an isomorphism between (F, f) and (F, f'), this correspondence is injective. $\square$

Instead of working directly with $\mathcal{F}(t)$, we pass to an equivalent category. For that purpose let $Q(t)$ be the quiver defined as follows: the set of vertices is simply $\{1, \ldots, t\}$ and the arrows of the quiver are $i \xrightarrow{\alpha_i} i+1$ and $i+1 \xrightarrow{\beta_i} i$ for $i = 1, \ldots, t-1$. Let J be the ideal in the *path algebra* $kQ(t)$ of this quiver given by the following relations: $\beta_1\alpha_1 = 0$ and $\beta_i\alpha_i = \alpha_{i-1}\beta_{i-1}$ for $1 < i < t$. Then we denote the finite-dimensional quotient algebra $kQ(t)/J$ by $\mathcal{A}(t)$.

We illustrate the example $Q(6)$ in Figure 5.1.

Let $\mathcal{A}(t)$–mod be the category of all finite-dimensional left $\mathcal{A}(t)$–modules and by $\mathcal{M}(t)$ we denote the full subcategory of modules M in $\mathcal{A}(t)$–mod subject to the condition that all maps M_{α_i} are injective for $1 \leq i \leq t-1$.

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha_1"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \overset{3}{\bullet} \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_3"] & \overset{4}{\bullet} \arrow[l, shift left, "\beta_3"] \arrow[r, shift left, "\alpha_4"] & \overset{5}{\bullet} \arrow[l, shift left, "\beta_4"] \arrow[r, shift left, "\alpha_5"] & \overset{6}{\bullet} \arrow[l, shift left, "\beta_5"] \end{tikzcd}$$

FIGURE 5.1. The quiver $Q(6)$

The following key observation is due to P. Gabriel.

LEMMA 5.2. *The category $\mathcal{F}(t)$ and the subcategory $\mathcal{M}(t)$ of $\mathcal{A}(t)$–mod are equivalent for each $t \in \mathbb{N}$.*

PROOF. Let (F, f) be in $\mathcal{F}(t)$ and define a representation $M(F, f)$ of the quiver $Q(t)$ above via the flag associated to F as follows:

$$V_1 \xrightarrow{\alpha_1} V_2 \ \cdots \ V_{t-1} \xrightarrow{\alpha_{t-1}} V_t,$$

where the maps M_{α_i} are simply the inclusions $V_i \hookrightarrow V_{i+1}$ and the maps M_{β_i} are the restrictions of f to V_{i+1}. One easily checks that $M(F, f)$ satisfies the conditions above, that is $M(F, f)$ is in $\mathcal{M}(t)$, and that the map $(F, f) \mapsto M(F, f)$ defines an equivalence of categories. $\square$

Crucial to our approach is the following fact, see [**26**, §6 - 7]:

LEMMA 5.3. *The subcategory $\mathcal{M}(t)$ of $\mathcal{A}(t)$–mod is precisely the category $\mathcal{F}(\Delta)$ of Δ–filtered modules over the quasi-hereditary algebra $\mathcal{A}(t)$, for $t \in \mathbb{N}$.*

REMARK 5.4. Let $t \in \mathbb{N}$. Then $\mathcal{F}(t)$ is a Krull-Schmidt category, that is every object in $\mathcal{F}(t)$ has a unique decomposition into a direct sum of indecomposable ones (up to the order of the summands). Being closed under taking direct summands, the subcategory $\mathcal{M}(t)$ inherits the Krull-Schmidt property from $\mathcal{A}(t)$–mod, whence so does $\mathcal{F}(t)$ by Lemma 5.2.

We observe that $\mathcal{A}(t)$ is the *Auslander algebra* of the representation-finite algebra $k[x]/(x^t)$, see [**26**, §6–7], see also [**17**]. The key result for our purpose is [**26**, Prop. 7.2]:

THEOREM 5.5. *Let $t \in \mathbb{N}$. Then the representation type of $\mathcal{M}(t)$ is finite precisely when $t \leq 5$, it is tame for $t = 6$, and wild if $t \geq 7$.*

The first part of the statement follows from the finiteness of the Auslander-Reiten quivers of $\mathcal{M}(t)$ for $t \leq 5$, as exhibited in [**26**].

We can now state the principal result of this section:

THEOREM 5.6. *Let $t \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$ and let V be a finite-dimensional k–vector space so that $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Then the number of P–orbits on $\mathfrak{p}_u$ is finite if and only if $t \leq 5$.*

PROOF. Note that $\ell(P_u) = t - 1$. Because of the equivalence above (Lemma 5.2), Theorem 5.5 also holds for $\mathcal{F}(t)$ in place of $\mathcal{M}(t)$. The finiteness statement follows from the first part of Theorem 5.5 and Lemmas 5.1 and 5.2. Since $\operatorname{mod} P > 0$ whenever $\ell(P_u) \geq 5$, that is when $t \geq 6$, by Proposition 4.4, the claim follows. □

REMARK. In [**46**, §3] representations of $Q(t)$ with the same relations but without the injectivity condition demanded for $\mathcal{M}(t)$ are used to describe closures of nilpotent conjugacy classes for general linear groups.

5.1.2. The Number of P–Orbits on $\mathfrak{p}_u$.

Let $\mathcal{I}(t)$ be a complete set of representatives of isomorphism classes of indecomposable objects in $\mathcal{F}(t)$. According to Lemma 5.2 and Theorem 5.5, the set $\mathcal{I}(t)$ is finite precisely when $t \leq 5$. By I_j denote the j-th member of $\mathcal{I}(t)$ for $1 \leq j \leq m := |\mathcal{I}(t)|$. By Remark 5.4, $\mathcal{F}(t)$ is a Krull-Schmidt category, thus each object in $\mathcal{F}(t)$ has a unique decomposition as a direct sum of indecomposable ones (up to the order of the summands). The number m of isomorphism classes of indecomposable objects in $\mathcal{M}(t)$ for $t \leq 5$ and their dimension vectors can be determined from the Auslander-Reiten quivers listed in [**26**]. There are 7, 16, and 45 such classes in $\mathcal{M}(t)$ for $t = 3, 4, 5$, respectively.

Next we present a formula for the number of orbits in all the finite cases. The proof follows at once from the first part of Theorem 5.6, Lemma 5.1, and the fact that $\mathcal{F}(t)$ is a Krull-Schmidt category, see Remark 5.4.

COROLLARY 5.7. *Let $t \leq 5$, $\mathbf{d} \in \mathbb{N}^t$, and V is a finite-dimensional k–vector space with $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Set $m = |\mathcal{I}(t)|$. Then the number of P–orbits on $\mathfrak{p}_u$ equals the number of m–tuples $(a_1, \ldots, a_m) \in \mathbb{N}_0^m$ such that*

$$\mathbf{d} = \sum_{j=1}^{m} a_j \, \underline{\dim}\, I_j. \tag{9}$$

The dimension vectors $\underline{\dim}\, I_j$ of the representatives of all indecomposable objects in $\mathcal{F}(t)$, for $t \leq 5$, can be determined, from the

Auslander-Reiten quivers in [**26**] or [**19**, §8]. Then formula (9) of Corollary 5.7 can be used to explicitly compute $N(\mathbf{d})$, the number of orbits of $P(\mathbf{d})$ on $\mathfrak{p}_u(\mathbf{d})$, for $\mathbf{d} \in \mathbb{N}^t$, algorithmically. An implementation based on this formula was also used to obtain the number of orbits in the examples of the appendix.

The parabolic subgroups $P, Q \subset \mathrm{GL}(V)$ are *associated* provided they admit G–conjugate Levi subgroups. Our next result shows that in case $t \leq 5$ the number of orbits only depends on the conjugacy class of a Levi subgroup of $P = P(\mathbf{d})$ in $\mathrm{GL}(V)$, that is on the *association class of* P, rather than on the conjugacy class of P. Let $N(\mathbf{d})$ denote the number of P–orbits on $\mathfrak{p}_u$.

Corollary 5.8. *Let V be a finite-dimensional k–vector space, $P = P(\mathbf{d})$, and $P' = P(\mathbf{d}')$ parabolic subgroups of* $\mathrm{GL}(V)$*, where both* $\mathbf{d}$ *and* $\mathbf{d}'$ *are of length $t \leq 5$. If P and P' are associated, then*

$$N(\mathbf{d}) = N(\mathbf{d}').$$

Proof. Since P and P' are associated, there is a permutation σ of $\{1, \dots, t\}$ such that $\mathbf{d}' = \sigma\mathbf{d} = (d_{\sigma 1}, \dots, d_{\sigma t})$. Considering the dimension vectors $\underline{\dim}\, I_j$ for I_j in $\mathcal{I}(t)$ and $t \leq 5$ (these can be worked out directly from the dimension vectors of the AR-quivers in [**26**], for instance, see the appendix of the preprint version of [**32**]), one observes that the multiplicity of each t–tuple $\underline{\dim}\, I_j$ equals that of $\sigma\,\underline{\dim}\, I_j$ for any permutation σ of $\{1, \dots, t\}$ and any I_j for $t \leq 5$. The claim now follows readily from formula (9) in Corollary 5.7. □

Remark 5.9. In $\mathrm{GL}(V)$ the Richardson class of a parabolic $P(\mathbf{d})$ depends only on the partition obtained from the composition $\mathbf{d}$ [**78**, II §5]. Since any two associated parabolics have the same Richardson class, for instance, [**78**, Prop. II.3.7], Corollary 5.8 states that the number of $P(\mathbf{d})$–orbits on $\mathfrak{p}_u(\mathbf{d})$ only depends on the Richardson class of $P(\mathbf{d})$. We emphasize that there is no known canonical bijection between the sets of orbits for any two associated parabolic subgroups of $\mathrm{GL}(V)$ in general. Although the number of orbits is the same, the difference of these two actions is stressed, for instance, by the fact that the closure posets may differ entirely. This feature is illustrated by a small example in the appendix (Figures 8.5 and 8.6).

Remark. For a parabolic subgroup P of $\mathrm{GL}(V)$ the map $x \mapsto 1+x$ is a P–equivariant morphism from $\mathfrak{p}_u$ to P_u. Thus we obtain the same results as above for the action of P on P_u instead of $\mathfrak{p}_u$.

Remark. The categories $\mathcal{F}(t)$ and $\mathcal{M}(t)$ can be defined for an *arbitrary* field k and the results relating to these categories are still valid. In particular, Theorem 5.6 and Corollary 5.7 are valid for an arbitrary infinite field k, see [**31, 32**].

Moreover, all the results above can be applied and interpreted for finite fields as well. Let k be algebraically closed of characteristic p and σ a *Frobenius endomorphism* of $G = \mathrm{GL}(V)$. Let $P = P(\mathbf{d})$ be a parabolic subgroup of G and suppose that $t \leq 5$. Then, by Theorem 5.6 and Corollary 5.7, the number of orbits of P^σ on $\mathfrak{p}_u^\sigma$ equals the number of orbits of P on $\mathfrak{p}_u$. It follows from [**84**, I.3.4] that the centralizer in P of an element in $\mathfrak{p}_u$ is connected; likewise for the action of P on P_u. This connectedness property is well known for the centralizers of unipotent elements in the ambient group G, see [**78**].

Remark 5.10. Let k be a field and $n \in \mathbb{N}$. Suppose that the equation $x^n = \mu$ can be solved in k for every μ in k. Then $\mathrm{GL}_n(k) = \mathrm{SL}_n(k) \cdot D$, where $D = \{\mathrm{diag}(\lambda, \dots, \lambda) \mid \lambda \in k^\sharp = k \setminus \{0\}\}$ is a one-dimensional central torus of $\mathrm{GL}_n(k)$ and $\mathrm{SL}_n(k) \cap D$ is finite. Since D is central in $\mathrm{GL}_n(k)$, it acts trivially on both $\mathfrak{p}_u$ and P_u. Therefore, in this case the statements of the results of this section also hold for $\mathrm{SL}(V)$ in place of $\mathrm{GL}(V)$. In general, if P is a parabolic subgroup of $\mathrm{GL}(V)$ such that P acts on $\mathfrak{p}_u$ with an infinite number of orbits, then so does $\mathrm{SL}_n(k) \cap P$. The converse, however, does not hold. For instance, let B be a Borel subgroup of $\mathrm{GL}_2(\mathbb{Q})$. Then B has two orbits on $\mathrm{Lie}\, B_u$, while $B \cap \mathrm{SL}_2(\mathbb{Q})$ admits an infinite number of orbits on the Lie algebra of its unipotent radical, as $\mathbb{Q}^\sharp/(\mathbb{Q}^\sharp)^2$ is infinite.

Remark. Matrix representatives of all the indecomposable objects in $\mathcal{F}(t)$ for $t \leq 5$ can be computed explicitly using the Auslander-Reiten quivers in [**26**]. For a fixed dimension vector $\mathbf{d}$ one can then obtain representatives for all orbits of $P = P(\mathbf{d})$ on $\mathfrak{p}_u$ by means of taking direct sums according to formula (9) in Corollary 5.7. This corresponds to taking direct sums of objects in $\mathcal{F}(t)$, see Lemma 5.1.

5.1.3. The Tame Case. For our purpose of classifying parabolics in classical groups of modality zero, we need more detailed information on the tame situation of Theorem 5.5, that is the instance when $t = 6$.

By Theorem 5.5 the module category $\mathcal{M}(6)$ is of tame representation type and so is $\mathcal{F}(6)$ by Lemma 5.2. Accordingly, a family of non-isomorphic indecomposable objects in $\mathcal{F}(6)$ depends only on a single parameter. Owing to [**26**, Prop. 7.2], $\mathcal{M}(6)$ is of *tubular* type. This implies that there exist two dimension vectors in $\mathcal{M}(6)$, say $\mathbf{d}_1$ and $\mathbf{d}_2$, with the property that whenever $\mathcal{M}(6)(\mathbf{d})$ admits a one-parameter family of indecomposable non-isomorphic objects, $\mathbf{d}$ is of the form $\mathbf{d} = a_1\mathbf{d}_1 + a_2\mathbf{d}_2$ for some non-negative integers a_1 and a_2, see [**66**,

§5]. In this case we simply say that $\mathbf{d}$ admits a one-parameter family of indecomposable non-isomorphic objects in $\mathcal{M}(6)$. It turns out that $\mathbf{d}_1$ and $\mathbf{d}_2$ are linearly dependent in this case [**32**, Prop. 4.8]. Thus the same holds in $\mathcal{F}(6)$ by Lemma 5.2. This fact is significant for our study of parabolics of modality zero in groups of type D_r, see Proposition 5.26.

LEMMA 5.11. *Any dimension vector in $\mathcal{F}(6)$ which admits a non-trivial one-parameter family of indecomposable objects is an integer multiple of* $(1,1,1,1,1,1)$.

PROOF. See [**32**, Prop. 4.8]. □

This property of $\mathcal{F}(6)$ allows us to still determine $\operatorname{mod} P$ in the tame case:

PROPOSITION 5.12. *Let $\mathbf{d} \in \mathbb{N}^6$ and let V be a finite-dimensional k–vector space so that $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Then*

$$\operatorname{mod} P = \min\{d_1, \dots , d_6\}.$$

PROOF. See [**32**, Prop. 4.10]. □

Our next corollary is a direct consequence of Theorem 3.8 and Proposition 5.12.

COROLLARY 5.13. *Let $t \geq 7$, $\mathbf{d} = (d_1, \dots , d_t) \in \mathbb{N}^t$, and V is a finite-dimensional k–vector space of dimension $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Let m be the minimum of the six largest components d_i of $\mathbf{d}$. Then*

$$\operatorname{mod} P \geq m.$$

PROOF. Let H be the standard Levi subgroup of $\mathrm{GL}(V)$ corresponding to these six largest parts of $\mathbf{d}$. Setting $Q = H \cap P$ we get, by construction, $\ell(Q_u) = 5$. By Theorem 3.8 and Proposition 5.12 we conclude

$$\operatorname{mod} P \geq \operatorname{mod} Q = m,$$

as claimed. □

5.1.4. Quasi-hereditary Algebras and Δ–Filtered Modules. In the following sections we address the main results from [**19**]. For representation-finite algebras it was shown by G. Zwara [**95**] that the geometric degeneration of modules can be characterized in terms of dimensions of morphism spaces. The aim of [**19**, §4] was to obtain the analogous result for the category of Δ–filtered modules of a quasi-hereditary algebra. Proposition 5.16 ensures that the relation defined by comparing dimensions of spaces of homomorphisms yields a partial order for Δ–filtered modules, the hom–*order*. We recall some definitions and relevant properties of quasi-hereditary algebras. Unless stated otherwise we refer to [**67**] for proofs of the statements listed below.

