Piecewise projective and piecewise Steinberg homomorphisms, and a formula in H_1

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This report is a part of a project with Vlad Sergiescu

1 Introduction

Let $\mathrm{CPP}_C(S^1)$ denote the group of C^1, piecewise $\mathrm{SL}_2\mathbb{R}$ homomorphisms of $S^1 = \mathbb{R}^2 - (0,0)/\mathbb{R}^+$ which are the identity near the ray through $(0,1)$ (see 2.1). A remarkable feature of this group ([G2]) is that its abelianization is isomorphic to $H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R});\mathbb{Z})$, where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal cover of the Lie group $\mathrm{SL}(2,\mathbb{R})$, considered as a discrete group. Our purpose in this report is to give an explicit formula: given $g \in \mathrm{CPP}_C(S^1)$ we describe an explicit $w(g) \in H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R});\mathbb{Z})$.

The isomorphism $H_1(\mathrm{CPP}_C(S^1)) \to H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R}))$ of [G2] has the following geometric description. Let $g \in \mathrm{CPP}_C(S^1)$, and consider the mapping torus of g, with its codimension–one foliation whose leaves contain the orbits of g. Since g is supported away from $(0,1)$, we can cut at $(0,1)$ to obtain a mapping sphere S^2_g (see [Ma]) with a codimension–one foliation whose leaves are cicles near the poles. This transversely CPP singular foliation determines a map from S^2_g to BCPP, the Haefliger classifying space ([H]) for such structures, and thus an element $[S^2_g] \in H_2(\mathrm{BCPP})$. Let $|\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}|$ denote the flat S^1 bundle over a $K(\mathrm{SL}(2)\mathbb{R}),\nVdash)$; this space is a $K(\widetilde{\mathrm{SL}}(2,\mathbb{R}),\nVdash)$ and possesses a natural map $\varphi : |\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}| \to \mathrm{BCPP}$ which induces an isomorphism in homology ([G2]) and so $[S^2_g] = \varphi_*(\chi_g)$, $\chi_g \in H_2|\mathrm{SL}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}| = \mathbb{H}_{\nvDash}(\widetilde{\mathrm{SL}}(\nvDash,\mathbb{R});\mathbb{Z})$. How can we describe χ_g derectly, in terms of g?

Recall ([A],[Mi]) that $H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R})) \simeq \mathrm{div}(\mathbb{K}_{\nvDash}\mathbb{R})$, the divisible part of $K_2\mathbb{R}$. This group is isomorphic to $\mathbb{R}^+ \wedge \mathbb{R}^+/\diagdown \wedge \nVdash - \diagdown$, the exterior product of $\mathbb{R}^+$ with itself over $\mathbb{Z}$, modulo the subgroup generated by $r \wedge 1 - r$, $0 < r < 1$. Given $g \in \mathrm{CPP}_C(S^1)$, we define an element $w(g) \in K_2\mathbb{R}$ as follows: Let $\chi_1, \ldots, \chi_n \in S^1$, listed in counterclockwise order, be the points where the germs $g^l_{\chi_i}, g^r_{\chi_i} \in \mathrm{SL}(2,\mathbb{R})$ (see 2.1) of g just clockwise and counterclockwise of χ_i differ, and set $D_{\chi_i}g = (g^r_{\chi_i})^{-1}g^l_{\chi_i}$. Clearly, we have

$$\mathrm{id} = D_{\chi_n}g \ldots D_{\chi_1}g \tag{1}$$

in $\mathrm{SL}(2,\mathbb{R})$. Since g is C^1, each $D_{\chi_i}g$ is parabolic, and may be expressed as

$$D_{\chi_i}g = \begin{pmatrix} 1 & 0 \\ -a_i & 1 \end{pmatrix} \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix} \qquad a_i, b_i \in \mathbb{R}$$

