The Pfaff–Saalschütz identity in hyperbolic geometry

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1 Introduction

Let $P_n(\mathbb{C})$ denote the complex vector space of all polynomials in z and $\bar{z}$ whose degree at z and at $\bar{z}$ is bounded by n. Let $P_n(\mathbb{H}^3)$ be the complex vector space of all polynomials in $z, \bar{z}$ and t whose total degree is less or equal to n. Both spaces are natural $PSL_2(\mathbb{C})$–modules: $P_n(\mathbb{C})$ gets its $PSL_2(\mathbb{C})$ structure from the canonical action of $SL_2(\mathbb{C})$ on $\mathbb{C}^2$, $P_n(\mathbb{H}^3)$ is acted on by $PSL_2(\mathbb{C})$ as a submodule of the ring of affine Klein coordinates on three dimensional hyperbolic space, see line (2) below.

$P_n(\mathbb{C})$ is irreducible. $P_2(\mathbb{H}^3)$ contains a non zero element which is invariant under $PSL_2(\mathbb{C})$, namely $r^2 := t - z\bar{z}$. This statement is commonly known as one of the exceptional isomorphisms between Lie groups. It identifies $PSL_2(\mathbb{C})$ and $SO(1,3)$.

There is a natural homomorphism

$$h_n : P_n(\mathbb{H}^3) \to P_n(\mathbb{C}).$$

It is defined by

$$h_n(t) = z\bar{z}$$

and it respects the action of $PSL_2(\mathbb{C})$ on both modules. It geometrically means restricting functions defined on a three dimensional ball to its boundary which is a two dimensional sphere. Here the interior of the ball plays the role of real Lobačevskij (or, equivalently, hyperbolic) space in dimension 3, and its boundary is identified with the Riemann sphere.

We, by explicitly exhibiting a homomorphism η_n inverse to h_n:

$$\begin{aligned} \eta_n : P_n(\mathbb{C}) &\to P_n(\mathbb{H}^3) \\ h_n \circ \eta_n &= id P_n(\mathbb{C}) \end{aligned}$$

show that h_n is surjective. η_n, of course, will be $PSL_2(\mathbb{C})$–equivariant, again. It will be uniquely determined by these properties, so $P_n(\mathbb{H}^3)$ contains $P_n(\mathbb{C})$ as an irreducible

$PSL_2(\mathbb{C})$–module exactly once. Surprisingly, the equivariance of η_n is equivalent to the validity of the so called Pfaff–Saalschütz identity. The kernel of h_n can easily be identified with $P_{n-2}(\mathbb{H}^3)$, so we get a decomposition

$$P_n(\mathbb{H}^3) = \bigoplus_{2j \leq n} P_{n-2j}(\mathbb{C})$$

into irreducible submodules.

We finally observe that η_n relates the Laplacian Δ_E on $\mathbb{C}$ as a Euclidean plane to the Laplacian Δ_L on three dimensional Lobačevskij space $\mathbb{H}^3$.

It seems to be worthwile studying algebraic geometric objects like the set of zeros of some $\varphi \in P_n(\mathbb{H}^3)$. If, in particular, φ has the form $\eta_n(t)$ for some $f \in P_n(\mathbb{C})$ then its set of zeros in $\mathbb{H}^3$ will be a geometric object which perpendicularly meets the boundary, its "trace" there being the set of zeros of $f(z, \bar{z})$ on the Riemann sphere. This observation comprises and generalizes well known facts from (linear) elementary hyperbolic geometry in dimension three, as, e.g., compiled in [2]. An example will be discussed.

2 Prerequisits

We collect a few facts from elementary geometry. Let $\mathcal{H}_2$ be the collection of all hermitean forms on $\mathbb{C}^2$. It is a fourdimensional real vector space. Taking Gram matrices

$$Gr(\beta) = \begin{pmatrix} \beta(e_1, e_1) & \beta(e_1, e_2) \\ \beta(e_2, e_1) & \beta(e_2, e_2) \end{pmatrix}$$

for $\beta \in \mathcal{H}_2$, with respect to some fixed basis (e_1, e_2) of $\mathbb{C}^2$, identifies $\mathcal{H}_2$ and

$$\left\{ \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} | \xi, \eta \in \mathbb{R}, \zeta \in \mathbb{C} \right\}.$$

$\mathcal{H}_2$ carries a geometry $\langle \quad , \quad \rangle$:

$$\langle \beta, \beta \rangle = det Gr(\beta)$$

is a quadratic form on $\mathcal{H}_2$ with signature (1,3).

