FILTERING MODULES OF FINITE PROJECTIVE DIMENSION

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Dedicated to Idun Reiten on the occasion of her sixtieth birthday.

Abstract. For a right artinian ring Λ we show that for every $n \geq 0$ there exists a pure-injective Λ-module P_n such that the Λ-modules of projective dimension at most n are precisely the direct factors of Λ-modules having a finite filtration in products of copies of P_n. This is a consequence of a general description of certain contravariantly finite resolving subcategories of $\operatorname{Mod}\Lambda$. It leads in addition to a one-to-one correspondence between equivalence classes of (not necessarily finitely generated) cotilting modules and resolving subcategories of $\operatorname{Mod}\Lambda$ which are closed under products and admit finite resolutions and special right approximations. As an application it is shown that every finitely presented partial cotilting module over an artin algebra admits a complement.

1. Introduction

Let Λ be a ring (associative with 1) and consider the category $\operatorname{Mod}\Lambda$ of (left) Λ-modules. In this paper we study the modules of finite projective dimension and prove the following result.

Theorem 1. *Let Λ be a right artinian ring. Then there exists for every integer $n \geq 0$ a pure-injective Λ-module P_n such that the Λ-modules of projective dimension at most n are precisely the direct factors of Λ-modules X having a filtration*

$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_l = 0$$

such that X_i/X_{i+1} is isomorphic to a product of copies of P_n for all i.

We denote by $\operatorname{pd} X$ the projective dimension of a Λ-module X and recall that

$$\operatorname{Fin.dim}\Lambda = \sup\{\operatorname{pd} X \mid X \in \operatorname{Mod}\Lambda \text{ and } \operatorname{pd} X < \infty\}$$

is the *finitistic dimension* of Λ. It is conjectured that this dimension is always finite, and we have now a countable test set for this conjecture.

Corollary. $\operatorname{Fin.dim}\Lambda = \sup\{\operatorname{pd} P_n \mid n \geq 0\}$.

Our analysis of modules with finite projective dimension is based on a number of formal properties. Recall that a class $\mathcal{X}$ of Λ-modules is *resolving* if $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and contains all projectives. Moreover, $\mathcal{X}$ is *definable* if $\mathcal{X}$ is closed under products, filtered colimits, and pure submodules. For every $n \geq 0$, the class of Λ-modules X with $\operatorname{pd} X \leq n$ is resolving and definable, provided that Λ is right artinian (since this is well-known to be true for $n = 0$). Therefore Theorem 1 is a consequence of the following result. It describes the objects of an arbitrary class which is resolving and definable. Recall that a ring Λ is said to be *semi-primary* if there exists a nilpotent ideal $\mathfrak{a}$ such that $\Lambda/\mathfrak{a}$ is semisimple.

Theorem 2. *Let Λ be a semi-primary ring with radical $\mathfrak{r}$ satisfying $\mathfrak{r}^l = 0$. Suppose that $\mathcal{X}$ is a class of Λ-modules which is resolving and definable. Then every Λ-module C has a minimal right $\mathcal{X}$-approximation $X_C \to C$, where X_C and $Y_C = \operatorname{Ker}(X_C \to C)$ are pure-injective if C is pure-injective. Moreover, for every Λ-module C the following are equivalent:*

(1) *C belongs to $\mathcal{X}$;*
(2) *C is the direct factor of a Λ-module X having a filtration*
$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_l = 0$$
such that X_i/X_{i+1} is isomorphic to a product of copies of $X_{\Lambda/\mathfrak{r}}$ for all i;
(3) $\operatorname{Ext}^1_\Lambda(C, Y_{\Lambda/\mathfrak{r}}) = 0$*;*
(4) $\operatorname{Ext}^j_\Lambda(C, Y_{\Lambda/\mathfrak{r}}) = 0$ *for all $j \geq 1$.*

We recall that a map $\phi\colon X \to C$ is a *right $\mathcal{X}$-approximation* of C if X belongs to $\mathcal{X}$ and every map $X' \to C$ with X' in $\mathcal{X}$ factors through ϕ.

The module $Y_{\Lambda/\mathfrak{r}}$ in Theorem 2 plays a very special role and it turns out that a class $\mathcal{X}$ of Λ-modules is resolving and definable if and only if there is a pure-injective Λ-module T such that $\mathcal{X} = {}^\perp T$ where
$${}^\perp T = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, T) = 0 \text{ for all } i \geq 1\}.$$
Therefore it is natural to ask to what extent a module T with $\mathcal{X} = {}^\perp T$ is determined by $\mathcal{X}$. Also, one can ask to what extent classes $\mathcal{X}$ of modules can be classified by modules T satisfying $\mathcal{X} = {}^\perp T$. For a complete answer to these questions some extra assumptions on $\mathcal{X}$ are needed. We obtain a one-to-one correspondence between subcategories of $\operatorname{Mod}\Lambda$ and equivalence classes of cotilting modules which is the analogue of a correspondence established by Auslander and Reiten for finitely presented modules over artin algebras [5]. Here, a Λ-module T is a *cotilting module* if

(T1) $\operatorname{id} T < \infty$;
(T2) $\operatorname{Ext}^i_\Lambda(\prod T, T) = (0)$ for all $i > 0$ and all products $\prod T$ of copies of T;
(T3) there exists an injective generator I and a long exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to I \to 0$ with T_i in $\operatorname{Prod} T$ for all $i = 0, 1, \ldots, n$.

Two cotilting modules T and T' are called *equivalent* if $\operatorname{Prod} T = \operatorname{Prod} T'$, where $\operatorname{Prod} T$ denotes the closure under products and direct factors of T. Moreover, for a subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ a right $\mathcal{X}$-approximation $X \xrightarrow{\phi} C$ of a module C is *special* if ϕ is an epimorphism and $\operatorname{Ext}^1_\Lambda(\mathcal{X}, \operatorname{Ker}\phi) = (0)$.

Theorem 3. *Let Λ be a ring. Then there is an one-to-one correspondence between equivalence classes of cotilting modules and resolving subcategories of* $\operatorname{Mod}\Lambda$ *which are closed under products and direct factors and admit finite resolutions and special right approximations.*

The correspondence is given by $T \longmapsto {}^\perp T$ and $\mathcal{X} \longmapsto \mathcal{X} \cap \mathcal{X}^\perp$.

The final part of this paper discusses complements for partial cotilting modules. Recall that T is a *partial cotilting module* if (T1) and (T2) hold. A Λ-module T' is a *complement* for T if $T \amalg T'$ is a cotilting module. Note that even for artin algebras such complements need not to exist if one restricts to the category of finitely presented modules. We provide various criteria for the existence of complements and get as a consequence the following result.