Let $\mathcal{A}$ be a finite-dimensional algebra over k and let $E(1), \dots, E(t)$ be a set of representatives of the isomorphism classes of simple $\mathcal{A}$–modules. For each i we fix a projective cover $P(i)$ of $E(i)$ and denote by $\Delta(i)$ the maximal factor module of $P(i)$ with composition factors in $\{E(1), \dots, E(i)\}$. The $\Delta(i)$'s are called the *standard* modules of $\mathcal{A}$ and $\mathcal{F}(\Delta)$ denotes the category of all $\mathcal{A}$–modules M which have a filtration $M = M_0 \supset \cdots \supset M_r = 0$ such that each factor M_{i-1}/M_i belongs to $\{\Delta(1), \dots, \Delta(t)\}$. The algebra $\mathcal{A}$ is called *quasi-hereditary* provided $\mathrm{End}(\Delta(i)) = k$ and $P(i)$ belongs to $\mathcal{F}(\Delta)$ for each i.

Suppose from now on that $\mathcal{A}$ is quasi-hereditary. Then the full subcategory $\mathcal{F}(\Delta)$ of $\mathcal{A}$–mod is closed under direct summands and extensions. A module X in $\mathcal{F}(\Delta)$ is called (relative) Ext-*projective* if $\mathrm{Ext}^1_A(X, M) = 0$ for all $M \in \mathcal{F}(\Delta)$; likewise for Ext-*injective* modules in $\mathcal{F}(\Delta)$. The indecomposable Ext-projective modules in $\mathcal{F}(\Delta)$ are just the projective $\mathcal{A}$–modules $P(1), \dots, P(t)$, whereas the indecomposable Ext-injective modules in $\mathcal{F}(\Delta)$ are the so-called *characteristic* modules $T(1), \dots, T(t)$; the direct sum $T := \oplus T(i)$ is a *tilting* module, see [**67**].

Denote by rad the Jacobson radical of the category $\mathcal{A}$–mod, see [**28**, §3.2]. An *almost split sequence* in $\mathcal{F}(\Delta)$ is a non-split exact sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

in $\mathcal{A}$–mod with X, Y, Z in $\mathcal{F}(\Delta)$ such that each radical morphism $\gamma \in \mathrm{rad}(M, Z)$ with $M \in \mathcal{F}(\Delta)$ factors through β and each radical morphism $\delta \in \mathrm{rad}(X, M)$ with $M \in \mathcal{F}(\Delta)$ factors through α. The category $\mathcal{F}(\Delta)$ *admits almost split sequences*; by that we mean: if X is indecomposable and not Ext-injective in $\mathcal{F}(\Delta)$, then there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$. Moreover, Z is determined uniquely up to isomorphism by X and is often denoted by $\tau_\Delta^- X$. Conversely, if Z is indecomposable and not Ext-projective in $\mathcal{F}(\Delta)$, then there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$; likewise, the module X is determined by Z up to isomorphism and is usually denoted by $\tau_\Delta Z$. The map τ_Δ is called the *Auslander-Reiten translation* and the sequence above is also called an *Auslander-Reiten sequence*. The quiver whose vertices are the isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ and whose arrows are given by the maps in almost split sequences is called the *Auslander-Reiten quiver* of the underlying module category $\mathcal{F}(\Delta)$, see [**1**], [**28**], or [**66**].

For a finite-dimensional $\mathcal{A}$–module M and an indecomposable module Z we denote the multiplicity of Z in M as a direct summand by $\mu(M,Z)$. Observe that $\mu(M,Z)$ can be computed via

$$\mu(M,Z) = \dim \operatorname{Hom}(M,Z) - \dim \operatorname{rad}(M,Z).$$

LEMMA 5.14. *Let* $0 \to X \to Y \to Z \to 0$ *be an almost split sequence in* $\mathcal{F}(\Delta)$ *and let* M *be in* $\mathcal{F}(\Delta)$. *Then we have*

(1) $\mu(M,Z) = \dim \operatorname{Hom}(M, X \oplus Z) - \dim \operatorname{Hom}(M,Y)$,
(2) $\mu(M,X) = \dim \operatorname{Hom}(X \oplus Z, M) - \dim \operatorname{Hom}(Y,M)$.

PROOF. (1) From the definition of an almost split sequence we obtain an exact sequence

$$0 \to \operatorname{Hom}(M,X) \to \operatorname{Hom}(M,Y) \to \operatorname{rad}(M,Z) \to 0.$$

Thus, we get $\dim \operatorname{Hom}(M, X\oplus Z) - \dim \operatorname{Hom}(M,Y) = \dim \operatorname{Hom}(M,Z) - \dim \operatorname{rad}(M,Z) = \mu(M,Z)$. We obtain (2) by duality. $\square$

Up to Morita-equivalence we may assume that the algebra $\mathcal{A}$ is presented in the form $\mathcal{A} = kQ/J$ where Q is a quiver and J is an *admissible* ideal of the *path algebra* kQ of Q see [**28**, §8]. The quiver $Q = (Q_0, Q_1, t, h)$ is described by its set of vertices Q_0, the set of arrows Q_1 and two maps $t, h : Q_1 \to Q_0$ which determine *tail* and *head* of each arrow.

An $\mathcal{A}$–module M is a family $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where each M_i is a finite-dimensional vector space and each $M_\alpha : M_{t\alpha} \to M_{h\alpha}$ is a k–linear map such that $M_\xi = 0$ for all $\xi \in J$, where M_ξ is defined as follows: the element $\xi \in J$ is a linear combination of paths w in kQ, say $\xi = \sum c_w w$ with $c_w \in k$; we set $M_w = M_{\alpha_m} \circ \cdots \circ M_{\alpha_1}$ for any path $w = \alpha_1 \cdots \alpha_m$ and define $M_\xi = \sum c_w M_w$.

For each $\mathcal{A}$–module M we denote by $\underline{\dim}\, M \in \mathbb{N}^t$ the dimension vector of M. There are several equivalent descriptions of the dimension vector:

$$(\underline{\dim}\, M)_i = \dim M_i = \dim \operatorname{Hom}(P(i), M)$$

for $i = 1, \ldots, t$.

REMARK. Note that for M and N in $\mathcal{F}(\Delta)$ we have $\underline{\dim}\, M = \underline{\dim}\, N$ precisely when $\dim \operatorname{Hom}(M, T(i)) = \dim \operatorname{Hom}(N, T(i))$ for each $i = 1, \ldots, t$.

Now fix k–spaces M_i for each $i = 1, \ldots, t$, set $e_i = \dim M_i$ and $\mathbf{e} = (e_1, \ldots, e_t)$. The affine variety $\mathcal{R}(\mathbf{e})$ of $\mathcal{A}$–modules with dimension vector $\mathbf{e}$ consists of tuples of k–linear maps $(M_\alpha : M_{t\alpha} \to M_{h\alpha})_{\alpha \in Q_1}$

such that $M_\xi = 0$ for all $\xi \in J$. Clearly, this variety contains representatives of all isomorphism classes of $\mathcal{A}$–modules of dimension vector $\mathbf{e}$. Throughout, we identify $\mathcal{A}$–modules or representations of the associated quiver of dimension vector $\mathbf{e}$ with points in the variety $\mathcal{R}(\mathbf{e})$.

The reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ acts on $\mathcal{R}(\mathbf{e})$ by conjugation in each vector space M_i and the orbits are precisely the isomorphism classes of $\mathbf{e}$–dimensional modules. The modules in $\mathcal{F}(\Delta)$ with dimension vector $\mathbf{e}$ form a $G(\mathbf{e})$–stable subset $\mathcal{R}(\Delta)(\mathbf{e})$ of $\mathcal{R}(\mathbf{e})$. We consider the restriction of the Bruhat-Chevalley order to the orbits in $\mathcal{R}(\Delta)(\mathbf{e})$.

Remark 5.15. For $i = 1, 2$ let N_i be in $\mathcal{F}(\Delta)(\mathbf{e})$ and let $\mathcal{O}_i$ be the $G(\mathbf{e})$–orbit of N_i in $\mathcal{R}(\Delta)(\mathbf{e})$. Whenever $\mathcal{O}_1 \leq \mathcal{O}_2$ in the closure order, by abuse of notation, we simply write $N_1 \leq N_2$ and call N_1 a *(geometric) degeneration* of N_2.

Proposition 5.16. *Let M and N be in $\mathcal{F}(\Delta)(\mathbf{e})$. Then M and N are isomorphic provided* $\dim \mathrm{Hom}(M, X) = \dim \mathrm{Hom}(N, X)$ *for all* $X \in \mathcal{F}(\Delta)$.

Proof. We argue by induction on $\mathbf{e}$. Let Z be a non-Ext-projective indecomposable module in $\mathcal{F}(\Delta)$ with almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{F}(\Delta)$. By Lemma 5.14 we obtain

$$\begin{aligned} \mu(N, Z) &= \dim \mathrm{Hom}(N, X \oplus Z) - \dim \mathrm{Hom}(N, Y) \\ &= \dim \mathrm{Hom}(M, X \oplus Z) - \dim \mathrm{Hom}(M, Y) = \mu(M, Z). \end{aligned}$$

If $\mu(N, Z) > 0$, then $N = N' \oplus Z$, $M = M' \oplus Z$ and we apply induction to N' and M'. On the other hand, if $\mu(N, Z) = 0 = \mu(M, Z)$ for *all* non-Ext-projective modules Z in $\mathcal{F}(\Delta)$, then M and N are projective $\mathcal{A}$–modules of the same dimension vector. Hence they are isomorphic, as the dimension vectors $\underline{\dim}\, P(1), \ldots, \underline{\dim}\, P(t)$ are linearly independent for any quasi-hereditary algebra. $\square$

Remark 5.17. Let M and N be in $\mathcal{F}(\Delta)(\mathbf{e})$ such that

$$\dim \mathrm{Hom}(N, X) \geq \dim \mathrm{Hom}(M, X) \text{ for all } X \in \mathcal{F}(\Delta).$$

Then we write

$$[N] \geq_{\mathrm{hom}} [M],$$

where $[Y]$ denotes the isomorphism class of the module Y in $\mathcal{A}$–mod. Thanks to Proposition 5.16, this defines a partial order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$, called the hom*–order*. If $[N] \geq_{\mathrm{hom}} [M]$, then, by abuse of notation, we write $N \geq_{\mathrm{hom}} M$, and call N a *Hom-degeneration* of M.

We say that a quasi-hereditary algebra $\mathcal{A}$ is Δ–*finite* provided the category $\mathcal{F}(\Delta)$ of Δ–filtered $\mathcal{A}$–modules is of finite representation type. The main result from [**19**] in this context is

THEOREM 5.18. *Suppose that $\mathcal{A}$ is a Δ–finite quasi-hereditary algebra. Then for each dimension vector $\mathbf{e}$ the following posets coincide:*

(i) *the closure order on the set of $G(\mathbf{e})$–orbits on $\mathcal{R}(\Delta)(\mathbf{e})$;*
(ii) *the poset opposite to the* hom*–order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$.*

For the long and intricate proof of Theorem 5.18 we refer the reader to [**19**, §4].

5.1.5. The Bruhat-Chevalley Order on the Set of P–Orbits. In this section we return to the setting of Section 5.1.1 and the discussion of parabolic groups P in general linear groups $\mathrm{GL}(V)$. The aim is a complete combinatorial description of the Bruhat-Chevalley order on the set of P–orbits on $\mathfrak{p}_u$, given this set is finite. In [**31, 32**] precisely all these instances are determined.

For $\mathbf{d} \in \mathbb{N}^t$ let $\mathbf{e} = \Sigma\mathbf{d}$ be the t–tuple defined by the partial sums $e_j := \sum_{i=1}^{j} d_i$ for $1 \leq j \leq t$. We denote by $\mathcal{F}(\Delta)(\mathbf{e})$ the subcategory of $\mathcal{F}(\Delta)$ of all modules of dimension vector $\mathbf{e}$ of the quasi-hereditary algebra $\mathcal{A}(t)$. Observe that the collection of all $\mathcal{A}(t)$–modules in $\mathcal{F}(\Delta)$ of *fixed* dimension vector $\mathbf{e}$ together with a *fixed* set of k–spaces V_i of dimension e_i, for $1 \leq i \leq t$, is an algebraic variety. We denote this variety by $\mathcal{R}(\Delta)(\mathbf{e})$. It is defined as the locally closed subvariety of the k–vector space

$$\mathcal{R}(\mathbf{e}) := \bigoplus_{\alpha_i} \mathrm{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{\beta_j} \mathrm{Hom}(V_{j+1}, V_j)$$

of all possible linear maps corresponding to the arrows α_i and β_j in the quiver $Q(t)$ satisfying the relations $\beta_1\alpha_1 = 0$ and $\beta_i\alpha_i = \alpha_{i-1}\beta_{i-1}$ for $1 < i < t$ from above (closed condition) and such that the collection of direct summands corresponding to the arrows α_i consist of injective linear maps (open condition). Since the injectivity of the linear maps M_{α_i} is preserved by isomorphisms, we infer that the natural action of the reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ on $\mathcal{R}(\mathbf{e})$ leaves $\mathcal{R}(\Delta)(\mathbf{e})$ invariant. The action of an element $g = (g_1, g_2, \dots)$ in $G(\mathbf{e})$ on $\mathcal{R}(\Delta)(\mathbf{e})$ is given by base change in each of the spaces V_i, that is $g \cdot M(F, f) = (\dots, g_{i+1}M_{\alpha_i}g_i^{-1}, \dots, g_iM_{\beta_i}g_{i+1}^{-1}, \dots)$, where $g_i \in \mathrm{GL}(e_i)$ for each i.

The principal result from [**19**] concerning the action of P on $\mathfrak{p}_u$ is

THEOREM 5.19. *Let* $t \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$ *and let* V *be a finite-dimensional* k*–vector space such that* $\dim V = \sum d_i$. *Let* $P = P(\mathbf{d})$ *be the standard parabolic subgroup of* $\mathrm{GL}(V)$ *associated to* $\mathbf{d}$. *Set* $\mathbf{e} = \Sigma\mathbf{d}$. *Suppose that the number of* P*–orbits on* $\mathfrak{p}_u$ *is finite. Then the following posets coincide:*

(i) *the closure order on the set of* P*–orbits on* $\mathfrak{p}_u$*;*
(ii) *the closure order on the set of* $G(\mathbf{e})$*–orbits on* $\mathcal{R}(\Delta)(\mathbf{e})$*;*
(iii) *the poset opposite to the* hom*–order on the set of isomorphism classes of* $\mathcal{F}(\Delta)(\mathbf{e})$.

For our setting the advantage of working with the hom–order is due to the fact that it is given purely by discrete invariants, more specifically, by the ordered tuples $\underline{\mathrm{hdim}}\, X$ whose j-th entry consists of $\dim \mathrm{Hom}(X, I_j)$, where I_j runs through a complete set of representatives of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. Consequently, once the square matrix $(\dim \mathrm{Hom}(I_i, I_j))_{i,j}$ is computed, the hom–order on $\mathcal{F}(\Delta)(\mathbf{e})$ can be computed explicitly in any finite instance. Thus Theorem 5.19 allows us to explicitly determine the closure relation for the P–orbits on $\mathfrak{p}_u$ purely combinatorially with the aid of this hom–order. We illustrate some examples in the appendix.

Theorem 5.19 follows readily from Theorem 5.18 and our next result [**19**, Thm. 1.2].