The group $PSL_2(\mathbb{C})$ acts, in a canonical way, on $\mathcal{H}_2$, which is expressed explicitly by

$$\begin{aligned} \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} &\to S \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} {}^t\bar{S} \\ &= \begin{pmatrix} a\bar{a}\xi + a\bar{b}\zeta + \bar{a}b\bar{\zeta} + b\bar{b}\eta & a\bar{c}\xi + a\bar{d}\zeta + b\bar{c}\bar{\zeta} + b\bar{d}\eta \\ \bar{a}c\xi + \bar{b}c\zeta + \bar{a}d\bar{\zeta} + \bar{b}d\eta & c\bar{c}\xi + c\bar{d}\zeta + \bar{c}d\bar{\zeta} + d\bar{d}\eta \end{pmatrix} \end{aligned} \tag{1}$$

on the Gram matrix model of $\mathcal{H}_2$. Here $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, and $\bar{S}, {}^tS$ denote complex conjugation at each entry and transposition, respectively. The quadratic form $\langle , \rangle$ on $\mathcal{H}_2$ is invariant under the action of $PSL_2(\mathbb{C})$; in fact $PSL_2(\mathbb{C})$ is identified with $SO(1,3)$, this way. The $PSL_2(\mathbb{C})$–action on $\mathcal{H}_2$ is described with respect to the $\mathcal{H}_2$–basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a\bar{a} & 2Re(a\bar{b}) & 2Im(\bar{a}b) & b\bar{b} \\ Re(a\bar{c}) & Re(a\bar{d}+b\bar{c}) & Im(\bar{a}d-\bar{b}c) & Re(b\bar{d}) \\ Im(a\bar{c}) & Im(a\bar{d}+b\bar{c}) & Re(\bar{a}d-\bar{b}c) & Im(b\bar{d}) \\ c\bar{c} & 2Re(c\bar{d}) & 2Im(\bar{c}d) & d\bar{d} \end{pmatrix}$$

.

Let

$$\begin{aligned} \mathcal{H}_2^0 &= \{\beta \in \mathcal{H}_2 | \langle \beta, \beta \rangle = 0, \beta \neq 0\}, \\ \mathcal{H}_2^+ &= \{\beta \in \mathcal{H}_2 | \langle \beta, \beta \rangle > 0\}, \\ \mathcal{H}_2^* &= \mathcal{H}_2^0 \cup \mathcal{H}_2^+. \end{aligned}$$

Both $\mathcal{H}_2^0$ and $\mathcal{H}_2^+$, and so $\mathcal{H}_2^*$, are $PSL_2(\mathbb{C})$–invariant cones in $\mathcal{H}_2$. The projectivation $\mathcal{H}_2^*/\mathbb{R}^*$ of $\mathcal{H}_2^*$ is a three dimensional closed ball. Classically, its boundary $\mathcal{H}_2^0/\mathbb{R}^*$ and its interiour $\mathcal{H}_2^+/\mathbb{R}^*$ are identified with the Riemann sphere $P^1\mathbb{C}$ and the Poincaré model $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+ = \{(z, r) | z \in \mathbb{C}, r > 0\}$ of three dimensional Lobačevskij (equivalent, hyperbolic) space, respectively:

$$\mathcal{H}_2^0 \ni \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} \quad \to \quad z = \frac{\zeta}{\eta} = \frac{\xi}{\bar{\zeta}} \in P^1\mathbb{C},$$

$$\mathcal{H}_2^+ \ni \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} \quad \to \quad (z, r) = \left(\frac{\zeta}{\eta}, \frac{\sqrt{\xi\eta - \zeta\bar{\zeta}}}{\eta} \right) \in \mathbb{C} \times \mathbb{R}_+.$$

An inspection of (1) shows that, on $\mathcal{H}_2^0$, for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, the mapping

$$\begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} \to S \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} {}^t\bar{S}$$

means, expressed in terms of the coordinate z on $\mathcal{H}_2^0/\mathbb{R}^*$:

$$z \to \frac{az+b}{cz+d}$$

which is the standard interpretation of S as a Möbius transformation. Analoguously, the action of $PSL_2(\mathbb{C})$ on $\mathcal{H}_2^+$ is transformed into the action of the group $Iso^+(\mathbb{H}^3)$ of orientation preserving isometries on hyperbolic space:

$$\mathbb{H}^3 \ni (z,r) \to \left(\frac{(az+b)(\overline{cz+d}) + a\bar{c}r^2}{j_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; z,r\right)}, \frac{r}{j_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; z,r\right)} \right) \in \mathbb{H}^3.$$

Here, for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, we define

$$j_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; z,r\right) = |cz+d|^2 + |c|^2 r^2.$$

Note that $j_3^{-3}(S;z,r)$ is the determinant of the Jacobian of the mapping $S \in SL_2(\mathbb{C})$ on $\mathbb{H}^3$. This is in analogy to the fact that if

$$j_2(S;z) = cz+d,$$

$j_2^{-2}(S;z)$ is the (determinant of the) Jacobian of $S \in SL_2(\mathbb{C})$, now regarded as a Möbius transformation on $\mathbb{C} \cup \{\infty\}$.

