Primer on CPP Geometry

Peter Greenberg

Institut Fourier, Grenoble

Our purpose is to describe some basic ideas and constructions in C^1, piecewise–projective (CPP for short) geometry on the circle. This geometry seems to provide a good context in which to discuss (see definition 1.1), the Higman–Thompson group G (see [Br], [GhS]) and certain of its subgroups (see [G2]).

This report was written during a visit to the SFB at the University of Bielefeld, to which I am very grateful. Conversations with Ross Geoghegan and Vlad Sergiescu were both encouraging and helpful.

0.1 Introduction

We require some notation. Consider the action of $\mathrm{PSL}_2\ \mathbb{R}$ on $\mathbb{H}$, the hyperbolic disk, by linear fractional transformations. The action extends to a transitive action on $S^1 = \mathbb{R} \cup \{\infty\}$. Let, for $x \in S^1$, A_x (respectively P_x) denote the subgroup fixing x (resp. fixing x and with derivative 1 at x). The elements of P_x are called parabolic, and

$$\begin{aligned} A_\infty &= \left\{ \begin{pmatrix} r & \frac{b}{r} \\ 0 & \frac{1}{r} \end{pmatrix} : r > 0, b \in \mathbb{R} \right\} = \{x \longmapsto r^2 x + b\} \\ P_\infty &= \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \{x \longmapsto x + t\} \simeq \mathbb{R}. \end{aligned}$$

The circle is given the usual counterclockwise orientation.

Definition 0.1.1 *A homeomorphism $g : V \longrightarrow \mho$ between open subsets of S^1 is C^1, piecewise–projective (we write $g \in CPP$) if g has a continuous derivative, and if there is a discrete subset S of V such that for every connected component of $V - S$, $g|_C \equiv g_c$, where $g_c \in \mathrm{PSL}_2\ \mathbb{R}$. Denote by $CPP(S^1)$ the group of CPP homeomorphismus of S^1.*

If $g \in CPP$, $g : V \longrightarrow \mho$ and $x \in V$, then g^l_x, g^r_x, the left and right germs of g at x, are the elements of $\mathrm{PSL}_2\ \mathbb{R}$ such that $g|_{(x-\varepsilon,x)} \equiv g^l_x$, $g|_{(x,x+\varepsilon)} \equiv g^r_x$, for some small ε. Define the change at x by $D_x g \equiv (g^r_x)^{-1} g^l_x$. Since g is C^1, $D_x g \in P_x$ for all x. The set $bk(g)$ of breakpoints of g is the set of x such that $D_x g \neq id$. Note that $bk(g)$ is a finite set if $g \in CPP(S^1)$.

Proposition 0.1.1.1 *Let $bk(g) = \{x_1, \ldots, x_n\}$ in counterclockwise order. Then $D_{x_n} g \circ \ldots \circ D_{x_1} g = id$.*

Proof: For all i, $1 \leq i < n$, we have $g^r_{x_i} = g^l_{x_{i+1}}$, and $g^r_{x_n} = g^l_{x_1}$.

The result follows.

The pseudogroup CPP of all CPP homeomorphisms was studied in [G1]. There a map $K(\widetilde{\mathrm{PSL}_2\ \mathbb{R}}, 1) \longrightarrow BCPP$ was constructed, which induces an isomorphism in homology. (Here $\widetilde{\mathrm{PSL}_2\ \mathbb{R}}$ is the universal cover of the Lie group $\mathrm{PSL}_2\ \mathbb{R}$, and $BCPP$ is the Haefliger classifying space of the pseudogroup CPP). Combining this map with techniques of Mather, McDiff, Segal and Thurston (see e.g. [Ma], [Mc]) we find a homology isomorphism $K(CPP(S^1), 1) \longrightarrow \mathcal{L}(K(\widetilde{\mathrm{PSL}_2\ \mathbb{R}}, 1)^+)$, where $^+$ is Quillens plus construction, and $\mathcal{L}X$ is the homotopy quotient $X^{S^1} x_{S^1} ES^1$, where S^1 acts on X^{S^1} on the right, by composition. As a consequence, the homology of the discrete group $CPP(S^1)$ is closely related to that of $\widetilde{\mathrm{PSL}_2\ \mathbb{R}}$; in particular $H_1 CPP(S^1) \simeq H_2(\widetilde{\mathrm{PSL}_2\ \mathbb{R}}) \simeq K_2\mathbb{R}/\{-⊮, -⊮\}$. Thus we can give (2.11) a direct geometric proof that $K_2\mathbb{R}/\{-⊮, -⊮\}$ is divisible.

Let $G = \{g \in CPP(S^1) : g^l_x, g^r_x \in \mathrm{PSL}_2\ \mathbb{Z}$, ⌢ ∈ 𝕊⊮, and ⌉(ð) ⊆ $\mathbb{Q} \cup \{\infty\}\}$. It was pointed out by Thurston that G is the Higman–Thompson group ([Br], [GhS]) studied in different guises in group theory, logic and dynamical systems. Part 3 of this report describes a space T_g on which G acts, with discrete orbits and finitely generated stabilizers. As an application, we obtain an explicit $K(F', 1)$, where $F' = \{g \in G, g^l_\infty = g^r_\infty = id\}$, and, in section 4, a map from $K(F', 1)$ to ΩS^2 which induces a homology isomorphism $K(F', 1) \longrightarrow \Omega S^3$.

But we begin with some elementary geometry.

0.2 Bracelets and CPP geometry

The main idea in CPP geometry is that $CPP(S^1)$ acts "at infinity" on the disk $\mathbb{H}$. Namely (fig. 2.1) let $g \in CPP(S^1)$, and C a component of $S^1 - bg(g)$. Then g admits an extension as an isometry on the convex closure of C in $\mathbb{H}$, namely as $g_c \in \mathrm{PSL}_2\ \mathbb{R}$. The complement of the union of these convex closures has finite area, so g is defined on almost all of $\mathbb{H}$.

figure 2.1

Though these convex closures are disjoint, they are "glued at infinity" in a way respected by $CPP(S^1)$. Recall (fig. 2.1) that a horocircle is a circle in the disk, tangent to S^1. The horocircles at $x \in S^1$ provide an identification between the points of geodesics with an endpoint at x. For any $x \in S^1$, an element of P_x maps a horocircle at x to itself. Thus we identify the set of horocircles with $\mathrm{PSL}_2\,\mathbb{R}/\mathbb{P}_\infty$: set h_∞ to be the horocircle at ∞ which passes through $i \in \mathbb{H}$; identify hP_∞, $h \in \mathrm{PSL}_2\,\mathbb{R}$, with the horocircle $h(h_\infty)$. Now if $g \in CPP(S^1)$ we define the action of g on horocircles by $g(hP_\infty) = (g_x^l h)P_\infty$, where $h_\infty = x$, and note that $g(hP_\infty) = (g_x^l h)P_\infty = (D_x g)^{-1}(g_x^l hP_\infty) = (g_x^r h)P_\infty$.