Theorem 4. *Let Λ be an artin algebra. Then every finitely presented partial cotilting module admits a complement.*

This describes the main results of this paper which is divided into two parts. The first part (Sections 2 – 3) contains the material on approximations and filtrations with respect to suitable subcategories $\mathcal{X}$ of $\operatorname{Mod}\Lambda$. The second part (Sections 4 – 6) discusses Ext-orthogonal complements and cotilting theory.

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2. Constructing approximations

Let Λ be an associative k-algebra over some commutative ring k. We denote by $\operatorname{Mod}\Lambda$ the category of (left) Λ-modules, and right modules over Λ are identified with the left modules over the opposite ring Λ^{op}. We fix a minimal injective cogenerator I for $\operatorname{Mod} k$ and denote by $D = \operatorname{Hom}_k(-, I)$ the corresponding functor $\operatorname{Mod} k \to \operatorname{Mod} k$. Note that D induces exact functors between $\operatorname{Mod}\Lambda$ and $\operatorname{Mod}\Lambda^{\mathrm{op}}$. We have for every Λ-module X a natural map $\phi_X \colon X \to D^2X$, defined by $\phi_X(x)(\alpha) = \alpha(x)$ for $x \in X$ and $\alpha \in DX$. The map ϕ_X is a split monomorphism if and only if X is *pure-injective*. In particular, a Λ-module is pure-injective if it is of the form DY for some Y in $\operatorname{Mod}\Lambda^{\mathrm{op}}$.

Let $\mathcal{X}$ be a class of Λ-modules. Given a Λ-module C, a map $\phi\colon X \to C$ is a *right $\mathcal{X}$-approximation* of C if X belongs to $\mathcal{X}$ and every map $X' \to C$ with X' in $\mathcal{X}$ factors through ϕ. The approximation ϕ is *minimal* if every endomorphism $\varepsilon\colon X \to X$ with $\phi \circ \varepsilon = \phi$ is an isomorphism. A minimal right $\mathcal{X}$-approximation of C is unique up to a non-canonical isomorphism, and it is often denoted by $X_C \to C$. Of course, there is the dual concept of a *left $\mathcal{X}$-approximation* of C, and a minimal one is usually denoted by $C \to X^C$.

In this section we construct $\mathcal{X}$-approximations, assuming that $\mathcal{X}$ satisfies some special conditions. We start with some preparations.

Lemma 2.1. *Let $\mathcal{X}$ be a class of pure-injective Λ-modules which is closed under products. Then every Λ-module has a left $\mathcal{X}$-approximation.*

Proof. We use the category $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$ of additive functors $\operatorname{mod}\Lambda^{\mathrm{op}} \to \mathrm{Ab}$ from finitely presented Λ^{op}-modules to abelian groups. The fully faithful functor

$$F\colon \operatorname{Mod}\Lambda \longrightarrow (\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab}), \quad C \mapsto - \otimes_\Lambda C$$

identifies the pure-injective Λ-modules with the injective objects of the abelian category $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$. The fact that $\mathcal{X}$ is closed under products implies the existence of a map $\phi\colon C \to X$ with X in $\mathcal{X}$ such that $K = \operatorname{Ker} F(\phi)$ is contained in $\operatorname{Ker} F(\phi')$ for all $\phi'\colon C \to X'$ with X' in $\mathcal{X}$. Now fix such a map ϕ'. Clearly, $F(\phi')$ factors through the canonical map $F(C) \to F(C)/K$. Using the injectivity of $F(X)$, we conclude that $F(\phi')$ factors through $F(\phi)$. Thus ϕ' factors through ϕ and ϕ is a left $\mathcal{X}$-approximation. $\square$

Lemma 2.2. *Let $\mathcal{X}$ be a class of Λ-modules which is closed under coproducts and satisfies $D^2\mathcal{X} \subseteq \mathcal{X}$.*

(a) *Every pure-injective Λ-module C has a right $\mathcal{X}$-approximation $\phi\colon X \to C$ such that X is pure-injective.*

(b) *If* $\phi\colon X_C \to C$ *is a minimal right* $\mathcal{X}$*-approximation of a pure-injective* Λ*-module* C*, then* X_C *and* $\operatorname{Ker}\phi$ *are pure-injective.*

Proof. (a) Let $\psi\colon DC \to Y$ be the left $D\mathcal{X}$-approximation which exists by Lemma 2.1. Now choose a left inverse $\pi\colon D^2C \to C$ for the natural map $C \to D^2C$ and put $\phi = \pi \circ D\psi$. Then it is easily checked that $\phi\colon DY \to C$ is a right $\mathcal{X}$-approximation of C.

(b) Let C be pure-injective, and suppose $\phi\colon X_C \to C$ is a minimal right $\mathcal{X}$-approximation. Since $C \to D^2C$ is a split monomorphism, X_C and $\operatorname{Ker}\phi$ are direct factors of D^2X and $D^2 \operatorname{Ker}\phi$ respectively. Thus X and $\operatorname{Ker}\phi$ are pure-injective. ☐

Minimal approximations do not exist in general. However, there is the following lemma which is due to Enochs.

Lemma 2.3 ([8, p. 207]). *Let* $\mathcal{X}$ *be a class of* Λ*-modules which is closed under filtered colimits. Then every* Λ*-module having a right* $\mathcal{X}$*-approximation has a minimal right* $\mathcal{X}$*-approximation.*

The next lemma is well-known as Wakamatsu's Lemma.

Lemma 2.4 ([5, Lemma 1.3]). *Let* $\mathcal{X}$ *be a class of* Λ*-modules which is closed under extensions, and let*

$$0 \longrightarrow Y \longrightarrow X \xrightarrow{\phi} C \longrightarrow 0$$

be an exact sequence of Λ*-modules.*

(a) *If* ϕ *is a minimal right* $\mathcal{X}$*-approximation, then* $\operatorname{Ext}^1_\Lambda(\mathcal{X}, Y) = 0$.

(b) *If* $\operatorname{Ext}^1_\Lambda(\mathcal{X}, Y) = 0$ *and* X *belongs to* $\mathcal{X}$*, then* ϕ *is a right* $\mathcal{X}$*-approximation.*

Recall that a class $\mathcal{X}$ of Λ-modules is *resolving* if $\mathcal{X}$ is closed under extensions, kernels of epimorphisms, and contains all projectives. Furthermore, a right $\mathcal{X}$-approximation $X \xrightarrow{\phi} C$ is *special* if ϕ is an epimorphism and $\operatorname{Ext}^1_\Lambda(\mathcal{X}, \operatorname{Ker}\phi) = (0)$. Hence Wakamatsu's Lemma states that a surjective minimal right $\mathcal{X}$-approximation for $\mathcal{X}$ extension closed, is special. An easy application of part (b) in Wakamatsu's Lemma gives the following lemma which is due to Auslander and Reiten.