In what follows we shall use, besides the geometric Poincaré coordinates (z,r) on $\mathbb{H}^3$, the more algebraic Klein coordinates $(z,\bar{z},t)$, defined by

$$\mathcal{H}_2^+ \ni \begin{pmatrix} \xi & \zeta \\ \bar{\zeta} & \eta \end{pmatrix} = \eta \begin{pmatrix} t & z \\ \bar{z} & 1 \end{pmatrix}. \tag{2}$$

Note that $t = z\bar{z} + r^2$.

3 Representations

Let $n \geq 0$ a fixed integer. We denote

$$P_n(\mathbb{C}) = \mathbb{C}[z]_n \otimes \overline{\mathbb{C}[z]_n}$$

the space of all polynomials with complex coefficients on the two variables $z, \bar{z}$, the exponents at both z and $\bar{z}$ being bounded by n. (Here, $\mathbb{C}[z]_n$ is the space of complex polynomials on the variable z with degree at most n.) $P_n(\mathbb{C})$ is a complex vector space with dimension $(n+1)^2$. $\mathbb{C}[z]_n$ is acted on by $SL_2(\mathbb{C})$ via

$$\mathbb{C}[z]_n \ni f \to f|_n S \in \mathbb{C}[z]_n$$

where, for $S \in SL_2(\mathbb{C})$, we set

$$(f|_n S)(z) = j_2^n(S;z)(f \circ S)(z).$$

Of course, $\mathbb{C}[z]_n$ equals the set of all elements $f \in \mathbb{C}[z]$ such that $j_2^n(S;z)(f \circ S)(z)$, as a rational function, is a polynomial, again. It is, as an $SL_2(\mathbb{C})$–module, the n^{th}

symmetric tensor power of $\mathbb{C}[z]_2$. $P_n(\mathbb{C})$ inherits a canonical $PSL_2(\mathbb{C})$ structure; the action will also be denoted by

$$P_n(\mathbb{C}) \ni f \rightarrow f|_n S \in P_n(\mathbb{C}).$$

Here, of course,

$$(f|_n S)(z, \bar{z}) = |j_2(S; z)|^{2n} f(S(z), \overline{S(z)}).$$

It is well known and an easy exercise to show that, for each $n \geq 0$, $P_n(\mathbb{C})$ is an irreducible $\mathbb{C}[PSL_2(\mathbb{C})]$–module.

We define $\mathbb{C}[z, \bar{z}, t]$ to be the ring of affine coordinates on $\mathbb{H}^3$. Note that the geometric variable $r = \sqrt{t - z\bar{z}}$ will not be considered a rational function on $\mathbb{H}^3$.

Let φ be a complex valued function, defined on $\mathbb{H}^3$. For $S \in SL_2(\mathbb{C})$ and n integral

$$j_3^n(S; z, r)(\varphi \circ S)(z, r)$$

will be complex valued on $\mathbb{H}^3$, again. We define, for $n \geq 0$,

$$P_n(\mathbb{H}^3) = \{\varphi \in \mathbb{C}[z, \bar{z}, t] \quad | \quad j_3^n(S; z, r)(\varphi \circ S)(z, r) \in \mathbb{C}[z, \bar{z}, t] \quad \forall S \in SL_2(\mathbb{C})\}.$$

The elements of $P_n(\mathbb{H}^3)$ will be called polynomials on $\mathbb{H}^3$ of degree at most of n. Observe that, for $\varphi \in \mathbb{C}[z, \bar{z}, t]$ the function $j_3^n(S; z, r)(\varphi \circ S)(z, r)$ will be a rational function on $\mathbb{H}^3$ in the strict sense that, for some integer $N \gg 0$,

$$j_3^N(S; z, r)\Big(j_3^n(S; z, r)(\varphi \circ S)(z, r)\Big)$$

will be a polynomial, again.

Proposition. $P_n(\mathbb{H}^3)$ is the set of all polynomials in $\mathbb{C}[z, \bar{z}, t]$ of total degree at most n. So,

$$\begin{aligned} P_0(\mathbb{H}^3) &= \mathbb{C}, \\ P_1(\mathbb{H}^3) &= \mathbb{C} \oplus \mathbb{C}z \oplus \mathbb{C}\bar{z} \oplus \mathbb{C}t, \end{aligned}$$

and $P_n(\mathbb{H}^3)$ is the n^{th} symmetric tensor power of $P_1(\mathbb{H}^3)$. In particular

$$dim_{\mathbb{C}} P_n(\mathbb{H}^3) = \binom{n+3}{3}.$$

We note that, by construction, $P_n(\mathbb{H}^3)$ is a $PSL_2(\mathbb{C})$–module as well. We denote the action again by

$$P_n(\mathbb{H}^3) \ni \varphi \rightarrow \varphi|_n S \in P_n(\mathbb{H}^3)$$

where now, for $S \in SL_2(\mathbb{C})$

$$(\varphi|_n S)(z, r) = j_3^n(S; z, r)(\varphi \circ S)(z, r).$$

In the proof of the proposition we make use of the fact that

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

generates $SL_2(\mathbb{C})$. We compute

$$\begin{aligned} j_3\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; z, r \right) &= 1, \\ j_3\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; z, r \right) &= z\bar{z} + r^2 = t. \end{aligned}$$

Let $\varphi_{klm}(z,r) = z^k \bar{z}^l t^m$ for k, l, m nonnegative integers. We compute, for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$\varphi_{klm}(S; z, r) = (-1)^{k+l} z^l \bar{z}^k t^{-(k+l+m)}.$$

So, in order the denominator of any linear combination of such functions to be cancelled one has to multiply by t^n where $n \geq k + l + m$. This settles the only not quite obvious part of the proposition.