Let h, h' be horocircles at $x, x' \in S^1$ respectively, and define a real number $d(h,h')$ as follows: let γ be the geodesic from x to x', and set $|d(h,h')|$ to be the distance (along γ) from $\gamma \cap h$ to $\gamma \cap h'$. Then set $d(h,h') > 0$ if $h \cap h' = \emptyset$, and $d(h,h') \le 0$ if $h \cap h' \neq \emptyset$.

Definition 0.2.1 *Let $x, y \in S^1$. A partial bracelet B from x to y is a set of horocircles h_i at points $x = x_0, x_1, \ldots, x_n = y$ so that h_i is tangent to h_{i+1}, $0 \le i \le n-1$. The set of tangent points of B is $\tau(B) = \{x_i\}$. Set $\delta(B) = d(h_0, h_n)$. If $\delta(B) = 0$, B is a bracelet.*

By the remarks preceeding 2.2, if B is a bracelet and $g \in CPP(S^1)$, then $g(B)$ makes sense as a set of horocircles.

Proposition 0.2.1.1 *Suppose $g \in CPP(S^1)$, B is a partial bracelet from x to y, and $bk(g) \subseteq \tau(B)$. Then $g(B)$ is a partial bracelet from gx to gy, $\tau g(B) = g\tau(B)$, and $\delta g(B) = \delta(B)$.*

Proof: For each i, $g_{x_i}^r = g_{x_{i+1}}^l$ are isometries, and so $d(h_i, h_{i+1}) = d(g_{x_i}^r h_i, g_{x_{i+1}}^l h_{i+1}) = d(gh_i, gh_{i+1})$; the proposition follows.

Basic facts about bracelets will yield certain elements of $CPP(S^1)$.

Proposition 0.2.1.2 **(a)** *Let $x_1, \ldots, x_{2k+1}$ be distinct points of S^1. There is a unique bracelet B with $\tau(B) = \{x_i\}$.*

(b) *Let $x_1, \ldots, x_{2k}$ be distinct points of S^1. Let B be a partial bracelet from x_1 to x_{2k}, with $\tau(B) = \{x_i\}$. Then for any partial bracelet B' from x_1 to x_{2k}, with $\tau(B') = \{x_i\}$, $\delta(B') = \delta(B)$.*

Proof: **(a)** Let $B = \{h_1, \ldots, h_{2k+1}\}$ be a partial bracelet from x_1 to x_{2k+1}, and suppose $d(h_1, h_{2k+1}) > 0$. Enlarge h_1 slightly, while fixing $\tau(B)$; in order for B to remain a partial bracelet, h_2 must shrink, h_3 grow, and so on: we see that h_{2k+1} grows. Eventually h_1 and h_{2k+1} are tangent, giving a bracelet, the unique bracelet with $\tau(B) = \{x_1, \ldots, x_{2k+1}\}$. (If $d(h_1, h_{2k+1}) < 0$ at the beginning, start by shrinking h_1 slightly).

(b) Let $B = \{h_1, \ldots, h_{2k}\}$ be a partial bracelet from x_1 to x_{2k} (shown as the undashed horocircles in figure 2.5), and suppose $\delta(B) > 0$. As in the proof of (a) enlarge h_1

figure 2.5

slightly and alter all of the h_i so that they still give a partial bracelet $B' = \{h'_1, \ldots, h'_{2k}\}$ from x_1 to x_{2k} (shown as the dashed circles). Let d_i be the distance from h_i to h'_i, measured along a geodesic with an endpoint at x_i; the number is independent of the geodesic chosen, so $d_1 = d_2 = \ldots = d_{2k}$. Now $\delta(B') = \delta(B) - d_1 + d_{2k} = \delta(B)$. q.e.d.

If $\{x_1, \ldots, x_{2k}\}$ are distinct points in S^1, let $\delta\{x_i\}$ denote the value $\delta(B)$ for any partial bracelet from x_1 to x_{2k}, with $\tau(B) = \{x_i\}$.

Corollary 0.2.1.1 *(of the proof and figure) Let $B = \{h_1, \ldots h_{2k}\}$ be a partial bracelet from x_1 to x_{2k}, and let $\delta = \delta(B)$. Let B' be a partial bracelet from x_i to x_{i-1}, with $\tau B' = \{x_i\} = \tau B$. Then $\delta(B') = (-1)^{i+1}\delta(B)$.*

Proposition 0.2.1.3 *Let $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ be two sets of n distinct points in S^1, each listed in counterclockwise order. Let B, B' be partial bracelets from x_1 to x_n, y_1 to y_n respectively, with $\tau(B) = \{x_i\}$, $\tau(B') = \{y_i\}$. Then there exists $f \in CPP(S^1)$ with $bk(f) \subseteq \{x_i\}$, $f(B) = B'$ if and only if $\delta(B) = \delta(B')$. If f exists it is unique.*

Proof: In fact, for each i, $1 \leq i \leq n-1$ there is a unique $f_i \in \mathrm{PSL}_2\,\mathbb{R}$ such that $f_i(h_i) = h'_i$, $f_i(h_{i+1}) = h'_{i+1}$, and there is a unique f_n such that $f_n(h_n) = h'_n$, $f_n(h_1) = h'_1$. Define $f^r_{x_i} = f_i = f^l_{x_{i+1}}$, $f^l_{x_1} = f^r_{x_n} = f$. Then $f \in CPP(S^1)$ if and only if the $f^{-1}_{i+1} f_i \in P_{x_{i+1}}$, and similary for $f^{-1} f_n$. But since $f^{-1}_{i+1} f_i(h_{i+1}) = h_{i+1}$, the result follows; similarly, $f_1^{-1} f_n \in P_{x_1}$.