Lemma 2.5 ([5, Proposition 3.7]). *Let* $\mathcal{X}$ *be a class of* Λ*-modules which is resolving, and let*

$$0 \longrightarrow C_1 \longrightarrow C \longrightarrow C_2 \longrightarrow 0$$

be an exact sequence of Λ*-modules. Suppose there are right* $\mathcal{X}$*-approximations* $\phi_i\colon X_i \to C_i$ *with* $\operatorname{Ext}^1_\Lambda(\mathcal{X}, \operatorname{Ker}\phi_i) = 0$ *for* $i = 1, 2$*. Then there exists a right* $\mathcal{X}$*-approximation* $\phi\colon X \to C$ *with* $\operatorname{Ext}^1_\Lambda(\mathcal{X}, \operatorname{Ker}\phi) = 0$*. Moreover, there are exact sequences*

$$0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow \operatorname{Ker}\phi_1 \longrightarrow \operatorname{Ker}\phi \longrightarrow \operatorname{Ker}\phi_2 \longrightarrow 0.$$

We are now in a position to prove the main result of this section. To this end fix a class $\mathcal{X}$ of Λ-modules and consider the following conditions on $\mathcal{X}$:

(X1) $\mathcal{X}$ is resolving;
(X2) $\mathcal{X}$ is closed under filtered colimits;
(X3) $\mathcal{X}$ is closed under D^2;
(X4) $\mathcal{X}$ is closed under products.

Note that (X2) implies that $\mathcal{X}$ is closed under coproducts.

Theorem 2.6. *Let Λ be a semi-primary ring and let $\mathcal{X}$ be a class of Λ-modules satisfying* (X1) – (X3). *Then every Λ-module C has a minimal right $\mathcal{X}$-approximation $X_C \to C$, where X_C and $Y_C = \operatorname{Ker}(X_C \to C)$ are pure-injective if C is pure-injective.*

Proof. We use induction on the number n such that $\mathfrak{r}^n C = 0$. If $n = 1$, then C is semisimple and therefore pure-injective. Thus C has a right $\mathcal{X}$-approximation by Lemma 2.2, which can be chosen to be minimal by Lemma 2.3. Now assume the assertion for $n-1$ and consider the exact sequence

$$0 \longrightarrow \mathfrak{r}C \longrightarrow C \longrightarrow C/\mathfrak{r}C \longrightarrow 0.$$

We have minimal approximations for the end terms and obtain an approximation for C by applying Lemma 2.5, in combination with part (a) of Lemma 2.4. Again, a minimal approximation for C exist by Lemma 2.3. The pure-injectivity of X_C and $\operatorname{Ker}(X_C \to C)$ follows from Lemma 2.2. This completes the proof. □

Recall that a class $\mathcal{X}$ of Λ-modules is *contravariantly finite* if every Λ-module has a right $\mathcal{X}$-approximation.

Corollary 2.7. *Let Λ be right artinian and denote by $\mathcal{X}$ the class of Λ-modules having finite projective dimension. Then the following are equivalent:*

(1) *$\mathcal{X}$ is contravariantly finite;*
(2) *$\mathcal{X}$ is closed under coproducts;*
(3) $\operatorname{Fin.dim}\Lambda < \infty$.

Proof. (1) $\Rightarrow$ (2) Let $\{X_i\}_{i\in I}$ be a set of modules in $\mathcal{X}$. Let $X \to \amalg_{i\in I} X_i$ be a right $\mathcal{X}$-approximation. Then X is in the category $\mathcal{X}_N$ of modules of projective dimension at most N for some N. Since every Λ-module has a minimal right $\mathcal{X}_N$-approximation, there is a minimal right $\mathcal{X}_N$-approximation $X_{\amalg_{i\in I} X_i} \to \amalg_{i\in I} X_i$. This approximation is a direct factor for the approximation $X \to \amalg_{i\in I} X_i$, therefore $X_{\amalg_{i\in I} X_i} \to \amalg_{i\in I} X_i$ also is a minimal right $\mathcal{X}$-approximaiton. It follows that $\amalg_{i\in I} X_i$ is a direct factor of $X_{\amalg_{i\in I} X_i}$ and consequently $\mathcal{X}$ is closed under coproducts.

The implication (2) $\Rightarrow$ (3) is straightforward. For the last implication observe now that the projective Λ-modules satisfy (X1) – (X3) since Λ is right artinian. Thus for every $n \geq 0$, the modules of projective dimension at most n satisfy (X1) – (X3). Therefore $\mathcal{X}$ is contravariantly finite if $\operatorname{Fin.dim}\Lambda < \infty$, by Theorem 2.6. □

The assumption on the ring Λ in Corollary 2.7 is not really needed. In fact, Aldrich et al. have shown that the modules of projective dimension at most n form a contravariantly finite subcategory of $\operatorname{Mod}\Lambda$ for any ring Λ and every $n \geq 0$; see [1]. Results which are closely related but different have been obtained more recently in [10] by Trlifaj.

3. Constructing filtrations

We fix again a class $\mathcal{X}$ of Λ-modules. In this section we use the construction of $\mathcal{X}$-approximations from the preceding section to construct for each object in $\mathcal{X}$ a special filtration.

Theorem 3.1. *Let Λ be a semi-primary ring with radical $\mathfrak{r}$ satisfying $\mathfrak{r}^l = 0$. Let $\mathcal{X}$ be a class of Λ-modules which is resolving and closed under products and direct factors. Suppose there exists a minimal right $\mathcal{X}$-approximation $X_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$ and let $Y_{\Lambda/\mathfrak{r}}$ be its kernel. Then $\mathcal{X}$ is contravariantly finite in* $\operatorname{Mod}\Lambda$. *Moreover, for every Λ-module C the following are equivalent:*

(1) *C belongs to $\mathcal{X}$;*
(2) *C is the direct factor of a Λ-module X having a filtration*
$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_l = 0$$
such that X_i/X_{i+1} is isomorphic to a product of copies of $X_{\Lambda/\mathfrak{r}}$ for all i;
(3) $\operatorname{Ext}^1_\Lambda(C, Y_{\Lambda/\mathfrak{r}}) = 0$*;*
(4) $\operatorname{Ext}^j_\Lambda(C, Y_{\Lambda/\mathfrak{r}}) = 0$ *for all $j \geq 1$.*

Proof. We fix a minimal right $\mathcal{X}$-approximation $\phi\colon X_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$. For every cardinal κ we get an exact sequence
$$0 \longrightarrow Y^\kappa_{\Lambda/\mathfrak{r}} \longrightarrow X^\kappa_{\Lambda/\mathfrak{r}} \xrightarrow{\phi^\kappa} (\Lambda/\mathfrak{r})^\kappa \longrightarrow 0$$
such that ϕ^κ is a right $\mathcal{X}$-approximation, since $\mathcal{X}$ is closed under products. Moreover, $\operatorname{Ext}^1_\Lambda(\mathcal{X}, Y^\kappa_{\Lambda/\mathfrak{r}}) = 0$ since $\operatorname{Ext}^1_\Lambda(\mathcal{X}, Y_{\Lambda/\mathfrak{r}}) = 0$ by Wakamatsu's lemma.

