FOXBY EQUIVALENCE, COMPLETE MODULES, AND TORSION MODULES

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0. Introduction

This manuscript considers classical Foxby equivalence for local, commutative, noetherian rings (see [2]), and generalizes it to an equivalence theory for derived categories over Differential Graded Algebras. It shows that both classical Foxby equivalence, and the Morita theory for complete modules and torsion modules developed by Dwyer and Greenlees in [7] arise as special cases. It also shows that a new instance of our theory which one can reasonably call "Matlis equivalence" gives a new characterization of Gorenstein rings.

(0.1) **Classical Foxby equivalence.** The dualizing complex over a commutative, local, noetherian ring R is a classical object in commutative algebra and algebraic geometry. If D is such a complex, then the functor $\mathrm{RHom}_R(-,D)$ is a contravariant equivalence between suitable derived categories. See [5, sec. A8] for the basics of this.

However, there is a less standard application of D, described by Avramov and Foxby in [2]: Consider the adjoint pair of functors

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R D}{\rightleftarrows}} \mathsf{D}(R). \tag{0.1.1}$$

They induce quasi-inverse equivalences of full subcategories

$$\mathcal{A}_D \underset{\mathrm{RHom}_R(D,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R D}{\rightleftarrows}} \mathcal{B}_D, \tag{0.1.2}$$

where $\mathcal{A}_D$ and $\mathcal{B}_D$ are the so-called *Auslander* and *Bass classes* of R. An important feature of $\mathcal{A}_D$ and $\mathcal{B}_D$ is that $\mathcal{A}_D$ contains all bounded complexes of flat modules, while $\mathcal{B}_D$ contains all bounded complexes of injective modules. This construction is known as *Foxby equivalence* [5], [6], [8], [17]. It generalizes Sharp's results from [16] which are restricted to the Cohen-Macaulay situation.

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Although it is not stated explicitly in [2], $\mathcal{A}_D$ and $\mathcal{B}_D$ are defined in terms of unit and counit, η and ϵ, of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R D, \mathrm{RHom}_R(D,-))$: The complex X is in $\mathcal{A}_D$ if η_X is an isomorphism, and the complex Y is in $\mathcal{B}_D$ if ϵ_Y is an isomorphism. ([2] adds some boundedness conditions to this, but the theory works without them.)

This theory has important applications to the theory of Gorenstein rings. E.g., see [5], [8], and [17]'s treatment of Gorenstein projective dimension, and [2] and [9]'s treatment of local ring homomorphisms of finite Gorenstein dimension. It has also been extended to semi-dualizing complexes in [6].

(0.2) **Dwyer and Greenlees' Morita theory.** This theory deals with complete modules and torsion modules, and is described in the preprint [7]. It generalizes Rickard's Morita theory from [15]. The setup is this: R is a ring, and A is a perfect complex of R-left-modules (i.e. a bounded complex of finitely generated projective R-left-modules). Dwyer and Greenlees construct the *endomorphism Differential Graded Algebra*, $\mathcal{E} = \mathrm{Hom}_R(A, A)$, and note that A aquires an $\mathcal{E}$-left-structure, hence becomes a Differential Graded R-left-$\mathcal{E}$-left-module. (For a few words about the theory of Differential Graded Algebras, and Differential Graded modules over them, see (0.4).) Then they consider the adjoint pair of functors

$$\mathsf{D}(\mathcal{E}^{\mathrm{opp}}) \underset{\mathrm{RHom}_R(A,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_{\mathcal{E}} A}{\rightleftarrows}} \mathsf{D}(R), \tag{0.2.1}$$

where $\mathcal{E}^{\mathrm{opp}}$ is the opposite algebra of $\mathcal{E}$, so $\mathsf{D}(\mathcal{E}^{\mathrm{opp}})$ is the derived category of Differential Graded $\mathcal{E}$-right-modules. They now prove that the adjoint pair induces quasi-inverse equivalences of full subcategories

$$\mathsf{D}(\mathcal{E}^{\mathrm{opp}}) \underset{\mathrm{RHom}_R(A,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_{\mathcal{E}} A}{\rightleftarrows}} \mathbf{A}_{\mathrm{tors}}, \tag{0.2.2}$$

where $\mathbf{A}_{\mathrm{tors}}$ is a certain full subcategory of $\mathsf{D}(R)$ consisting of what Dwyer and Greenlees call torsion complexes. (This is only the left half of the diagram from [7, thm. 2.1], but the right half, which deals with complete complexes, can be understood similarly.)

Just as above, the full subcategories in the second diagram can be characterized by the unit and the counit of the adjunction being isomorphisms.

(0.3) **This paper.** Note the typograpical resemblance between diagrams (0.1.1) and (0.2.1) and between diagrams (0.1.2) and (0.2.2). In both instances one considers an adjoint pair of functors of the form $(- \overset{\mathrm{L}}{\otimes}$

$M, \mathrm{RHom}(M, -))$, and in both instances one obtains induced equivalences of full subcategories.

This observation led us to the main construction of this manuscript: We let R and S be Differential Graded Algebras and let M be a Differential Graded R-left-S-left-module, and consider the adjoint pair of functors

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(S). \tag{0.3.1}$$

Defining full subcategories $\mathcal{A}_M$ and $\mathcal{B}_M$ in terms of unit and counit being isomorphisms, these restrict to quasi-inverse equivalences of categories

$$\mathcal{A}_M \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M.$$

Our first main remark is that this construction can be specialized to both classical Foxby equivalence (see (1.6)), and Dwyer and Greenlees' theory (see (1.7)).

But more is true: We can also prove general results about the size of the classes $\mathcal{A}_M$ and $\mathcal{B}_M$, by using so-called *evaluation morphisms* from the theory of Differential Graded modules (see section 3). Moreover, it turns out that other instances of our construction than the ones considered by Foxby and Dwyer and Greenlees are interesting for ring theory (see (1.8)).

It is also a point that we systematically keep track of the two module structures on M, thereby making the interplay between the derived categories of the two Differential Graded Algebras R and S clearer. This is in the spirit of non-commutative homological ring theory, see e.g. [18].

In the forthcoming [10], we use our theory to introduce Gorenstein Differential Graded Algebras.

The outline of the manuscript after the present introduction is as follows.

- Section 1 sets up the equivalence theory described above, and shows how classical Foxby equivalence and Dwyer and Greenlees' theory are special cases (see (1.6) and (1.7)).

  It also in (1.8) considers the situation where $R = S$ is just a commutative, local, noetherian ring, and where M is $\mathrm{E}(k)$, the injective hull of the residue class field k. It is proved that this theory can recognize Gorenstein rings by the conditions $k \in \mathcal{A}_{\mathrm{E}(k)}$ and $k \in \mathcal{B}_{\mathrm{E}(k)}$. As proved by Foxby, the same statement is valid for the Auslander and Bass classes of classical Foxby equivalence where M is the dualizing complex D, see [5, (3.1.12) and (3.2.10)]. However, $\mathrm{E}(k)$ is a simpler and more canonical object than D (which does not even exist over all rings).

- Section 2 proves theorem (2.2) on the existence of "small" K-projective resolutions, which improves and clarifies a result from [1]. The result is necessary in section 3 for our results on the size of Auslander and Bass classes, and in the forthcoming [10] on Gorenstein Differential Graded Algebras. It states that over a nice Differential Graded Algebra, any sufficiently "small" Differential Graded module has a "small" K-projective resolution. The point of this result is that [1, sec. 1, thms. 1 and 2] shows, among other things, that when the Differential Graded module A has a "small" K-projective resolution, then the evaluation morphisms

$$\mathrm{RHom}(A,B) \overset{\mathrm{L}}{\otimes} F \xrightarrow{\omega} \mathrm{RHom}(A, B \overset{\mathrm{L}}{\otimes} F)$$

and

$$A \overset{\mathrm{L}}{\otimes} \mathrm{RHom}(B,I) \xrightarrow{\theta} \mathrm{RHom}(\mathrm{RHom}(A,B),I)$$

are isomorphisms.
- Section 3 proceeds with the abstract theory, and proves results on the size of the classes $\mathcal{A}_M$ and $\mathcal{B}_M$. The idea is to rewrite unit and counit of the adjoint pair (0.3.1) in terms of the above mentioned evaluation morphisms (see lemma (3.3)). This implies that $\mathcal{A}_M$ and $\mathcal{B}_M$ can be characterized by the evaluation morphisms being isomorphisms (see theorem (3.4)).

  We then apply the results from section 2 and [1, sec. 1, thms. 1 and 2] to get conditions under which the evaluation morphisms are isomorphisms. This leads to corollaries (3.5) and (3.9) which state that
  - If M is "very nice" over S, respectively over R, then $\mathcal{A}_M$ is all of $\mathsf{D}(R^{\mathrm{opp}})$, respectively $\mathcal{B}_M$ is all of $\mathsf{D}(S)$,
  - If M is "nice" over S, respectively over R, then $\mathcal{A}_M$ contains $\mathcal{F}(R^{\mathrm{opp}})$, respectively $\mathcal{B}_M$ contains $\mathcal{I}(S)$. Here $\mathcal{F}(R^{\mathrm{opp}})$ and $\mathcal{I}(S)$ are classes which generalize the classes of bounded complexes of flats and bounded complexes of injectives known from classical ring theory.