We note the representation of $PSL_2(\mathbb{C})$ on $P_1(\mathbb{H}^3)$, taking the basis $(1, z, \bar{z}, t)$:

$$\begin{pmatrix} t|_1 S \\ z|_1 S \\ \bar{z}|_1 S \\ 1|_1 S \end{pmatrix} = \begin{pmatrix} a\bar{a} & a\bar{b} & \bar{a}b & b\bar{b} \\ a\bar{c} & a\bar{d} & b\bar{c} & b\bar{d} \\ \bar{a}c & \bar{b}c & \bar{a}d & \bar{b}d \\ c\bar{c} & c\bar{d} & \bar{c}d & d\bar{d} \end{pmatrix} \begin{pmatrix} t \\ z \\ \bar{z} \\ 1 \end{pmatrix}$$

where $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$.

We note that $P_0(\mathbb{H}^3) = \mathbb{P}_0(\mathbb{C})$ and $P_1(\mathbb{H}^3) = \mathbb{P}_1(\mathbb{C})$ are irreducible $PSL_2(\mathbb{C})$–modules. This remark will allow to decompose $P_n(\mathbb{H}^3)$ into irreducible summands.

4 Results

The main statement of this paper is contained in

Theorem. $P_n(\mathbb{H}^3) = \bigoplus_{2j \leq n} \mathbb{P}_{n-2j}(\mathbb{C})$ is the decomposition of $P_n(\mathbb{H}^3)$ into irreducible $PSL_2(\mathbb{C})$–summands.

In order to prove the theorem we establish an exact sequence of $PSL_2(\mathbb{C})$–modules

$$0 \longrightarrow P_{n-2}(\mathbb{H}^3) \xrightarrow{\mu_n} \mathbb{P}_n(\mathbb{H}^3) \xrightarrow{\tilde{\pi}_n} \mathbb{P}_n(\mathbb{C}) \longrightarrow 0. \tag{3}$$

So

$$P_n(\mathbb{H}^3) \cong \mathbb{P}_n(\mathbb{C}) \oplus \mathbb{P}_{n-2}(\mathbb{H}^3) \tag{4}$$

for $n \geq 2$. (For $n = 0$ or 1 we know already: $P_0(\mathbb{H}^3) = \mathbb{P}_0(\mathbb{C}), \mathbb{P}_1(\mathbb{H}^3) = \mathbb{P}_1(\mathbb{C})$.)

From semisimplicity of $PSL_2(\mathbb{C})$ it follows that there exists a $PSL_2(\mathbb{C})$–eqivariant homomorphism

$$\eta_n : P_n(\mathbb{C}) \to \mathbb{P}_n(\mathbb{H}^3). \tag{5}$$

All $PSL_2(\mathbb{C})$–modules $P_n(\mathbb{C})$ being irreducible and pairwise not isomorphic one reads from (4) that η_n is unique up to a complex scalar factor, so may and will be normalized by

$$h_n \circ \eta_n = id P_n(\mathbb{C}). \tag{6}$$

We write down η_n explicitly.

Each polynomial $\varphi \in P_n(\mathbb{H}^3)$ is a well defined function on the closure $\mathbb{H}^3 \cup \mathbb{P}^1\mathbb{C}$ of Lobačevskij space. We define h_n to be the restriction of $\varphi \in P_n(\mathbb{H}^3)$ to the boundary:

$$\begin{array}{rcl} h_n : P_n(\mathbb{H}^3) & \to & P_n(\mathbb{C}) \\ h_n(\varphi)(z) & = & \varphi(z, 0) \end{array}$$

in terms of the coordinates (z, r) on $\mathbb{H}^3$, or, equivelently

$$h_n(\varphi)(z, \bar{z}) = \varphi(z, \bar{z}, z\bar{z})$$

in terms of the affine coordinates on $\mathbb{H}^3$. Obviously, if for $k, l, m \geq 0$ the inequality $k + l + m \leq n$ holds then

$$h_n(z^k \bar{z}^l t^m) = z^{k+m} \bar{z}^{l+m}$$

is in $P_n(\mathbb{C})$. By the discussion in the first paragraph, h_n is $PSL_2(\mathbb{C})$–equivariant:

$$h_n(\varphi|_n S) = h_n(\varphi)|_n S$$

for $\varphi \in P_n(\mathbb{H}^3), \mathbb{S} \in \mathbb{SL}_2(\mathbb{C})$.