Now fix a Λ-module C and consider the filtration
$$C = \mathfrak{r}^0 C \supseteq \mathfrak{r}^1 C \supseteq \ldots \supseteq \mathfrak{r}^l C = 0.$$
Each factor $\mathfrak{r}^i C/\mathfrak{r}^{i+1} C$ is semi-simple and therefore a direct factor of $(\Lambda/\mathfrak{r})^\kappa$ for some cardinal κ. Thus we can add a semi-simple module C' and get a new filtration
$$C \amalg C' = C_0 \supseteq C_1 \supseteq \ldots \supseteq C_l = 0$$
such that $C_i/C_{i+1} \cong (\Lambda/\mathfrak{r})^\kappa$ for all i. We get from Lemma 2.5 a right $\mathcal{X}$-approximation $\psi\colon X \to C \amalg C'$ with a filtration
$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_l = 0$$
such that $X_i/X_{i+1} \cong X^\kappa_{\Lambda/\mathfrak{r}}$ for all i. Clearly, the composite of ψ with the projection $C \amalg C' \to C$ is a right $\mathcal{X}$-approximation of C.

(1) $\Rightarrow$ (2) Suppose that C belongs to $\mathcal{X}$ and let $X \to C$ be the approximation which has been constructed in the first part of the proof. The identity $C \to C$ factors through $X \to C$ and therefore C is a direct factor of X which has a special filtration.

(2) $\Rightarrow$ (1) This is clear since $\mathcal{X}$ is closed under extensions and direct factors, and every product of copies of $X_{\Lambda/\mathfrak{r}}$ belongs to $\mathcal{X}$.

(1) $\Rightarrow$ (4) We have $\operatorname{Ext}^1_\Lambda(\mathcal{X}, Y_{\Lambda/\mathfrak{r}}) = 0$ by Lemma 2.4. Using the fact that $\mathcal{X}$ is resolving, we get by dimension shift that $\operatorname{Ext}^j_\Lambda(\mathcal{X}, Y_{\Lambda/\mathfrak{r}}) = 0$ for all $j \geq 1$.

(4) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) Fix a Λ-module C. The construction in the first part of this proof shows that for some approximation $\psi\colon X \to C \amalg C'$ the kernel $Y = \operatorname{Ker}\psi$ has a filtration
$$Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_l = 0$$
such that $Y_i/Y_{i+1} \cong Y^\kappa_{\Lambda/\mathfrak{r}}$ for all i. Now suppose that $\operatorname{Ext}^1_\Lambda(C, Y_{\Lambda/\mathfrak{r}}) = 0$. Thus $\operatorname{Ext}^1_\Lambda(C, Y) = 0$ and therefore the inclusion $C \to C \amalg C'$ factors through ψ. It follows that C is a direct factor of X. We conclude that C belongs to $\mathcal{X}$. □

Remark 3.2. Let $\mathcal{X}$ be a resolving class of Λ-modules with a minimal right $\mathcal{X}$-approximation $X_{\Lambda/\mathfrak{r}} \to \Lambda/\mathfrak{r}$. Then $\mathcal{X}$ is closed under products if and only if every product of copies of $X_{\Lambda/\mathfrak{r}}$ belongs to $\mathcal{X}$.

We are now in a position to prove our result about modules of finite projective dimension.

Proof of Theorem 1. Let Λ be right artinian and denote by $\mathcal{X}$ the class of modules having projectives dimension at most n. Then the class of projective Λ-modules satisfies (X1) – (X4), and this carries over to $\mathcal{X}$. We have therefore a minimal right $\mathcal{X}$-approximation $P_n \to \Lambda/\mathfrak{r}$ by Theorem 2.6. Now apply Theorem 3.1. □

4. Ext-orthogonal classes

Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of Λ-modules. Then we define

$$\mathcal{X}^{\perp} = \{Y \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,Y) = 0 \text{ for all } X \in \mathcal{X} \text{ and } i \geq 1\},$$

$$^{\perp}\mathcal{Y} = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,Y) = 0 \text{ for all } Y \in \mathcal{Y} \text{ and } i \geq 1\}.$$

For a Λ-module T we write $T^{\perp} = \{T\}^{\perp}$ and $^{\perp}T = {}^{\perp}\{T\}$.

We have seen in Theorem 3.1 that every class $\mathcal{X}$ satisfying (X1) – (X4) is of the form $\mathcal{X} = {}^{\perp}T$ for some appropriate module T. Next we study the modules T having the property that $^{\perp}T$ satisfies (X1) – (X4).

Lemma 4.1. *Let T be a pure-injective Λ-module. Then $^{\perp}T$ is closed under filtered colimits.*

Proof. If T is pure-injective, then $\Omega^{-i}(T)$ is pure-injective for all $i \geq 1$. Therefore $\operatorname{Ext}^i_\Lambda(\varinjlim X_j, T) \cong \varprojlim \operatorname{Ext}^i_\Lambda(X_j, T)$ for any filtered system $\{X_j\}$. Thus $^{\perp}T$ is closed under filtered colimits. □

Lemma 4.2. *Let T be a pure-injective Λ-module. Then $^{\perp}T$ is closed under pure submodules and pure factor modules.*

Proof. Since $^{\perp}T$ is resolving, it is enough to show that $^{\perp}T$ is closed under pure factor modules. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with B in $^{\perp}T$.

A Λ-module X belongs to $^{\perp}T$ if and only if $\operatorname{Ext}^1_\Lambda(X, \overline{T}) = 0$ for $\overline{T} = \prod_{i=0}^{\infty} \Omega^{-i}(T)$. Clearly, $\operatorname{Ext}^1_\Lambda(X, \overline{T}) = 0$ implies $\operatorname{Ext}^i_\Lambda(X, \overline{T}) = 0$ for all $i \geq 1$. Using the fact that $\overline{T}$ is pure-injective, the assertion follows by applying $\operatorname{Hom}_\Lambda(-, \overline{T})$ to the pure exact sequence $0 \to A \to B \to C \to 0$. □

Recall that a class $\mathcal{X}$ of Λ-modules is *definable* if there exists a family of coherent functors $F_i\colon \operatorname{Mod}\Lambda \to \mathrm{Ab}$ such that a Λ-module C belongs to $\mathcal{X}$ if and only if $F_i(C) = 0$ for all i. Here, a functor $F\colon \operatorname{Mod}\Lambda \to \mathrm{Ab}$ is *coherent* if there exists an exact sequence

$$\operatorname{Hom}_\Lambda(Y, -) \longrightarrow \operatorname{Hom}_\Lambda(X, -) \longrightarrow F \longrightarrow 0$$

where X and Y are finitely presented Λ-modules. The following lemma shows that the definition coincides with the one given in the introduction.