THEOREM 5.20. *Let* $t \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$, *and* V *is a finite-dimensional* k*–vector space so that* $\dim V = \sum d_i$. *Let* $P = P(\mathbf{d})$ *be the parabolic subgroup of* $\mathrm{GL}(V)$ *associated to* $\mathbf{d}$. *Set* $\mathbf{e} = \Sigma\mathbf{d}$. *Then the following posets coincide:*

(i) *the closure order on the set of* P*–orbits on* $\mathfrak{p}_u$*;*
(ii) *the closure order on the set of* $G(\mathbf{e})$*–orbits on* $\mathcal{R}(\Delta)(\mathbf{e})$.

PROOF. Observe that $\mathcal{R}(\Delta)(\mathbf{e})$ and $\mathfrak{p}_u(\mathbf{d})$ are *not* isomorphic varieties; $\mathfrak{p}_u(\mathbf{d})$ can be identified with the proper subvariety of $\mathcal{R}(\Delta)(\mathbf{e})$ consisting of all those representations M of $Q(t)$ with a *fixed* set of injective maps M_{α_i}. For our purpose, however, it is enough to construct morphisms between these varieties in both directions preserving the orbit structure, as such morphisms do preserve orbit closures.

We fix vector spaces V_i together with injections $V_i \longrightarrow V_{i+1}$, where $\dim V_i = e_i$. Let P be the stabilizer in $\mathrm{GL}(V)$ of the flag $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{t-1} \subseteq V_t = V$. For f in $\mathfrak{p}_u$ we define a representation $M(f) := \{M_{\alpha_i}, M_{\beta_i}\}$ for $1 \leq i < t$ in $\mathcal{F}(\Delta)(\mathbf{e})$ of $\mathcal{A}(t)$ by setting $M_{\beta_j} := f|_{V_{j+1}}$ and M_{α_i} is just the fixed injection in the flag. The map $f \mapsto M(f)$ obviously defines a morphism from $\mathfrak{p}_u(\mathbf{d})$ to $\mathcal{R}(\Delta)(\mathbf{e})$.

It remains to construct a morphism in the opposite direction. For that purpose let $\{M_{\alpha_i}, M_{\beta_j}\}$ be a representation in $\mathcal{F}(\Delta)(\mathbf{e})$ with vector spaces $V_1, \dots, V_t$ with $\dim V_i = e_i$, then $M_{\alpha_{t-1}} M_{\beta_{t-1}}$ is an endomorphism of V_t lying in $\mathfrak{p}_u(\mathbf{d})$. Both morphisms preserve orbits which is easily deduced from the equivalences of the corresponding categories constructed above. $\square$

We stress that the equivalence of Theorem 5.20 is valid independent of the representation type of $\mathcal{F}(\Delta)$, that is it does not require that $\mathfrak{p}_u$ is a finite orbit module for P, as needed for Theorem 5.19.

Remark 5.21. In Section 5 of [**19**] we discuss a more conceptual approach to degenerations of Δ–filtered modules of an arbitrary Δ–finite quasi-hereditary algebra. The crucial concept here is the notion of a *global minimal* hom–*degeneration* in $\mathcal{F}(\Delta)$; that is a pair of modules X and Y in $\mathcal{F}(\Delta)$ of the same dimension vector without common direct summand such that the difference $\underline{\mathrm{hdim}}\, X - \underline{\mathrm{hdim}}\, Y$ is a standard basis vector in $\mathbb{Q}^m$, where m is the number of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. The main result in [**19**, §5] shows that such pairs correspond bijectively to almost split sequences in the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$. In that section we give a definition of a globally minimal degeneration which applies to both, the geometric setting given by orbit closures, as well as to the combinatorial setup of the hom–order. Finally, we refer to Appendix B in [**19**] for a list of the Auslander-Reiten quivers of $\mathcal{F}(t)$ in the significant finite instances along with normal forms for the indecomposable modules.

5.1.6. Hasse Diagrams. For every $\mathbf{d} \in \mathbb{N}^t$, and $t \leq 5$, the data consisting of the orbits, their dimensions, and closure relations, that is the Hasse diagram, can be generated by machine calculations based on formula (9) from Corollary 5.7 and the $m \times m$–matrix

$$\mathbf{D} := (\dim \mathrm{Hom}(I_i, I_j))_{i,j}.$$

Given an m–tuple $\mathbf{a} := (a_1, \dots, a_m)$ such that $\mathbf{d} = \sum a_j \underline{\dim}\, I_j$ as in Corollary 5.7, let $F_{\mathbf{a}}$ be the associated object in $\mathcal{F}(t)$, that is

$$F_{\mathbf{a}} := \sum a_j I_j.$$

Then we readily obtain the m–vector $\underline{\mathrm{hdim}}\, F_{\mathbf{a}}$ whose j-th entry consists of $\dim \mathrm{Hom}(F_{\mathbf{a}}, I_j)$ simply by matrix multiplication

$$\underline{\mathrm{hdim}}\, F_{\mathbf{a}} = \mathbf{a} \cdot \mathbf{D}.$$

Comparing these m–vectors in the resulting finite set obtained from all m–tuples $\mathbf{a}$ from Corollary 5.7 then allows us to determine the exact poset structure of the hom–order. By Theorem 5.19 we then obtain the desired poset of the closure order simply by taking the opposite of the hom–poset. Our examples in the appendix were generated in this fashion; see [**19**, §7] for additional ones.

5.2. The Classical Groups

In this section we address the principal result from [**32**, Thm. 1.1]. Under the assumption that $\operatorname{char} k$ is zero a partial classification was obtained in [**31**]. Throughout, for G of type D_r we denote by τ the usual graph automorphism of G of order 2 as above.

Theorem 5.22. *Let G be simple classical and $P \subseteq G$ is parabolic. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Then* $\operatorname{mod} P = 0$ *if and only if one of the following holds:*

(i) $\ell(P_u) \leq 4$*;*
(ii) *G is of type D_r, $\ell(P_u) = 5$, $\tau P \neq P$, and the semisimple part of L_P consists of two simple components.*

The proof of Theorem 5.22 essentially consists in a reduction to the case of general linear groups, that is to Theorem 5.6, by means of taking fixed points of graph automorphisms.

Remark 5.23. Since k is algebraically closed, $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ only differ by central elements, and thus by Remark 5.10 the statements of the previous section also hold for $\mathrm{SL}(V)$ in place of $\mathrm{GL}(V)$. If $P = P(\mathbf{d})$ is a parabolic subgroup of $\mathrm{SL}(V)$ with $\mathbf{d} \in \mathbb{N}^t$, then $\ell(P_u) = t-1$. Hence, Theorem 5.6 implies the desired finiteness statement of Theorem 5.22 for $\mathrm{SL}(V)$.

Lemma 5.24. *Let $G = \mathrm{SL}(V)$. Then each of the classical groups $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$ can be realized as a fixed point subgroup of G for a suitable semisimple automorphism Θ of G, that is G^Θ equals $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$. Moreover, each parabolic subgroup of G^Θ can be obtained as a fixed point subgroup P^Θ for some Θ–invariant parabolic subgroup P of G. In case G^Θ is $\mathrm{SO}(V)$ and $\dim V$ is even, P^Θ is only determined up to equivalence under the graph automorphism of G^Θ.*

For a proof of the assertion on G^Θ, see [**85**, §11 p. 169]. The remaining statements concerning parabolics follow easily from the explicit description of Θ in [**85**]. Note that $\ell(P_u^\Theta) \leq \ell(P_u)$. In general, this inequality may be strict.

We formulate a first consequence for the other simple classical groups from the finiteness result for $\mathrm{SL}(V)$. In case H is of type D_r let τ be again the graph automorphism of H of order 2.

COROLLARY 5.25. *Let H be simple classical and $Q \subseteq H$ is parabolic. Suppose* $\operatorname{char} k \neq 2$. *Then* $\operatorname{mod} Q = 0$ *provided*

(a) *H is of type B_r or C_r and $\ell(Q_u) \leq 4$; or*
(b) *H is of type D_r, and one of the following holds:*
  (i) *$\ell(Q_u) \leq 3$, or*
  (ii) *$\ell(Q_u) = 4$, and $\tau Q = Q$.*

PROOF. Let $G = \mathrm{SL}(V)$ and let Θ be the automorphism of G as in Lemma 5.24 such that G^Θ equals H. Since Θ has order two, the assumption on the characteristic of k ensures that Θ is semisimple. Thus Corollary 3.4 applies in this instance. By Lemma 5.24 we can obtain Q as a fixed point subgroup P^Θ for some Θ–invariant parabolic subgroup P of G. In each of the cases considered this can be done for a suitable P with $\ell(P_u) \leq 4$. The desired result for H then follows from Corollary 3.4 and the finiteness result for G. We illustrate a few examples below. □

Next we address the remaining incidences of Theorem 5.22 of type D_r.

PROPOSITION 5.26. *Let H be of type D_r, $r \geq 5$, and $Q \subseteq H$ is parabolic. Suppose* $\operatorname{char} k \neq 2$. *Then* $\operatorname{mod} Q = 0$ *provided*

(i) *$\ell(Q_u) = 4$, and $\tau Q \neq Q$; or*
(ii) *$\ell(Q_u) = 5$, $\tau Q \neq Q$, and the semisimple part of L_Q consists of two simple components.*

PROOF. We argue as in the proof of Corollary 5.25. Let $G = \mathrm{SL}(V)$ and let Θ be the semisimple automorphism of G as above such that H is equal to G^Θ. As asserted by Lemma 5.24, we can obtain Q as a fixed point subgroup P^Θ for some Θ–invariant parabolic subgroup P of G. However, in all of these cases $\ell(P_u) = 5$ and $\operatorname{mod} P > 0$ by Proposition 4.4. Whence, Corollary 3.4 does not yield the desired finiteness statement for $P^\Theta = Q$.

The parabolic subgroups $P = P(\mathbf{d})$ of G that occur in this way have the feature that $\mathbf{d}$ is of the form $(1, a, b, b, a, 1)$ or $(a, 1, b, b, 1, a)$, with $a, b \in \mathbb{N}$, depending on whether $\ell(Q_u)$ equals 4 or 5, respectively. Thus $\operatorname{mod} P = 1$, by Proposition 5.12, and the only minimal dimension vector admitting a one-parameter family of orbits which is a summand of such a particular $\mathbf{d}$ is $(1, 1, 1, 1, 1, 1)$, by Lemma 5.11. This dimension vector corresponds to the standard Borel subgroup B in $\mathrm{SL}_6(k)$. In this instance there is a unique one-parameter family of B–orbits on $\mathfrak{b}_u$ with representatives of the form

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \lambda & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where $\lambda \in k$ and the dots represent zero entries. A conjugate one-parameter family already appears in Zalesskiĭ's paper [**94**]; see also [**20**]. One readily checks that no B–conjugate of any member of this family is invariant under $d\Theta_e$, the differential of Θ; a necessary condition for $d\Theta_e$–invariance is that the entries of the second main diagonal are all equal to zero. But the 1 in position $(3, 4)$ cannot be removed using elements from B. Embedding this family into $\mathfrak{p}_u$ by taking direct sums gives rise to a family of P–orbits on $\mathfrak{p}_u$ with the same property, namely, that no P–conjugate of any of its members is $d\Theta_e$–invariant. Consequently, the intersection of the single non-trivial one-parameter family of P–orbits on $\mathfrak{p}_u$ with $\mathfrak{q}_u$ is empty. According to the proof of Corollary 3.4, a P–orbit on $\mathfrak{p}_u$ intersects $\mathfrak{q}_u$ either trivially or in a finite union of Q–orbits. Thus, Q in turn only has a finite number of orbits on $\mathfrak{q}_u$. This completes the proof of the proposition. □

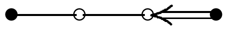
Finally, we do have all the ingredients to complete the

Proof of Theorem 5.22. The result follows from Proposition 4.4, Theorem 5.6 (and Remark 5.23), Corollary 5.25, and Proposition 5.26. □

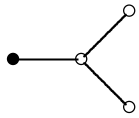
Remark 5.27. We may also obtain the result of Corollary 5.25 for groups of type D_r from the result for groups of type B_r. For, let G be of type B_r and let H be the simple regular subgroup of G whose root system consists of the set of long roots in Ψ. Then H is of type D_r, e.g., see [**85**, §11]. Let Q be a parabolic subgroup of H as in Corollary 5.25. There exists a parabolic subgroup P of G with $\ell(P_u) \leq 4$ such that $P \cap H$ is either Q or the image of Q under the graph automorphism τ of H. The desired result for H then follows from the result for G and Theorem 3.10.

We illustrate the procedures in the proofs of Corollary 5.25, Proposition 5.26, and of Remark 5.27 by some examples in Figures 5.2 and 5.3.

$G = \mathrm{SL}_8(k)$, P, $\ell(P_u) = 4$

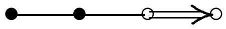


$G^\Theta = \mathrm{Sp}_8(k)$, P^Θ, $\ell(P_u^\Theta) = 4$

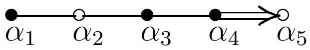


$G^{\Theta'} = \mathrm{SO}_8(k)$, $P^{\Theta'}$, $\ell(P_u^{\Theta'}) = 4$

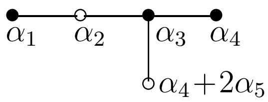
$G = \mathrm{SL}_9(k)$, P, $\ell(P_u) = 4$



$G^\Theta = \mathrm{SO}_9(k)$, P^Θ, $\ell(P_u^\Theta) = 4$



$G = \mathrm{SO}_{11}(k)$, P, $\ell(P_u) = 4$

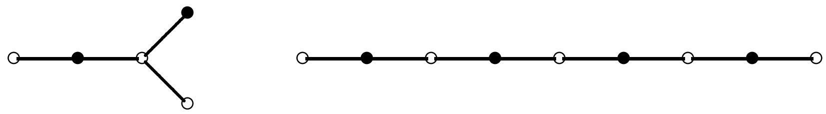


$H = \mathrm{SO}_{10}(k)$, Q, $\ell(Q_u) = 3$

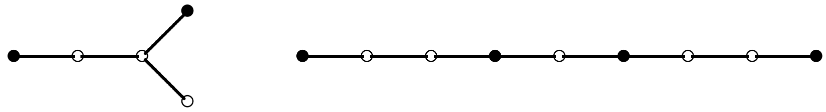
FIGURE 5.2. Some examples of fixed point configurations

EXAMPLES 5.28. Let G and P be as in Corollary 5.25. The colored nodes in Figure 5.2 indicate the simple roots in $\Pi(P)$ and likewise for $\Pi(H)$. In our first example different automorphisms lead to different fixed point subgroups. For an explicit description of the automorphisms in terms of matrices consult [**85**, §11]. The last pair of diagrams in Figure 5.2 demonstrates the method outlined in Remark 5.27.

EXAMPLES 5.29. Our examples in Figure 5.3 demonstrate the minimal rank incidences of Proposition 5.26. Suppose G, P, $H = G^\Theta$, and $Q = P^\Theta$ are as in Proposition 5.26. As before the colored nodes indicate the simple roots in $\Pi(P)$ and $\Pi(Q)$, respectively. In our first example the class of nilpotency of Q_u is 4, and 5 in the second, see (3) on page 4. The corresponding dimension vectors $\mathbf{d}$ of the groups $P(\mathbf{d})$



$H = \mathrm{SO}_{10}(k)$, Q, $\ell(Q_u) = 4$$G = \mathrm{SL}_{10}(k)$, $P(\mathbf{d})$, $\mathbf{d} = (1, 2, 2, 2, 2, 1)$



$H = \mathrm{SO}_{10}(k)$, Q, $\ell(Q_u) = 5$$G = \mathrm{SL}_{10}(k)$, $P(\mathbf{d})$, $\mathbf{d} = (2, 1, 2, 2, 1, 2)$

FIGURE 5.3. The significant cases from Proposition 5.26

in $\mathrm{SL}_{10}(k)$ are $\mathbf{d} = (1, 2, 2, 2, 2, 1)$ and $\mathbf{d} = (2, 1, 2, 2, 1, 2)$, respectively. The class of nilpotency of P_u equals 5 in both events. The general cases only differ from those shown by larger Levi components.

5.3. The Exceptional Groups

In this section we present the classification of all parabolic subgroups P of exceptional algebraic groups of modality zero from [**37**]. This was achieved by means of the GAP share package MOP referred to in Section 3.2. Partial results for exceptional groups were obtained in [**35**], [**58**], and [**69**].