Corollary 0.2.1.2 *Let $x_1, \ldots, x_n$ be a set of distinct points of S^1, in counterclockwise order. If n is odd, or if n is even and $\delta\{x_i\} = 0$, there exists a unique $e = e_{\{x_i\}} \in CPP(S^1)$ with $bk(e) \subseteq \{x_i\}$, $ex_i = x_{i+1}$, $ex_n = x_1$. In the case where n is even, e exists only if $\delta\{x_i\} = 0$.*

Proof: Apply 2.4 to construct a bracelet with $\tau(B) = \{x_i\}$, and then apply 2.7, and 2.6 for the even case.

Definition 0.2.2 (a) *The $e_{\{x_i\}}$ are called the (generalized) elliptic elements of $CPP(S^1)$.*

(b) *Let $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ be $2n$ distinct points of S^1, listed in counterclockwise order, and let $u \in \mathbb{R}, \approxeq > \nvdash$. For any $a, b \in S^1$ let ${}^u h^b_a \in \mathrm{PSL}_2\,\mathbb{R}$ be the hyperbolic element fixing a and b, with derivative u at a, u^{-1} at b. Define $h = {}^u h^{\{y_i\}}_{\{x_i\}}$, the (generalized) hyperbolic by $bk(h) \subseteq \{x_i, y_i\}$, $h^l_{\{x_i\}} = {}^u h^{y_{i-1}}_{x_i}$, $h^r_{x_i} = {}^u h^{y_i}_{x_i}$.*

Note that $\left({}^u h^{\{y_i\}}_{\{x_i\}}\right)^{-1} = {}^{u^{-1}} h^{\{y_i\}}_{\{x_i\}} = {}^u h^{\{x_i\}}_{\{y_i\}}$. We obtain:

Corollary 0.2.2.1 *If $\delta\{x_i, y_i\} = 0$, then $e_{\{x_i, y_i\}}$ conjugates ${}^u h^{\{y_i\}}_{\{x_i\}}$ to its inverse.*

Proposition 0.2.2.1 *$H_2(\widetilde{\mathrm{PSL}_2\,\mathbb{R}})$ is divisible.*

Proof: As remarked in the introduction, it suffices to prove that $H_1 CPP(S^1)$ is divisible. To this end, we prove that $CPP(S^1)$ is generated by the elements of $\mathrm{PSL}_2\,\mathbb{R}$, together with the ${}^u h^{\{y_i\}}_{\{x_i\}}$, $n = 2$. The elements of $\mathrm{PSL}_2\,\mathbb{R}$ are commutators, and the generalized hyperbolics are divisible, proving the proposition. In fact, this shows that there is an $\mathbb{R}^+$ action on $H_2(\widetilde{\mathrm{PSL}_2\,\mathbb{R}})$.

We first show that if $g \in CPP(S^1)$, with 4 breakpoints x, y, z and w, in counterclockwise order, then $g = {}^u h^{y,w}_{x,z} \circ f$, for some $u \in \mathbb{R}$, $f \in \mathrm{PSL}_2\,\mathbb{R}$. For we can assume (precomposing with an element of $\mathrm{PSL}_2\,\mathbb{R}$) that $g^r_w = g^l_x = id$. Now the elements ${}^u g = {}^u h^{y,w}_{x,z} \circ {}^{u^{-1}} h^w_x$ share the property that ${}^u g^r_w = {}^u g^l_x = id$. A little geometry convinces one that for all α between x and w there is a u such that ${}^u g(y) = \alpha$. It is now easy to see that g (given $g^r_w = g^l_x = id$) is uniquely determined by $g(y)$, hence $g = {}^u g$ for some u.

Finally we prove by induction on n that an element of $CPP(S^1)$ with n breakpoints is a product of elements of $\mathrm{PSL}_2\ \mathbb{R}$ and elements ${}^u h^{yw}_{xz}$. For $n \leq 4$ it is proved. Let g be an element of $CPP(S^1)$ with $bk(g) = \{x_1, \ldots, x_n\}$; assume as above that $g^r_{x_n} = g^l_{x_1} = id$. Let $f \in CPP(S_1)$ such that $bk(f) \subseteq \{x_1, g(x_2), g(x_3), x_n\}$, such that $f^r_{x_n} = f^l_{x_1} = id$, and with $fg(x_2) = x_2$. (Such an f can be constructed, as in the preceeding paragraph, as a ${}^u g$). Then $(fg)^l_{x_2} = (fg)^l_{x_1} = id$, so $bk(fg) \subseteq \{x_2, \ldots, x_n\}$ and induction concludes the proof.

0.3 A contractible space for G

Recall that $G = \{g \in CPP(S^1) : bk(g) \subseteq \mathbb{Q} \cup \{\infty\}, \eth^{<}_{\curvearrowleft}, \eth^{>}_{\curvearrowleft} \in \mathrm{PSL}_{\nvDash}\ \mathbb{Z}$ for all $\curvearrowleft \in \mathbb{S}^{\nVdash}\}$; define also $F = \{g \in G : g_(\infty) = \infty\}$. Identifying $P_\infty \cap \mathrm{PSL}_2\ \mathbb{Z} \simeq \mathbb{Z}, (\eth^{<}_{\infty}, \eth^{>}_{\infty})$ defines a homomorphism $f \longrightarrow \mathbb{Z}$ whose kernel is F', the commutator subgroup of F. More generally, let $\Gamma \subseteq \mathrm{PSL}_2\ \mathbb{Z}$ be a subgroup of finite index, and let $\Sigma \subseteq \mathbb{Q} \cup \{\infty\}$ be a Γ–invariant set. Then $G_{\Gamma,\Sigma} = \{g \in G : g^l_x, g^r_x \in \Gamma$ for all $x \in S^1, bk(g) \subseteq \Sigma\}$. The geometry of these groups, studied briefly in [G2], is related to that of the surface $\Gamma \setminus \mathbb{H}$. In particular $G_{\Gamma,\Sigma}$ is torsion free if either $\Sigma \neq \mathbb{Q} \cup \{\infty\}$, or if the genus of $\Gamma \setminus \mathbb{H}$ is nonzero.