  When specializing our theory to classical Foxby equivalence, the second result specializes to the classical result that the Auslander class contains the bounded complexes of flats, while the Bass class contains the bounded complexes of injectives (see (3.10)). And when specializing our theory to the part of Dwyer and Greenlees' theory sketched above, the first result specializes to their result from [7, (2.9)] that $\mathcal{A}_M$ is all of $\mathsf{D}(R^{\mathrm{opp}})$ (see (3.6), and note that our notation differs from Dwyer and Greenlees', so our $\mathsf{D}(R^{\mathrm{opp}})$ is their $\mathsf{mod}\text{-}\mathcal{E}$).

(0.4) **Notation.** We shall use the theory of *Differential Graded Algebras* (abbreviated *DGAs*) and *Differential Graded modules* (abbreviated *DG-modules*) throughout, and use [3] and [11] for standard references.

Differential Graded Algebras (DGAs). To us, a *DGA* is a graded algebra R over the fixed commutative ground ring $\Bbbk$, equipped with a differential ∂ of degree -1 satisfying $\partial^2 = 0$ and

$$\partial(rr') = \partial(r)r' + (-1)^{|r|}r\partial(r')$$

where $|r|$ denotes the degree of a homogeneous element r. By $R^\natural$ we denote the graded $\Bbbk$-algebra obtained by forgetting R's differential.

For the rest of this section, let R and S be DGAs.

Differential Graded modules (DG-modules). A *DG-R-left-module* M is a graded R-left-module equipped with a differential ∂^M of degree -1 satisfying $(\partial^M)^2 = 0$, which is compatible with scalar multiplication by R-elements,

$$\partial^M(rm) = \partial(r)m + (-1)^{|r|}r\partial^M(m).$$

To emphasize the left-action of R, we often denote such an M by ${}_RM$.

Similarly, a DG-R-right-module could be denoted N_R. Note that a DG-R-right-module is "the same thing" as a DG-R^{opp}-left-module, where R^{opp} is the opposite algebra of R, which has the product $r_1 \cdot r_2 = (-1)^{|r_1||r_2|}r_2r_1$.

If R and S are DGAs, then a DG-R-left-S-left-module M is a graded R-left-S-left-module satisfying the compatibility condition

$$r(sm) = (-1)^{|r||s|}s(rm),$$

equipped with a differential of degree -1 satisfying $(\partial^M)^2 = 0$, which is compatible with scalar multiplication by R- and S-elements. To emphasize the actions of R and S, we often denote such an M by ${}_{R,S}M$. Similarly, a DG-R-left-S-right-module could be denoted ${}_RN_S$, and a DG-R-right-S-right-module could be denoted $P_{R,S}$.

If M is a DG-R-module, then $M^\natural$ denotes the graded $R^\natural$-module obtained by forgetting M's differential.

Homology modules. A DG-module M can be viewed as a complex of $\Bbbk$-modules and as such has a *cycle module* $\mathrm{Z}M$, a *boundary module* $\mathrm{B}M$, and a *homology module* $\mathrm{H}M = \mathrm{Z}M/\mathrm{B}M$ which are a priori graded $\Bbbk$-modules. If z is a cycle in the complex M, then the homology class of z is denoted $\operatorname{cls} z$. Further, the multiplication on R induces a multiplication on $\mathrm{H}R$, which becomes a graded $\Bbbk$-algebra, and if M is a DG-R-module, then the scalar multiplication of R on M induces a scalar multiplication of $\mathrm{H}R$ on $\mathrm{H}M$, which becomes a graded $\mathrm{H}R$-module.

Shifts. If X is a graded $R^\natural$-left-module, then the *n'th shift of* X is denoted by $\mathcal{S}^nX$. As a graded $\Bbbk$-module, it is defined by $(\mathcal{S}^nX)_i = X_{i-n}$. If x is in X_i, then $\mathcal{S}^nx$ denotes x viewed as an element in $(\mathcal{S}^nX)_{i+n}$. The shift $\mathcal{S}^nX$ is made into a graded $R^\natural$-left-module by $r\cdot\mathcal{S}^nx = (-1)^{n|r|}\mathcal{S}^n(rx)$.

If M is a DG-R-left-module, then the shift $\mathcal{S}^n(M^\natural)$ of the graded module $M^\natural$ can be equipped with the differential $\partial^{\mathcal{S}^nM}(\mathcal{S}^nm) = (-1)^n\mathcal{S}^n(\partial^Mm)$, and becomes a DG-$R$-left-module which we denote $\mathcal{S}^nM$.

A similar procedure is used for shifting graded $R^\natural$-right-modules and DG-R-right-modules.

Morphisms. A *morphism of DG-modules* is a degree preserving module homomorphism $M \longrightarrow N$ which is compatible with the differentials. It induces a morphism of homology modules $\mathrm{H}M \longrightarrow \mathrm{H}N$, and if this is an isomorphism, then we call the original morphism a *quasi-isomorphism*, and denote it by $M \xrightarrow{\simeq} N$. A genuine isomorphism is of course also a quasi-isomorphism.

A morphism $M \xrightarrow{\mu} N$ has a *mapping cone*, C, defined as a graded module by

$$C^\natural = \mathcal{S}^1(M^\natural) \oplus N^\natural,$$

and turned into a DG-module by the differential

$$\partial^C(\mathcal{S}^1 m, n) = (\mathcal{S}^1(-\partial^M m), \partial^N n + \mu m).$$

Note that the multiplication by algebra elements on C is twisted by a sign in the M-variable.

Categories and functors of DG-modules. There is a category of DG-R-left-modules and morphisms. From it, one obtains the *homotopy category* $\mathsf{K}(R)$ by identifying homotopic morphisms (see [11, sec. 2]), and from this one obtains the *derived category* $\mathsf{D}(R)$ by inverting the classes of the quasi-isomorphisms (see [11, sec. 4]). Both the homotopy category and the derived category are triangulated, with the triangulation coming from the mapping cone construction (see [11]). Similar constructions can be made for DG-modules with structures over more than one DGA. We use "$\cong$" to denote categorical isomorphisms.

If ${}_RM_S$ is a DG-R-left-S-right-module and ${}_RN_T$ is a DG-R-left-T-right-module, then $\mathrm{Hom}_{R^\natural}(M^\natural, N^\natural)$ has a canonical differential turning it into a DG-S-left-T-right-module denoted $\mathrm{Hom}_R(M,N)$. Viewing this as a complex, its cycles are precisely the morphisms of DG-R-left-modules, and its boundaries are exactly the null homotopic morphisms of DG-R-left-modules.

And if ${}_SM_R$ is a DG-S-left-R-right-module and ${}_RN_T$ is a DG-R-left-T-right-module, then $M^\natural \otimes_{R^\natural} N^\natural$ has a canonical differential turning it into a DG-S-left-T-right-module denoted $M \otimes_R N$.

The functors $\mathrm{Hom}_R(-,-)$ and $- \otimes_R -$ are inherited to the homotopy categories, and can be used to define derived functors $\mathrm{RHom}_R(-,-)$ and $- \overset{\mathrm{L}}{\otimes}_R -$ on derived categories. They can be computed by using appropriate K-projective, K-injective, and K-flat resolutions, see [3] and [11, sec. 6].

1. Generalized Foxby equivalence

This section describes the most general version of our theory. It starts with a very general equivalence in theorem (1.1), and then immediately proceeds to look at DGAs. In (1.5), Foxby equivalence is given in the version with DGAs described in the introduction.

In (1.6) it is shown that our theory contains classical Foxby equivalence known from [2] and [6], and in (1.7) it is shown that our theory contains the Dwyer and Greenless theory from [7].

In (1.8) to (1.10), we consider a new instance of our theory, where the dualizing complex from classical Foxby equivalence is replaced with $\mathrm{E}(k)$, the injective hull of the residue class field k. This theory turns out to be able to detect Gorensteinness in the same way as classical Foxby equivalence, namely by k being in the Auslander and Bass classes (see [5, (3.1.12) and (3.2.10)]).

(1.1) **Theorem.** *Consider categories* C, D *and an adjoint pair of functors* (F, G),

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] & \mathsf{D}. \arrow[l, shift left, "G"] \end{tikzcd}$$

Denote unit and counit of the adjunction by η *and* ϵ*. Define full subcategories of* C *and* D,

$$\mathcal{A} = \{A \in \mathsf{C} \mid \eta_A \text{ is an isomorphism}\},$$
$$\mathcal{B} = \{B \in \mathsf{D} \mid \epsilon_B \text{ is an isomorphism}\}.$$

Then the functors F *and* G *restrict to a pair of quasi-inverse equivalences of categories,*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B}. \arrow[l, shift left, "G"] \end{tikzcd}$$

Proof. This is an easy exercise in adjoint functors. □

(1.2) **Definition (Auslander and Bass classes).** In the situation of theorem (1.1), we call $\mathcal{A}$ the Auslander class, and $\mathcal{B}$ the Bass class. These names are due to [2].