THEOREM 5.30. *Suppose G is of exceptional type and that* $\operatorname{char} k$ *is either zero or a good prime for G. Let P be a parabolic subgroup of G. Then P acts on $\mathfrak{p}_u$ with a finite number of orbits if and only if one of the following holds:*

(i) $\ell(P_u) \leq 4$;
(ii) *G is of type E_6, $\ell(P_u) = 5$, and P is of type $A_1^2 A_2$ or A_3;*
(iii) *G is of type E_7, $\ell(P_u) = 5$, and P is of type $A_1 A_4$.*

Theorems 5.22 and 5.30 give a complete classification of parabolic subgroups P of reductive groups with a finite number of orbits on $\mathfrak{p}_u$.

PROOF. It follows from Proposition 4.5 that $\operatorname{mod} P > 0$ provided none of the conditions of Theorem 5.30 is satisfied.

In each of the cases of Theorem 5.30 when the Dynkin diagram of G is simply laced the desired finiteness statements were obtained directly using MOP. The classification of modality zero parabolics in G_2 already follows from [**20**] and [**58**, Thm. 4.2].

Thus, only the instances of F_4 remain. Let G be of type E_6 and let τ be the graph automorphism of G of order 2. The fixed point subgroup G^τ is of type F_4. Let Q be a parabolic subgroup of G^τ. Then, $\operatorname{mod} Q > 0$ provided $\ell(Q_u) \geq 5$, by Proposition 4.5(i). In order to show the converse it suffices to prove that $\operatorname{mod} Q = 0$ provided Q is minimal with respect to satisfying $\ell(Q_u) \leq 4$, by Lemma 3.1(ii). This leads to the three instances when Q is of type B_2, $A_1\widetilde{A}_2$, or $\widetilde{A}_1 A_2$, where $\widetilde{A}_i$ represents a subsystem of type A_i consisting of short roots. Each such Q can be realized as the τ-fixed point subgroup of a parabolic subgroup P of G; see Figure 5.4 below. Each occurring P satisfies $\ell(P_u) \leq 4$ and thus $\operatorname{mod} P = 0$ by the finiteness result for E_6. The desired result for F_4 then follows by Corollary 3.4. $\square$

Figure 5.4 presents the crucial F_4 cases from the proof of Theorem 5.30. As before, the solid nodes indicate the simple roots in the Levi subgroup of P and P^τ, respectively.

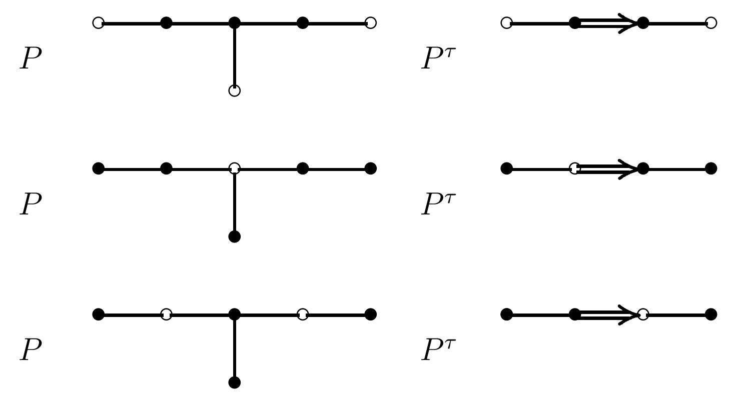


FIGURE 5.4. The crucial F_4 cases from Theorem 5.30

5.4. Modality for Associated Parabolic Groups

Suppose that G is reductive. The parabolic subgroups $P, Q \subset G$ are *associated* provided they admit G–conjugate Levi subgroups. It is a common phenomenon that a certain construction defined for a conjugacy class of P only depends on the *association class of* P, that is on the equivalence class of parabolics associated to P, rather than the class of P. A typical example is the induction of unipotent classes [**48**], a special case of which is the fact that two associated parabolics have the same Richardson class, see [**34**], or [**9**, Cor. 5.18].

We discuss some results concerning the modality of associated parabolic subgroups of G. In Corollary 5.8 we have seen that associated parabolics in general linear groups of modality zero do in fact have the same number of orbits on the unipotent radical.

PROPOSITION 5.31. *Let G be reductive and P, Q associated parabolic subgroups of G. Then* $\operatorname{mod} P = 0$ *if and only if* $\operatorname{mod} Q = 0$.

PROOF. This follows from Theorems 5.22 and 5.30 and the classification of conjugacy classes of Levi subgroups of G from [**4**, Prop. 6.2, 6.3], based on E.B. Dynkin's classification of regular subalgebras of $\mathfrak{g}$ [**27**, Thm. 5.4]. □

For general linear groups Proposition 5.31 extends to

PROPOSITION 5.32. *Let P, Q be associated parabolic subgroups of* $\mathrm{GL}(V)$. *If $\ell(P_u) \leq 5$, then*

$$\operatorname{mod} P = \operatorname{mod} Q.$$

PROOF. Set $\ell = \ell(P_u) = \ell(Q_u)$. For $\ell \leq 4$ this is evident from Theorem 5.6. For $\ell = 5$ this follows immediately from Proposition 5.12. Since the Levi subgroups of P and Q are conjugate, we see that $P = P(\mathbf{d})$ and $Q = P(\mathbf{d}')$, where $\mathbf{d} = (d_1, \ldots, d_6)$ and $\mathbf{d}' = \sigma\mathbf{d}$ for some permutation σ of $\{1, \ldots, 6\}$. Thus the modality of both groups equals $\min\{d_1, \ldots, d_6\}$ by Proposition 5.12. □

See Proposition 6.8 for some instances in exceptional groups where $\operatorname{mod} P$ only depends on the association class of P.

REMARK. These results suggest that the map which assigns to a given parabolic P its modality $\operatorname{mod} P$ is constant on association classes. Moreover, several examples indicate that this map is in fact constant on classes of parabolics sharing the same Richardson class. For general linear groups these two equivalence relations on the set of parabolics are identical, see Remark 5.9.

5.5. Further Consequences

We discuss some additional consequences of the classification results, Theorems 5.22 and 5.30. Throughout this section, G is reductive.

COROLLARY 5.33. *Suppose P is a non-maximal parabolic subgroup of G and* $\operatorname{mod} P > 0$. *Then there exists a* proper *P–submodule $\mathfrak{s}$ of $\mathfrak{p}_u$ such that* $\operatorname{mod}(P : \mathfrak{s}) > 0$.

PROOF. Without loss, we may suppose that G is simple. The statement of the corollary is a consequence of the inductive construction of the cases when P is of positive modality, see [**69**, Thm. 3.1], [**35**, Lem. 3.13], and [**32**, Lem. 2.3]. Thanks to Theorems 5.22 and 5.30, G and P are as in Propositions 4.4 or 4.5. The desired statement follows inductively from the method employed in the proofs of these results, see [**69**, Thm. 6.3]. For, by construction, this follows for G as in Table 4.1 and $P = B$, by Lemma 4.1. If G and P are as in Propositions 4.4 or 4.5 and $P \neq B$, then, by induction, there exists a simple regular subgroup H of G of type A_5, B_3, C_3, or D_4, such that $Q = H \cap P$ is the Borel subgroup of H. Let $\mathfrak{n}$ be as in Table 4.1 and define $\mathfrak{s}$ to be the minimal P–submodule of $\mathfrak{p}_u$ containing $\mathfrak{n}$. By choice, we have $\operatorname{mod}(Q : \mathfrak{n}) > 0$. Thus, according to Theorem 3.8 and Remark 3.11, we derive $\operatorname{mod}(P : \mathfrak{s}) > 0$ as desired. The fact that $\mathfrak{s}$ is proper in $\mathfrak{p}_u$, provided P is not maximal, does not follow *a priori*. This is clear for classical groups from the proof of Proposition 4.4. For exceptional groups it follows from inspection of the tables and the construction in [**69**, §5]. $\square$

The following E_8 example from [**36**] illustrates that the statement of Corollary 5.33 is false if the non-maximality condition on P is relaxed.

REMARK 5.34. Suppose G is of type E_8 and P is conjugate to P_J, where $J = \Pi \setminus \{\sigma_5\}$. It was shown in [**69**] that $\operatorname{mod} P > 0$. Since P is a maximal parabolic subgroup of G, the various members of the descending central series of $\mathfrak{p}_u$ are the only P–submodules of $\mathfrak{p}_u$. Using the GAP package MOP it is shown in [**36**] that in fact $\operatorname{mod}(P : \mathfrak{p}_u') = 0$. In particular, $\operatorname{mod}(P : \mathfrak{n}) = 0$ for every proper P–submodule $\mathfrak{n}$ of $\mathfrak{p}_u$.

As a consequence, P admits a dense orbit on $\mathfrak{p}_u^{(i)}$ for each $i \geq 1$. By Richardson's Dense Orbit Theorem [**62**] P also has a dense orbit on $\mathfrak{p}_u$ itself. Consequently, every P–invariant linear subspace of $\mathfrak{p}_u$ is a prehomogeneous vector space for P, but nevertheless, $\operatorname{mod} P > 0$.

5.6. Some Generalizations

We close this chapter by glancing at a generalization of Theorem 5.22. In [**14**] all parabolic subgroups P of $\mathrm{GL}(V)$ with a finite number of orbits on $\mathfrak{p}_u^{(l)}$, the l-th member of the descending central series of $\mathfrak{p}_u$ were classified for $l \geq 1$. This was extended to all classical groups in [**18**]. Here we present the combined results [**18**, Thm. 1.1]:

THEOREM 5.35. *Let G be classical and $P \subseteq G$ parabolic. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Then* $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) = 0$ *for $l \geq 1$ precisely if one of the following conditions holds:*

(i) *G is of type A_r and $\ell(P_u) \leq 5 + 2l$;*
(ii) *G is of type B_r and $\ell(P_u) \leq 4 + 2l$;*
(iii) *G is of type C_r and $\ell(P_u) \leq 5 + 2l$;*
(iv) *G is of type D_r, either $\ell(P_u) \leq 4 + 2l$, or $\ell(P_u) = 5 + 2l$ and $\tau P \neq P$.*

The method of proof of the classification of all parabolic subgroups P of $\mathrm{GL}(V)$ satisfying $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) = 0$ for $l \geq 1$ from [**14**] is similar to the one of Theorem 5.6. There Brüstle and Hille generalize the machinery from Section 5.1 to this situation; in particular, this involves extensions of results from [**26**] concerning the category $\mathcal{F}(\Delta)$ of Δ–filtered modules of a particular quasi-hereditary algebra.

The extension of these results to other classical groups, Theorem 5.35, is analogous to the one of Section 5.2 using folding techniques in order to reduce the problem effectively to general linear groups. For details of the discussion and a proof of Theorem 5.35 we refer to [**18**].

CHAPTER 6

Parabolic Groups of Higher Modality

This chapter complements our preceeding discussion and provides some explicit results concerning parabolic groups of higher modality.

6.1. Lower Bounds for Modality

The aim of this section is to discuss lower bounds for the modality of parabolics of classical groups. These are constructed by means of Lemma 4.1, Theorem 3.10, and other results from Section 3.1. Throughout this chapter, G is a connected simple algebraic group and $r = \operatorname{rank} G$. In case $\operatorname{char} k$ is zero parts of these results appeared in [**70**].

PROPOSITION 6.1. *Let G be classical. Then we have*

$$\operatorname{mod} B \geq f(r),$$

where $f \in \mathbb{Q}[t]$ has degree 2. Here f may be chosen as in Table 6.1, depending on the type of G.

Type of G	$f(r)$
A_r	$(r^2 - 4r)/12$
B_r	$(r^2 - r - 2)/6$
C_r	$(r^2 - r - 2)/6$
D_r	$(r^2 - 2r - 5)/6$

TABLE 6.1. Lower bounds for $\operatorname{mod} B$

PROOF. In view of Lemma 4.1, Proposition 6.1 follows from the information provided in Table 6.2 below. For, let $\mathfrak{n}$ be the minimal B–submodule of $\mathfrak{b}_u$ containing the root spaces $\mathfrak{g}_\sigma$ relative to the simple roots σ from column 3 of Table 6.2. By Lemma 4.1 we have $\operatorname{mod} B \geq \mu(\mathfrak{n})$. Thus it suffices to calculate the values $\mu(\mathfrak{n})$ for the chosen ideals. It turns out that $\mu(\mathfrak{n})$ is a quadratic polynomial in r. The computations are omitted. For a fixed classical type we choose for $f(r)$ the polynomial $\mu(\mathfrak{n})$ which is minimal for that type. These yield the lower bounds in Table 6.1. $\square$

Type of G	r	$\mathfrak{n}$	$\mu(\mathfrak{n})$
A_r	$6n$	$\sigma_n, \sigma_{3n}, \sigma_{5n}$	$\frac{1}{12}(r^2-4r)$
	$6n+1$	$\sigma_n, \sigma_{3n+1}, \sigma_{5n+2}$	$\frac{1}{12}(r^2-4r+3)$
	$6n+2$	$\sigma_n, \sigma_{3n+1}, \sigma_{5n+2}$	$\frac{1}{12}(r^2-4r+4)$
	$6n+3$	$\sigma_n, \sigma_{3n+1}, \sigma_{5n+3}$	$\frac{1}{12}(r^2-4r+3)$
	$6n+4$	$\sigma_n, \sigma_{3n+2}, \sigma_{5n+4}$	$\frac{1}{12}(r^2-4r)$
	$6n+5$	$\sigma_{n+1}, \sigma_{3n+3}, \sigma_{5n+5}$	$\frac{1}{12}(r^2-4r+7)$
B_r	$6n$	σ_n, σ_{4n}	$\frac{1}{6}(r^2-r)$
	$6n+1$	σ_n, σ_{4n+1}	$\frac{1}{6}(r^2-r)$
	$6n+2$	$\sigma_{n+1}, \sigma_{4n+2}$	$\frac{1}{6}(r^2-r-2)$
	$6n+3$	σ_n, σ_{4n+2}	$\frac{1}{6}(r^2-r)$
	$6n+4$	$\sigma_{n+1}, \sigma_{4n+3}$	$\frac{1}{6}(r^2-r)$
	$6n+5$	$\sigma_{n+1}, \sigma_{4n+4}$	$\frac{1}{6}(r^2-r-2)$
C_r	$3n$	σ_n, σ_{3n}	$\frac{1}{6}(r^2-r)$
	$3n+1$	σ_n, σ_{3n+1}	$\frac{1}{6}(r^2-r)$
	$3n+2$	σ_n, σ_{3n+2}	$\frac{1}{6}(r^2-r-2)$
D_r	$6n$	σ_n, σ_{4n}	$\frac{1}{6}(r^2-2r)$
	$6n+1$	σ_n, σ_{4n+1}	$\frac{1}{6}(r^2-2r-5)$
	$6n+2$	σ_n, σ_{4n+1}	$\frac{1}{6}(r^2-2r)$
	$6n+3$	σ_n, σ_{4n+2}	$\frac{1}{6}(r^2-2r-3)$
	$6n+4$	σ_n, σ_{4n+2}	$\frac{1}{6}(r^2-2r-2)$
	$6n+5$	σ_n, σ_{4n+3}	$\frac{1}{6}(r^2-2r-3)$

TABLE 6.2. Lower bounds for $\operatorname{mod} B$ in classical groups

REMARKS. For G classical, the dimension of B_u (which equals the number of positive roots of G) grows quadratically with the rank of G. Thus the polynomial bounds in Proposition 6.1 are optimal in terms of their degrees.

We extend Proposition 6.1 to parabolics:

PROPOSITION 6.2. *Let G be classical and P a parabolic subgroup of G with $s = \operatorname{rank}_{\mathrm{s}} P$. Then we have*

$$\operatorname{mod} P \geq f(r-s),$$

where $f \in \mathbb{Q}[t]$ has degree 2. Again, f may be chosen as in Table 6.1, according to the type of G.