It has been suggested that G behaves like an infinite–dimensional arithmetic group. With this idea in mind, we construct a contradictible space T_G on which G acts with discrete orbits, and such that the isotropy subgroups are finite cyclic, or conjugates of $\mathrm{PSL}_2\ \mathbb{Z}$. As a result, T_G/F' is a $K(F', 1)$. In ¶4 we construct an explicit map, $T_G/F' \longrightarrow \Omega S^2$ inducing an isomophism in homology $K(F', 1) \longrightarrow \Omega S^3$. (But we use a priori knowledge of such a map, [GhS], to prove the isomorphism in homology).

The definition of T_G is motivated by "universal Teichmüller space" ideas ([Le]).

Definition 0.3.1 *Let $X = \{f \in CPP(S^1), bk(f) \subseteq \mathbb{Q} \cup \{\infty\}\}$. Define $T_G = \mathrm{PSL}_2\ \mathbb{R} \setminus \mathbb{X}$, the quotient of X by the free left action by composition. Then G acts on T_G on the right, by composition.*

We now describe T_G as a topological space; it will have the topology of a direct limit $\lim_{\rightarrow} V_S$ of components of real algebraic varieties W_S, which we now define.

Recall that $\mathrm{PSL}_2\ \mathbb{Z}$ provides an identification of P_r with P_∞, for any $r \in \mathbb{Q} \cup \{\infty\}$. For $\mathrm{PSL}_2\ \mathbb{Z}$ acts transitively on $\mathbb{Q} \cup \{\infty\}$, and $P_r \cap \mathrm{PSL}_2\ \mathbb{Z}$ is the full stabilizer of r. Thus any $g \in \mathrm{PSL}_2\mathbb{Z}$ such that $gr = \infty$ induces the same isomorphism of P_r with $P_\infty = \{x \longmapsto x + t\} \simeq \mathbb{R}$. We write $p^t_r \in P_r$ as the element corresponding to $x \longmapsto x + t$.

Suppose $g \in X$, with breakpoints $x_1, \ldots, x_n$ in counterclockwise order; we clearly have $D_{x_n} g \circ \ldots \circ D_{x_1} g = id$. The action of $\mathrm{PSL}_2\mathbb{R}$ does not change the $D_{x_i} g$; indeed an element of T_G is uniquely described by the $D_{x_i} g$, for any g representing that element.

Let $S \subseteq \mathbb{Q} \cup \{\infty\}$ be a finite subset, and define $W_S = \{(t_s) \in \mathbb{R}^{\mathbb{S}} : \not\succ\imath\approx\!\sim\, = \,\daleth$, where the product is taken in the counterclockwise order of the $\sim\, \in \mathbb{S}\}$. An element of W_S describes a CPP covering of S^1 by S^1, with breakpoints in S. Let V_S be the set of elements of W_S of degree one.

Evidently, $T_G = \bigcup V_S$ as a set; from now on (t_s) denotes an element of T_G, for example (0_s) is the class of the identity map.

Proposition 0.3.1.1 *W_S is a codimension 3 real analytic subvariety of $\mathbb{R}^{\mathbb{S}}$, and (0_s) is a nonsingular point of W_S, when the cardinaltity of S is at least* 3.

Proof: Define $\Pi : \mathbb{R}^{\mathbb{S}} \longrightarrow \mathrm{PSL}_{\ncong} \mathbb{R}$ by $\Pi(t_S) = \Pi p_s^{t_S}$ (the product in counterclockwise order); then $\Pi^{-1}(id) = W_S$. Since Π is clearly real analytic, it suffices to show that (0_s) is a nonsingular point.

Let $\{s_1, \ldots, s_n\} = S$, listed in counterclockwise order. Conjugating by an element of $\mathrm{PSL}_2\, \mathbb{R}$, we assume $s_1 = 1, s_2 = \infty, s_n = 0$; thus we can write

$$p_{s_1}^{t_{s_1}} = \begin{pmatrix} 1-x & x \\ -x & 1+x \end{pmatrix} \qquad p_{s_2}^{t_{s_2}} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \qquad p_{s_3}^{t_{s_3}} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Taking the derivative of Π with respect to x, y and z, and using $\Pi = id$, we obtain

$$\frac{\partial \Pi}{\partial x} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \frac{\partial \Pi}{\partial y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \frac{\partial \Pi}{\partial z} = \begin{pmatrix} x - x^2 & (1-x)^2 \\ -x^2 & x^2 - x \end{pmatrix}$$

at a point of W_S. These vectors span $sl_2\, \mathbb{R}$ except when $x = 1$, and in particular when $x = y = z = 0$.

If $S \subseteq S'$ there is an inclusion $V_S \subseteq V_S'$. The proof of 3.2 shows that the points of V_S are nonsingular in $V_{S'}$, if the cardinality of S' is at least three more than that of S. We give T_G the direct limit topology $T_G = \lim_{\substack{\to \\ S}} V_S$.

Proposition 0.3.1.2 *Every $(t_s) \in T_G$ prossesses an open neighbourhood $(t_s) \in \mathcal{V} \subseteq T_G$ such that if $g \in G$, $\mathcal{V}g \cap \mathcal{V} \neq \emptyset$ if and only if $(t_s)g = (t_s)$.*

Proof: If $g \in G$ we will show below that

$$(tg)_s = t_{g(s)} + D_s g \tag{0.1}$$

Recall that $D_s g \in \mathbb{Z}$, $s \in S^1$, and let Y be the set of nonzero elements of $\{|t_s|; s \in \mathbb{Q} \cup \{\infty\}\} \cup \{|\approx_\sim - \approx_{\sim'} + \ltimes; \sim, \sim' \in \mathbb{Q} \cup \{\infty\}, \ltimes \in \mathbb{Z}\}$. For each $(t_s) \in T_G$, the number of nonzero t_s is finite, and so Y is a finite set and contains a minimal $\varepsilon > 0$. Let $\mathcal{V} = \{(x_s) : |t_s - x_s| < \frac{\varepsilon}{3}, s \in \mathbb{Q} \cup \{\infty\}\}$. By 3.4, $\mathcal{V} \cap \mathcal{V}g$ is nonempty only if $(t_s)g = (t_s)$.