(or simply as $D_{\chi_i}g = \begin{pmatrix} 1 & 0 \\ -c_i & 1 \end{pmatrix}$ if $\chi_i = (0,-1)$). Recall ([Mi],[A]) the generators of the Steinberg group $\chi_{12}(\lambda), \chi_{21}(\mu)$, $\lambda, \mu \in \mathbb{R}$ which map to $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \in$ $\mathrm{SL}(2,\mathbb{R})$, and set

$$y_i = \begin{cases} x_{21}(-a_i)x_{12}(b_i)x_{21}(a_i) & chi_i \neq (0,-1) \\ x_{21}(c_i) & chi_i = (0,-1) \end{cases}$$

By equation (1,1), $y_n \dots y_1$ lies in the kernel of the projection $\mathrm{St}(2,\mathbb{R}) \to \mathrm{SL}(\nvDash,\mathbb{R})$, and so $w(g) \equiv y_n \dots y_1 \in K_2\mathbb{R}$.

Theorem 1.1 *$w(g) \in divK_2\mathbb{R}$, and $\varphi(\precsim(\eth)) = [\mathbb{S}^{\nvDash}_{\eth}]$.*

In section 2, the definition of a group of "piecewise Steinberg" homeomorphisms of S^1 implies that w is a homomorphism. In section 3, theorem 1.1 is proved by calculating $w(g)$ and $[S^2_g]$ for a family of g which generates $\mathrm{CPP}_C(S^1)$. A different approach in section 4 shows directly that w is an isomorphism $H_1\mathrm{CPP}_C(S^1) \to$div $K_2\mathbb{R}$. Sundry remarks conclude the paper.

This research was conducted in part during a visit to the SFB for discrete math at the University of Bielefeld, for whose hospitality the author is very grateful. Conversations with Bjorn Jahren, Roger Alperin and Keith Dennis were particularly helpful.

2 Piecewise projective and piecewise Steinberg groups

We regard $\mathrm{SL}(2,\mathbb{R})$ as acting on the circle of rays $S^1 = \mathbb{R}^{\nvDash} - (\nvdash,\nvdash)/\mathbb{R}^+$; we will abuse notation and denote points in S^1 by ordered pairs, so that $(1,0) = (2,0)$, for example.

Definition 2.1 *Let $CPP_C(S^1)$ denote the group of C^1 homeomorphisms of S^1, supported away from $(0,1)$, such that there is a finite set $S \subset S^1$ such that for every connected component C of S^1–S, $g|_C \equiv g_C \in SL(2,\mathbb{R})$. If $s \in S^1$, $g \in CPP_C(S^1)$ let $g^l_s, g^r_s \in SL(2,\mathbb{R})$ be the restrictions of g to small neighborehoods clockwise and counterclockwise of s, and set $D_sg = (g^r_s)^{-1}g^l_s$.*

Of course, $D_s g(s) = s$, but furthermore, since g is C^1, $D_s g'(s) = 1$. That is, $D_s g$ is parabolic, conjugate to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Let P_s, $s \in S^1$ denote the group of parabolics fixing s.

Note (eqn. (1.1)) that if $\chi_1 \ldots \chi_n \in S^1$ are the points, listed in counterclockwise order, where $D_{\chi_i} g \neq$id, then $D_{\chi_n} g \ldots D_{\chi_1} g =$id.

We recall some facts about the Steinberg group (see [Mi], [A] for details). The group $\mathrm{St}(2,\mathbb{R})$ is generated by $x_{12}(\lambda), x_{21}(\mu), \lambda, \mu \in \mathbb{R}$, and has a surjection $\pi : \mathrm{St}(2,\mathbb{R}) \to \mathrm{SL}(\nvDash,\mathbb{R})$ defined by $\pi(x_{12}(\lambda)) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\pi(x_{21}(\mu)) = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$. The kernel $K_2(2,\mathbb{R})$ of π is the center of $\mathrm{St}(2,\mathbb{R})$ and $K_2(2,\mathbb{R}) = \mathrm{div}\mathbb{K}_{\nvDash}\mathbb{R} \oplus \mathbb{Z}$; the $\mathbb{Z}$ is generated by the symbol $\{-1,-1\}$ (see [Mi]).