Proof. The proof consists in an inductive procedure similar to the one encountered in Proposition 4.4. We may assume that P is standard, i.e., $P = P_J$, where $J \subseteq \Pi$. The special case when J is empty is the one from Proposition 6.1. So we may suppose that J is non-empty. We are going to construct a sequence of simple regular subgroups G_t of G together with parabolic subgroups P_t of G_t for $t \geq 1$ with certain properties. This sequence will be obtained by means of an iteration process whose initial step is defined as follows:

Since J is non-empty, there exists a pair of simple roots which are adjacent in the Dynkin diagram of G, where precisely one of them is in J. We fix such a pair which is either of the form $\{\sigma_i, \sigma_{i+1}\}$ for some $i < r$, or $\{\sigma_{r-2}, \sigma_r\}$ in case G is of type D_r. For the definition of G_1 we distinguish three cases:

(i) For G of type A_r, B_r, or C_r and $i < r-1$, or D_r and $i < r-2$, let G_1 be the connected simple regular subgroup of G defined by the set of 'simple' roots

$$\Pi(G_1) := \{\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_r\}.$$

(ii) If G is of type C_r and $i = r-1$, then let G_1 be the connected simple regular subgroup of G defined by

$$\Pi(G_1) := \{\sigma_1, \ldots, \sigma_{r-2}, 2\sigma_{r-1} + \sigma_r\}.$$

(iii) In case G is of type D_r and the chosen pair of consecutive simple roots is either $\{\sigma_{r-2}, \sigma_{r-1}\}$ or $\{\sigma_{r-2}, \sigma_r\}$ we let G_1 be the connected simple regular subgroup of G given by

$$\Pi(G_1) := \{\sigma_1, \ldots, \sigma_{r-2} + \sigma_{r-1}, \sigma_{r-2} + \sigma_r\}.$$

In each case the subgroup G_1 is of the same classical type as that of G and

$$\operatorname{rank} G_1 = \operatorname{rank} G - 1.$$

Define $P_1 := P \cap G_1$. As, by construction, only one of the two chosen simple roots is in J, we see that either

$$\operatorname{rank} G_1 - \operatorname{rank}_{\mathrm{s}} P_1 = \operatorname{rank} G - \operatorname{rank}_{\mathrm{s}} P, \quad \text{or} \tag{10}$$

$$\operatorname{rank} G_1 - \operatorname{rank}_{\mathrm{s}} P_1 = \operatorname{rank} G - \operatorname{rank}_{\mathrm{s}} P + 1. \tag{11}$$

The latter equality only occurs in case (iii) above when $\{\sigma_{r-1}, \sigma_r\} \subseteq J$. If P_1 is not a Borel subgroup of G_1, we may repeat the same procedure now with G_1 in place of G, etc. Iterating this process defines the desired sequence of simple subgroups G_t of rank $r-t$ and corresponding parabolic subgroups $P_t := G_t \cap P$ for $t \geq 1$. This procedure stops once we have arrived at the standard Borel subgroup of the corresponding simple subgroup. It follows from (10) and (11) that the length of our sequence of simple subgroups G_t is either s or $s-1$. The later case, illustrated by an example below, occurs precisely when G is of type D_r and $\{\sigma_{r-1}, \sigma_r\} \subseteq J$. In the first case set $H := G_s$ and $Q := P_s$, while in the second we let H be the simple regular subgroup of G_{s-1} corresponding to the usual subsystem of type D_{r-s} and we set $Q := H \cap P$.

Observe that G_{t-1} is the derived subgroup of a Levi subgroup of G_t for each $t \geq 1$, setting $G_0 := G$. Thus Theorem 3.8 applies to each consecutive pair in this sequence.

Since H is again a classical simple group, Q a Borel subgroup of H, and $\operatorname{rank} H = r - s$, using Theorem 3.8 and Proposition 6.1 we infer by induction that

$$\operatorname{mod} P \geq \operatorname{mod} Q \geq f(r-s)$$

for some $f \in \mathbb{Q}[t]$ from Table 6.1 according to the type of H. □

We demonstrate the procedure in the proof of Proposition 6.2 with two examples in Figure 6.1 below.

Examples 6.3. In our first example in Figure 6.1 the group G is of type B_7 and $\operatorname{rank}_{\mathrm{s}} P = 3$. As before the colored nodes label the simple roots in the standard Levi subgroup of P. We indicate a regular embedding of G_3 in G. Here $H = G_3$. This example describes the generic situation (10), while the second one shows an instance when (11) applies.

Among the bounds in Table 6.1 the one for type A_r is minimal (for $r \geq 4$). Thus we may formulate a uniform lower bound for $\operatorname{mod} P$ independent of the type of G:

Corollary 6.4. *Suppose G and P are as in Proposition 6.2. Then we have*

$$\operatorname{mod} P \geq \frac{1}{12}((r-s)^2 - 4(r-s)).$$

From Corollary 6.4 we immediately derive

Corollary 6.5. *Let G be classical. Suppose G admits a parabolic subgroup P with $\operatorname{rank}_{\mathrm{s}} P = s$ and $\operatorname{mod} P = m$. Then we have*

$$\operatorname{rank} G \leq 2\sqrt{3m+1} + s + 2.$$

Since there is only a finite number of isomorphism classes of exceptional algebraic groups, we readily conclude a finiteness result due to V.L. Popov [**57**, §2].

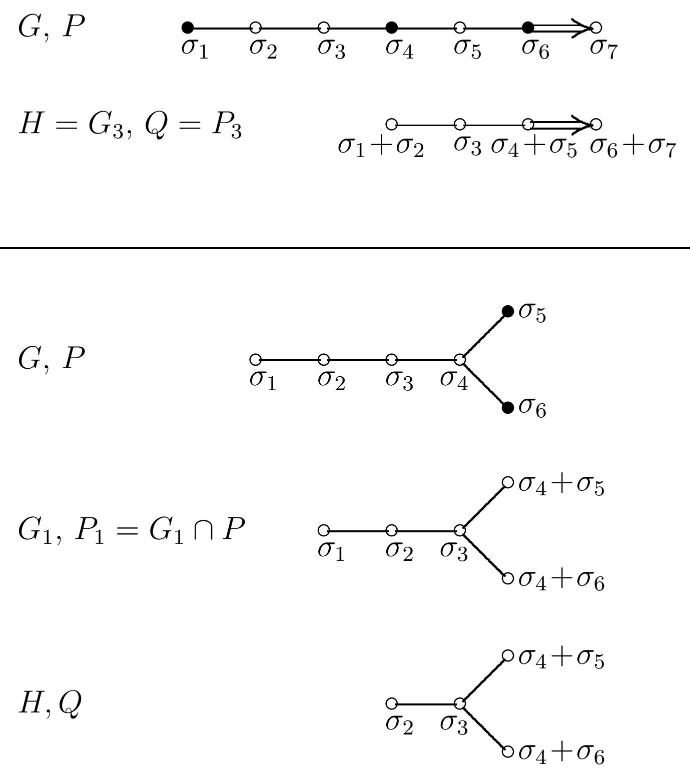


FIGURE 6.1. Illustrating the proof of Proposition 6.2

COROLLARY 6.6. *There is only a finite number of simple algebraic groups (up to isomorphism) admitting parabolic subgroups with fixed semisimple rank and prescribed modality.*

As in the case of Borel subgroups, the degree of the lower bound $f(r-s)$ in Proposition 6.2 is optimal, since $\dim P_u$ is a quadratic polynomial in r for fixed s.

6.2. Some Explicit Examples

It follows from work in [**35, 36**] that for G of type A_r for $r \leq 10$, B_r for $r \leq 6$, or D_r, C_r for $r \leq 7$, the bounds given in Table 6.2 are also upper bounds for $\operatorname{mod} B$. Thus we have $\operatorname{mod} B = \mu(\mathfrak{n})$ in these instances. We list these cases in Table 6.3 below together with the ideals $\mathfrak{n}$ from Table 6.2. For G of type A_r, for $r \leq 7$, B_3, B_4, and C_3, the modality of Borel subgroups can also be determined from the information in Table 1 in [**20**]. The upper bounds for $\operatorname{mod} B$ were computed by MOP or its precursor, cf. [**35**].

Type of G	$\mathfrak{n}$	$\dim \mathfrak{n}$	$\operatorname{mod} B$
A_5	$\sigma_1, \sigma_3, \sigma_5$	13	1
A_6	$\sigma_1, \sigma_3, \sigma_5$	18	1
A_7	$\sigma_1, \sigma_4, \sigma_7$	22	2
A_8	$\sigma_1, \sigma_4, \sigma_7$	29	3
A_9	$\sigma_1, \sigma_4, \sigma_8$	35	4
A_{10}	$\sigma_1, \sigma_5, \sigma_9$	42	5
B_3	σ_2	7	1
B_4	σ_1, σ_3	14	2
B_5	σ_1, σ_4	21	3
B_6	σ_1, σ_4	29	5
C_3	σ_1, σ_3	8	1
C_4	σ_1, σ_4	13	2
C_5	σ_1, σ_5	19	3
C_6	σ_2, σ_6	29	5
C_7	σ_2, σ_7	38	7
D_4	σ_2	9	1
D_5	σ_3	15	2
D_6	σ_1, σ_4	25	4
D_7	σ_1, σ_5	34	5

TABLE 6.3. Modality of B for classical G

Type of G	$\mathfrak{n}$	$\dim \mathfrak{n}$	$\operatorname{mod} B$
G_2	σ_1	5	1
F_4	σ_2	20	4
E_6	σ_4	29	5
E_7	σ_5	50	≥ 10
E_8	σ_2	92	≥ 20

TABLE 6.4. Modality of B for exceptional G

We emphasize that even in type A_r it is not known whether $\operatorname{mod} B$ is in fact a polynomial in r as suggested by the information in Table 6.2.

REMARK 6.7. The modality of a Borel subgroup of a simple group G of small rank is given as in Tables 6.3 and 6.4. The fact that the lower bounds $\mu(\mathfrak{n})$ are also upper bounds can be checked directly with MOP or follow from earlier work in [**35**]. In these tables we record those roots σ such that $\mathfrak{n}$ is the minimal B–invariant submodule of $\mathfrak{b}_u$ containing the root spaces $\mathfrak{g}_\sigma$. For E_7 and E_8 the modality of B is not known; we only have lower bounds, as shown.

Next we give some examples where MOP is used to calculate the modality of parabolic groups $P(\neq B)$ in some exceptional groups.

PROPOSITION 6.8. *Let G be simple and $P \subseteq G$ parabolic. Suppose* $\operatorname{char} k$ *is not a bad prime for G. Then* $\operatorname{mod} P = 1$ *provided one of*

(i) *G is of type E_6 and P is of type A_1A_2, or A_2^2;*
(ii) *G is of type E_7 and P is of type A_4 or D_4.*

PROOF. The fact that $\operatorname{mod} P \geq 1$ in each of the instances shown follows from Proposition 4.5. MOP can be employed directly to show that $\operatorname{mod} P \leq 1$ in each case. $\square$

REMARK 6.9. It follows from Proposition 6.8 and [**4**, Prop. 6.2, 6.3] that any two associated parabolic subgroups of G as in the statement of Proposition 6.8 have the same modality.

Further explicit modality computations in E_6 obtained with the aid of MOP and more instances of associated parabolic subgroups with matching higher modality can be found in [**36**, Table 1].

CHAPTER 7

Abelian Ideals

Throughout this chapter, G denotes a (connected) reductive complex algebraic group with Lie algebra $\operatorname{Lie} G = \mathfrak{g}$. Let B be a Borel subgroup of G. The group B acts on any ideal of $\operatorname{Lie} B = \mathfrak{b}$ by means of the adjoint representation. In this chapter we present the results from [**56**] where we study the relationship between spherical nilpotent orbits and abelian ideals $\mathfrak{a}$ of $\mathfrak{b}$, using the structure theory for these orbits from [**55**]. Our chief result is that, for an abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$, any nilpotent orbit meeting $\mathfrak{a}$ is a spherical G–variety, see Theorem 7.3.

As a consequence of this we obtain a short conceptual proof of a finiteness theorem from [**72**, Thm. 1.1]. Namely, for a parabolic subgroup P of G and an abelian ideal $\mathfrak{a}$ of $\mathfrak{p}$ in the nilpotent radical $\mathfrak{p}_u$, the group P operates on $\mathfrak{a}$ with finitely many orbits. The proof of this fact in [**72**] involved long and tedious case by case considerations.

In Proposition 7.7 we also prove a partial converse to the result just mentioned. We say that an ideal of $\mathfrak{b}$ is ad–*nilpotent* whenever it consists of nilpotent elements. In case G is simply laced, we show that an ad–nilpotent ideal $\mathfrak{c}$ of $\mathfrak{b}$ is abelian provided any nilpotent orbit meeting $\mathfrak{c}$ is spherical.

Let $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ be a $\mathbb{Z}$–grading of $\mathfrak{g}$. The largest integer n so that $\mathfrak{g}(n) \neq \{0\}$ is called the *height* of the grading. In this context write $\Psi(i)$ instead of $\Psi(\mathfrak{g}(i))$ for each $i \in \mathbb{Z}$. It is well-known that $\mathfrak{g}(0)$ is reductive, e.g., see [**93**]. By $W(0)$ we denote the Weyl group of $\mathfrak{g}(0)$.

A grading is said to be *standard* if $\bigoplus_{i>0} \mathfrak{g}(i)$ is contained in $\mathfrak{b}_u$. Any choice of a standard parabolic subgroup P of G canonically defines a standard $\mathbb{Z}$–grading of $\mathfrak{g}$ as follows. Let $P = LP_u$ be the Levi decomposition of P with standard Levi subgroup L. Let $\Pi(L)$ be the set of simple roots of L. Define the function $d : \Psi \longrightarrow \mathbb{Z}$ by setting $d(\sigma) := 0$ if σ is in $\Pi(L)$ and $d(\sigma) := 1$ if σ is in $\Pi \setminus \Pi(L)$, and extend d linearly to all of Ψ. Then for $i \neq 0$ we define $\mathfrak{g}(i) := \bigoplus\limits_{d(\alpha)=i} \mathfrak{g}_\alpha$ and $\mathfrak{g}(0) := \mathfrak{t} \oplus \bigoplus\limits_{d(\alpha)=0} \mathfrak{g}_\alpha$. Thus we have $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$ and moreover, $\mathfrak{l} = \mathfrak{g}(0)$, $\mathfrak{p} = \bigoplus\limits_{i\geq 0} \mathfrak{g}(i)$, and $\mathfrak{p}_u = \bigoplus\limits_{i>0} \mathfrak{g}(i)$. Clearly, $d(\varrho) = \sum_{\sigma\in\Pi(L)} n_\sigma$ is the height of this grading.

7.1. Abelian Ideals and Spherical Orbits

A nilpotent orbit (conjugacy class) $\mathcal{O}$ in $\mathfrak{g}$ is said to be *spherical* whenever it is a spherical G–variety, that is B acts on it with an open orbit. Thus, by a fundamental theorem, due to M. Brion [**12**] and E.B. Vinberg [**91**] independently, B acts on $\mathcal{O}$ with a finite number of orbits. Since $\mathcal{O}$ is quasi-affine, it is spherical if and only if the algebra of polynomial functions $\mathbb{C}[\mathcal{O}]$ is a *multiplicity free* G–module [**92**]. The following characterization of spherical nilpotent orbits can be found in [**53**, §3.1] and [**55**, Thm. 3.2].

Theorem 7.1. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. The following statements are equivalent:*

(i) *$\mathcal{O}$ is spherical;*
(ii) *$(\operatorname{ad} x)^4 = 0$ for every $x \in \mathcal{O}$;*
(iii) *$\mathcal{O}$ contains a representative of the form $e_{\alpha_1} + \cdots + e_{\alpha_t}$, where $\{\alpha_1, \dots, \alpha_t\} \subseteq \Pi$ is a set of mutually orthogonal* simple *roots.*

It is not hard to prove that the number t in Theorem 7.1(iii) does not depend on the choice of a representative for $\mathcal{O}$. Also, the number of long and short roots among the α_i's is an invariant of the orbit. This property means that a minimal Levi subalgebra of $\mathfrak{g}$ meeting $\mathcal{O}$ is the sum of t copies of $\mathfrak{sl}_2$. This subalgebra is unique up to conjugation. If $\{\alpha_1, \dots, \alpha_t\}$ consists of s short and l long roots, then we say that $\mathcal{O}$ *is of type* $s\widetilde{A}_1 + lA_1$. This notation is consistent with the one used for denoting nilpotent orbits in the exceptional Lie algebras [**24, 27**].