Now we prove (3.4). Let $p = p_{g(s)}^{t_{g(s)}}$; the term on the left is $(g_s^r)^{-1} p g_s^l \in P_s$. This is conjugate by an element of $\mathrm{PSL}_2\,\mathbb{Z}$, to $p g_s^l (g_s^r)^{-1} \in P_{g(s)}$. But $g_s^l (g_s^r)^{-1}$ is conjugate, in $\mathrm{PSL}_2\,\mathbb{Z}$ to $D_s g$. These conjugations preserve the isomorphisms $P_r \longrightarrow P_\infty$, proving (3.4).

In order to prove that T_G is contractible, we describe a particular filtration of T_G. A reference for the facts quoted about Ford circles and Farey fractions is [Ra].

Definition 0.3.2 (The Ford Bracelet B_F) *Let B_1 be the unique bracelet with $\tau(B_1) = \{\infty, 0, 1\}$; define H_∞ to be the horocircle at ∞, and h_1^l the horocircle at 1. Given B_{n-1} and h_{n-1}^l, we define B_n and h_n^l as follows: B_n is the union of B_{n-1} with the horocircles tangent to succeeding horocircles in B_{n-1} (see fig. 3.6); h_n^l is the horocircle between h_{n-1}^l and h_∞.*

figure 3.6

The Farey fractions describe how to calculate the $\tau(B_n)$; in particular h_n^l is tangent to S^1 at n. Let $S_n = \tau(B_n)$; the cardinality of S_n is $3 \cdot 2^{n-1}$. The <u>Ford bracelet</u> B_F is $\bigcup B_n$, as a union of horocircles. Some facts:

Facts 3.7 **(a)** $\tau B_F = \bigcup S_n = \mathbb{Q} \cup \{\infty\}$

(b) *B_F is* $\mathrm{PSL}_2\,\mathbb{Z}$ *invariant.*

Corollary 3.8 *If $g \in G$, $gB_F = B_F$.*

Corollary 3.8 should be interpreted as saying: let $g \in G$, and let $n > 0$ so that $bk(g) \subseteq S_n$. Then $g(B_N)$ is a bracelet contained in some B_m, $m \geq N$, for all $N \geq n$.

Filter X by $X_n = \{f \in X, bg(f) \subseteq S_n\}$, and filter T_G by $(T_G)_n = \{(t_s) : t_s = 0$ if $s \notin S_n\}$, so that $(T_G)_n = \mathrm{PSL}_2\,\mathbb{R} \setminus \mathbb{X}_{\ltimes}$.

Proposition 0.3.2.1 *T_G is contractible.*

Proof: Let $Y = \{f \in X, f^l_\infty = id\}$. Then $Y \longrightarrow T_G$ is one–to–one. We give Y the topology of T_G, and prove that Y is contractible. Filter Y by $Y_n = \{f \in Y, bk(f) \subseteq S_n\}$. We will show that Y_n deforms in itself to Y_{n-1}. Since Y_1 is a point, Y is contractible.

To each $f \in Y_n$ we associate the bracelet $f(B_n)$; since $f^l_\infty = id$, $h_\infty, h^l_n \in f(B_n)$. Indeed for any bracelet B such that h_∞ and h^l_n are neighboring horocircles belonging to B, and such that $\tau(B)$ has $3 \cdot 2^{n-1}$ points, 2.1.3 says that there is a unique $f \in Y_n$ such that $f(B_n) = B$, $bk(f) \subseteq S_n$. So we will deform Y_n into Y_{n-1} by deforming bracelets: $B = f(B_n)$, for an $f \in Y_{n-1}$ only if the horocircles obtained from B by omitting every other one, beginning with h^l_n, yields a bracelet which contains H_∞ and h^l_{n-1} as neighboring horocircles (see firgure 3.10).

figure 3.10

Thus we will describe a well defined deformation of any bracelet $f(B_n)$ to a bracelet with the form in fig. 3.10. Figure 3.11 illustrates the deformation, which has three steps.

(a) dotted horocircles indicate new positions

(b) horocircles with dots come from B_{n-1}

(c) the new location of y is $y' = n - 1$

figure 3.11

<ins>**Step (a)**</ins> Changing only the tangent points of the first and third horocircles counterclockwise from h_∞, enlarge or shrink the second until it is tangent to h_∞.

<ins>**Step (b)**</ins> Repeat step (a), but with the third, fifth and fourth horocircles, then the fifth, seventh and sixth, and so on, until (3.10(b)) the horocircles from B_{n-1} form a partial bracelet from ∞ to the point y in 3.10(b).

Step (c) We must now deform the bracelet until the horocircle at y is tangent to h_∞. Apply a path of parabolics in P_∞, starting with the identity, to alter all of the horocircles but h_∞, h_n^l and those at y, and x (in 3.11(b)). Let the parabolics change the point y, but deform the horocircle at the point y so that it remains tangent to h_n^l and to that at z, and until it is tangent to h_∞ as well. During this deformation, the change of the horocircle at x is defined by requiring that all horocircles define a bracelet. At this point we have the situation in 3.11(c). Let y' be the new tangent point of the horocircle which in step (b) arrived to y. Since h_∞, h_n^l and the horocircle at y' form a bracelet, $y' = n - 1$ and we have a bracelet $f(B_n)$, for some $f \in Y_{n-1}$.

Proposition 0.3.2.2 *Let T_G^- denote T_G with the G–orbit of (0_s) removed. Then T_G^- is weakly contractible.*

Proof: Since the orbit of (0_S) is discrete, it suffices to show that $T_g - (0_S)$ is weakly contactible, which, since T_G is contractible come down to showing that a compact polyhedron $K \subseteq T_G$ may be deformed off (0_S). Any compact K lies in $(T_G)_n$ for some n, and hence in $(T_G)_n + 1$ where, since (0_S) is nonsingular, it may be deformed away from (0_S).

Theorem 0.3.2.1 *Let $g \in G$ and suppose $xg = x, x \in T_G$. Then either g is of finite order, or else g is conjugate in G to an element of* $\mathrm{PSL}_2\ \mathbb{Z}$, *in which case x is in the orbit of* (0_s).

Corollary 0.3.2.1 **(i)** *T_G/F' is a $K(F', 1)$.*

(ii) *Let H be a torsion free subgroup of G. Then T_G^-/H is a $K(H, 1)$.*

Recall, for example, that F and certain of the $G_{\Gamma,\Sigma}$ are torsion free.

The proof of the theorem 3.2.1 reduces to two lemmas.