(1.3) **Setup.** In the rest of this section, R and S are DGAs, and ${}_{R,S}M$ is a DG-R-left-S-left-module.

(1.4) **Remark (Tensor and Hom over a DGA).** From [3, sec. 4.4] we know that there is an adjoint pair of functors

$$\begin{tikzcd} \mathsf{K}(R^{\mathrm{opp}}) \arrow[r, shift left, "- \otimes_R M"] & \mathsf{K}(S). \arrow[l, shift left, "\mathrm{Hom}_S(M,-)"] \end{tikzcd}$$

The unit η of the adjunction is given by

$$\mathrm{id}_{\mathsf{K}(R^{\mathrm{opp}})}(L) \xrightarrow{\eta_L} \mathrm{Hom}_S(M, L \otimes_R M), \quad \eta_L(\ell) = (m \mapsto \ell \otimes m),$$

and the counit ϵ is given by

$$\operatorname{Hom}_S(M,N) \otimes_R M \xrightarrow{\epsilon_N} \operatorname{id}_{\mathsf{K}(S)}(N), \quad \epsilon_N(\mu \otimes m) = \mu(m).$$

Since all modules have K-projective and K-injective resolutions by [3, thms. 7.1.1 and 8.1.1], we can get $- \overset{\mathrm{L}}{\otimes}_R M$ from $- \otimes_R M$ by using a K-projective resolution in the first variable, and we can get $\operatorname{RHom}_S(M,-)$ from $\operatorname{Hom}_S(M,-)$ by using a K-injective resolution in the second variable. The adjointness described above is inherited by the derived functors in a straightforward way.

(1.5) **Foxby equivalence.** Consider the adjoint pair of functors,

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\operatorname{RHom}_S(M,-)}{\overset{- \overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(S).$$

Theorem (1.1) now says: Denoting unit and counit of the adjunction by η and ϵ, there are Auslander and Bass classes,

$$\mathcal{A}_M = \{ L \in \mathsf{D}(R^{\mathrm{opp}}) \mid \eta_L \text{ is an isomorphism} \},$$
$$\mathcal{B}_M = \{ N \in \mathsf{D}(S) \mid \epsilon_N \text{ is an isomorphism} \},$$

and the functors $- \overset{\mathrm{L}}{\otimes}_R M$ and $\operatorname{RHom}_S(M,-)$ restrict to a pair of quasi-inverse equivalences of categories,

$$\mathcal{A}_M \underset{\operatorname{RHom}_S(M,-)}{\overset{- \overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M.$$

(1.6) **Classical Foxby equivalence.** Classical Foxby equivalence in the setup of [2, sec. 3] is a special case of the above: Let R be a commutative noetherian ring, viewed as a DGA concentrated in degree zero, and let S equal R. Let M be a dualizing complex over R, that is, M is a bounded complex of injective modules with finitely generated homology, so that the canonical morphism $R \longrightarrow \operatorname{RHom}_R(M,M)$, given by $r \mapsto (m \mapsto rm)$, is a quasi-isomorphism. Clearly, M is a DG-R-left-S-left-module.

So (1.1) applies, and our adjoint pair

$$\mathsf{D}(R) \underset{\operatorname{RHom}_R(M,-)}{\overset{- \overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(R)$$

is simply the pair of functors from the classical Foxby equivalence theorem, [2, thm. (3.2)], and our Auslander and Bass classes,

$$\mathcal{A}_M = \{ L \in \mathsf{D}(R) \mid \eta_L \text{ is an isomorphism} \},$$
$$\mathcal{B}_M = \{ N \in \mathsf{D}(R) \mid \epsilon_N \text{ is an isomorphism} \},$$

are simply the Auslander and Bass classes of [2, def. (3.1)], except that we have avoided the (unnecessary) boundedness conditions in [2]. Furthermore, our equivalence result, (1.5), essentially specializes to [2]'s equivalence theorem, [2, thm. (3.2)].

Moreover, we show in section (3.10) how our general theory of evaluation morphisms can be used to prove that $\mathcal{A}_M$ contains all bounded complexes of flat modules, and that $\mathcal{B}_M$ contains all bounded complexes of injective modules, as already proved in [2].

Finally, the above way of viewing classical Foxby equivalence also applies to the more general Foxby equivalence theory with semi-dualizing complexes constructed in [6, sec. 4].

(1.7) **Dwyer and Greenlees' theory.** Dwyer and Greenless' theory from [7] is a special case of the above: Let S be any ring, viewed as a DGA concentrated in degree zero, and let M be a perfect complex of S-left-modules, that is, a bounded complex of finitely generated projective S-left-modules. Set R equal to $\operatorname{Hom}_S(M,M)$. It is not difficult to check that this is a DGA, that M aquires the structure of DG-R-left-module, and that this structure is compatible with M's S-structure, so that M is in fact a DG-R-left-S-left-module, ${}_{R,S}M$.

So (1.5) applies, and our quasi-inverse equivalences between the Auslander and Bass classes,

$$\mathcal{A}_M \underset{\operatorname{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M,$$

is identical to the right half of the following diagram from Dwyer and Greenlees' Morita theorem, [7, thm. 2.1]:

$$\mathbf{A}_{\mathrm{comp}} \underset{C}{\overset{E}{\rightleftarrows}} \mathsf{mod}\text{-}\mathcal{E} \underset{E}{\overset{T}{\rightleftarrows}} \mathbf{A}_{\mathrm{tors}}$$

([7] denotes our R by $\mathcal{E}$, and our $\mathsf{D}(R^{\mathrm{opp}})$ by $\mathsf{mod}\text{-}\mathcal{E}$).

This can be seen trough the following steps:

- Observe that our functors $-\overset{\mathrm{L}}{\otimes}_R M$ and $\operatorname{RHom}_S(M,-)$ are the same as [7]'s functors T and E.
- Prove that the Bass class $\mathcal{B}_M$ equals $\mathbf{A}_{\mathrm{tors}}$. This is done in [7, thm. 2.1].
- Prove that the Auslander class $\mathcal{A}_M$ equals $\mathsf{mod}\text{-}\mathcal{E}$ in [7]'s notation, which is $\mathsf{D}(R^{\mathrm{opp}})$ in our notation. This is also done in [7, thm. 2.1]; we show in section (3.6) how it follows from our general theory of evaluation morphisms.

Finally, replacing M by $\operatorname{Hom}_S(M,S)$, our theory can be specialized to the other half of [7, thm. 2.1].

(1.8) **The Auslander and Bass classes for** $\mathrm{E}(k)$**.** Let R be a commutative, local, noetherian ring, with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$, and let $\mathrm{E}(k)$ denote the injective hull of k. We want to consider our theory in the case where $S = R$ and $M = \mathrm{E}(k)$. In this setup it turns out that the corresponding Auslander and Bass classes contain k precisely when R is Gorenstein.

Recall that the same statement is true for the Auslander and Bass classes of classical Foxby equivalence where M is the dualizing complex D (see [5, (3.1.12) and (3.2.10)]). However, not all commutative, local, noetherian rings admit a dualizing complex.

Note that, since the duality theory involving the functor $\mathrm{RHom}_R(-, \mathrm{E}(k))$ is just classical Matlis duality, it seems reasonable that one should call the theory treated in this and the next two paragraphs "Matlis equivalence".

(1.9) **Lemma.** *Let* R *be as in* (1.8). *Then the following statements are equivalent:*

1. R *is Gorenstein.*
2. $\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \mathcal{S}^{-d}k$ *for some* d.
3. $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \mathcal{S}^{d}k$ *for some* d.

If the equivalent statements hold, then $d = \dim R$.

Proof. Let $-^{\vee} = \mathrm{RHom}_R(-, \mathrm{E}(k))$ denote the Matlis duality functor.

It is not difficult to see that each of the numbered statements is equivalent to the same statement for $\widehat{R}$, the completion of R in the $\mathfrak{m}$-adic topology. Hence, we can suppose that R is complete. For this, one uses that the artinian R-module $\mathrm{E}(k)$ can be viewed as an $\widehat{R}$-module, which satisfies the isomorphisms of $\widehat{R}$-modules $\mathrm{E}(k) \otimes_R \widehat{R} \cong E_{\widehat{R}}(k) \cong \mathrm{E}(k)$, see [4, ex. 3.2.14].

We first show that (1) is equivalent to (2): There are isomorphisms

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \mathrm{RHom}_R(k^{\vee}, \mathrm{E}(k)^{\vee}) \cong \mathrm{RHom}_R(k, R), \tag{1.9.1}$$

which are hyperhomological generalizations of formulae from classical Matlis duality (see [14, thm. 18.6]). But R is Gorenstein precisely if $\mathrm{RHom}_R(k, R)$ is isomorphic to $\mathcal{S}^{-d}k$ for some d, by the hyperhomological version of [14, thm. 18.1]. So the result follows.