Consider the action of $\mathrm{St}(2,\mathbb{R})$ on S^1 via π. Let $\mathrm{At}_\infty = \{g \in \mathrm{St}(2,\mathbb{R}), \eth(\nvDash,\nVdash) = (\nvDash,\nVdash) \in \mathbb{S}^{\nVdash}\}$ and let $\mathrm{Pt}_\infty = \{x_{21}(\lambda), \lambda \in \mathbb{R}\}$. Since $x_{21}(\lambda)x_{21}(\mu) = x_{21}(\lambda+\mu)$, Pt_∞ is a subgroup of At_∞.

Keith Dennis told me how to prove:

Lemma 2.2 *Pt_∞ is a normal subgroup of At_∞.*

Proof: We use the standard notation and relations in $\mathrm{St}(2,\mathbb{R})$, conveniently summarized in [A]. There is an element $h_{12}(r) \in \mathrm{St}(2,\mathbb{R})$, for all $r > 0$, with $\pi h_{12}(r) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$. An element of At_∞ can thus be written $g = z x_{21}(t) h_{12}(r)$, with $t \in \mathbb{R}$, $r > 0$, and $z \in K_2(2,\mathbb{R})$. Then

$$\begin{aligned} g^{-1}x_{21}(s)g &= h_{12}(r)^{-1}x_{21}(s)h_{12}(r) && (\text{since } [x_{21}(t), x_{21}(s)] = 1 \\ &= x_{21}(r^2 s) && \text{and } z \text{ is in the center)} \\ & && \text{(see relations in [A])}, \end{aligned}$$

and so Pt_∞ is normal.

To consider the action on S^1, it is better to pass to a quotient. Let $\underline{\mathrm{St}}(2,\mathbb{R}) = \mathrm{St}(\nvDash,\mathbb{R})/\{-\nVdash,-\nVdash\}$, where $\{-1,-1\}$ generates the (nondivisible) $\mathbb{Z}$ in $K_2(2,\mathbb{R})$. There is a diagram

$$
\begin{tikzcd}
 & \mathbb{Z}\{-\nVdash,-\nVdash\} \arrow[d] \arrow[r, phantom, "="] & \mathbb{Z} \arrow[d] \\
\mathrm{div}K_2\mathbb{R} \arrow[r, tail] \arrow[d, equal] & \mathrm{St}(2,\mathbb{R}) \arrow[r, two heads, "\pi"] \arrow[d] & \widetilde{\mathrm{SL}}(2,\mathbb{R}) \arrow[d] \\
\mathrm{div}K_2\mathbb{R} \arrow[r, tail] & \underline{\mathrm{St}}(2,\mathbb{R}) \arrow[r, "\underline{\pi}"] & \mathrm{SL}(2,\mathbb{R})
\end{tikzcd}
$$

whose lower right–hand square is a pullback. So $\mathrm{St}(2,\mathbb{R})$ is the "topological universal cover" of the group $\underline{\mathrm{St}}(2,\mathbb{R})$. Consider the action of $\underline{\mathrm{St}}(2,\mathbb{R})$ on S^1 via $\underline{\pi}$. Let $\underline{\mathrm{At}}_\infty = \{g \in \underline{\mathrm{St}}(2,\mathbb{R}), \eth(\nvdash,\nVdash) = (\nvdash,\nVdash) \in \mathbb{S}^{\nVdash}\}$, and let $\underline{\mathrm{Pt}}_\infty = \{x_{21}(\lambda), \lambda \in \mathbb{R}\}$ regarded in $\underline{\mathrm{St}}(2,\mathbb{R})$. Then (2.2) implies immediately:

Lemma 2.3 *$\underline{Pt}_\infty$ is normal in $\underline{At}_\infty$.*

Let $s \in S^1$, and define $\underline{\mathrm{Pt}}_s = g^{-1}\underline{\mathrm{Pt}}_\infty g$, where $g \in \underline{\mathrm{St}}(2,\mathbb{R})$ and $g(s) = (0,1)$; lemma 2.4 proves that $\underline{\mathrm{Pt}}_s$ is defined independently of g, and is normal in the stabilizer of $s \in S^1$ in $\underline{\mathrm{St}}(2,\mathbb{R})$.