The kernel of h_n will consists of all elements in $P_n(\mathbb{H}^3)$ which are divisible by $r^2 = t - z\bar{z}$. This special polynomial is in $P_2(\mathbb{H}^3)$, and, by construction, is an eigenvector for all of $PSL_2(\mathbb{C})$:

$$(t - z\bar{z})|_2 S = t - z\bar{z}$$

for all $S \in SL_2(\mathbb{C})$. So

$$\mu_n : P_{n-2}(\mathbb{H}^3) \to \mathbb{P}_n(\mathbb{H}^3)$$

defined by

$$\mu_n(\varphi) = (t - z\bar{z}) \cdot \varphi$$

is a $PSL_2(\mathbb{C})$–injection. It identifies the kernel of h_n with $P_{n-2}(\mathbb{H}^{\nVDash})$.

We see that the isomorphism stated in the theorem may be formulated in the following way:

$$\bigoplus_{2j \leq n} P_{n-2j}(\mathbb{C}) \ni (\mho_{\ltimes}, \mho_{\ltimes - \nvDash}, \cdots) \to \sum_{\gimel \leq \nvDash \ltimes} (\approx - F\bar{\nvDash})^{\gimel} \eta_{\ltimes - \nvDash\gimel}(\mho_{\ltimes - \nvDash\gimel}) \in \mathbb{P}_{\ltimes}(\mathbb{H}^{\nVDash}).$$

We now define

$$\eta_n(f)(z,r) := \frac{1}{n!} \sum_{0 \leq j \leq n} \frac{(n-j)!}{j!} \frac{\partial^{2j} f(z,\bar{z})}{\partial z^j \partial \bar{z}^j} r^{2j}$$

for $f \in P_n(\mathbb{C})$. We shall prove (5), (6), and the equivariance of η_n:

$$\eta_n(f|_n S) = \eta_n(f)|_n S \quad \forall S \in SL_2(\mathbb{C}). \tag{7}$$

As for (5) we refer to the next paragraph; (6) then is trivial. Due to the multiplicativity of (7) in $S \in SL_2(\mathbb{C})$ we can and shall restrict ourselves to $S \in \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$.

Let first $S = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} j_2(S;z) &= 1, \\ j_3(S;z,r) &= 1. \end{aligned}$$

We compute, denoting $f_b(z,\bar{z}) = f(z+b, \bar{z}+\bar{b})$ for $f \in P_n(\mathbb{C})$

$$\begin{aligned} \eta_n(f|_n S)(z,r) &= \eta_n(f_b)(z,r) \\ &= \frac{1}{n!} \sum_{0 \leq j \leq n} \frac{(n-j)!}{j!} \frac{\partial^{2j} f_b(z,\bar{z})}{\partial z^j \partial \bar{z}^j} r^{2j} \\ &= \eta_n(f)(z+b, r) \\ &= (\eta_n(f)|_n S)(z,r) \end{aligned}$$

Now, let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We compute

$$\begin{aligned} j_2(S;z) &= z, \\ j_3(S;z,r) &= z\bar{z} + r^2 = t. \end{aligned}$$

We establish the validity of (7) for $f_{k,l}(z,\bar z) = z^k\,\bar z^l, \quad 0 \le k,l \le n$. Assume with no loss of generality, $k \le l$. Some elementary calculation has to be done:

$$\begin{aligned}
\eta_n(f_{k,l}|_n S)(z,r) &= (-1)^{k+l}\eta_n(f_{n-k,n-l})(z,r)\\
&= (-1)^{k+l}\frac{1}{n!}\sum_{0\le j\le n}\frac{(n-j)!}{j!}j!\binom{n-k}{j}z^{n-k-j}j!\binom{n-l}{j}\bar z^{n-l-j}r^{2j}\\
&= (-1)^{k+l}\sum_{0\le j\le n-l}\frac{1}{\binom{n}{j}}\binom{n-k}{j}\binom{n-l}{j}z^{n-k-j}\,\bar z^{n-l-j}r^{2j}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(\eta_n(f_{k,l})|_n S)(z,r)\\
=\;& (z\bar z+r^2)^n\sum_{0\le\nu\le n}\frac{(-1)^{k+l}}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\left(\frac{\bar z}{z\bar z+r^2}\right)^{k-\nu}\left(\frac{z}{z\bar z+r^2}\right)^{l-\nu}\left(\frac{r}{z\bar z+r^2}\right)^{2\nu}\\
=\;& (-1)^{k+l}(z\bar z+r^2)^{n-k-l}\sum_{0\le\nu\le k}\frac{1}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}z^{l-\nu}\,\bar z^{k-\nu}r^{2\nu}\\
=\;& (-1)^{k+l}\sum_{\substack{0\le\nu\le k\\ 0\le\mu\le n-k-l}}\frac{1}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\binom{n-k-l}{\mu}z^{l-\nu}\,\bar z^{k-\nu}(z\bar z)^{n-k-l-\mu}r^{2(\mu+\nu)}\\
=\;& (-1)^{k+l}\sum_{\substack{0\le\nu\le k\\ 0\le\mu\le n-k-l}}\frac{1}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\binom{n-k-l}{\mu}z^{n-k-(\mu+\nu)}\bar z^{n-l-(\mu+\nu)}r^{2(\mu+\nu)}\\
=\;& (-1)^{k+l}\sum_{0\le j\le n-l}\left(\sum_{\substack{0\le\nu\le k\\ 0\le\mu\le n-k-l\\ \mu+\nu=j}}\frac{1}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\binom{n-k-l}{\mu}\right)z^{n-k-j}\,\bar z^{n-l-j}r^{2j}
\end{aligned}$$