Lemma 0.3.2.1 *Let $g \in$ $\mathrm{PSL}_2\ \mathbb{Z}$, g of infinite order. Then (0_s) is the unique point fixed by g.*

Lemma 0.3.2.2 *If $g \in G$ of infinite order and $hf = hg$ for some $f \in X$, $h \in \mathrm{PSL}_2\ \mathbb{R}$ then there is a $\gamma \in \mathrm{PSL}_2\ \mathbb{R}$ so that $\gamma f \in G$.*

Proof of 3.2.1: Suppose $fg = hf$, $g \in \mathrm{PSL}_2\ \mathbb{Z}$, $h \in \mathrm{PSL}_2\ \mathbb{R}$, $f \in X$; then $fgf^{-1} = h$, and we want to show that $f \in \mathrm{PSL}_2\ \mathbb{R}$, that is that $bk(f) = \emptyset$. Since $g \in \mathrm{PSL}_2\ \mathbb{Z}$ of infinite order, g is parabolic or hyperbolic; the two cases will be treated seperately.

(a) $\underline{g \text{ parabolic}}$ We assume $g_\infty = \infty$ and $gx > x, x \in S^1 - \infty = \mathbb{R}$. If $f \notin \mathrm{PSL}_2\,\mathbb{R}$ then f has breakpoints distinct from ∞; let $x_1 \in \mathbb{R}$ be the minimum. Since g and h have no breakpoints, $g^{-1}x_1$ must also be a breakpoint of f (see 3.17(a)). Since $g^{-1}x_1 < x_1$, this contradicts the assumption on x_1.

(b) $\underline{g \text{ hyperbolic}}$ (see 3.17(b)) Then g has fixed points $p, q \in S^1$, and we can assume q is expansive, p contracting. Recall that q and p are irrational and thus $p, q, \notin bk(f)$. Let x_1 be the first breakpoint of f counterclockwise of q. Then $q, g^{-1}x_1, x_1$ are in counterclockwise order, but as in case (a) $g^{-1}x_1$ must be a breakpoint for f, a contradiction.

figure 3.17

Proof of 3.2.2: Write $h = fgf^{-1}$. A result of Ghys and Sergiescu ([GhS]) asserts that the rotation number of an element of G is rational. Hence if g is conjugate to $h \in \mathrm{PSL}_2\,\mathbb{R}$, and g is of infinte order, then h is parabolic or hyperbolic. We will prove, in each case, that $D_x f \in \mathrm{PSL}_2\,\mathbb{Z}$ for all $x \in S^1$.

(a) $\underline{h \text{ parabolic}}$ (see fig. 3.18(a)) Since g is conjugate to h, g has a single point which, without altering the argument, we can take to be ∞. We suppose then that $gx > x, x \in \mathbb{R} = \mathbb{S}^{\not{1}} - \infty$. Let $x_1 < x_2 < \ldots < x_n = \infty$ include $bk(f)$, and set $f_i = f^l_{x_i}$. We show by induction that $f^{-1}_{i+1} f_i \in \mathrm{PSL}_2\,\mathbb{Z}$.

To begin, we show $f_2^{-1} f_1 \in \mathrm{PSL}_2\,\mathbb{Z}$. Since $g^{-1}x_1 < x_1, g^{-1}x_1$ is not a breakpoint of f, and so since $bk(h) = \emptyset$, $f_1 \circ g^l_{g^{-1}x_1} = f_2 \circ g^r_{g^{-1}x_1}$, whence $f_2^{-1} f_1 \in \mathrm{PSL}_2\,\mathbb{Z}$. Now suppose that for all $j < i$, $f_j^{-1} f_{j-1} \in \mathrm{PSL}_2\,\mathbb{Z}$. Then in fact for all $x < x_i, f_x^{r\,-1} f_x^l \in \mathrm{PSL}_2\,\mathbb{Z}$, and this is true in particular for $x = g^{-1}x_i$. Let $y = fg^{-1}x_i$. Since h has no breakpoints, $f_i \circ g^l_{g^{-1}x_i} \circ f_y^{-1l} = f_{i+1} g^r_{g^{-1}x_i} \circ f_y^{-1r}$, so $f^{-1}_{i+1} f_i = g^r_{g^{-1}x_i} \circ f_y^{-1r} (f_y^{-1l})^{-1} g^l_{g^{-1}x_i}$. So since $f_y^{-1r} (f_y^{-1l})^{-1} = (f^{r\,-1}_{g^{-1}x_i} \circ f^l_{g^{-1}x_i})^{-1}$, $f^{-1}_{i+1} f_i \in \mathrm{PSL}_2\,\mathbb{Z}$.

figure 3.18

(b) h hyperbolic (see 3.18(b)) Then since g is C^1 conjugate to h, g has two fixed points q and p; we assume $g'(q) > 1 > g'(p)$, so that p and q are irrational, and $p, q \notin bk(f)$. Let $x_{-m}, \ldots, x_{-1}, x_1, \ldots, x_n$ be the breakpoints of f, in counterclockwise order, and such that x_{-1}, q, x_1, are in counterclockwise order (3.18(b)). The proof above, starting with x_1, shows that $D_{x_i} f \in \mathrm{PSL}_2\ \mathbb{Z}, \sqsupset \geq \nVdash$. Applying the symetric argument, starting with x_{-1} and working down to x_{-m} proves $D_{x_i} f \in \mathrm{PSL}_2\ \mathbb{Z}, \sqsupset \geq -\nVdash$, q.e.d.

Theorem 0.3.2.2 *Let* $\underline{f} \in T_G$, $\underline{f}$ *not in the g-orbit of* (0_s). *Then stab(*$\underline{f}$*)=*$\{g \in G, \underline{f}g = \underline{f}\}$ *is a finite cyclic group, whose order is bounded by the cardinality of* $bk(\underline{f})$.

Proof: Let $f \in X$ be a choice of representative for $\underline{f}$. We first show that stab($\underline{f}$) is a subgroup of $\mathbb{R}/\mathbb{Z}$. Suppose $g, g' \in$ stab($\underline{f}$). Then $h = fgf^{-1}, h' = fg'f^{-1}$ are elements of $\mathrm{PSL}_2\ \mathbb{R}$ of finite order, by 3.13. If $[h, h'] \neq id$, then $[h, h']$ is parabolic or hyperbolic, whence, since $[h, h'] = f[g, g']f^{-1}$, $\underline{f}$ must be in the orbit of (0_S), by 3.13. Thus f conjugates all elements of stab($\underline{f}$) to the same subgroup $\mathbb{R}/\mathbb{Z}$ of $\mathrm{PSL}_2\ \mathbb{R}$. It remains to show that the order of an element of stab($\underline{f}$) is bounded by the cardinality of $bk(f)$.