Lemma 4.3 ([6, Section 2.3])**.** *A class $\mathcal{X}$ of Λ-modules is definable if and only if $\mathcal{X}$ is closed under products, filtered colimits, and pure submodules.*

Lemma 4.4. *Let $\mathcal{X}$ be a class of Λ-modules which is definable. Then $D^2\mathcal{X} \subseteq \mathcal{X}$.*

Proof. Given Λ-modules X and C with X finitely presented, we have

$$D^2 \operatorname{Hom}_\Lambda(X, C) \cong \operatorname{Hom}_\Lambda(X, D^2 C).$$

If $F\colon \operatorname{Mod}\Lambda \to \mathrm{Ab}$ is a coherent functor, we have therefore $D^2(F(C)) \cong F(D^2C)$. Thus $D^2\mathcal{X} \subseteq \mathcal{X}$ since $\mathcal{X}$ is definable. □

Corollary 4.5. *Let Λ be a semi-primary ring and $\mathcal{X}$ be a class of Λ-modules. Then the following are equivalent:*

(1) *$\mathcal{X} = {}^{\perp}T$ for some pure-injective module T, and $\mathcal{X}$ is closed under products;*
(2) *$\mathcal{X}$ is resolving and definable;*
(3) *$\mathcal{X}$ satisfies* (X1) – (X4).

Proof. (1) $\Rightarrow$ (2) Clearly, ${}^{\perp}T$ is resolving. Lemma 4.1 and 4.2 imply that $\mathcal{X}$ is closed under filtered colimits and pure submodules. Thus $\mathcal{X}$ is definable by Lemma 4.3.

(2) $\Rightarrow$ (3) Use Lemma 4.3 and 4.4.

(3) $\Rightarrow$ (1) The module $\Lambda/\mathfrak{r}$ has a minimal right $\mathcal{X}$-approximation $\phi\colon X \to \Lambda/\mathfrak{r}$ by Theorem 2.6. Applying Theorem 3.1, we obtain $\mathcal{X} = {}^{\perp}T$ for $T = \operatorname{Ker}\phi$. ☐

Remark 4.6. For the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in Corollary 4.5, no assumption on Λ is needed.

We collect now our findings and obtain the proof of Theorem 2. This result describes the objects in a class of modules which is resolving and definable.

Proof of Theorem 2. Let $\mathcal{X}$ be a class of modules which is resolving and definable. Then $\mathcal{X}$ satisfies (X1) – (X4). Now apply Theorem 2.6 and 3.1. ☐

5. Cotilting modules

In the previous sections the subcategories of prime interest have been subcategories of $\operatorname{Mod}\Lambda$ satisfying (X1) – (X4). Over a semi-primary ring such subcategories were characterised in Corollary 4.5. It is then natural to ask if this characterisation is valid outside the class of semi-primary rings. We do not know if any such extension exists.

Over a semi-primary ring Λ a subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ satisfying (X1) – (X4) is shown to be a contravariantly finite subcategory of $\operatorname{Mod}\Lambda$, where every Λ-module has a minimal (in particular a special) right $\mathcal{X}$-approximation. Even when adding the condition of contravariantly finiteness of $\mathcal{X}$ in $\operatorname{Mod}\Lambda$ to the conditions (X1) – (X4), a characterisation of subcategories $\mathcal{X}$ satisfying these conditions is unknown to us in general. However, adding in addition that the resolution dimension of $\operatorname{Mod}\Lambda$ with respect to the subcategory $\mathcal{X}$ is finite, we prove that $\mathcal{X}$ corresponds to pure-injective cotilting modules T over Λ via $\mathcal{X} = {}^{\perp}T$.

More generally, this section is devoted to finding a one-to-one correspondence between equivalence classes of cotilting modules and resolving subcategories $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ closed under products where $\operatorname{Mod}\Lambda$ has finite resolution dimension with respect to $\mathcal{X}$ and every Λ-module has a special right $\mathcal{X}$-approximation. This yields an analogue of the characterisation of finitely generated cotilting modules over artin algebras given in Theorem 5.5 in [5]. Furthermore, we characterise the subcategories corresponding to pure-injective cotilting modules.

Let Λ be a ring. Recall from [3] that a Λ-module T is a cotilting module if

(T1) $\operatorname{id} T < \infty$;
(T2) $\operatorname{Ext}^i_\Lambda(\prod T, T) = (0)$ for all $i > 0$ and all products $\prod T$ of copies of T;
(T3) there exists an injective generator I and a long exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to I \to 0$ with T_i in $\operatorname{Prod} T$ for all $i = 0, 1, \ldots, n$.

The following characterisation of cotilting modules in terms of subcategories of $\operatorname{Mod}\Lambda$ is given in Theorem 4.2 in [3], where we note the dependence on two subcategories, $\mathcal{X}$ and $\mathcal{X}^{\perp}$.

Theorem 5.1 ([3, Theorem 4.2])**.** *Let $\mathcal{X}$ be class of modules in* $\operatorname{Mod}\Lambda$ *closed under kernels of epimorphisms and such that $\mathcal{X}\cap\mathcal{X}^{\perp}$ is closed under products. The following are equivalent.*

(1) *There exists a cotilting module T with $\operatorname{id}T \leq n$ such that $\mathcal{X} = {}^{\perp}T$;*
(2) *Every left Λ-module has a special $\mathcal{X}$-approximation and all modules Y in $\mathcal{X}^{\perp}$ have $\operatorname{id}Y \leq n$.*

To give our characterisation of a cotilting module in terms of properties of one subcategory of $\operatorname{Mod}\Lambda$ we need to recall the following notions and Proposition 1.8 with remark from [5].

Let $\mathcal{X}$ be a subcategory of $\operatorname{Mod}\Lambda$. Recall that the *resolution dimension* of a Λ-module C with respect to $\mathcal{X}$, $\operatorname{resdim}_{\mathcal{X}}(C)$, is the smallest positive integer n such that there exists a long exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to C \to 0$ with X_i in $\mathcal{X}$ for all $i = 0, 1, \ldots, n$. If no such integer exists, $\operatorname{resdim}_{\mathcal{X}}(C) = \infty$. The resolution dimension of $\operatorname{Mod}\Lambda$ with respect to $\mathcal{X}$ is defined as

$$\operatorname{resdim}_{\mathcal{X}}(\operatorname{Mod}\Lambda) = \sup\{\operatorname{resdim}_{\mathcal{X}}(C) \mid C \in \operatorname{Mod}\Lambda\}.$$

Lemma 5.2 ([5, Proposition 1.8])**.** *Let $\mathcal{X}$ be a subcategory of* $\operatorname{Mod}\Lambda$ *containing all projective Λ-modules and closed under extensions and direct factors, where all Λ-modules have a special right $\mathcal{X}$-approximation. Let $\mathcal{Y} = \{Y \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{X}, Y) = (0)\}$.*

(a) *The subcategory $\mathcal{Y}$ is a covariantly finite extension closed subcategory of* $\operatorname{Mod}\Lambda$ *containing all injective modules. Moreover, for any Λ-module C there exists a left $\mathcal{Y}$-approximation $0 \to C \to Y^C \to X^C \to 0$, such that X^C is in $\mathcal{X}$.*
(b) $\mathcal{X} = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(X, \mathcal{Y}) = (0)\}$.