The equivalence between parts (i) and (ii) of Theorem 7.1 is proved in [**53**, §3.1]. There it is shown *a priori* that whenever $(\operatorname{ad} x)^3 = 0$, then $\mathcal{O}$ is spherical and also when $(\operatorname{ad} x)^4 \neq 0$, then $\mathcal{O}$ is not spherical. Case by case considerations are only required to show that $\mathcal{O}$ is spherical if $(\operatorname{ad} x)^4 = 0$ and $(\operatorname{ad} x)^3 \neq 0$ for every $x \in \mathcal{O}$.

Making use of Theorem 7.1, we set up a direct link between the abelian ideals in $\mathfrak{b}$ and spherical nilpotent orbits. It is easy to show that any abelian ideal $\mathfrak{a} \subset \mathfrak{b}$ contains no semisimple elements, that is $\mathfrak{a} \subset \mathfrak{b}_u$. Therefore, such an $\mathfrak{a}$ is completely determined by the corresponding subset $\Psi(\mathfrak{a})$ of Ψ.

Proposition 7.2. *Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{b}$ and let $\mu_i \in \Psi(\mathfrak{a})$ for $i = 1, \dots, 4$. Define the operator $\Upsilon : \mathfrak{g} \longrightarrow \mathfrak{g}$ by $\Upsilon := \prod_{i=1}^{4} \operatorname{ad} e_{\mu_i}$. Then $\Upsilon \equiv 0$.*

Proof. Since $\mathfrak{a}$ is abelian, Υ does not depend on the ordering of the μ_i's.

1. We first show that Υ annihilates the lowest weight space of $\mathfrak{g}$, i.e., $\Upsilon\, e_{-\varrho} = 0$.

Assume this is not the case. Then $[e_{\mu_i}, e_{-\varrho}] \neq 0$ and hence $(\mu_i, \varrho) > 0$ for each i (since ϱ is long). More precisely, $(\mu_i, \varrho^\vee) = 2$ in case $\mu_i = \varrho$ and otherwise $(\mu_i, \varrho^\vee) = 1$. Since $\Upsilon\, e_{-\varrho} \in \mathfrak{g}_{-\varrho+\mu_1+\cdots+\mu_4}$ and $(-\varrho+\mu_1+\cdots+\mu_4, \varrho^\vee) \leq 2$, the only possibility is that $(\mu_i, \varrho^\vee) = 1$ for each $i = 1, \ldots, 4$ and therefore we have $-\varrho+\mu_1+\cdots+\mu_4 = \varrho$; that is,

$$2\varrho = \mu_1 + \cdots + \mu_4. \tag{12}$$

Observe also that $\operatorname{ad} e_{\mu_i} \operatorname{ad} e_{\mu_j}(e_{-\varrho}) \neq 0$ for $i \neq j$ and, since $\mu_i + \mu_j$ is not a root, we have $\varrho - \mu_i - \mu_j \in \Psi$. It follows from (12) that

$$\sum_{1\leq i<j\leq 4} (\varrho - \mu_i - \mu_j) = 6\varrho - 3(\mu_1 + \cdots + \mu_4) = 0.$$

Therefore, the set $\{\varrho - \mu_i - \mu_j\}_{i,j}$ contains a positive root. Without loss, we may suppose that $\varrho - \mu_1 - \mu_2 \in \Psi^+$. Then $\varrho - \mu_1 = (\varrho - \mu_1 - \mu_2) + \mu_2 \in \Psi(\mathfrak{a})$, since $\mathfrak{a}$ is an ideal in $\mathfrak{b}$. Thus both, μ_1 and $\varrho - \mu_1$ are in $\Psi(\mathfrak{a})$ contradicting the fact that $\mathfrak{a}$ is abelian. Consequently, we have $\Upsilon e_{-\varrho} = 0$, as claimed.

2. Here we show that $\Upsilon e_\gamma = 0$ for all remaining $\gamma \in \Psi \cup \{0\}$. (If $\gamma = 0$, then e_γ stands for an arbitrary element in $\mathfrak{t}$.) We argue by induction on the sum of the coefficients of the simple roots of the difference $\gamma - (-\varrho) = \sum_\sigma k_\sigma \sigma$ ($\sigma \in \Pi$), i.e., on $\sum_\sigma k_\sigma$. The case when this sum is zero is just the one studied in part 1 above. Suppose that $e_\gamma = [e_\sigma, x]$, where $\sigma \in \Pi$ and either $x = e_{\gamma'}$ for some $\gamma' \in \Psi$ (such an equality exists provided $\gamma \neq -\varrho$), or, in the case $\gamma = \sigma$ is simple, we may choose a suitable element $x \in \mathfrak{t}$ that satisfies this relation. By Υ_i we denote the operator corresponding to the quadruple of roots where μ_i is replaced by $\mu_i + \sigma$. (If $\mu_i + \sigma \notin \Psi$, then $\Upsilon_i \equiv 0$.) One checks that

$$\Upsilon\, e_\gamma = [e_\sigma, \Upsilon\, x] + \sum_{i=1}^{4} \Upsilon_i\, x.$$

By induction assumption for the operators Υ and Υ_i, we have $\Upsilon\, x = 0$ and $\Upsilon_i\, x = 0$. Thus $\Upsilon\, e_\gamma = 0$, as desired. $\square$

THEOREM 7.3. *If $\mathfrak{a}$ is an abelian ideal in $\mathfrak{b}$, then any G–orbit meeting $\mathfrak{a}$ is spherical and $G \cdot \mathfrak{a}$ is the closure of a spherical nilpotent orbit.*

PROOF. If $x = \sum e_{\mu_i} \in \mathfrak{a}$, then $(\operatorname{ad} x)^4$ is the sum of operators of the form described in Proposition 7.2. Therefore, $(\operatorname{ad} x)^4 = 0$, and thus $G \cdot x$ is spherical, by Theorem 7.1. Because $G \cdot \mathfrak{a}$ is irreducible and the number of nilpotent orbits is finite, $G \cdot \mathfrak{a}$ is the closure of a single nilpotent orbit. $\square$

COROLLARY 7.4. *Let $\mathfrak{a}$ be an abelian ideal in $\mathfrak{b}$. Then B has finitely many orbits in $\mathfrak{a}$.*

PROOF. The desired finiteness follows readily from Theorem 7.3 and the finiteness property for spherical varieties. ☐

We obtain [**72**, Thm. 1.1] as an immediate consequence of Corollary 7.4:

COROLLARY 7.5. *Let P be a parabolic subgroup of G and let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{p}$ in $\mathfrak{p}_u$. Then P acts on $\mathfrak{a}$ with finitely many orbits.*

PROOF. Observe that $\mathfrak{a} \subseteq \mathfrak{p}_u \subseteq \mathfrak{b}_u$ is also an ideal of $\mathfrak{b}$. Thus, by Corollary 7.4, B acts on $\mathfrak{a}$ with a finite number of orbits and thus, so does P. ☐

REMARKS 7.6. The particular case when $\mathfrak{a}$ is in the center of $\mathfrak{p}_u$ is well-known. Then the action factors through a Levi subgroup of P. Here the finiteness follows from a result of E.B. Vinberg [**90**, §2] (see also V.G. Kac [**39**] or R.W. Richardson [**63**, §3]). For a detailed account of the orbit structure in this situation, see [**51**] or [**65**, §2, §5].

Observe that for abelian P–invariant sub-factors in $\mathfrak{p}_u$, the analogous statement of Corollary 7.5 is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on $\mathfrak{p}_u$, see Chapter 4. Examples in this context also show that a parabolic subgroup may have an infinite number of orbits on ideals in $\mathfrak{p}_u$ of nilpotency class two.

Corollary 7.5 was first proved in [**72**] in a long case by case analysis. More specifically, it was shown in [**72**] that for A a closed normal unipotent subgroup of P the number of P–orbits on A is finite provided A is abelian; the proof in [**72**] is valid in arbitrary characteristic.

EXAMPLE. Abelian ideals of $\mathfrak{b}$ are readily constructed by means of gradings. Let $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ be a standard $\mathbb{Z}$–grading of $\mathfrak{g}$ of height d. Define $m := [d/2]+1$ and set $\mathfrak{a} := \bigoplus_{i \geq m} \mathfrak{g}(i)$. Then $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$. Obviously, m is the least possible value ensuring that $\bigoplus_{i \geq m} \mathfrak{g}(i)$ is abelian. Therefore, any nilpotent orbit in $\mathfrak{g}$ meeting $\mathfrak{a}$ is spherical. In the context of gradings this can be derived by a shorter argument than the one used in the proof of Proposition 7.2. For, let x be in $\mathfrak{a}$. As the components of x have degree at least m, we have

$$(\operatorname{ad} x)^4 \mathfrak{g}(j) \subseteq \bigoplus_{i \geq j+4m} \mathfrak{g}(i) = \{0\}$$

for each $j \in \mathbb{Z}$. Consequently, $(\operatorname{ad} x)^4 \equiv 0$ on all of $\mathfrak{g}$.

We close this section with a partial converse to Theorem 7.3.

PROPOSITION 7.7. *Suppose G is simply laced. Let $\mathfrak{c}$ be an* ad–*nilpotent ideal of $\mathfrak{b}$ such that any nilpotent orbit meeting $\mathfrak{c}$ is spherical. Then $\mathfrak{c}$ is abelian.*

Proof. Suppose $\mathfrak{c}$ is not abelian. Then there exist $\alpha, \beta, \gamma \in \Psi(\mathfrak{c})$ so that $[e_\alpha, e_\beta] = e_\gamma$. By the assumption on G, the roots α and β span a subsystem of Ψ of type A_2. Let H be the corresponding simple subgroup of G of type A_2. Then $x := e_\alpha + e_\beta$ is regular nilpotent in $\mathfrak{h}$. By direct matrix calculation, one obtains $(\operatorname{ad}_{\mathfrak{h}} x)^4 \not\equiv 0$. Consequently, $(\operatorname{ad} x)^4 \not\equiv 0$ on all of $\mathfrak{g}$. It follows from Theorem 7.1 that the corresponding nilpotent orbit in $\mathfrak{g}$ is not spherical, a contradiction. □

Remark. It is worth noting that Proposition 7.7 is false if G has two root lengths. For instance, let G be of type C_r $(r \geq 2)$ and let P be the stabilizer of the 1-dimensional space $\mathfrak{g}_\varrho$. Then P is parabolic and $\mathfrak{p}_u$ is the Heisenberg Lie algebra of dimension $2r-1$, which is not abelian. We have, however, $(\operatorname{ad} x)^4 = 0$ for all $x \in \mathfrak{p}_u$.

7.2. Maximal Abelian Ideals

Throughout this section suppose that G is simple. We recall the classification of the maximal abelian ideals of $\mathfrak{b}$ from [**72**] and record it in Tables 7.1 and 7.2 below.

Theorem 7.8. *Every maximal abelian ideal of* $\operatorname{Lie} B = \mathfrak{b}$ *is listed in Tables 7.1 and 7.2.*

The fact that each ideal $\mathfrak{a}$ listed in these tables is abelian follows from the observation that the sum of any two roots in $\Psi(\mathfrak{a})$ is not a root, because it exceeds ϱ in some coefficient. The fact that each of these ideals is maximal among the abelian ones and that this list is complete consists of a detailed case by case analysis.

The proof of Theorem 7.8 from [**72**], involving case by case considerations, is rather unsatisfactory. It would be very desirable to have a uniform proof of this result.

Remarks. We are going to explain the various pieces of notation in Tables 7.1 and 7.2 associated to each maximal abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$. In the second column we specify the set of generating roots $\Gamma_{\mathfrak{a}}$ for $\mathfrak{a}$, that is $\Gamma_{\mathfrak{a}}$ is the minimal set of roots α such that $\mathfrak{a}$ is the smallest $\mathfrak{b}$-submodule of $\mathfrak{b}_u$ containing the root spaces $\mathfrak{g}_\alpha$. The simple roots σ_i are labeled as in [**11**]. We abbreviate some roots as follows: in type B_r set $\beta_i = \sigma_1 + \cdots + \sigma_i$ and $\gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_r$, where $2 \leq i \leq r$. Similarly, for type D_r we define $\beta_i = \sigma_1 + \cdots + \sigma_i$ and $\gamma_i = \sigma_{i-1} + 2\sigma_i + \cdots + 2\sigma_{r-2} + \sigma_{r-1} + \sigma_r$ for $3 \leq i \leq r-2$, also $\beta = \beta_{r-2} + \sigma_{r-1}$, $\gamma = \beta_{r-2} + \sigma_r$, and $\delta = \sigma_{r-2} + \sigma_{r-1} + \sigma_r$.

G	$\Gamma_{\mathfrak{a}}$	$P_{\mathfrak{a}}$	$\dim \mathfrak{a}$	$d_{\mathfrak{a}}$	$\sigma_{\mathfrak{a}}$
A_r	$\sigma_i \ \ (1 \leq i \leq r)$	σ_i	$i(r-i+1)$	1	σ_i
B_r	σ_1	σ_1	$2r-1$	1	σ_1
	$\beta_i, \gamma_i \ \ (3 \leq i \leq r)$	σ_1, σ_i	$(4r+i^2-5i+2)/2$	3	σ_{i-1}
C_r	σ_r	σ_r	$(r^2+r)/2$	1	σ_r
D_r	σ_1	σ_1	$2r-2$	1	σ_1
	σ_{r-1}/σ_r	σ_{r-1}/σ_r	$(r^2-r)/2$	1	σ_{r-1}/σ_r
	$\beta_i, \gamma_i \ (3 \leq i \leq r-2)$	σ_1, σ_i	$(4r-5i+i^2)/2$	3	σ_{i-1}
	β, γ, δ	$\sigma_1, \sigma_{r-1}, \sigma_r$	$(r^2-3r+6)/2$	3	σ_{r-2}

TABLE 7.1. The maximal abelian ideals of $\mathfrak{b}$ for classical $\mathfrak{g}$

The normalizer of $\mathfrak{a}$ in G is a parabolic subgroup of G, since it contains B. In the third column of the tables we indicate the standard Levi subgroup $L_{\mathfrak{a}}$ of $P_{\mathfrak{a}} := N_G(\mathfrak{a})$ by listing the complementary simple roots $\Pi \setminus \Pi(L_{\mathfrak{a}})$.

In the next two columns we list $\dim \mathfrak{a}$ and $d_{\mathfrak{a}} := d(\varrho)$, the height of the grading afforded by $P_{\mathfrak{a}}$, respectively.

It follows from Theorem 7.8 that the number of maximal abelian ideals of $\mathfrak{b}$ equals the number of long simple roots of G. In Theorem 7.10 we define a canonical bijection between these two sets. The simple root $\sigma_{\mathfrak{a}}$ corresponding to $\mathfrak{a}$ under this bijection is indicated in column 6 of the tables.

Since $\mathfrak{a}$ is an irreducible subvariety of $\mathfrak{b}_u$, there exists a unique nilpotent orbit $\mathcal{O}_{\mathfrak{a}}$ such that $\mathcal{O}_{\mathfrak{a}} \cap \mathfrak{a}$ is dense in $\mathfrak{a}$. In the last column of Table 7.2 we present the label of $\mathcal{O}_{\mathfrak{a}}$ following the labeling of the nilpotent classes according to E.B. Dynkin [**27**], see also [**24**].

Using the description of $P_{\mathfrak{a}}$ furnished in the third column in Tables 7.1 and 7.2, the height $d_{\mathfrak{a}} = d(\varrho)$ of the grading afforded by $P_{\mathfrak{a}}$ is readily determined. Note that $d_{\mathfrak{a}}$ is always odd and for $m = [d_{\mathfrak{a}}/2] + 1$ we have $\mathfrak{a} = \bigoplus_{i \geq m} \mathfrak{g}(i)$. According to Theorem 7.3, the orbit $\mathcal{O}_{\mathfrak{a}}$ is always spherical. If the label of $\mathcal{O}_{\mathfrak{a}}$ is $s\widetilde{A}_1 + lA_1$, then the sum $s + l$ is the number t from Theorem 7.1(iii). It is also possible to determine the labeling of the weighted Dynkin diagram defining $\mathcal{O}_{\mathfrak{a}}$.