First, by the result of Ghys and Sergiescu quoted above, any element of stab($\underline{f}$) is of finite order. Thus for any $x \in S^1, \{g^n(x)\}$ is a finite set, whose cardinality is the order of g. Thus is suffices to prove that if $x \in bk(f)$, $g \in$ stab($\underline{f}$), then either $\{g^n(x)\} \subseteq bk(f)$, or $D_x f \in \mathrm{PSL}_2\ \mathbb{Z}$. Suppose $g^k x \notin bk(f), k \geq 1$. Let $z = f(x)$. Since $fg^k f^{-1}$ has no breakpoints, $id = D_z(fg^k f^{-1}) = D_z(g^k f^{-1}) = id$. Since $d_x g^k \in \mathrm{PSL}_2\ \mathbb{Z}$, the same is true for f.

0.4 Application: A map $T_G/F' \longrightarrow \Omega S^2$

It is known, via pseudogroup methods ([GhS]) that a $K(F', 1)$ admits a map to ΩS^3, the space of based loops on the 3-sphere, which induces an isomorphism in homology.

In this section we construct (4.1) an explicit map $w : T_G/F' \longrightarrow \Omega S^2$. Recall the Hopf fibration $\Omega S^3 \xrightarrow{j} \Omega S^2 \xrightarrow{p} S^1$; as $H^1(F';\mathbb{Z}) = \nvdash$, the map w is homotopic to jw', where $w' : T_G/F' \longrightarrow \Omega S^3$ is a lift of w. We will prove (4.4) that this w' provides the isomorphism in homology that one knows exists.

In order to construct w, we use James' model [J] for $\Omega S^2 = \Omega\Sigma(S^1)$, namely as the free topological monoid on $\mathbb{R}/\mathbb{Z} = \mathbb{S}^{\nVdash}$, with $1 \equiv 0$ as the identity. Thus (identifying by abuse ΩS^2 with this monoid) a pointin ΩS^2 is a word $U = \vartheta_1\vartheta_2\ldots\vartheta_n, \vartheta_i \in \mathbb{R}/\mathbb{Z}$. (The projection $p : \Omega S^2 \longrightarrow S^1$ in the Hopf fibration sends U to $\vartheta_1 + \ldots + \vartheta_n \in \mathbb{R}/\mathbb{Z}$). To define $w : T_G/F' \longrightarrow \Omega S^2$, we define an explicit F'–equivariant map $\overline{w} : Y \longrightarrow \Omega S^2$, where $Y = \{f \in X, f^l_\infty = id\}$ is considered as a homeomorphic copy of T_G.

Definition 0.4.1 (of $\overline{w}$) *(see fig. 4.2) Given $g \in Y$, let $n \in \mathbb{N}$ such that $bk(f) \subseteq \tau(B_n)$, and consider the bracelet $g(B_n)$. Let d_0 be the lenght of the segment of h_∞ from i to the point of tangency of h_∞ with the next horocircle counterclockwise; let d_1 be the distance along this horocircle to the point of tangenly with the next, and so on; let $d_{3\cdot 2^{n-1}}$ be the distance along h_∞ from i to the point of tangency with h^l_n. Let θ_i be the reduction modulo 1 of d_i, and define $\overline{w}(f) = \theta_0\theta_1\ldots\theta_{3\cdot 2^{n-1}}$.*

figure 4.2

To prove that $\overline{w}$ is well defined we must show that the definition is independent of n.

Lemma 0.4.1.1 *Let B be a bracelet of three horocircles. The length of the segment of one horocircle between the point of tangency with the order two is equal to 1.*

Proof: Apply an element of $\mathrm{PSL}_2\ \mathbb{R}$ so that $\tau(B) = \{0, 1, \infty\}$ and such that the segment in question is $I = \{i + x, x \in [0, 1]\}$ in the upper half plane model of $\mathbb{H}$. Now

$$\text{length}(I) = \int_i \frac{\sqrt{dx^2 + dy^2}}{y} = \int_0^1 \frac{dx}{y} = 1$$

.

Now consider (see the dotted horocircles in 4.2) using B_{n+1} instead of B_n to define $\overline{w}$; one obtains the lengths $(d_0+1), 1, d_1+2, 1, d_2+2, 1, \ldots, d_{2^{n-1}3-1}+2, 1, d_{2^{n-1}3}+1 = d_{3\cdot 2^n}$. hus in Ωs^2 (reducing modulo 1) we have the same word. By Corollary 3.8 $\overline{w}$ is F'–equivariant and defines $w : Y/F' \longrightarrow \Omega S^2$.

Theorem 0.4.1.1 *A lift of w to ΩS^3 induces an isomorphism in homology.*

Remark 0.4.1.1 **(i)** *We shall only sketch the proof of* 4.4; *the proof is somewhat unsatisfying because it uses a priori knowledge of the homology of F', which came from the less geometric map $K(F', 1) \longrightarrow \Omega S^3$ arising from pseudogroup methods.*

(ii) *Let $F^+ = \{g \in F; g^l_\infty = id\}$, so that there is an exact sequence $F' \longrightarrow F^+ \longrightarrow \mathbb{Z}$. The definition of $\overline{w}$ defines a map $Y/F^+ \longrightarrow \Omega S^2$. The methods used in the proof of* 4.4 *show that this map induces zero on $H_1(F^+; \mathbb{Z}) \simeq \mathbb{H}_{\nVdash}(\nleq\mathbb{S}^{\nvDash}) = \mathbb{Z}$.*

(iii) *A slightly different but homotopically the same $\overline{w}$, say $\tilde{w}$, can be defined as follows: Let $f \in Y$, comsidered as $(t_s) \in T_G$. Let $\tilde{w}(t_s) = \theta_\infty \theta_{s_1} \ldots \theta_{s_n}$ where $\infty, S_1, \ldots, S_n$ in counterclockwise order contain the breakpoints of f, and θ_{s_i} is the reduction modulo* 1 *of t_{s_i}.*