To conclude the proof, we show that (2) is equivalent to (3). Consider the following chain of isomorphisms,

$$\begin{aligned}
\mathrm{RHom}_R(\mathrm{E}(k),k) &\overset{(a)}{\cong} \mathrm{RHom}_R(\mathrm{E}(k),k^\vee)\\
&= \mathrm{RHom}_R(\mathrm{E}(k),\mathrm{RHom}_R(k,\mathrm{E}(k)))\\
&\overset{(b)}{\cong} \mathrm{RHom}_R(k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k),\mathrm{E}(k))\\
&= (k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))^\vee,
\end{aligned}$$

where "(a)" is again by Matlis duality, and "(b)" is by adjointness. This shows that (3) implies (2). Since $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$ has artinian homology, it also shows by Matlis duality that

$$\mathrm{RHom}_R(\mathrm{E}(k),k)^\vee \cong k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k), \tag{1.9.2}$$

and this shows that (2) implies (3).

Finally, in case the numbered conditions hold so R is Gorenstein, we know $\mathrm{RHom}_R(k,R) \cong \mathcal{S}^{-\dim R}k$, again by the hyperhomological version of [14, thm. 18.1]. Hence (1.9.1) proves $\mathrm{RHom}_R(\mathrm{E}(k),k) \cong \mathcal{S}^{-\dim R}k$, and (1.9.2) proves $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \mathcal{S}^{\dim R}k$. So we conclude $d = \dim R$. □

(1.10) **Theorem (Gorenstein sensitivity).** *Let R be as in (1.8). Then the following statements are equivalent:*

1. *R is Gorenstein.*
2. *$k \in \mathcal{A}_{\mathrm{E}(k)}$.*
3. *$k \in \mathcal{B}_{\mathrm{E}(k)}$.*

Proof. (1) ⇒ (2). When R is Gorenstein, we have $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \mathcal{S}^{\dim R}k$ by lemma (1.9)(3). Hence

$$\mathrm{RHom}_R(\mathrm{E}(k),k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)) \cong \mathrm{RHom}_R(\mathrm{E}(k),\mathcal{S}^{\dim R}k) \cong k,$$

where the second "$\cong$" uses lemma (1.9)(2).

To see $k \in \mathcal{A}_{\mathrm{E}(k)}$ we must see that the unit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k), \mathrm{RHom}_R(\mathrm{E}(k),-))$ evaluated on k is an isomorphism, that is, that $k \xrightarrow{\eta_k} \mathrm{RHom}_R(\mathrm{E}(k),k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))$ is an isomorphism. This is the same as seeing that $\mathrm{H}\eta_k$ is an isomorphism. But by the above computation, both source and target of η_k have homology given by k in degree 0, and 0 in all other degrees, so since k is a simple module, it suffices to see that $\mathrm{H}_0\eta_k$ is non-zero.

To compute η_k, we replace $\mathrm{E}(k)$ by a free resolution F. Thus, η_k is the chain map $k \longrightarrow \mathrm{Hom}_R(F,k \otimes_R F)$ given by $x \mapsto (f \mapsto x \otimes f)$. In particular we have $1_k \mapsto (f \mapsto 1_k \otimes f)$, so all we need to see is that the cycle $(f \mapsto 1_k \otimes f)$ in the complex $\mathrm{Hom}_R(F,k \otimes_R F)$ is non-zero, that is, not a boundary. But the boundaries in a Hom complex are exactly the null homotopic chain maps, so we must check that $f \mapsto 1_k \otimes f$ is not null homotopic.

But if it were null homotopic, then it would remain so upon tensoring with k. That is, $k \otimes_R F \longrightarrow k \otimes_R k \otimes_R F$ given by $y \otimes f \mapsto y \otimes 1_k \otimes f$ would be null homotopic. But using $k \otimes_R k \cong k$, this map can be identified with the identity on $k \otimes_R F$, hence cannot be null homotopic because $k \otimes_R F \cong k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$ has non-vanishing homology by lemma (1.9)(3).

(1) $\Rightarrow$ (3). This is seen by a computation similar to the one above.

(2) $\Rightarrow$ (1). If $k \in \mathcal{A}_{\mathrm{E}(k)}$ then $k \xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{E}(k), k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))$. And it is easy to see that R's maximal ideal $\mathfrak{m}$ annihilates the modules in a suitable representative of $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$, so

$$k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \bigoplus_{i \in I} \mathcal{S}^{\beta_i} k. \tag{1.10.1}$$

Combining this gives $\mathrm{RHom}_R(\mathrm{E}(k), \bigoplus_{i\in I} \mathcal{S}^{\beta_i} k) \cong k$. Suppose that $\bigoplus_{i\in I} \mathcal{S}^{\beta_i} k$ contained more than one summand, say $\mathcal{S}^{\beta_1} k \oplus \mathcal{S}^{\beta_2} k \oplus (\bigoplus_{i\in I'} \mathcal{S}^{\beta_i} k)$. Then we would have

$$\mathrm{RHom}_R(\mathrm{E}(k), \mathcal{S}^{\beta_1} k) \oplus \mathrm{RHom}_R(\mathrm{E}(k), \mathcal{S}^{\beta_2} k) \oplus \mathrm{RHom}_R(\mathrm{E}(k), \textstyle\bigoplus_{i\in I'} \mathcal{S}^{\beta_i} k) \cong k. \tag{1.10.2}$$

However, using $\mathrm{E}(k) \otimes_R \widehat{R} \cong \mathrm{E}_{\widehat{R}}(k) \cong \mathrm{E}(k)$ again, it is not difficult to see

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \mathrm{RHom}_{\widehat{R}}(\mathrm{E}_{\widehat{R}}(k), k),$$

and by Matlis duality, this is again $\mathrm{RHom}_{\widehat{R}}(k, \widehat{R})$ which is non-zero. As k is an indecomposable object in $\mathsf{D}(R)$, this gives a contradiction with equation (1.10.2), and thus there can only be one summand in (1.10.1), so $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \mathcal{S}^{\beta_j} k$. By (1.9)(3), R is Gorenstein.

(3) $\Rightarrow$ (1). If $k \in \mathcal{B}_{\mathrm{E}(k)}$ then $\mathrm{RHom}_R(\mathrm{E}(k), k) \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \xrightarrow{\cong} k$. Again it is easy to see that $\mathfrak{m}$ annihilates the modules in a suitable representative of $\mathrm{RHom}_R(\mathrm{E}(k), k)$. Thus

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \bigoplus_{i\in I} \mathcal{S}^{\beta_i} k.$$

Combining this gives $(\bigoplus_{i\in I} \mathcal{S}^{\beta_i} k) \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \bigoplus_{i\in I} (\mathcal{S}^{\beta_i} k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)) \cong k$.

Again, using that k is an indecomposable object in $\mathsf{D}(R)$, the only possibility is that there is only one summand, so $k \cong \mathcal{S}^{\beta_j} k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$. By (1.9)(3), R is Gorenstein. $\square$

2. Small K-projective resolutions

This section proves a result, theorem (2.2), on the existence of "small" K-projective resolutions. It is an improvement and clarification of [1, sec. 1, prop. 1] [1] (we weaken the assumption that the DGA in question is positively graded), and is necessary in section 3 for our results on the size of Auslander and Bass classes, and in the forthcoming [10] on Gorenstein DGAs.

Consider a DGA, R, satisfying

- $\mathrm{H}_i R = 0$ for $i < 0$.
- $\mathrm{H}_0 R$ is left-noetherian.
- $\mathrm{H}_i R$ is finitely generated as a left-module over $\mathrm{H}_0 R$ for all $i \in \mathbb{Z}$.

We prove in theorem (2.2) that any DG-R-left-module M with right-bounded $\mathrm{H}M$ and with each $\mathrm{H}_i M$ finitely generated over $\mathrm{H}_0 R$ has a semi-free resolution $P \xrightarrow{\simeq} M$ such that

$$P^\natural = \bigoplus_{i \geq u} \mathcal{S}^i (R^\natural)^{\gamma_i},$$

with $\gamma_i \in \mathbb{N}_0$.

The point of this result is that [1, sec. 1, thms. 1 and 2] shows that when such a resolution exists for a DG-module A, then the evaluation morphisms

$$\mathrm{RHom}(A, B) \overset{\mathrm{L}}{\otimes} F \xrightarrow{\omega} \mathrm{RHom}(A, B \overset{\mathrm{L}}{\otimes} F)$$

and

$$A \overset{\mathrm{L}}{\otimes} \mathrm{RHom}(B, I) \xrightarrow{\theta} \mathrm{RHom}(\mathrm{RHom}(A, B), I)$$

are isomorphisms under some boundedness restrictions on B, on the K-flat resolution of F, and on the K-injective resolution of I.