Definition 2.4 *Let $g \in CPP_C(S^1)$. A $\underline{\text{lift}}$ $\overline{g}$ of g is a choice, for each $s \in S^1$, of lifts $\overline{g}^l_s, \overline{g}^r_s \in \underline{St}(2,\mathbb{R})$ of g^l_s, g^r_s such that $D_s\overline{g} \equiv (\overline{g}^r_s)^{-1}(\overline{g}^l_s)$ is not the identity only if $D_s g \neq id$ or $s = (0,1)$, and such that $D_s\overline{g} \in \underline{Pt}_s$ for all $s \in S^1$, $s \neq (0,1)$.*

The set of lifts forms a group $\mathrm{CPSt}_K(S^1)$ with a surjection to $\mathrm{CPP}_C(S^1)$. Denote by $(l,r) : \mathrm{CPSt}_K(S^1) \to \mathrm{div}K_2\mathbb{R} \oplus \mathrm{div}\mathbb{K}_{\nvdash}\mathbb{R}$ the map defined by $(l,r)(\overline{g}) = (\overline{g}^l_{(0,1)}, \overline{g}^r_{(0,1)})$, and define $\mathrm{CPSt}_C(S^1) \equiv \mathrm{Ker}(l,r)$.

Proposition 2.5 *$w : CPP_C(S^1) \to K_2\mathbb{R}$ is a homomorphism.*

Proof: A section L of $\pi : \mathrm{CPSt}_K(S^1) \to \mathrm{CPP}_C(S^1)$ is uniquely defined by $rL(g) = \mathrm{id}$, $g \in \mathrm{CPP}_C S^1$. Now observe, by 2.2 and 2.4, that $w(g) = lL(g)$. Indeed (in the notation of section 1) $y_i = (\overline{g}^r_{\chi_i})^{-1}\overline{g}^l_{\chi_i}$, for all i.

3 Examples

In this section we define a family of elements $g_{a,\lambda,w} \in \mathrm{CPP}_C(S^1)$. Then we calculate that for $g = g_{a,\lambda,w}$, $[S^2_g] = \varphi(w(g))$, to prove 1.2.

3.1 The examples.

We define $g = g_{a,\lambda,w} \in \mathrm{CPP}_C(S^1)$, $a, \lambda, -w \in \mathbb{R}^+$ as a commutator $g = [h,k]$. Here $k = \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and h is defined as follows:

(i) On the left half circle from $(0,1)$ to $(0,-1)$, $h \equiv \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$

(ii) from $(0,-1)$ to $(1,w)$ $h \equiv \begin{pmatrix} 1/a & 0 \\ w - aw & a \end{pmatrix}$

(iii) from $(1,w)$ to $(1,0)$ $h \equiv \begin{pmatrix} 1 & \frac{1-a}{wa} \\ 0 & 1 \end{pmatrix}$

(iv) from $(1,0)$ to $(0,1)$, $h \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note that h is CPP except at $(0,1)$, and that $g = g_{a,\lambda,w} \equiv [h,k]$ is supported in the quarter circle from $(0,-1)$ to $(1,0)$ and has just 4 breakpoints. In [G3] it is shown (in a slightly different context) that the conjugates of $g_{a,\lambda,w}$ by elements of $\mathrm{SL}_2\mathbb{R}$ generate $\mathrm{CPP}_C(S^1)$.

Recall the isomorphism $\mathrm{div}K_2\mathbb{R} \to \mathbb{R}^+ \wedge \mathbb{R}^+ / \setminus \wedge (\nVdash - \setminus)$, defined by $\{u,v\} \to u \wedge v$ ([Mi]).

Proposition 3.1 *$w(g_{a,\lambda,w}) = \lambda \wedge a$.*

Proof: Regarding h as CPP except at $(0,1)$, let $\overline{h}$ be the lift with $\overline{h} = \mathrm{id} \in \underline{\mathrm{St}}(2,\mathbb{R})$, from $(1,0)$ counterclockwise to $(0,1)$. Let $\overline{k} \in \underline{\mathrm{St}}(2,\mathbb{R})$ be any lift of k, e.g. $\overline{k} = h_{12}(1/\lambda)$. Then $w(g) = w([h,k]) = -r[\overline{h},\overline{k}] = -\{\frac{1}{a}, \frac{1}{\lambda}\} = \lambda \wedge a$.