The coefficient of $z^{n-k-j}\,\bar z^{n-l-j}$ therefore is

$$(-1)^{k+l}\sum_{\substack{0\le\nu\le k\\ 0\le j-\nu\le n-k-l}}\frac{1}{\binom{n}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\binom{n-k-l}{j-\nu}$$

which is, because of

$$\frac{1}{\binom{n}{\nu}} = \frac{1}{\binom{n}{j}\binom{j}{\nu}}\binom{n-\nu}{j-\nu},$$

equal to

$$\frac{(-1)^{k+l}}{\binom{n}{j}}\sum_{0\le\nu\le j}\frac{1}{\binom{j}{\nu}}\binom{k}{\nu}\binom{l}{\nu}\binom{n-\nu}{j-\nu}\binom{n-k-l}{j-\nu} = \frac{(-1)^{k+l}}{\binom{n}{j}}\binom{n-k}{j}\binom{n-l}{j}$$

because of the Pfaff–Saalschütz–identity (see next paragraph). So the equivariance of η_n is proven.

We include a list of $\eta_n(z^k\,\bar z^l), \quad 0 \le k,l \le 4$. Remember $t = z\bar z + r^2$.

$n = 0:$

$$\eta_0(1) = 1$$

$n = 1:$

$$\eta_1(1) = 1$$
$$\eta_1(z) = z \qquad \eta_1(\bar{z}) = \bar{z}$$
$$\eta_1(z\bar{z}) = t$$

$n = 2:$

$$\eta_2(1) = 1$$
$$\eta_2(z) = z \qquad \eta_2(\bar{z}) = \bar{z}$$
$$\eta_2(z^2) = z^2 \qquad \eta_2(\bar{z}^2) = \bar{z}^2$$
$$\eta_2(z\bar{z}) = \tfrac{1}{2}(z\bar{z} + t)$$
$$\eta_2(z^2\bar{z}) = z^2\bar{z} \qquad \eta_2(z\bar{z}^2) = z\bar{z}^2$$
$$\eta_2(z^2\bar{z}^2) = t^2$$

$n = 3:$

$$\eta_3(1) = 1$$
$$\eta_3(z) = z \qquad \eta_3(\bar{z}) = \bar{z}$$
$$\eta_3(z^2) = z^2 \qquad \eta_3(\bar{z}^2) = \bar{z}^2$$
$$\eta_3(z^3) = z^3 \qquad \eta_3(\bar{z}^3) = \bar{z}^3$$
$$\eta_3(z\bar{z}) = \tfrac{1}{3}(2z\bar{z} + t)$$
$$\eta_3(z^2\bar{z}) = \tfrac{1}{3}(z^2\bar{z} + 2zt) \qquad \eta_3(z\bar{z}^2) = \tfrac{1}{3}(z\bar{z}^2 + 2\bar{z}t)$$
$$\eta_3(z^3\bar{z}) = z^2 t \qquad \eta_3(z\bar{z}^3) = \bar{z}^2 t$$
$$\eta_3(z^2\bar{z}^2) = \tfrac{1}{3}(2z\bar{z}t + t^2)$$
$$\eta_3(z^3\bar{z}^2) = zt^2 \qquad \eta_3(z^2\bar{z}^3) = \bar{z}t^2$$
$$\eta_3(z^3\bar{z}^3) = t^3$$