(2.1) **Lemma.** *Let R be a DGA with $\mathrm{H}_{-1} R = 0$, let $L \xrightarrow{\alpha} M$ be a morphism of DG-R-left-modules, and let n be in $\mathbb{N}$.*

We can then construct a commutative diagram of morphisms of DG-R-left-modules,

$$\begin{tikzcd} \mathcal{S}^n R^{(Y)} \arrow[r, "\Delta"] & L \arrow[r, "\lambda"] \arrow[d, "\alpha"'] & \widetilde{L}. \arrow[dl, "\widetilde{\alpha}"] \\ & M & \end{tikzcd}$$

so that

1. $\mathrm{H}_n \widetilde{\alpha}$ *is injective, and if* $\mathrm{H}_n \alpha$ *is surjective then* $\mathrm{H}_n \widetilde{\alpha}$ *is bijective.*
2. $\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L \xrightarrow{\lambda} \widetilde{L} \longrightarrow$ *is a mapping cone triangle, in particular,* λ *is injective.*

[1] Since submitting the present version of the manuscript, we were made aware by H.-B. Foxby that a proof of theorem (2.2) is to appear in [3]. Hence the proof appearing below of theorem (2.2) will not be included in the final version of the manuscript.

3. *$\widetilde{L}/L$ is isomorphic to $\mathcal{S}^{n+1}R^{(Y)}$, and is in particular free on a basis of cycles.*

in the following way: Pick any data Y and $\{m_y\}_{y\in Y}$ satisfying:

- *$Y \subseteq Z_n L$ is so that $\operatorname{cls} Y$ generates $\operatorname{Ker} \mathrm{H}_n\alpha$.*
- *$\{m_y\}_{y\in Y}$ is a system in M_{n+1} so that $\partial^M m_y = \alpha y$ for each $y \in Y$.*

(such data can always be picked), and

- *Define $\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L$ by $e_y \longmapsto y$, where e_y is the generator of the y'th copy of $\mathcal{S}^n R$.*
- *Define $L \xrightarrow{\lambda} \widetilde{L}$ as the mapping cone of Δ. That is,*

$$\widetilde{L}^\natural = (\mathcal{S}^1\mathcal{S}^n R^{(Y)})^\natural \oplus L^\natural, \quad \partial^{\widetilde{L}}(\mathcal{S}^1 \sum r_y e_y, \ell) = (\mathcal{S}^1 \sum -\partial(r_y)e_y, \partial^L \ell + \sum r_y y),$$

and λ is just the inclusion of L into $\widetilde{L}$.

- *Define $\widetilde{L} \xrightarrow{\widetilde{\alpha}} M$ by*

$$\widetilde{\alpha}(\mathcal{S}^1 \sum r_y e_y, \ell) = \sum (-1)^{|r_y|} r_y m_y + \alpha\ell.$$

Proof. To see that the commutative diagram exists, let us first check that $\widetilde{\alpha} : \widetilde{L} \longrightarrow M$ is a morphism of DG-R-left-modules. It is easy to check that $\widetilde{\alpha}$ is R-linear, if one remembers the sign convention for shifts. So we need to see that $\widetilde{\alpha}$ commutes with the differentials in question. We start by considering the expression

$$\begin{aligned}
\widetilde{\alpha}\partial^{\widetilde{L}}(\mathcal{S}^1 \sum r_y e_y, \ell) &= \widetilde{\alpha}(\mathcal{S}^1 \sum -\partial(r_y)e_y, \partial^L \ell + \sum r_y y) \\
&= \sum (-1)^{|\partial r_y|}(-\partial(r_y))m_y + \alpha(\partial^L \ell + \sum r_y y) \\
&= \sum (-1)^{|r_y|}\partial(r_y)m_y + \alpha\partial^L \ell + \sum r_y \alpha y
\end{aligned}$$

Next we need to calculate

$$\begin{aligned}
\partial^M \widetilde{\alpha}(\mathcal{S}^1 \sum r_y e_y, \ell) &= \partial^M (\sum (-1)^{|r_y|} r_y m_y + \alpha\ell) \\
&= \sum (-1)^{|r_y|} \partial^M (r_y m_y) + \partial^M \alpha\ell \\
&= \sum (-1)^{|r_y|} (\partial(r_y)m_y + (-1)^{|r_y|} r_y \partial^M(m_y)) + \partial^M \alpha\ell \\
&= \sum (-1)^{|r_y|}\partial(r_y)m_y + \partial^M \alpha\ell + \sum r_y \partial^M(m_y).
\end{aligned}$$

And since $\{m_y\}_{y\in Y}$ is a system in M_{n+1} such that $\partial^M m_y = \alpha y$ for all $y \in Y$, we conclude that $\widetilde{\alpha}$ commutes with the differentials, so $\widetilde{\alpha}$ is a morphism of DG-R-left-modules.

It is clear that $\widetilde{\alpha}\lambda = \alpha$.

(1) Since $\widetilde{L}$ is the mapping cone of $\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L$, we have the following distinguished triangle,

$$\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L \longrightarrow \widetilde{L} \longrightarrow,$$

producing a long exact homology sequence fitting into the following commutative diagram of $\mathrm{H}_0 R$-left-modules.

$$\begin{tikzcd}
 & \operatorname{Ker} \mathrm{H}_n \alpha \arrow[d, hook] & & \\
\mathrm{H}_n \mathcal{S}^n R^{(Y)} \arrow[r, "\mathrm{H}_n \Delta"] & \mathrm{H}_n L \arrow[r] \arrow[d, "\mathrm{H}_n \alpha"'] & \mathrm{H}_n \widetilde{L} \arrow[r] \arrow[dl, "\mathrm{H}_n \widetilde{\alpha}"] & \mathrm{H}_{n-1} \mathcal{S}^n R^{(Y)}. \\
 & \mathrm{H}_n M & &
\end{tikzcd}$$

Note that by construction, $\mathrm{H}_n \Delta$ surjects onto $\operatorname{Ker} \mathrm{H}_n \alpha$. And $\mathrm{H}_{n-1} \mathcal{S}^n R^{(Y)} = \mathrm{H}_{-1} R^{(Y)} = 0$. Thus the above diagram is in fact

$$\begin{tikzcd}
 & \operatorname{Ker} \mathrm{H}_n \alpha \arrow[d, hook] & & \\
\mathrm{H}_n \mathcal{S}^n R^{(Y)} \arrow[ur, two heads] \arrow[r, "\mathrm{H}_n \Delta"] & \mathrm{H}_n L \arrow[r, two heads] \arrow[d, "\mathrm{H}_n \alpha"'] & \mathrm{H}_n \widetilde{L} \arrow[r] \arrow[dl, "\mathrm{H}_n \widetilde{\alpha}"] & 0. \\
 & \mathrm{H}_n M & &
\end{tikzcd}$$

A small diagram chase shows that $\mathrm{H}_n \widetilde{\alpha}$ is injective, and that $\mathrm{H}_n \widetilde{\alpha}$ is bijective when $\mathrm{H}_n \alpha$ is surjective.

(2) By construction, λ is the mapping cone of Δ, so by construction, $\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L \xrightarrow{\lambda} \widetilde{L} \longrightarrow$ is a mapping cone triangle.

(3) $\widetilde{L}$ is the mapping cone of $\mathcal{S}^n R^{(Y)} \xrightarrow{\Delta} L$. This makes it easy to compute $\widetilde{L}/L$ which is just $\mathcal{S}^n R^{(Y)}$ with the usual differential. $\square$

(2.2) **Theorem.** *Let R be a DGA satisfying the following conditions:*

- $\mathrm{H}_i R = 0$ *for* $i < 0$.
- $\mathrm{H}_0 R$ *is left-noetherian.*
- $\mathrm{H}_i R$ *is finitely generated as a left-module over* $\mathrm{H}_0 R$ *for all* $i \in \mathbb{Z}$.

Suppose M is a DG-R-left-module with the following properties

- $\mathrm{H}_i M$ *is finitely generated over* $\mathrm{H}_0 R$ *for all* $i \in \mathbb{Z}$.
- *There exists a* $u \in \mathbb{Z}$ *such that* $\mathrm{H}_i M = 0$ *for all* $i < u$.

Then there exists a semi-free DG-R-left-module P and a quasi-isomorphism $P \xrightarrow{\simeq} M$ where

$$P^\natural = \bigoplus_{i \geq u} \mathcal{S}^i (R^\natural)^{\gamma_i},$$

and $\gamma_i \in \mathbb{N}_0$.

Proof. Shifting if necessary we may assume $\mathrm{H}_i M = 0$ for $i < 0$. We then construct the claimed DG-R-left-module by induction.

We start by setting $P^{(-1)} = 0$ and by letting $P^{(-1)} \xrightarrow{\alpha^{(-1)}} M$ be the zero map.

Suppose now that $P^{(n-1)}$ and $P^{(n-1)} \xrightarrow{\alpha^{(n-1)}} M$ have been defined, and that all $\mathrm{H}_i P^{(n-1)}$ are finitely generated over $\mathrm{H}_0 R$. Let $X_n \subseteq \mathrm{Z}_n M$ be a finite set such that $\operatorname{cls} X_n$ generates $\mathrm{H}_n M$ over $\mathrm{H}_0 R$. There is a canonical map

$$\beta^{(n)} : \mathcal{S}^n R^{(X_n)} \longrightarrow M,$$

sending the generator e_x to x for $x \in X_n$, and hence a map

$$(\alpha^{(n-1)}, \beta^{(n)}) : P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)} \longrightarrow M,$$

and $\mathrm{H}_n(\alpha^{(n-1)}, \beta^{(n)})$ is a surjection, since already $\mathrm{H}_n \beta^{(n)}$ is a surjection. Moreover, all $\mathrm{H}_i(P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})$ are clearly finitely generated over $\mathrm{H}_0 R$.