By Theorem 7.8, the number of maximal abelian ideals equals the number of long simple roots of $\mathfrak{g}$. This numerical coincidence suggests that there should exist a canonical one-to-one correspondence between these two sets. We show that this correspondence can be obtained in an axiomatic way. It is presented in column 6 of Tables 7.1 and 7.2.

G	$\Gamma_{\mathfrak{a}}$	$P_{\mathfrak{a}}$	$\dim \mathfrak{a}$	$d_{\mathfrak{a}}$	$\sigma_{\mathfrak{a}}$	$\mathcal{O}_{\mathfrak{a}}$
E_6	σ_1/σ_6	σ_1/σ_6	16	1	σ_1/σ_6	$2A_1$
	$01\underset{1}{2}10$	σ_4	11	3	σ_2	$3A_1$
	$11\underset{0}{1}10,\ 01\underset{1}{2}21$	σ_1, σ_5	13	3	σ_3	$3A_1$
	$01\underset{0}{1}11,\ 12\underset{1}{2}10$	σ_3, σ_6	13	3	σ_5	$3A_1$
	$11\underset{0}{1}11,\ 01\underset{1}{2}11,\ 11\underset{1}{2}10$	$\sigma_1, \sigma_4, \sigma_6$	12	5	σ_4	$3A_1$
E_7	σ_7	σ_7	27	1	σ_7	$[3A_1]''$
	$12\underset{1}{2}100$	σ_3	17	3	σ_1	$[3A_1]'$
	$01\underset{1}{2}210$	σ_5	20	3	σ_2	$[3A_1]'$
	$00\underset{1}{1}111, 12\underset{2}{3}210$	σ_2, σ_7	22	3	σ_6	$4A_1$
	$01\underset{1}{2}221, 12\underset{1}{2}110$	σ_3, σ_6	18	5	σ_3	$[3A_1]'$
	$01\underset{1}{2}111, 12\underset{1}{3}210$	σ_4, σ_7	20	5	σ_5	$4A_1$
	$01\underset{1}{2}211, 12\underset{1}{2}210, 12\underset{1}{2}111$	$\sigma_3, \sigma_5, \sigma_7$	19	7	σ_4	$4A_1$
E_8	$01\underset{1}{2}2221$	σ_7	29	3	σ_8	$3A_1$
	$12\underset{2}{3}2100$	σ_2	36	3	σ_1	$4A_1$
	$12\underset{1}{3}3210$	σ_5	34	5	σ_2	$4A_1$
	$11\underset{1}{2}2221,\ 23\underset{2}{4}3210$	σ_1, σ_7	30	5	σ_7	$4A_1$
	$12\underset{1}{2}2221,\ 13\underset{2}{4}3210$	σ_3, σ_7	31	7	σ_6	$4A_1$
	$12\underset{1}{3}3321,\ 12\underset{2}{3}2210$	σ_2, σ_6	34	7	σ_3	$4A_1$
	$12\underset{1}{3}2221,\ 12\underset{2}{4}3210$	σ_4, σ_7	32	9	σ_5	$4A_1$
	$12\underset{1}{3}3221, 12\underset{2}{3}2221,\ 12\underset{2}{3}3210$	$\sigma_2, \sigma_5, \sigma_7$	33	11	σ_4	$4A_1$
F_4	1220	σ_2	8	3	σ_1	$\widetilde{A}_1 + A_1$
	1221, 0122	σ_2, σ_4	9	5	σ_2	$\widetilde{A}_1 + A_1$
G_2	21	σ_1	3	3	σ_2	$\widetilde{A}_1$

TABLE 7.2. The maximal abelian ideals for exceptional $\mathfrak{g}$

Let $\Delta := \Delta(\mathfrak{g})$ be the Dynkin diagram of $\mathfrak{g}$. We identify the nodes of Δ with the simple roots Π of $\mathfrak{g}$ and write Δ^σ for the Dynkin diagram which is obtained from Δ by removing $\sigma \in \Pi$ together with the edges linked to it. By $\pi_0(\Delta^\sigma)$ we denote the set of connected components of Δ^σ and by $\Delta^\sigma = \bigcup_c \Delta^\sigma_c$ for $c \in \pi_0(\Delta^\sigma)$ the decomposition of Δ^σ into its components. We write Ψ_c^σ for the root system corresponding to Δ_c^σ and ϱ_c^σ for the highest (long) root in Ψ_c^σ for each c. Observe that if we consider the standard grading of $\mathfrak{g}$ corresponding to $\sigma \in \Pi$, then, using the previous notation, we have $\Psi(0) = \sqcup_c \Psi_c^\sigma$.

Let Π_ℓ denote the set of long simple roots and $\mathcal{A}_{max}$ the set of all maximal abelian ideals in $\mathfrak{b}$. Associated with any $\mathfrak{a} \in \mathcal{A}_{max}$, we have the following data: the set of generators $\Gamma_\mathfrak{a} \subset \Psi^+$ and the height $d_\mathfrak{a}$ of the grading determined by $P_\mathfrak{a} = N_G(\mathfrak{a})$. The following observation giving a more precise form for the equality $\#\Pi_\ell = \#\mathcal{A}_{max}$ is indicative for our construction. Recall the decomposition of ϱ as the sum of simple roots $\varrho = \sum n_\sigma \sigma$ from above. The number of times a fixed integer occurs as the value for $d_\mathfrak{a}$, as $\mathfrak{a}$ varies over $\mathcal{A}_{max}$, equals the number of times it occurs as the expression $2n_\sigma - 1$, as σ runs through Π_ℓ. Therefore, it is just to require that the sought after bijection

$$\psi : \Pi_\ell \longrightarrow \mathcal{A}_{max}, \quad \sigma \mapsto \psi(\sigma) =: \mathfrak{a}_\sigma,$$

does satisfy the condition $d_{\mathfrak{a}_\sigma} = 2n_\sigma - 1$ for each $\sigma \in \Pi_\ell$.

Ideally, starting with a long simple root, an explicit *a priori* procedure should yield the corresponding maximal abelian ideal. Indeed, we are able to state such a construction when $n_\sigma \leq 2$. It is worth noting that this is sufficient to cover all classical instances.

The case when $n_\sigma = 1$ is straightforward. Here the simple root σ (which is always long) determines a grading $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, and we merely set $\mathfrak{a}_\sigma = \mathfrak{g}(1)$. It is easily seen that $\mathfrak{g}(1)$ is a maximal abelian ideal. Notice that in this case $\mathfrak{a}_\sigma$ is the nilpotent radical of the parabolic subalgebra corresponding to σ and $\Gamma_{\mathfrak{a}_\sigma} = \{\sigma\}$.

The case $n_\sigma = 2$ is the subject of the following theorem.

THEOREM 7.9. *Let $\sigma \in \Pi_\ell$ such that $n_\sigma = 2$. Let $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(i)$ be the corresponding $\mathbb{Z}$–grading. Let e_γ be a highest weight vector in the $\mathfrak{g}(0)$–module $\mathfrak{g}(1)$. Then we have*

(i) *$\mathfrak{a}_\sigma := [e_\gamma, \mathfrak{g}(0)] \oplus \mathfrak{g}(2)$ is an abelian ideal in $\mathfrak{b}$;*
(ii) *$\Gamma_{\mathfrak{a}_\sigma} = \{\gamma - \varrho_c^\sigma \mid c \in \pi_0(\Delta^\sigma)\}$; in particular, $\#\Gamma_{\mathfrak{a}_\sigma} = \#\pi_0(\Delta^\sigma)$;*
(iii) *$\mathfrak{a}_\sigma$ is maximal and $d_{\mathfrak{a}_\sigma} = 3$.*

PROOF. (i) Notice that σ and γ are the lowest and highest weight in the $\mathfrak{g}(0)$–module $\mathfrak{g}(1)$, respectively. It follows that γ is $W(0)$–conjugate to σ and therefore γ is long.

It is easily seen that $\mathfrak{a}_\sigma$ is an ideal of $\mathfrak{b}$ in $\mathfrak{b}_u$. Set $V = [e_\gamma, \mathfrak{g}(0)]$. Clearly, $\mathfrak{a}_\sigma$ is abelian if and only if $[V, V] = \{0\}$; that is, if $\mu_1, \mu_2 \in \Psi(V)$, then $\mu_1 + \mu_2$ is not a root. By the definition of V, we have

$\mu_i = \gamma - \beta_i$ for some $\beta_i \in \Psi(0)^+ \cup \{0\}$, $i = 1, 2$. We distinguish various possibilities for β_1 and β_2.

(a) $\beta_1 \neq 0$, $\beta_2 = 0$:
Since γ is long and $\gamma \neq \beta_1$, we have $(\gamma, \gamma) > (\gamma, \beta_1)$. Therefore, $(\gamma, \gamma - \beta_1) > 0$ and hence $\gamma + (\gamma - \beta_1) \notin \Psi$.

(b) $\beta_1 \neq 0$, $\beta_2 \neq 0$:
Since γ is long, the condition $\gamma - \beta_i \in \Psi$ means that $(\gamma, \beta_i) > 0$ and then $(\gamma, \beta_i) = \frac{1}{2}(\gamma, \gamma)$, $i = 1, 2$. Therefore, we have

(∗) $\qquad (\gamma - \beta_1, \beta_2) = \frac{1}{2}(\gamma, \gamma) - (\beta_1, \beta_2) \geq 0$, since $\beta_1 \neq \beta_2$, and

(∗∗) $\qquad (\gamma - \beta_1, \gamma - \beta_2) = (\beta_1, \beta_2)$.

(b_1) At least one of β_1 and β_2, say β_2, is long.
Then $\gamma - \beta_2$ is long as well. Since $\gamma - \beta_1 \in \Psi(1)$ and $\beta_2 \in \Psi(0)^+$, we have $\gamma - \beta_1 \neq \beta_2$ and hence $(\gamma - \beta_1, \beta_2) < (\beta_2, \beta_2) = (\gamma, \gamma)$. It then follows from the equality in (∗) that $(\beta_1, \beta_2) > -\frac{1}{2}(\gamma, \gamma)$ and, consequently, $(\beta_1, \beta_2) \geq 0$. Now using (∗∗), we obtain $(\gamma - \beta_1) + (\gamma - \beta_2) \notin \Psi$, since $\gamma - \beta_2$ is long.

(b_2) Both β_1 and β_2 are short.
Then $|(\beta_1, \beta_2)| \leq \frac{1}{2}(\beta_1, \beta_1) < \frac{1}{2}(\gamma, \gamma)$ and (∗) shows that $(\gamma - \beta_1, \beta_2) > 0$. Therefore, $\gamma - \beta_1 - \beta_2$ is a root in $\Psi(1)$. Since γ is long and $(\gamma, \beta_i) = \frac{1}{2}(\gamma, \gamma)$ for $i = 1, 2$, we conclude that $(\gamma - \beta_1 - \beta_2, \gamma) = 0$ and therefore, $(\gamma - \beta_1 - \beta_2) + \gamma \notin \Psi$.

(ii) Using the notation of part (i), we have $\Psi(\mathfrak{a}_\sigma) = \Psi(V) \cup \Psi(2)$. First we show that none of the generators in $\Gamma_{\mathfrak{a}_\sigma}$ lies in $\Psi(2)$. For this end, it suffices to show that the lowest weight δ in $\Psi(2)$, is not a generator. (Recall that $\mathfrak{g}(2)$ is an irreducible $\mathfrak{g}(0)$–module and therefore δ is uniquely determined in $\Psi(2)$.) Since $\delta - \sigma$ is a root (in $\Psi(1)$), it is enough to show that it lies in $\Psi(V)$. Because γ is the highest weight in $\Psi(1)$, we see that $\gamma + \sigma$ is a root, and hence $(\gamma, \sigma) < 0$. Since $\Psi(3)$ is empty, $\gamma + \delta$ is not a root. Thus, $(\gamma, \delta) \geq 0$ and then $(\gamma, \delta - \sigma) > 0$. This implies that $\gamma - (\delta - \sigma)$ is a root lying in $\Psi(0)^+$. By the very construction of V, this means $\delta - \sigma \in \Psi(V)$, as desired.

Now we consider the elements of $\Psi(V)$. Let w_0 be the longest element in $W(0)$. Then $w_0(\varrho_c^\sigma) = -\varrho_c^\sigma$ for each $c \in \pi_0(\Delta^\sigma)$ and $w_0(\sigma) = \gamma$. Since ϱ_c^σ is the highest root in Ψ_c^σ (but not in Ψ), we have $\varrho_c^\sigma + \sigma \in \Psi$. Hence $w_0(\varrho_c^\sigma + \sigma) = \gamma - \varrho_c^\sigma$ is also a root. According to the construction of part (i), the corresponding root space lies in $\mathfrak{a}_\sigma$. Moreover, since $\{\varrho_c^\sigma\}_c$ are clearly the maximal possible elements of $\Psi(0)^+$ that can be subtracted from γ, i.e., that $\{\gamma - \varrho_c^\sigma\}_c$ are the elements of $\Psi(V)$ of minimal height, we obtain $\{\gamma - \varrho_c^\sigma\}_c \subseteq \Gamma_{\mathfrak{a}_\sigma}$. On the other hand, suppose $\gamma - \mu \in \Psi(V)$, where $\mu \in \Psi(0)^+ \setminus \{\varrho_c^\sigma\}_c$. Then $\mu \in \Psi_c^\sigma$ for some $c \in \pi_0(\Delta^\sigma)$ and hence $\varrho_c^\sigma - \mu$ is a sum of positive roots from Ψ_c^σ and so is $(\gamma - \mu) - (\gamma - \varrho_c^\sigma) = \varrho_c^\sigma - \mu$. Therefore, $\gamma - \mu \notin \Gamma_{\mathfrak{a}_\sigma}$.

(iii) Using the information in Tables 7.1 and 7.2 this is readily verified. □

REMARK. Utilizing Proposition 7.9, we can describe the map ψ in all classical cases.

In the following theorem we axiomatize the properties of this mapping. Whenever ϱ is fundamental (this refers to all simple Lie algebras except for those of type A_r and C_r) there is a unique simple root σ^* such that $(\varrho, \sigma^*) \neq 0$, see [**11**]. Observe that σ^* is always long.

THEOREM 7.10. *There is a unique bijection* $\psi : \Pi_\ell \longrightarrow \mathcal{A}_{max}$ *(*$\mathfrak{a}_\sigma := \psi(\sigma)$*) satisfying the following conditions:*

1. $d_{\mathfrak{a}_\sigma} = 2n_\sigma - 1$.
2. *If* $n_\sigma = 1$, *then* $\Gamma_{\mathfrak{a}_\sigma} = \{\sigma\}$.
3. *If* $n_\sigma = 2$, *then* $\mathfrak{a}_\sigma$ *is defined as in Proposition 7.9.*
4. $\#\Gamma_{\mathfrak{a}_\sigma} = \#\pi_0(\Delta^\sigma)$ *provided* $\mathfrak{g}$ *is not of type* A_r.
5. *Suppose* ϱ *is fundamental. Then for any sequence* $(\sigma^*, \alpha, \beta, \dots)$ *of simple roots, adjacent in* Δ *(and mutually distinct), we have* $\dim \mathfrak{a}_{\sigma^*} < \dim \mathfrak{a}_\alpha < \dim \mathfrak{a}_\beta < \dots$.

PROOF. The proof consists of a case by case argument. One only needs to exploit the second and fifth columns in Tables 7.1 and 7.2. The resulting correspondence is presented in Figures 7.1 and 7.2 where we label each node $\sigma \in \Pi_\ell$ with $\dim \mathfrak{a}_\sigma$. □

Observe that conditions 4 and 5 of Theorem 7.10 follow from the first three for B_r, D_r, E_6, and F_4. In fact, condition 5 is required only to construct ψ for E_7 and E_8.

In the diagrams in Figures 7.1 and 7.2 the marked node indicates the one corresponding to the simple root σ^*. Because there is a unique long simple root in C_r and G_2, these cases are omitted.