$n = 4:$

$$
\begin{gathered}
\eta_4(1) = 1 \\
\eta_4(z) = z \qquad \eta_4(\bar z) = \bar z \\
\eta_4(z^2) = z^2 \qquad \eta_4(\bar z^2) = \bar z^2 \\
\eta_4(z^3) = z^3 \qquad \eta_4(\bar z^3) = \bar z^3 \\
\eta_4(z^4) = z^4 \qquad \eta_4(\bar z^4) = \bar z^4 \\
\eta_4(z\bar z) = \tfrac{1}{4}(3z\bar z + t) \\
\eta_4(z^2\bar z) = \tfrac{1}{2}(z^2\bar z + zt) \qquad \eta_4(z\bar z^2) = \tfrac{1}{2}(z\bar z^2 + \bar z t) \\
\eta_4(z^3\bar z) = \tfrac{1}{4}(z^3\bar z + 3z^2 t) \qquad \eta_4(z\bar z^3) = \tfrac{1}{4}(z\bar z^3 + 3\bar z^2 t) \\
\eta_4(z^4\bar z) = z^3 t \qquad \eta_4(z\bar z^4) = \bar z^3 t \\
\eta_4(z^2\bar z^2) = \tfrac{1}{6}(z^2\bar z^2 + 4z\bar z t + t^2) \\
\eta_4(z^3\bar z^2) = \tfrac{1}{2}(z^2\bar z t + z t^2) \qquad \eta_4(z^2\bar z^3) = \tfrac{1}{2}(z\bar z^2 t + \bar z t^2) \\
\eta_4(z^4\bar z^2) = z^2 t^2 \qquad \eta_4(z^2\bar z^4) = \bar z^2 t^2 \\
\eta_4(z^3\bar z^3) = \tfrac{1}{4}(3z\bar z t^2 + t^3) \\
\eta_4(z^4\bar z^3) = z t^3 \qquad \eta_4(z^3\bar z^4) = \bar z t^3 \\
\eta_4(z^4\bar z^4) = t^4
\end{gathered}
$$

Let

$$
\begin{aligned}
\Delta_E &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar z}, \\
\Delta_H &= r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}
\end{aligned}
$$

denote the Laplacian in euclidean dimension 2 and in hyperbolic dimension 3, respectively.

Theorem. We have, for $f \in P_n(\mathbb{C})$,

$$\Delta_H \eta_n(f) = r^2 \eta_{n-1}(\Delta_E f) \quad (n \geq 1)$$

.

The proof requires some elementary computation, only:

$$
\begin{aligned}
\Delta_H \eta_n(f) &= \frac{1}{n!}\sum_{0\le j\le n} \frac{(n-j)!}{j!}\left(r^2\left(4\Delta_E + \frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}\right)\frac{\partial^{2j} f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j} \\
&= \frac{1}{n!}\sum_{0\le j\le n} \frac{(n-j)!}{j!}\left(4\frac{\partial^{2(j+1)} f(z,\bar z)}{\partial z^{j+1}\partial \bar z^{j+1}} r^{2j+2} + 2j(2j-2)\frac{\partial^{2j} f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j}\right) \\
&= \frac{1}{n!} r^2 \sum_{0\le j\le n} \frac{(n-j)!}{j!}\frac{\partial^{2j}\Delta_E f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j} + \frac{1}{n!} r^2 \sum_{2\le j\le n} \frac{(n-j)!}{(j-2)!}\frac{\partial^{2j} 4 f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j} \\
&= \frac{1}{n!} r^2 \sum_{0\le j\le n-1} \frac{(n-j)!}{j!}\frac{\partial^{2j}\Delta_E f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j} + \frac{1}{n!} r^2 \sum_{0\le j\le n-1} \frac{(n-j-1)!}{j!} j \frac{\partial^{2j}\Delta_E f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j} \\
&= \frac{1}{(n-1)!} r^2 \sum_{0\le j\le n-1} \frac{(n-1-j)!}{j!}\frac{\partial^{2j}\Delta_E f(z,\bar z)}{\partial z^j \partial \bar z^j} r^{2j}
\end{aligned}
$$

$$= \quad \eta_{n-1}(\Delta_E f).$$

5 Lemmas

We show, in this paragraph that η_n, defined on $P_n(\mathbb{C})$, has values in $P_n(\mathbb{H}^{\nVdash})$. In addition, we give a straightforward proof of the Pfaff–Saalschütz identity in a form which is needed above. We frequently make use of the identities

$$\begin{aligned} \binom{x+y}{p} &= \sum_{0\le\nu\le p} \binom{x}{\nu}\binom{y}{p-\nu}, & (8) \\ \binom{x-y}{p} &= \sum_{0\le\nu\le p} (-1)^\nu \binom{x-\nu}{p-\nu}\binom{y}{\nu}, & (9) \end{aligned}$$

both as identities in $\mathbb{Q}[\frown,\frown]$. p is a nonnegative integer.

Lemma. For $0 \le k, l \le n$ we have

$$\eta_n(z^k \bar{z}^l) = \frac{1}{n!}\sum_j \frac{k!(n-k)!l!(n-l)!}{j!(k-j)!(l-j)!(n-k-l+j)!} z^{k-j}\bar{z}^{l-j}t^j$$

The summation extends over all j with

$$max\{0, k+l-n\} \le j \le min\{k,l\}.$$

This Lemma obviously implies (5).