We now use lemma (2.1) on the map $(\alpha^{(n-1)}, \beta^{(n)})$, noting that the set Y can be chosen finite, because it generates a submodule of the finitely generated module $\mathrm{H}_n(P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})$ over the left-noetherian ring $\mathrm{H}_0 R$. This results in

$$\begin{tikzcd}
\mathcal{S}^n R^{(Y_n)} \arrow[r, "\Delta_n"] & P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)} \arrow[r, "\lambda^{(n-1)}"] \arrow[d, "(\alpha^{(n-1)}, \beta^{(n)})"'] & (P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})^{\sim} \arrow[dl, "(\alpha^{(n-1)}, \beta^{(n)})^{\sim}"] \\
& M &
\end{tikzcd}$$

By the lemma, $\mathrm{H}_n((\alpha^{(n-1)}, \beta^{(n)})^{\sim})$ is bijective. Moreover, $\lambda^{(n-1)}$ is injective, and

$$\frac{(P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})^{\sim}}{P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)}}$$

is $\mathcal{S}^{n+1} R^{(Y_n)}$, and is free on a basis of cycles. Finally, all $\mathrm{H}_i(P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})^{\sim}$ are finitely generated over $\mathrm{H}_0 R$, as follows from the long exact sequence associated to the mapping cone triangle used in lemma (2.1) to define $(P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})^{\sim}$.

Defining

$$\begin{aligned} P^{(n)} &= (P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)})^{\sim}, \\ \alpha^{(n)} &= (\alpha^{(n-1)}, \beta^{(n)})^{\sim}, \end{aligned}$$

the above facts translate to: $\mathrm{H}_n \alpha^{(n)}$ is bijective; the map $P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)} \xrightarrow{\lambda^{(n-1)}} P^{(n)}$ is injective and

$$\frac{P^{(n)}}{P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)}}$$

is $\mathcal{S}^{n+1} R^{(Y_n)}$, and is free on a basis of cycles; and each $\mathrm{H}_i P^{(n)}$ is finitely generated over $\mathrm{H}_0 R$.

Iterating the above construction yields the commutative diagram

$$\begin{tikzcd}
P^{(-1)} \arrow[r, hook, "\iota^{(-1)}"] \arrow[d, "\alpha^{(-1)}"'] & P^{(-1)} \oplus \mathcal{S}^0 R^{(X_0)} \arrow[r, hook, "\lambda^{(-1)}"] \arrow[dl, "(\alpha^{(-1)},\beta^{(0)})"] & P^{(0)} \arrow[r, hook, "\iota^{(0)}"] \arrow[dll, "\alpha^{(0)}"] & P^{(0)} \oplus \mathcal{S}^1 R^{(X_1)} \arrow[r, hook, "\lambda^{(0)}"] \arrow[dlll, "(\alpha^{(0)},\beta^{(1)})"] & \cdots \\
M, & & & \cdots &
\end{tikzcd} \tag{2.2.1}$$

where the ι's are inclusions of direct summands. We let P be the union (i.e. the colimit) of the DG-modules in the diagram's first row, and let $P \xrightarrow{\alpha} M$ be the morphism induced by the morphisms in the diagram. We claim that this is the desired resolution.

P is semi-free: The first row in diagram (2.2.1) provides a filtration of P. The quotients are of the form

$$\frac{P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)}}{P^{(n-1)}} \cong \mathcal{S}^n R^{(X_n)}$$

and

$$\frac{P^{(n)}}{P^{(n-1)} \oplus \mathcal{S}^n R^{(X_n)}} \cong \mathcal{S}^{n+1} R^{(Y_n)}$$

(the second isomorphism was proved above), and these are free on bases of cycles, so P is semi-free.

α is a quasi-isomorphism: To prove that $\mathrm{H}_n\alpha$ is an isomorphism, consider step n in the construction that gave P and α. At this step, we proved that $\mathrm{H}_n\,\alpha^{(n)}$ was an isomorphism. We propose to show that in fact, $\mathrm{H}_n\alpha^{(n+i)}$ is an isomorphism for each $i \geq 0$; this will clearly force $\mathrm{H}_n\alpha$ to be an isomorphism, since α is induced by the system $\alpha^{(0)}, \alpha^{(1)}, \ldots$.

So consider our construction at step $n+i$: Going from $P^{(n+i)}$ to $P^{(n+i+1)}$ involves the maps

$$\begin{tikzcd}
P^{(n+i)} \arrow[r, hook, "\iota^{(n+i)}"] \arrow[dr, "\alpha^{(n+i)}"'] & P^{(n+i)} \oplus \mathcal{S}^{n+i+1} R^{(X_{n+i+1})} \arrow[r, hook, "\lambda^{(n+i)}"] \arrow[d, "(\alpha^{(n+i)},\beta^{(n+i+1)})"] & P^{(n+i+1)} \arrow[dl, "\alpha^{(n+i+1)}"] \\
& M. &
\end{tikzcd}$$

We will show that when $\mathrm{H}_n\alpha^{(n+i)}$ is an isomorphism, then the same is true for $\mathrm{H}_n\alpha^{(n+i+1)}$. For this, it is sufficient to see that H_n applied to the above diagram's horizontal arrows gives isomorphisms.

For the first arrow, this is clear, for it is just the inclusion of a direct summand, and $\mathrm{H}_n\,\mathcal{S}^{n+i+1} R^{(X_{n+i+1})} = \mathrm{H}_{-i-1}\,R^{(X_{n+i+1})} = 0$.

For the second arrow, observe that it is constructed using lemma (2.1), hence sits in a mapping cone triangle

$$\mathcal{S}^{n+i+1}R^{(Y_{n+i+1})} \longrightarrow P^{(n+i)} \oplus \mathcal{S}^{n+i+1}R^{(X_{n+i+1})} \xrightarrow{\lambda^{(n+i)}} P^{(n+i+1)} \longrightarrow .$$

As both $\mathrm{H}_n\mathcal{S}^{n+i+1}R^{(Y_{n+i+1})} = \mathrm{H}_{-i-1}R^{(Y_{n+i+1})}$ and $\mathrm{H}_{n-1}\mathcal{S}^{n+i+1}R^{(Y_{n+i+1})} = \mathrm{H}_{-i-2}R^{(Y_{n+i+1})}$ are zero, it follows that $\mathrm{H}_n\lambda^{(n+i)}$ is an isomorphism as desired.

$P^\natural$ has the desired form: If we take $\natural$ of diagram (2.2.1), then all the injections become split:

The injection

$$P^{(n)} \xhookrightarrow{\iota^{(n)}} P^{(n)} \oplus \mathcal{S}^{n+1}R^{(X_{n+1})}$$

is already split, and by the lemma, applying $\natural$ gives a split injection with

$$\frac{(P^{(n)} \oplus \mathcal{S}^{n+1}R^{(X_{n+1})})^\natural}{(P^{(n)})^\natural} \cong \mathcal{S}^{n+1}(R^\natural)^{(X_{n+1})}.$$

The injection

$$P^{(n)} \oplus \mathcal{S}^{n+1}R^{(X_{n+1})} \xhookrightarrow{\lambda^{(n)}} P^{(n+1)}$$

comes from lemma (2.1) where it was constructed as the mapping cone of $\mathcal{S}^{n+1}R^{(Y_{n+1})} \longrightarrow P^{(n)} \oplus \mathcal{S}^{n+1}R^{(X_{n+1})}$, so applying $\natural$ gives a split injection $(\lambda^{(n)})^\natural$ with

$$\frac{(P^{(n+1)})^\natural}{(P^{(n)} \oplus \mathcal{S}^{n+1}R^{(X_{n+1})})^\natural} \cong \mathcal{S}^{n+2}(R^\natural)^{(Y_{n+1})}.$$

So the diagram obtained from diagram (2.2.1) by taking $\natural$ builds $P^\natural$ by simply adding $\mathcal{S}^nR^{(X_n)}$'s and $\mathcal{S}^{n+1}R^{(Y_n)}$'s, so

$$P^\natural = (\bigoplus_{n\geq 0}\mathcal{S}^n(R^\natural)^{(X_n)}) \oplus (\bigoplus_{n\geq 0}\mathcal{S}^{n+1}(R^\natural)^{(Y_n)}).$$

As all X_n's and Y_n's are finite, this shows our claim on $P^\natural$. $\square$

3. Size of Auslander and Bass classes

We resume working under setup (1.3).

This section recalls some facts about the evaluation morphisms ω and θ from [1], and in lemma (3.3) rewrites unit and counit of the adjoint pair $(-\overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M,-))$ in terms of ω and θ. This is used in theorem (3.4) which under certain conditions characterizes the Auslander class by ω being an isomorphism, and the Bass class by θ being an isomorphism. This leads to corollaries (3.5) and (3.9) which under appropriate conditions state that the Auslander and Bass class contain many DG-modules.