Sketch of the proof of 4.4 The homology ring $H_*(\Omega S^2; \mathbb{Z})$ is a polynomial ring $\mathbb{Z}[\curvearrowleft_{\nVdash}]$, $x_1 \in H_1(\Omega S^2; \mathbb{Z})$, and the image of $H_*(\Omega S^3; \mathbb{Z})$ in $H_*(\Omega S^2; \mathbb{Z})$ is $\mathbb{Z}[\curvearrowleft_{\nVdash}^{\nvDash}]$. Knowing a priori the homology ring of F', it suffices to prove that $w_2 : H_2(F'; \mathbb{Z}) \rightarrow \mathbb{H}_{\nvDash}(\nleq\mathbb{S}^{\nvDash}; \mathbb{Z})$ is an isomorphism. We will construct an explicit $\overline{\mu} : I^2 \rightarrow Y$, where $I^2 = \{(s,t); 0 \leq s, t \leq 1\}$ which projects to $\mu : T^2 \rightarrow Y/F'$, where T^2 is the torus, such that $\mu_2[T^2]$ generates $H_2(F'; \mathbb{Z})$. Then we calculate that $w_2\mu_2[T^2]$ generates $H_2(\Omega S^2; \mathbb{Z})$, to complete the proof.

To define $\overline{\mu} : I^2 \rightarrow Y$ we must give an I^2–parameter family of bracelets. Consider figure 4.6(a); let h_1, h_2, h_3 (respectively h'_1, h'_2, h'_3) be the horocircles tangent to a, b, c (resp. a', b', c'). Let s (respectively t) parameterize the arc from a to b, and t the arc from a' to b'. Let $\overline{\mu}(s,t)$ be the bracelet shown in 4.6(b).

figure 4.6

Thus when $s = 0$, the two horocircles clockwise of 0 are at h_1, h_2; when $s = 1$ they are at h_2, h_3, and likewise for t, h'_1, h'_2 and h'_3. Now there is an element $g \in F'$ supported in $[-1, 0]$, with breakponts $-1, a, b, 0$ such that $g(a) = b, g(b) = c, g(h_1) = h_2, g(h_2) = h_3$, and there is a $g' \in F$ supported in $[0, 1]$, with breakpoints at $0, a', b', 1$ so that $g'(h'_1) = h'_2, g'(h'_2) = h'_3$. One knows ([GhS]) that the relation $[g, g'] = 1$ generates (via Hopfs formula) $H_2(F'; \mathbb{Z})$. Clearly $\mu_2[T^2]$ is this generator.

Thus we must prove that $w_2\mu_2[T^2]$ generates $H_2(\Omega S^3; \mathbb{Z})$. First we calculate $w\mu T^2$ (see figure 4.7).

figure 4.7

From 4.7 we see that $w\mu(s,t) = (1)(1 + e_1)(d_2)(d_3)(e_4 + 2 + e_5)(d_6)(d_7)(1 + e_8)1$ in ΩS^2, where $(\cdot)$ denotes reduction modulo 1. Note that:

as	s	goes monotonically from	0	to	1,
	e_1		2		1
	d_2		1		2
	d_3		2		1
	e_4		1		2

and similarly for t, so $w\mu(s,t)$ is homotopic to

$$\begin{aligned} w\mu(s,t) &\sim (1)(3-s)(1+s)(2-s)(5+s-t)(1+t)(2-t)(1+t) \\ &= (-s)(s)(-s)(s-t)(t)(-t)(t). \end{aligned}$$

To calculate $w\mu[T^2]$, we use the fact that $w\mu$ factors through the map $(\Theta_1, \ldots, \Theta_7) \mapsto \Theta_1 \cdots \Theta_7$ from the 7–torus to ΩS^2; each canonical T^2 in T^7 is sent to a generator. So $(w\mu)_2[T^2]$ is calculated by taking all exterior products between terms in $w\mu(s,t)$, evaluating in $\mathbb{Z} \simeq \mathbb{Z} \wedge \mathbb{Z}$: we have

$$\begin{aligned} (w\mu)_2[T^2] = \; & -s \wedge [s - s + (s - t) + t - t + t] \\ & +s \wedge [-s + s - t + t - t + t] \\ & -s \wedge [s - t + t - t + t] + \\ & +(s - t) \wedge [t - t + t] \qquad \longleftarrow \end{aligned}$$

$$
\begin{aligned}
&+t \wedge [-t+t] \\
&+(-t) \wedge t \\
= \quad & s \wedge t
\end{aligned}
$$

(all terms, besides that indicated by the arrow, vanish)

Since $s \wedge t$ generates $\mathbb{Z} \wedge \mathbb{Z} = \mathbb{Z}$, $(w\mu)_2[T^2]$ generates $H_2(\Omega S^2; \mathbb{Z})$, completing the sketch of the proof of 4.4.

0.5 Bibliography

[Br] K.Brown: *Finiteness Properties of Groups*, J. Pure and Applied Algebra 44(1987), 45–75.

[GhS] E. Ghys, V. Sergiescu: *Sur un group remarquable de difféomorphismes du cercle*, Comment. Math. Helv. 62(1987), 185–239.

[G1] P. Greenberg: *Pseudogroups of C^1, piecewise projective homeomorphisms*, Pac. Math. J. 129(1987), 67–75.

[G2] P. Greenberg: *Torsion and Periods for "piecewise" groups*, Institut Fourier preprint 171, 1991.

[G,S] P. Greenberg, V. Sergiescu: *An acyclic extension of the braid group*, Comment. Math. Helv. 66(1991), 109–138.

[J] I. James: *Reduced Product Spaces*, Ann. of Math. 62(1955), 170–197.

[Le] I. Letho: *Univalent Functions and Teichmüller Spaces*, GTM 109, Springer–Verlag, Berlin, 1987.

[Ma] J. Mather: *Integrability in Codimension One*, Comment. Math. Helv. 48(1973), 195–233.

[Mc] D. McDuff: *Foliations and Monoids of Embeddings* in *Geometric Topology*, Ed. Cantrell, Academic Press (1979), 429–444.

[Ra] H. Rademacher: *Topics in Analytic Number Theory*, Springer–Verlag, Berlin, 1973.

[CV] M. Culler, K. Vogtmann: *Moduli of Graphs and automorphisms of free groups*, Invent. Math. 84(1986), 91–119.