REMARKS. We close this chapter by referring to some recent work of B. Kostant [**43, 44**], extending earlier results from [**42**], where the family of *all* abelian ideals $\mathfrak{a}$ of the Borel subalgebra $\mathfrak{b}$ of a simple complex Lie algebra $\mathfrak{g}$ plays an important rôle. Motivated by Mal'cev's work [**50**], Kostant constructs inequivalent irreducible G–submodules in the exterior algebra $\wedge\mathfrak{g}$ of $\mathfrak{g}$, one for each abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ in [**42**], where $\operatorname{Lie} G = \mathfrak{g}$. In his recent summary [**43**], Kostant gives an account of D. Peterson's theorem that the number of abelian ideals in $\mathfrak{b}$ equals 2^r, where $r = \operatorname{rank} \mathfrak{g}$. See also [**56**, §3], where we give a natural bijection between the set of all abelian ideals of $\mathfrak{b}$ and the set

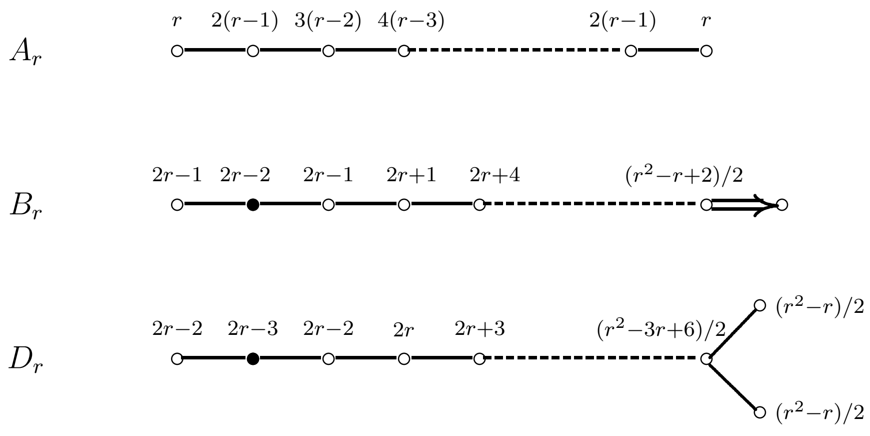


FIGURE 7.1. The function $\sigma \mapsto \dim \mathfrak{a}_\sigma$ for classical $\mathfrak{g}$

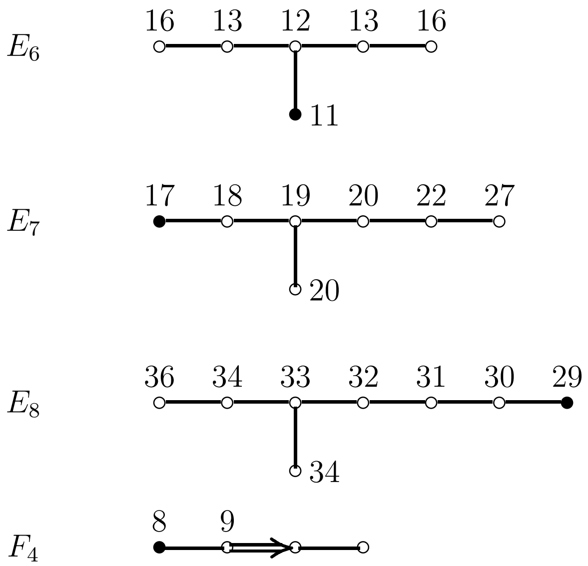


FIGURE 7.2. The function $\sigma \mapsto \dim \mathfrak{a}_\sigma$ for exceptional $\mathfrak{g}$

of standard parabolic subgroups of G in case G is of type A_r or C_r. This bijection is afforded by the canonical map $\mathfrak{a} \mapsto P_{\mathfrak{a}}$ which assigns to each abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ its normalizer $P_{\mathfrak{a}}$. This in particular implies Peterson's theorem in these instances. However, observe that this map fails to be a bijection in all other instances, cf. [**56**, §3].

CHAPTER 8

Appendix: Some Examples of Hasse Diagrams

In this section we present several examples of Hasse diagrams of the Bruhat-Chevalley order of the action of P on $\mathfrak{p}_u$ in general linear groups in some finite cases studied above, see [**19**]. Each individual poset was computed by the method outlined in 5.1.6.

In each of the figures below the vertices indicate the orbits, the labels give their dimensions, and the edges represent the minimal degenerations in the sense of Remark 5.21.

The Hasse diagrams associated to the closure posets of the actions of the Borel subgroup B on $\mathfrak{b}_u$ in the finite instances were determined first by V.V. Kashin [**41**], see Theorem 4.2. We present these in our first three examples; the underlying groups are $\mathrm{GL}_3(k)$, $\mathrm{GL}_4(k)$, and $\mathrm{GL}_5(k)$; there are 5, 16, and 61 orbits, respectively (Figures 8.1 - 8.3). In Figure 8.4 we show the poset of the action of the parabolic subgroup $P(\mathbf{d})$ in $\mathrm{GL}_{10}(k)$ on $\mathfrak{p}_u(\mathbf{d})$, where $\mathbf{d} = (1, 2, 3, 4)$. There are 151 orbits.

In Figures 8.5 and 8.6 we demonstrate an instance of the phenomenon described in Remark 5.9. There we consider two associated, but non-conjugate parabolic subgroups $P(\mathbf{d})$ in $\mathrm{GL}_5(k)$. While for $\mathbf{d} = (2, 1, 2)$, the poset is *rank unimodal*, its counterpart for the permuted tuple $\mathbf{d} = (2, 2, 1)$ is not even *ranked*. This example illustrates that, although the number of orbits in both cases is the same, see Corollary 5.7, the corresponding posets are rather different.

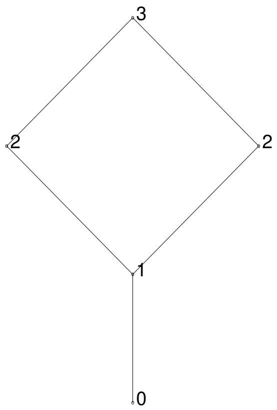


FIGURE 8.1. The poset of the B–orbits on $\mathfrak{b}_u$ in $\mathrm{GL}_3(k)$

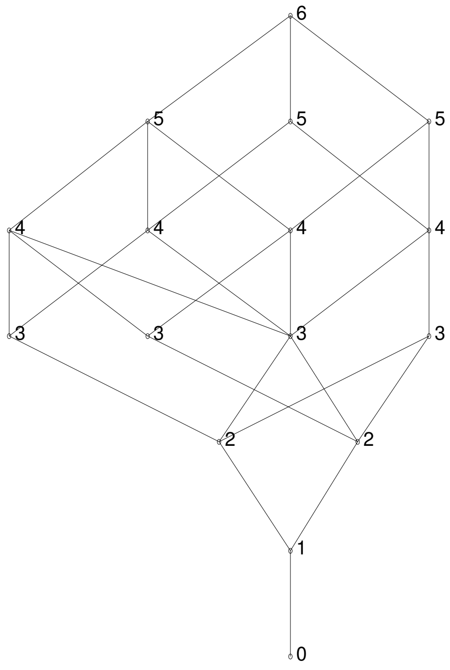


FIGURE 8.2. The poset of the B–orbits on $\mathfrak{b}_u$ in $\mathrm{GL}_4(k)$

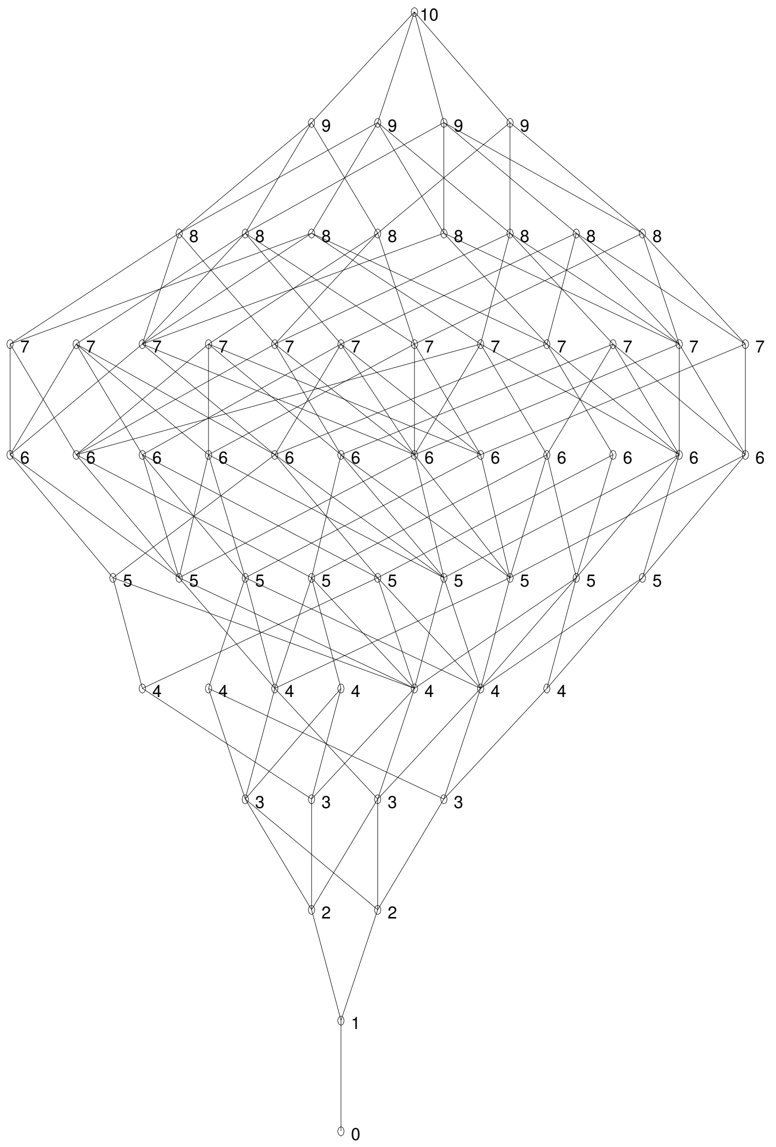


FIGURE 8.3. The poset of the B–orbits on $\mathfrak{b}_u$ in $\mathrm{GL}_5(k)$

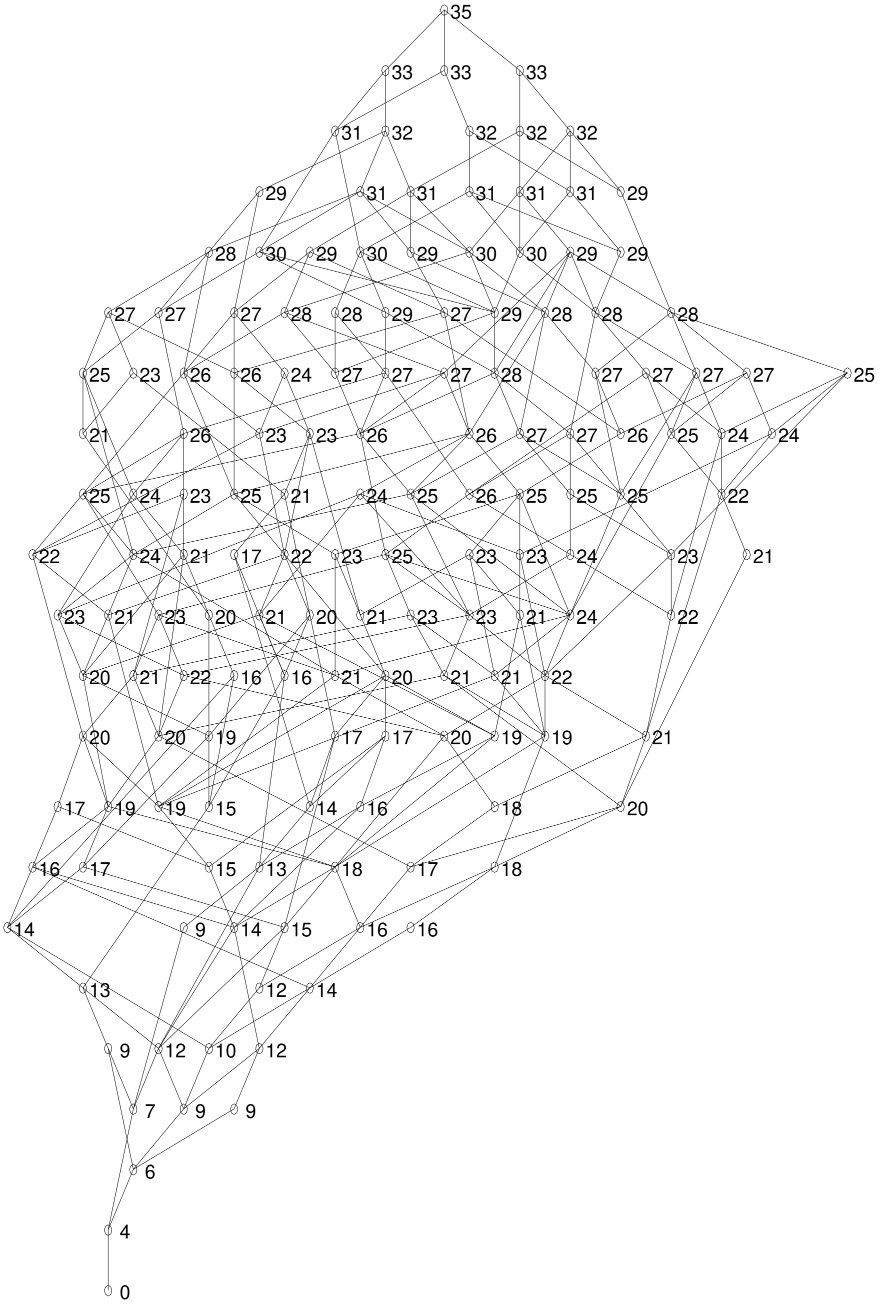


FIGURE 8.4. The Bruhat-Chevalley poset of the $P(\mathbf{d})$–orbits on $\mathfrak{p}_u(\mathbf{d})$ in $\mathrm{GL}_{10}(k)$, where $\mathbf{d} = (1, 2, 3, 4)$

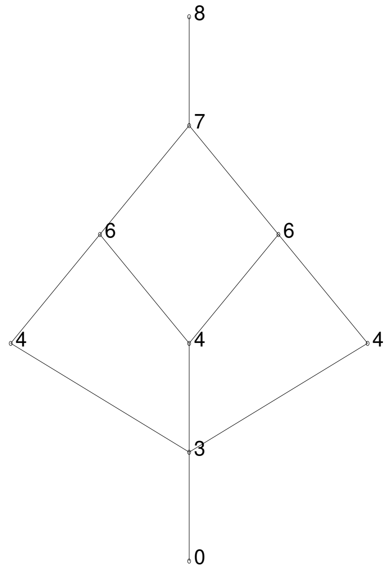


FIGURE 8.5. The Bruhat-Chevalley poset of the $P(\mathbf{d})$–orbits on $\mathfrak{p}_u(\mathbf{d})$ in $\mathrm{GL}_5(k)$, where $\mathbf{d} = (2, 1, 2)$

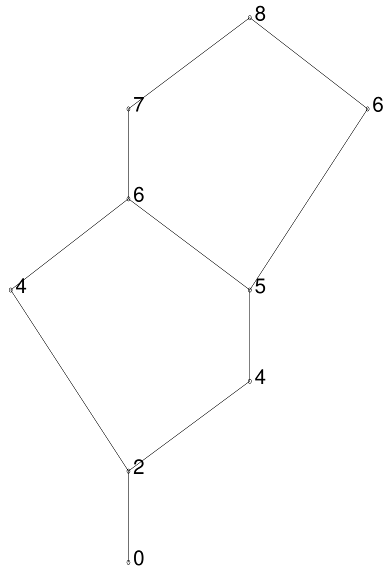


FIGURE 8.6. The Bruhat-Chevalley poset of the $P(\mathbf{d})$–orbits on $\mathfrak{p}_u(\mathbf{d})$ in $\mathrm{GL}_5(k)$, where $\mathbf{d} = (2, 2, 1)$

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