Proof. (a) It is clear from the definition of $\mathcal{Y}$ that $\mathcal{Y}$ is closed under extension and contains all injective Λ-modules.

Let C be in $\operatorname{Mod}\Lambda$, and let $0 \to C \to I(C) \to \Omega^{-1}(C) \to 0$ be the injective envelope of C. Then we obtain the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Y_{\Omega^{-1}(C)} \arrow[r, equal] \arrow[d] & Y_{\Omega^{-1}(C)} \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X_{\Omega^{-1}(C)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & I(C) \arrow[r] \arrow[d] & \Omega^{-1}(C) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $0 \to Y_{\Omega^{-1}(C)} \to X_{\Omega^{-1}(C)} \to \Omega^{-1}(C) \to 0$ is a special right $\mathcal{X}$-approximation of $\Omega^{-1}(C)$. Since $\mathcal{Y}$ is extension closed and contain all injective modules, it follows that E is in $\mathcal{Y}$ and that $0 \to C \to E \to X_{\Omega^{-1}(C)} \to 0$ is a left $\mathcal{Y}$-approximation.

(b) Let $\mathcal{X}' = \{X \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(X, \mathcal{Y}) = (0)\}$. It is clear that $\mathcal{X}$ is contained in $\mathcal{X}'$. Let X' be in $\mathcal{X}'$, and let $0 \to Y_{X'} \to X_{X'} \to X' \to 0$ be a special right $\mathcal{X}$-approximation of X'. Since $Y_{X'}$ is in $\mathcal{Y}$, it follows immediately that X' is a direct summand of $X_{X'}$, hence in $\mathcal{X}$ and therefore $\mathcal{X} = \mathcal{X}'$. $\square$

Now we describe the subcategories $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ corresponding to cotilting modules T such that $\mathcal{X} = {}^{\perp}T$.

Proposition 5.3. *Let Λ be a ring, and let $\mathcal{X}$ be a resolving subcategory of $\operatorname{Mod}\Lambda$ closed under products and direct factors with $\operatorname{resdim}_{\mathcal{X}}(\operatorname{Mod}\Lambda) < \infty$, such that every Λ-module has a special right $\mathcal{X}$-approximation. Then there exists a cotilting module T such that $\mathcal{X} = {}^{\perp}T$.*

Proof. Let I be an injective cogenerator for $\operatorname{Mod}\Lambda$, and let $\mathcal{Y} = \{Y \in \operatorname{Mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{X}, Y) = (0)\}$. Let $\operatorname{resdim}_{\mathcal{X}}(\operatorname{Mod}\Lambda) = n$. Since $\operatorname{Ext}^i_\Lambda(C, Y)$ can be computed using a (finite) resolution of C in $\mathcal{X}$ for Y in $\mathcal{Y}$, it follows that $\operatorname{Ext}^{n+1}_\Lambda(C, Y) = (0)$ for all Λ-modules C. Hence $\mathcal{Y}$ is contained in the full subcategory of $\operatorname{Mod}\Lambda$ consisting of all modules of injective dimension at most n. It follows from this and Lemma 5.2 (b) that there exists an exact sequence

$$0 \to T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} I \to 0$$

of special right $\mathcal{X}$-approximations. The modules $\operatorname{Ker} f_i$ for $i = 0, 1, \ldots, n$ and I are in $\mathcal{Y}$, consequently T_i for $i = 0, 1, \ldots, n$ are in $\mathcal{X} \cap \mathcal{Y}$. Let $T = \coprod_{i=0}^n T_i$. Since T is in $\mathcal{Y}$, the injective dimension of T is at most n.

Since T is in $\mathcal{X} \cap \mathcal{Y}$ and $\mathcal{X}$ is resolving and closed under all products, we obtain that $\operatorname{Ext}^i_\Lambda(\prod T, T) = (0)$ for all $i \geq 1$ and all products $\prod T$ of T. This shows that T is a cotilting module.

Since T is in $\mathcal{Y}$, the subcategory $\mathcal{X}$ is contained in ${}^{\perp}T$.

Before proving the converse inclusion we show that ${}^{\perp}T$ is cogenerated by $\operatorname{Prod} T$. Let X be in ${}^{\perp}T$. Then we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K_1 \arrow[r, equal] \arrow[d] & K_1 \arrow[d] & & \\
0 \arrow[r] & E \arrow[r] \arrow[d, "f"] & T_0 \arrow[r] \arrow[d] & \Omega^{-1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & I(X) \arrow[r] \arrow[d] & \Omega^{-1}(X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $X \to I(X)$ is the injective envelope of X in $\operatorname{Mod}\Lambda$ and $T_0 \to I(X)$ is a special right $\operatorname{Prod} T$-approximation with K_1 in $\mathcal{Y}$ that exists by construction of T. Hence the morphism $f\colon E \to X$ is a split epimorphism, and therefore X is a submodule of a product of T. Let $0 \to X \to T^X \to X' \to 0$ be a left $\operatorname{Prod} T$-approximation of X. Since T is a cotilting module, the long exact sequence induced by this short exact sequence shows that X' is in ${}^{\perp}T$ again. This shows that $\operatorname{Prod} T$ is an injective cogenerator for ${}^{\perp}T$. Since $\operatorname{resdim}_{{}^{\perp}T}(\operatorname{Mod}\Lambda)$ is finite, it follows from [4] that any Λ-module C has a special right ${}^{\perp}T$-approximation. This is also shown in Proposition 3.3 in [3], but we have included the argument since we need the construction later.

Let W be in $\mathcal{X} \cap \mathcal{X}^{\perp}$. Since W is in ${}^{\perp}T$ there exists an exact sequence $0 \to W \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$, where f_i induces a left $\operatorname{Prod} T$-approximation of $\operatorname{Coker} f_{i-1}$. Denote by L_i the kernel $\operatorname{Ker} f_{i+1}$. Since W is in $\mathcal{X}^{\perp}$ and all modules in $\mathcal{X}^{\perp}$ have finite injective dimension, $\operatorname{Ext}^i_{\Lambda}(L, W) = (0)$ for all $i > m$ and any Λ-module L, where $m = \operatorname{id}_{\Lambda} W$. In particular for $L = L_{m+1}$, hence $(0) = \operatorname{Ext}^{m+1}_{\Lambda}(L_{m+1}, W) \simeq \operatorname{Ext}^1_{\Lambda}(L_{m+1}, L_m)$ and L_m is in $\operatorname{Prod} T$. Since $\operatorname{Ext}^i_{\Lambda}(\operatorname{Prod} T, W) = (0)$ for all $i > 0$, it follows that $\operatorname{Ext}^1_{\Lambda}(L_m, L_{m-1}) \simeq \operatorname{Ext}^m_{\Lambda}(L_m, W) = (0)$. Therefore L_{m-1} is in $\operatorname{Prod} T$. Inductively W is in $\operatorname{Prod} T$, and $\mathcal{X} \cap \mathcal{X}^{\perp} = \operatorname{Prod} T$.