Finally, this is applied in sections (3.6) and (3.10) to recover previously known results about the size of the Auslander and Bass classes in the case of Dwyer and Greenlees' theory (described in (1.7)) and classical Foxby equivalence (described in (1.6)).

(3.1) **Remark.** In [1, sec. 1] two so-called evaluation morphisms are given: If

$$ {}_TF_R,\ {}_{U,S}A,\ {}_{R,S}B $$

are DG-modules with structures as indicated, then there is a natural morphism of DG-T-left-U-right-modules

$$ {}_TF_R \otimes_R \operatorname{Hom}_S({}_{U,S}A, {}_{R,S}B) \xrightarrow{\omega} \operatorname{Hom}_S({}_{U,S}A, {}_TF_R \otimes_R {}_{R,S}B) $$

given by

$$ (\omega(f \otimes \alpha))(a) = f \otimes \alpha(a). $$

Moreover, if F can be resolved by a DG-T-left-R-right-module which is K-flat over R, and A can be resolved by a DG-S-left-U-left-module which is K-projective over S, then ω induces a natural morphism of derived functors,

$$ {}_TF_R \overset{\mathrm{L}}{\otimes}_R \operatorname{RHom}_S({}_{U,S}A, {}_{R,S}B) \xrightarrow{\omega} \operatorname{RHom}_S({}_{U,S}A, {}_TF_R \overset{\mathrm{L}}{\otimes}_R {}_{R,S}B). $$

Note that it is not necessary that F and A have structures over T and U. That is, omitting T or U or both, there would still be morphisms given by the same prescriptions.

And if

$$ {}_{R,T}A,\ {}_{R,S}B,\ {}_SI_U $$

are DG-modules with structures as indicated, then there is a natural morphism of DG-T-left-U-right-modules

$$ \operatorname{Hom}_S({}_{R,S}B, {}_SI_U) \otimes_R {}_{R,T}A \xrightarrow{\theta} \operatorname{Hom}_S(\operatorname{Hom}_R({}_{R,T}A, {}_{R,S}B), {}_SI_U) $$

given by

$$ (\theta(\beta \otimes a))(\alpha) = (-1)^{|a||\alpha|}\beta\alpha(a). $$

And if A can be resolved by a DG-R-left-T-left-module which is K-projective over R, and I can be resolved by a DG-S-left-U-right-module which is K-injective over S, then θ induces a natural morphism of derived functors,

$$ \operatorname{RHom}_S({}_{R,S}B, {}_SI_U) \overset{\mathrm{L}}{\otimes}_R {}_{R,T}A \xrightarrow{\theta} \operatorname{RHom}_S(\operatorname{RHom}_R({}_{R,T}A, {}_{R,S}B), {}_SI_U). $$

Note that again, T or U or both could be omitted in both morphisms.

(3.2) **Remark.** If we allow ourselves to be sloppy for a moment, then remark (3.1) can be applied to the Auslander class $\mathcal{A}_M$ as follows. Suppose that $\operatorname{RHom}_S(M, M)$ is isomorphic to R. To check whether $L \in \mathsf{D}(R^{\mathrm{opp}})$ is in $\mathcal{A}_M$, we must check whether $\operatorname{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$ gives us L back. But we have

$$ \begin{aligned} \operatorname{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M) &\overset{\omega}{\to} L \overset{\mathrm{L}}{\otimes}_R \operatorname{RHom}_S(M, M) \\ &\cong L \overset{\mathrm{L}}{\otimes}_R R \\ &\cong L. \end{aligned} $$

So $\operatorname{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$ gives L back precisely if ω is an isomorphism. There is a dual remark for $\mathcal{B}_M$ and θ.

Of course, this is very imprecise. To actually check whether L is in $\mathcal{A}_M$ amounts to checking whether the specific map η_L is an isomorphism, and to actually check whether N is in $\mathcal{B}_M$ amounts to checking whether the specific map ϵ_N is an isomorphism. However, we shall see in lemma (3.3) and theorem (3.4) that this is really what the above computations do.

(3.3) **Lemma.** 1. *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over S. Let ρ denote the canonical morphism in the derived category of DG-R-left-R-right-modules,*

$$R \xrightarrow{\rho} \operatorname{RHom}_S(M, M), \quad 1_R \longmapsto \mathrm{id}_M\,.$$

For any DG-R-right-module, L, there is a commutative diagram,

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[drr, "\eta_L"'] & L \overset{\mathrm{L}}{\otimes}_R R \arrow[r, "1_L \overset{\mathrm{L}}{\otimes}_R \rho"] & L \overset{\mathrm{L}}{\otimes}_R \operatorname{RHom}_S(M, M) \arrow[d, "\omega"] \\ & & \operatorname{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M), \end{tikzcd}$$

where φ is the canonical isomorphism, and where η_L is the unit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R M, \operatorname{RHom}_S(M, -))$, evaluated on L.

2. *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over R. Let σ denote the canonical morphism in the derived category of DG-S-left-S-right-modules,*

$$S \xrightarrow{\sigma} \operatorname{RHom}_R(M, M), \quad 1_S \longmapsto \mathrm{id}_M\,.$$

For any DG-S-left-module, N, there is a commutative diagram,

$$\begin{tikzcd} \operatorname{RHom}_S(M, N) \overset{\mathrm{L}}{\otimes}_R M \arrow[d, "\theta"'] \arrow[drr, "\epsilon_N"] & & \\ \operatorname{RHom}_S(\operatorname{RHom}_R(M, M), N) \arrow[r, "\operatorname{RHom}_R(\sigma, 1_N)"'] & \operatorname{RHom}_S(S, N) \arrow[r, "\varphi"'] & N, \end{tikzcd}$$

where φ is the canonical isomorphism, and where ϵ_N is the counit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R M, \operatorname{RHom}_S(M, -))$, evaluated on N.

Proof. The proofs of (1) and (2) are similar, so we only show (1).

Replace L by a K-flat resolution (this is always possible, since L only has R-right-structure), and replace M by a DG-R-left-S-left-module which is K-projective over S. This enables us to write Hom_S and $\otimes_R$ rather than RHom_S and $\overset{\mathrm{L}}{\otimes}_R$.

Now let ℓ be in L, and consider the lemma's composition of morphisms evaluated on ℓ:

$$(\omega\circ(1_L\otimes\rho)\circ\varphi)(\ell) = (\omega\circ(1_L\otimes\rho))(\ell\otimes 1_R) = \omega(\ell\otimes \mathrm{id}_M) = (m\mapsto \ell\otimes m).$$

This is indeed $\eta_L(\ell)$, as one sees in remark (1.4). $\square$

(3.4) **Theorem.** 1. *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over S, and that the canonical morphism in the derived category of DG-R-left-R-right-modules $R \xrightarrow{\rho} \mathrm{RHom}_S(M,M)$ is a quasi-isomorphism. Then M's Auslander class is*

$$\mathcal{A}_M = \left\{ L\in \mathsf{D}(R^{\mathrm{opp}}) \;\middle|\; \begin{array}{l} L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M,M) \xrightarrow{\omega} \\ \mathrm{RHom}_S(M, L\overset{\mathrm{L}}{\otimes}_R M) \\ \text{ is an isomorphism} \end{array} \right\}$$

2. *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over R, and that the canonical morphism in the derived category of DG-S-left-S-right-modules $S \xrightarrow{\sigma} \mathrm{RHom}_R(M,M)$ is a quasi-isomorphism. Then M's Bass class is*

$$\mathcal{B}_M = \left\{ N\in \mathsf{D}(S) \;\middle|\; \begin{array}{l} \mathrm{RHom}_S(M,N)\overset{\mathrm{L}}{\otimes}_R M \xrightarrow{\theta} \\ \mathrm{RHom}_S(\mathrm{RHom}_R(M,M),N) \\ \text{ is an isomorphism} \end{array} \right\}$$

Proof. Again, the proofs of (1) and (2) are similar, so we only show (1).

Since M can be resolved by a DG-R-left-S-left-module which is K-projective over S, we are in the situation of lemma (3.3)(1). The lemma's composition $\omega\circ(1_L \overset{\mathrm{L}}{\otimes}_R \rho)\circ\varphi$ is the unit of the adjoint pair $(-\overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M,-))$, evaluated at L. Hence L is in $\mathcal{A}_M$ precisely if the composition is an isomorphism.

But since $R\xrightarrow{\rho}\mathrm{RHom}_S(M,M)$ is a quasi-isomorphism, both maps φ and $1_L\overset{\mathrm{L}}{\otimes}_R \rho$ are isomorphisms. Hence η_L is an isomorphism precisely when ω is. $\square$

(3.5) **Corollary.** 1. *Make the same assumptions as in theorem (3.4)(1): M can be resolved by a DG-R-left-S-left-module which is K-projective over S, and $R\xrightarrow{\rho}\mathrm{RHom}_S(M,M)$ is a quasi-isomorphism. Suppose moreover that when we forget the R-structure on M, we can resolve M by a DG-S-left-module, A, so that $({}_S A)^\natural$ is a direct summand in a finite coproduct of shifts of $S^\natural$.*

Then M's Auslander class $\mathcal{A}_M$ is all of $\mathsf{D}(R^{\mathrm{opp}})$.