Proof of 1.1: Since the $g_{a,\lambda,w}$ and their conjugates generate $\mathrm{CPP}_C(S^1)$, it suffices that $\varphi(w(g)) = [S^2_g]$, $g = g_{a,\lambda,w}$. What is required, then, is that S^2_g with its CPP foliation be concordant to an oriented 2–manifold with a CPP–structure (which, in this case, will be a "foliated microbundle") which represents $\lambda \wedge a$. Such a 2–manifold is the following disjoint union $T_1 \amalg T_2$ of tori:

Give T_2 the trivial CPP structure, by embedding T_2 as $T_2 \times (1,1)$ in the product $T_2 \times S^1$, foliated by the $T_2 \times c$, $c \in S^1$.

Let x, y generate $\pi_1 T_1$, and give T_1 the orientation $y \wedge x$. Give T_1 the CPP structure as the $(-1,0)$–section of the flat S^1–bundle induced by $x \mapsto \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$,

$y \mapsto \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Then it is clear that $[T_1 \amalg T_2] = \lambda \wedge a + 0 = \lambda \wedge a \in H_2(|\mathrm{SL}(2,\mathbb{R})\backslash \mathbb{S}^{\Bbbk}|) = \mathbb{H}_{\not{\Bbbk}}(\mathrm{BCPP})$.

We now construct a concordance from S^2_g to $T_1 \amalg T_2$, namely a 3–manifold M with boundary $S^2_g - (T_1 \amalg T_2)$ and a CPP foliation which restricts to the respective structures on the boundary. M is formed by gluing two copies D_1 and D_2 of $I \times \Delta_2$, where $I \subseteq S^1$ is the closed interval from $(-1, 0)$ counterclockwise to $(1, 1)$ and Δ_2 is a 2–simplex with vertices a, b, c.

The D_i are foliated as follows (fig. 3.3):

fig. 3.3

The leaves in D_1 are triangles formed as the convex closures of (t, a), $(k(t), b)$ and $(hk(t), c)$ in $I \times \Delta_2$. The leaves in D_2 are the convex closures of (t, a), $(h(t), b)$, $(kh(t), c) \in I \times \Delta_2$.

To form M, first glue the ab–face of D_1 to the bc–face of D_2, and the bc–face of D_1 to the ab–face of D_2. Recall that for t near $(-1, 0)$ and $(1, 1)$, $hk(t) = kh(t)$, so glue the ac–faces of D_1 and D_2 near their tops and bottoms. The "hole" left over gives S^2_g, and the tops and bottoms of D_1 and D_2 give T_2 and T_1, respectively.

So M is the required concordance, whose existence proves 1.1.

4 The Piecewise–Steinberg groupoid

Recall the subgroup $\mathrm{CPSt}_C(S^1) = \ker(l, r) \subseteq \mathrm{CPSt}_K(S^1)$. Evidently, L sends the commutator subgroup of $\mathrm{CPP}_C(S^1)$ to $\mathrm{CPSt}_C(S^1)$. Thus we can give an alternate proof that $w : H_1\mathrm{CPP}_C(S^1) \to \mathrm{div} K_2\mathbb{R}$ is an isomorphism, by proving directly:

Theorem 4.1 $H_1 CPSt_C(S^1) = 0$

Techniques of [G1] are invoked for the proof of 4.1. We define a groupoid CPSt; it follows from Mathers theorem ([Ma], see also [G2]) that $H_1\mathrm{CPSt}_C(S^1) = H_2\mathrm{BCPSt}$, where BCPSt, the classifying space of the groupoid, is weakly homotopic to a space which has the homology of $\mathrm{St}(2,\mathbb{R})$. Hence $H_1\mathrm{CPSt}_C(S^1) = 0$.

To begin, we define the groupoid.