For any Λ-module C there is an exact sequence

$$0 \to K_n \to X_{n-1} \to X_{n-2} \to \cdots \to X_1 \to X_0 \to C \to 0$$

of special right $\mathcal{X}$-approximations. Then K_n is in ${}^{\perp}(\mathcal{X}^{\perp}) = \mathcal{X}$ by dimension shift, so that K_n is in $\mathcal{X} \cap \mathcal{X}^{\perp}$. For C in ${}^{\perp}T$ the extension groups $\operatorname{Ext}^i_{\Lambda}(C, K_n) = (0)$ for all $i > 0$. By dimension shift the exact sequence $0 \to K_n \to X_{n-1} \to K_{n-1} \to 0$ splits, and therefore K_{n-1} is in $\mathcal{X} \cap \mathcal{X}^{\perp} = \operatorname{Prod} T$. By induction the exact sequence $0 \to K_1 \to X_0 \to C \to 0$ splits and C is in $\mathcal{X}$. This shows that ${}^{\perp}T$ is contained in $\mathcal{X}$ and consequently $\mathcal{X} = {}^{\perp}T$. This completes the proof of the proposition. $\square$

Next we prove that every cotilting module T in $\operatorname{Mod} \Lambda$ gives rise to a subcategory $\mathcal{X}$ of $\operatorname{Mod} \Lambda$ as described in the previous result.

Proposition 5.4. *Let Λ be a ring, and let T be a cotilting module. Then ${}^{\perp}T$ is a resolving subcategory of $\operatorname{Mod} \Lambda$ closed under products and direct factors with $\operatorname{resdim}_{{}^{\perp}T}(\operatorname{Mod} \Lambda) < \infty$, such that every Λ-module has a special right ${}^{\perp}T$-approximation.*

Proof. The subcategory ${}^{\perp}T$ of $\operatorname{Mod} \Lambda$ is clearly resolving and closed under direct factors by definition. Let the injective dimension of T be n. Then any n-th syzygy is in ${}^{\perp}T$, hence $\operatorname{resdim}_{{}^{\perp}T}(\operatorname{Mod} \Lambda) \leq n < \infty$. By Proposition 3.3 in [3] every Λ-module has a special right ${}^{\perp}T$-approximation.

Let $\mathcal{X}_T$ be the full subcategory of $\operatorname{Mod} \Lambda$ consisting of Λ-modules X which fit into an exact sequence

$$0 \to X \xrightarrow{g_0} T_0 \xrightarrow{g_1} T_1 \xrightarrow{g_2} \cdots$$

with T_i in $\operatorname{Prod} T$ for all i. Since T is a cotilting module, for any X in $\mathcal{X}_T$ with a coresolution as above we have that $\operatorname{Ext}^i_{\Lambda}(X, T) \simeq \operatorname{Ext}^{i+j}_{\Lambda}(\operatorname{Coker} g_{j-1}, T)$ for all $j \geq 1$. For $j \geq \operatorname{id} T = n$ these groups are zero, hence X is in ${}^{\perp}T$ and $\mathcal{X}_T \subseteq {}^{\perp}T$.

Using similar arguments as in the second last part of the proof of the previous result we obtain that ${}^{\perp}T \subseteq \mathcal{X}_T$ and consequently ${}^{\perp}T = \mathcal{X}_T$.

Assume that $\{X_i\}_{i \in \Gamma}$ is in ${}^{\perp}T$ for some index set Γ. Then for each i we have an exact sequence

$$0 \to X_i \to T_0^{X_i} \to T_1^{X_i} \to T_2^{X_i} \to \cdots$$

with $T_j^{X_i}$ in $\operatorname{Prod} T$. Then the sequence

$$0 \to \prod X_i \to \prod T_0^{X_i} \to \prod T_1^{X_i} \to \prod T_2^{X_i} \to \cdots,$$

is exact. By the above description of ${}^{\perp}T$ it is immediate that $\prod_{i \in \Gamma} X_i$ is in ${}^{\perp}T$, hence ${}^{\perp}T$ is closed under products. $\square$

The next result that we quote from [3] is the final piece we need to give the characterisation of cotilting modules in terms of subcategories of $\operatorname{Mod}\Lambda$.

Proposition 5.5 ([3, Lemma 2.4])**.** *Let Λ be a ring, and let T be a cotilting module. Then ${}^{\perp}T \cap ({}^{\perp}T)^{\perp} = \operatorname{Prod} T$.*

Two cotilting modules T and T' are called *equivalent* if $\operatorname{Prod} T = \operatorname{Prod} T'$. Combining the previous results we have the following characterisation of cotilting modules.

Theorem 5.6. *Let Λ be a ring. Then there is an one-to-one correspondence between resolving subcategories $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ closed under products and direct factors with $\operatorname{resdim}_{\mathcal{X}}(\operatorname{Mod}\Lambda) < \infty$, such that every Λ-module has a special right $\mathcal{X}$-approximation and equivalence classes of cotilting modules over Λ.*

The correspondence is given by $\mathcal{X} \longmapsto \mathcal{X} \cap \mathcal{X}^{\perp}$ and $T \longmapsto {}^{\perp}T$.

All known examples of cotilting modules are pure-injective. It is an open problem whether or not all cotilting modules are pure-injective. Mantese et al. have shown that a cotilting module T with injective dimension at most 1 is pure-injective if and only if $\operatorname{Cogen}(T)$ is closed under filtered colimits [9]. Here $\operatorname{Cogen}(T)$ denotes the full subcategory of $\operatorname{Mod}\Lambda$ consisting of modules which are cogenerated by T.

Using the previous results and results from Sections 2 and 4 we obtain the following characterisation of when a cotilting module is pure-injective.

Proposition 5.7. *Let Λ be a ring, and let T be a cotilting Λ-module. Then the following are equivalent:*

(1) *${}^{\perp}T$ is closed under pure factor modules;*
(2) *${}^{\perp}T$ is closed under filtered colimits and pure submodules;*
(3) *every Λ-module has a minimal right ${}^{\perp}T$-approximation and $D^2({}^{\perp}T) \subseteq {}^{\perp}T$;*
(4) *T is pure-injective.*

Proof. (1) $\Rightarrow$ (2) Clearly, ${}^{\perp}T$ is closed under coproducts. Given a filtered system $\{X_i\}$ of Λ-modules, the canonical map $\coprod_i X_i \to \varinjlim X_i$ is a pure epimorphism. Thus ${}^{\perp}T$ is closed under filtered colimits.