2. *Make the same assumptions as in theorem (3.4)(2): M can be resolved by a DG-R-left-S-left-module which is K-projective over R,*

and $S \xrightarrow{\sigma} \mathrm{RHom}_R(M,M)$ is a quasi-isomorphism. Suppose moreover that when we forget the S-structure on M, we can resolve M by a DG-R-left-module, A, so that $({}_R A)^\natural$ is a direct summand in a finite coproduct of shifts of $R^\natural$.

Then M's Bass class $\mathcal{B}_M$ is all of $\mathsf{D}(S)$.

Proof. Again the proofs of (1) and (2) are similar, so we only show (1).

We can use theorem (3.4)(1) to get $\mathcal{A}_M$. So to prove the corollary, we must see that

$$L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M,M) \xrightarrow{\omega} \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$$

is an isomorphism for any L.

Now, to see whether ω is an isomorphism, there is no need to remember the R-structure on the M's appearing in the first variable of the RHom's. Hence we can use the DG-S-left-module ${}_S A$ which is a resolution of ${}_S M$ to compute the two RHom's. But when ${}_S A$ has the special form required in the corollary, ω is an isomorphism by [1, sec. 1, thm. 2]. (Note that [1] actually requires $A^\natural$ itself to be a finite coproduct of shifts of $S^\natural$, but gives a proof which also applies to direct summands.)

So any L is in $\mathcal{A}_M$. □

(3.6) **Dwyer and Greenlees' theory (continued from section (1.7)).** Consider again section (1.7). It actually starts with a complex M which is perfect over S, hence corollary (3.5)(1)'s conditions on existence of resolutions hold. And the corollary's condition that $R \xrightarrow{\rho} \mathrm{RHom}_S(M,M)$ is a quasi-isomorphism is automatic, since we have in effect defined R to be $\mathrm{RHom}_S(M,M)$.

So corollary (3.5)(1) says that $\mathcal{A}_M$ is all of $\mathsf{D}(R^{\mathrm{opp}})$, as claimed already in example (1.7). (Recall that [7] denotes our $\mathsf{D}(R^{\mathrm{opp}}$ by $\mathsf{mod}\text{-}\mathcal{E}$.)

(3.7) **Definition.** If Q is a DGA, then we define two classes of DG-Q-left-modules by

$$\mathcal{F}(Q) = \left\{ L \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} L \text{ is quasi-isomorphic to a} \\ K\text{-flat left-bounded DG-module} \end{array} \right\}$$

and

$$\mathcal{I}(Q) = \left\{ N \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} N \text{ is quasi-isomorphic to a} \\ K\text{-injective right-bounded DG-module} \end{array} \right\}.$$

(3.8) **Definition.** Let Q be a DGA. We say that a DG-Q-left-module A is locally finite if $\mathrm{H}_i A$ is finitely generated as an $\mathrm{H}_0 Q$-left-module for each i.

(3.9) **Corollary.** 1. *Make the same assumptions as in theorem (3.4)(1): M can be resolved by a DG-R-left-S-left-module which is K-projective* *over S, and $R \xrightarrow{\rho} \mathrm{RHom}_S(M, M)$ is a quasi-isomorphism. Suppose moreover the following:*

- *R and S are non-negatively graded.*
- *$\mathrm{H}_0 S$ is left-noetherian, and S is locally finite as a DG-S-left-module.*
- *$\mathrm{H}M$ is bounded, and M is locally finite as a DG-S-left-module (here we have forgotten about M's R-structure).*

Then

$$\mathcal{F}(R^{\mathrm{opp}}) \subseteq \mathcal{A}_M.$$

2. *Make the same assumptions as in theorem (3.4)(2): M can be resolved by a DG-R-left-S-left-module which is K-projective over R, and $S \xrightarrow{\sigma} \mathrm{RHom}_R(M, M)$ is a quasi-isomorphism. Suppose moreover the following:*
   - *R and S are non-negatively graded.*
   - *$\mathrm{H}_0 R$ is left-noetherian, and R is locally finite as a DG-R-left-module.*
   - *$\mathrm{H}M$ is bounded, and M is locally finite as a DG-R-left-module (here we have forgotten about M's S-structure).*

   Then

$$\mathcal{I}(S) \subseteq \mathcal{B}_M.$$

Proof. Again, the proofs of (1) and (2) are similar, so we only show (1).

We can use theorem (3.4) to get $\mathcal{A}_M$. So to prove the corollary's claim, we must see that

$$L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M, M) \xrightarrow{\omega} \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$$

is an isomorphism when L is in $\mathcal{F}(R^{\mathrm{opp}})$.

Now, to see whether ω is an isomorphism, there is no need to remember the R-structure on the M's appearing in the first variable of the RHom's. Hence we can replace these M's by any quasi-isomorphic DG-S-left-module, ${}_S P$. We have made the assumptions that S is non-negatively graded and locally finite, that $\mathrm{H}_0 S$ is left-noetherian, that $\mathrm{H}M$ is bounded, so in particular right-bounded, and that ${}_S M$ is locally finite. Hence theorem (2.2) says that we can choose an ${}_S P$ which is semi-free and in particular K-projective, and has

$$({}_S P)^{\natural} = \bigoplus_{j \geq i} \mathcal{S}^j (S^{\natural})^{\gamma_j}$$

for certain finite numbers i and γ_j.

We can also replace the M's appearing in the second variable of the RHom's by any quasi-isomorphic DG-R-left-S-left-module ${}_{R,S}B$. And B can be chosen left-bounded: Since both R and S are concentrated on one side of degree 0 (namely, they are both non-negatively graded), it makes sense to truncate DG-R-left-S-left-modules, and since $\mathrm{H}M$ is bounded, and so in particular left-bounded, we can truncate M to the left to get a left-bounded B.

Finally, when L_R is in $\mathcal{F}(R^{\mathrm{opp}})$, we can replace L_R by a quasi-isomorphic K-flat left-bounded DG-R-right-module, F_R.

So what we need to see is in fact that

$$F \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(P, B) \xrightarrow{\omega} \mathrm{RHom}_S(P, F \overset{\mathrm{L}}{\otimes}_R B)$$

is an isomorphism. But P is K-projective and F is K-flat, so this is really

$$F \otimes_R \mathrm{Hom}_S(P, B) \xrightarrow{\omega} \mathrm{Hom}_S(P, F \otimes_R B).$$

And P, B, and F being as they are, this is known to be an isomorphism from [1, sec. 1, thm. 2]. $\square$

(3.10) **Classical Foxby equivalence (continued from section** (1.6)**).** Consider again section (1.6). The conditions of corollary (3.9)(1) hold: Since R is a commutative local noetherian ring and S equals R, we can resolve M by a DG-R-left-S-left-module which is K-projective over S simply by resolving it by a K-projective resolution of M as an R-complex. And we have that $R \longrightarrow \mathrm{RHom}_S(M, M)$ is a quasi-isomorphism by assumption on M. Finally, the three itemized requirements in the corollary are immediate by the assumptions on R, S, and M.

So corollary (3.9)(1) says that $\mathcal{A}_M$ contains $\mathcal{F}(R)$ (note $R^{\mathrm{opp}} = R$). In particular, $\mathcal{A}_M$ contains all bounded complexes of flat modules, as claimed in section (1.6).

Likewise, corollary (3.9)(2) says that $\mathcal{B}_M$ contains $\mathcal{I}(R)$. In particular, $\mathcal{B}_M$ contains all bounded complexes of injective modules, as claimed in section (1.6).

(3.11) **Remark.** Suppose that, of the data required in corollary (3.9)(1), only S and M are given. Replacing M with a K-projective resolution and constructing the endomorphism DGA, $\mathcal{E} = \mathrm{Hom}_S(M, M)$, the module M becomes a DG-$\mathcal{E}$-left-S-left-module.

Now, we can generally not hope to use corollary (3.9)(1) with $R = \mathcal{E}$, for this algebra is almost never non-negatively graded. However, it is a point that we can sometimes let R be a truncation of $\mathcal{E}$: If $\mathcal{E}$ only has homology in non-negative degrees, then we can let R be the truncation to non-negative degrees of $\mathcal{E}$. This truncation embeds into $\mathcal{E}$, so M becomes a DG-R-left-S-left-module. The canonical map $R \longrightarrow \mathrm{RHom}_S(M, M)$ is now just the embedding $R \hookrightarrow \mathcal{E}$, so is a quasi-isomorphism.

Note by [12, prop. III.4.2] that the quasi-isomorphism $R \xrightarrow{\simeq} \mathcal{E}$ induces an equivalence of derived categories of DG-right-modules, $\mathsf{D}(R^{\mathrm{opp}}) \xrightarrow{\simeq} \mathsf{D}(\mathcal{E}^{\mathrm{opp}})$.

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