Definition 4.2 *(Recall [G2]). Let $g : \mho \to V$ be a C^1 homeomorphism between open subsets of S^1. Then $g \in CPP$ if there is a descrete subset $S \subseteq \mho$ such that, for every component C of $\mho - S$, $g|_C \equiv g_C \in SL(2,\mathbb{R})$. As before, define $g^l_s, g^r_s \in SL(2,\mathbb{R})$, and $D_s g = (g^r_s)^{-1} g^l_s$, $s \in S^1$. A* lift *$\overline{g}$ of $g \in CPP$, $g : \mho \to V$ is a choice, for all $s \in \mho$ of lifts $\overline{g}^l_s, \overline{g}^r_s \in \underline{St}(2,\mathbb{R})$ of g^l_s, g^r_s such that $D_s\overline{g} = (\overline{g}^r_s)^{-1}\overline{g}^l_s$ is different from the identity only if $D_s g$ is different from the identity, and such that $D_s\overline{g} \in \underline{Pt}_s$, $s \in \mho$.*

Two lifts $\overline{g}, \overline{h}$ define the same germ *at $s \in S^1$ if s belongs to their domains, and $\overline{g}^l_s = \overline{h}^l_s$, $\overline{g}^r_s = \overline{h}^r_s$. Let $CPSt_1$ denote the space of germs of lifts, with the sheaf topology, and let CPSt be the toplogical groupoid ("etale over objects", [G1]) with morphisms $CPSt_1$ and objects S^1. Denote by BCPSt the classifying space of CPSt, with the "thick" realization ([Se]).*

Proof of 4.1 By the theorem of Mather ([Ma], [G2]) it suffices to prove that (i) CPSt is "germ–connected to the identity", ([G2]) and that (ii) BCPSt is 2–connected. For then (i) implies that $H_2\mathrm{CPSt}_C(S^1) = H_2\mathrm{BCPSt} = 0$ by (ii). Now (i) is a technical condition assured by the existence of the examples $g_{a,\lambda,w}$ of the previous section; we suppress the proof. For (ii), the main result of [G1] implies:

Facts 4.3 BCPSt is weakly homotopy equivalent to a pushout

$$\begin{array}{ccc} \mathrm{B}\underline{\mathrm{At}}_\infty & \xrightarrow{B_\Delta} & R \\ \downarrow & & \\ |\underline{\mathrm{St}}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}| & & \end{array}$$

Where

(i) $\underline{\mathrm{At}}_\infty = \{g \in \underline{\mathrm{St}}(2,\mathbb{R}), \eth(\nvdash,\nVdash) = (\nvdash,\nVdash) \in \mathbb{S}^{\nVdash}\}$

(ii) $|\underline{\mathrm{St}}(2,\mathbb{R})\backslash\mathbb{S}^{\nVdash}|$ is the total space of the flat S^1 bundle on $\mathrm{B}\underline{\mathrm{St}}(2,\mathbb{R})$ induced by $\underline{\mathrm{St}}(2,\mathbb{R}) \to \mathrm{SL}(\nvDash,\mathbb{R})$, and

(iii) R is a pushout

$$\begin{tikzcd} \mathrm{B}\underline{\mathrm{At}}^p_\infty \arrow[r, "B_{p_2}"] \arrow[d, "B_{p_1}"] & B\underline{\mathrm{At}}_\infty \\ B\underline{\mathrm{At}}_\infty & \end{tikzcd}$$

where $\underline{\mathrm{At}}^p_\infty = \{(g_1, g_2) \in \underline{\mathrm{At}}_\infty, g_1^{-1}g_2 \in \underline{\mathrm{Pt}}_\infty\}$ and $p_i(g_1, g_2) = g_i, \Delta(g) = (g, g)$.

By (2.2), $|\underline{\mathrm{St}}(2,\mathbb{R})\backslash\mathbb{S}^{⊮}|$ is a $K(\mathrm{St}(2,\mathbb{R}), ⊮)$, so $H_1|\underline{\mathrm{St}}(2,\mathbb{R})\backslash\mathbb{S}^{⊮}| = \mathbb{H}_{⊭}|\underline{\mathrm{St}}(⊬,\mathbb{R})\backslash\mathbb{S}^{⊮}| =$ ⊬. And $\pi_1\mathrm{BCPSt} = 0$ by (4.4) (or "germ–connected to the identity"). So the proof of 4.1 concludes with:

Lemma 4.4 *$|\underline{St}(2,\mathbb{R})\backslash\mathbb{S}^{⊮}| \to BCPSt$ induces an isomorphism in homology.*

Proof: Let $A = \{g \in \mathrm{SL}(2,\mathbb{R}), ð(⊬,⊮) = (⊬,⊮) \in \mathbb{S}^{⊮}\}$ and let $A^p = \{(g_1, g_2) \in A \times A, g_1^{-1}g_2 \in P_\infty\}$. One knows ([G2]) that the maps $A \to A^p \to A$ induce isomorphism in homology; 4.5 follows when we show the same is true for $\underline{\mathrm{At}}_\infty \to \underline{\mathrm{At}}^p_\infty \to \underline{\mathrm{At}}_\infty$.

Consider the pullback squares

and note that A and A^p act trivially on the kernels of the vertical projections. The Lemma then follows, for example, by comparing the spectral sequences of the projections.

5 Remarks

Let R be a subring of $\mathbb{R}$; define a group $\mathrm{CPP}^R_C(S^1)$ as in 2.1, but requiring in addition that

(i)

$$g^l_s, g^r_s \in \mathrm{SL}(2, R), s \in S' \text{ and}$$

(ii) $g_s^l = g_s^r$ unless s is a ray whose slope is in the quotient field of R.

For example, if $R = \mathbb{Z}$ one obtains $\mathrm{CPP}_C^{\mathbb{Z}}(S^1) = F'$, the commutator subgroup of the Higman–Thompson group F of homeomorphisms of an interval. To what extent do the results of this report apply to $\mathrm{CPP}_C^R(S^1)$?

First of all, sections 1 and 2 go through without change, so that there is a well–defined homomorphism $w : \mathrm{CPP}_C^R(S^1) \to H_2(\widetilde{\mathrm{SL}}(2,R);\mathbb{Z})$ (where $\widetilde{\mathrm{SL}}(2,R)$ is the pull-back over $\mathrm{SL}(2,R)$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{SL}(\nvDash,\mathbb{R})$). Whether w defines an isomorphism $w : H_1\mathrm{CPP}_C^R(S^1) \to H_2(\widetilde{\mathrm{SL}}(2,\mathbb{R});\mathbb{Z})$ depends, first of all, on whether indeed $H_1\mathrm{CPP}_C^R(S^1) \simeq H_2(\widetilde{\mathrm{SL}}(2,R))$. Considerations from [G2] show that in fact:

Fact 5.1 *Let R be a subring of $\mathbb{R}$. Let A be the group of affine transformations $x \mapsto ux + a$, $u \in A^*$, $a \in A$ and let $A^p = \{(g_1, g_2) \in A \times A, g_1' = g_2'\}$. If the maps $\Delta : A \to A^p$ and $p^i : A^p \to A$ induce isomorphisms in H_1 and H_2, then $H_1 CPP_C^R(S^1) \simeq H_2(\widetilde{SL}(2,R))$.*

Then, regarding 4.5 and its proof, we conclude as well that:

Corollary 5.2 *If the maps $\Delta : A \to A^p$ and $p_i : A^p \to A$ of 5.1 induce isomorphisms in H_1, H_2 and H_3, then w is an isomorphism.*

I do not know whether the $g_{a,\lambda,w}$ (appropriately defined) generate $\mathrm{CPP}_C^R(S^1)$, $R \neq \mathbb{R}$.

In any case, the conditions of 5.2 hold for a wide class of rings:

Example 5.3 Suppose $\frac{1}{2} \in R$. Then hypotheses of 5.2 hold, so w is an isomorphism. For we have

$$3^1 = 2$$

$$3^2 - 1 = 2^3$$

$$3 \cdot 5^3 - 7^3 - 3^3 - 1 = 2^2$$

and so $H_p(R^*; H_q R) = H_p(R^*; H_q(R \oplus R)) = 0$, $q = 1, 2, 3$, by the "Center Kills" lemma, implying that $H_p(A) \simeq H_p(A^P) \simeq H_p\mathbb{R}^*$, $p = 1, 2, 3$.

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