Since ${}^{\perp}T$ is resolving and closed under pure factor modules, ${}^{\perp}T$ is closed under pure submodules.

(2) $\Rightarrow$ (3) Every Λ-module has a right ${}^{\perp}T$-approximation by Proposition 5.4. This can be chosen to be minimal by Lemma 2.3. In addition, $D^2({}^{\perp}T) \subseteq {}^{\perp}T$ by Lemma 4.4, since ${}^{\perp}T$ is definable by Lemma 4.3.

(3) $\Rightarrow$ (4) First recall from Theorem 5.6 how the cotilting module corresponding to ${}^{\perp}T$ is constructed. We take an injective cogenerator of $\operatorname{Mod}\Lambda$ and take a sequence of special right ${}^{\perp}T$-approximations

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to I \to 0.$$

This sequence of approximations we can choose to be a sequence of minimal right ${}^{\perp}T$-approximations. Let $T' = \coprod_{i=0}^{n} T_i$. Then $\operatorname{Prod} T = \operatorname{Prod} T'$. By Lemma 2.2 all the modules T_i for $i = 0, 1, \ldots, n$ are pure-injective, since I is pure-injective. Hence T is pure-injective.

(4) $\Rightarrow$ (1) Use Lemma 4.1. $\square$

6. Complements of partial cotilting modules

Let Λ be a ring. A Λ-module T is called a *partial cotilting module* if

(T1) $\operatorname{id} T < \infty$;
(T2) $\operatorname{Ext}^i_\Lambda(\prod T, T) = (0)$ for all $i > 0$ and all products $\prod T$ of copies of T.

A Λ-module X is said to be a *complement* of a partial cotilting module T, if $T \amalg X$ is a cotilting module.

Restricting to the category of finitely presented modules over an artin algebra, a partial cotilting module does not always have a complement. However, in tilting theory various criteria are known for a partial tilting module to have a (possibly infinitely generated) complement [2].

This section is devoted to characterising when a partial cotilting module has a complement. The first result is an easy consequence of our characterisation of cotilting modules, and the proof follows the proof for the case of finitely presented partial cotilting modules over artin algebras.

Proposition 6.1. *Let Λ be a ring, and let T be a partial cotilting module. Then T has a complement if and only if ${}^\perp T$ contains a resolving subcategory $\mathcal{X}$ containing* $\operatorname{Prod} T$ *and closed under products and direct factors with* $\operatorname{resdim}_{\mathcal{X}}(\operatorname{Mod}\Lambda) < \infty$*, such that every Λ-module has a special right $\mathcal{X}$-approximation.*

Proof. Assume that $\mathcal{X}$ is a subcategory of ${}^\perp T$ as described above. Then $({}^\perp T)^\perp \subseteq \mathcal{X}^\perp$. Since T is in $({}^\perp T)^\perp$, the module T is in $\mathcal{X} \cap \mathcal{X}^\perp$, which corresponds to a cotilting module by Theorem 5.6. Hence T has a complement.

The other implication is immediate using Theorem 5.6. $\square$

Let $\mathcal{X}$ be an extension closed subcategory of $\operatorname{Mod}\Lambda$. A module I in $\mathcal{X}$ is called Ext-*injective in $\mathcal{X}$* if $\operatorname{Ext}^1_\Lambda(X, I) = (0)$ for all X in $\mathcal{X}$.

The following result characterises when a pure-injective partial cotilting module T has a complement which is Ext-injective in ${}^\perp T$. The corresponding result for partial tilting modules is proved in [2].

Theorem 6.2. *Let Λ be a ring, and let T be a partial cotilting module. Then the following are equivalent.*

(1) *T has a complement which is* Ext*-injective in ${}^\perp T$;*
(2) *${}^\perp T$ is closed under products and each Λ-module has a special right ${}^\perp T$-approximation.*

Proof. Assume that T has an Ext-injective complement X in ${}^\perp T$. Then ${}^\perp(T \amalg X) \subseteq {}^\perp T$. Since X is in $({}^\perp T)^\perp$, we have that ${}^\perp(T \amalg X) = {}^\perp T$. Since $T \amalg X$ is a cotilting module, the subcategory ${}^\perp(T \amalg X) = {}^\perp T$ is closed under products and each Λ-module has a special right ${}^\perp T$-approximation.

Conversely, since T has finite injective dimension, ${}^\perp T$ is a resolving subcategory of $\operatorname{Mod}\Lambda$ closed under products with $\operatorname{resdim}_{{}^\perp T}(\operatorname{Mod}\Lambda) < \infty$, where each Λ-module has a special right ${}^\perp T$-approximation. Hence $({}^\perp T) \cap ({}^\perp T)^\perp$, which contains $\operatorname{Prod} T$, corresponds to a cotilting module T' by Theorem 5.6 where T is a direct factor of a product of copies of T'. We conclude that T has an Ext-injective complement in ${}^\perp T$. $\square$

Next we use that each Λ-module has a special right ${}^{\perp}T$-approximation provided that T is pure-injective [7]. Combining this with Lemma 4.2, Theorem 5.6 and Proposition 5.7, we obtain the following consequence of Theorem 6.2.

Corollary 6.3. *Let Λ be a ring, and let T be a pure-injective partial cotilting module. Then the following are equivalent.*

(1) *T admits a complement which is pure-injective and* Ext*-injective in ${}^{\perp}T$;*
(2) *T admits a complement which is* Ext*-injective in ${}^{\perp}T$;*
(3) *${}^{\perp}T$ is closed under products.*

It is well-known that for an artin algebra Λ a finitely presented partial cotilting module does not necessarily have a finitely presented complement. If one passes to arbitrary modules, it is shown in [2] that a finitely presented partial tilting module has a complement provided that the ring Λ is left coherent. We have the dual result for artin algebras.

Corollary 6.4. *Let Λ be an artin algebra, and let T be a finitely presented Λ-module. If T is a partial cotilting module, then T has a complement which is pure-injective and* Ext*-injective in ${}^{\perp}T$.*

Proof. Let X be a finitely presented Λ-module. Then X is pure-injective and the functor

$$\operatorname{Ext}^i_\Lambda(-,X)\colon \operatorname{Mod}\Lambda \longrightarrow \operatorname{Ab}$$

is isomorphic to $D\,\underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr}D(\Omega^{-i+1}(X)),-)$, where $\underline{\operatorname{Hom}}_\Lambda(\operatorname{Tr}D(\Omega^{-i+1}(X)),-)$ is a coherent functor for all $i>0$. Here, we denote by $\operatorname{Tr}Y$ the transpose for a finitely presented right Λ-module Y . This shows that the subcategory ${}^{\perp}X$ is definable for any finitely presented X. Now the claim follows directly from the above result. $\square$

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