Proof of the Lemma. We have, assuming $l \le k$,

$$\begin{aligned} \eta_n(z^k \bar{z}^l) &= \frac{1}{n!}\sum_{0\le j\le l} \frac{(n-j)!k!l!}{j!(k-j)!(l-j)!} z^{k-j}\bar{z}^{l-j}(t-z\bar{z})^j \\ &= \frac{1}{n!}\sum_{0\le j\le l} (-1)^j \frac{k!l!}{j!}\left(\sum_{j\le\nu\le l} (-1)^\nu \frac{(n-\nu)!}{(\nu-j)!(k-\nu)!(l-\nu)!}\right) z^{k-j}\bar{z}^{l-j}t^j. \end{aligned}$$

The interior summation yields, after changing ν to $\nu+j$:

$$\begin{aligned} & (-1)^j \sum_{0\le\nu\le l-j} (-1)^\nu \frac{(n-j-\nu)!}{\nu!(k-j-\nu)!(l-j-\nu)!} \\ =\ & (-1)^j \frac{(n-l)!}{(k-j)!}\sum_{0\le\nu\le l-j} (-1)^\nu \binom{n-j-\nu}{l-j-\nu}\binom{k-j}{\nu} \\ =\ & (-1)^j \frac{(n-l)!}{(k-j)!}\binom{n-k}{l-j} \end{aligned}$$

by using (3). So the Lemma is proven.

Lemma (Pfaff–Saalschütz–identity) ([3], [4]). The following identity in $\mathbb{Q}[x, y, z]$ is valid for each integral $j \geq 0$:

$$\sum_{0 \leq \nu \leq j} \frac{1}{\binom{j}{\nu}} \binom{x}{\nu} \binom{y}{\nu} \binom{z-\nu}{j-\nu} \binom{z-(x+y)}{j-\nu} = \binom{z-x}{j} \binom{z-y}{j}.$$

This identity is easily transformed into more classical ones, e. g.

$$\sum_{0 \leq \nu \leq j} \frac{(-j)_\nu (a)_\nu (b)_\nu}{\nu! (c)_\nu (d)_\nu} = \frac{(c-a)_j (c-b)_j}{(c)_j (c-a-b)_j} \tag{10}$$

where $c + d + j = a + b + 1$ and

$$(w)_\nu = (-1)^\nu \nu! \binom{-w}{\nu}$$

is the product of ν ascending factors $w, w+1, \cdots, w+\nu-1$. We simply have to substitute

$$\begin{aligned} a &= -x \\ b &= -y \\ d &= -z \end{aligned}$$

in order to get the identity of the Lemma. (For (10), see [1]).

Proof of the Lemma. Let

$$F_j(x, y, z) = \binom{z-x}{j} \binom{z-y}{j}.$$

We write

$$\binom{z-y}{j} = \frac{1}{\binom{j}{\mu}} \binom{z-y}{\mu} \binom{z-y-\mu}{j-\mu}$$

for $0 \leq \mu \leq j$, and we make use of (9) in

$$\binom{z-x}{j} = \sum_{0 \leq \mu \leq j} (-1)^\mu \binom{z-\mu}{j-\mu} \binom{x}{\mu}.$$

We get

$$F_j(x, y, z) = \sum_{0 \leq \mu \leq j} \frac{(-1)^\mu}{\binom{j}{\mu}} \binom{z-\mu}{j-\mu} \binom{x}{\mu} \binom{z-y}{\mu} \binom{z-y-\mu}{j-\mu}.$$

We apply (8) to

$$\binom{z-y-\mu}{j-\mu} = \binom{(z-(x+y)) + (x-\mu)}{j-\mu}$$

getting

$$\begin{aligned} F_j(x, y, z) &= \sum_{\substack{0 \leq \mu \leq j \\ 0 \leq \nu \leq j-\mu}} \frac{(-1)^\mu}{\binom{j}{\mu}} \binom{z-\mu}{j-\mu} \binom{x}{\mu} \binom{z-y}{\mu} \binom{z-(x+y)}{j-\mu-\nu} \binom{x-\mu}{\nu} \\ &= \sum_{0 \leq \mu \leq \nu \leq j} \frac{(-1)^\mu}{\binom{j}{\mu}} \binom{z-\mu}{j-\mu} \binom{x}{\mu} \binom{z-y}{\mu} \binom{z-(x+y)}{j-\nu} \binom{x-\mu}{\nu-\mu} \end{aligned}$$

after having shifted ν to $\nu-\mu$. We combine the two factors $\binom{x}{\mu}$ and $\binom{x-\mu}{\nu-\mu}$ to

$$\binom{x}{\nu}\binom{x-\mu}{\nu-\mu}=\binom{\nu}{\mu}\binom{x}{\nu},$$

write

$$\binom{z-\mu}{j-\mu}=\frac{1}{\binom{j-\mu}{j-\nu}}\binom{z-\mu}{\nu-\mu}\binom{z-\nu}{j-\nu}$$

and get

$$\begin{aligned}
F_j(x,y,z) &= \sum_{0\le\mu\le\nu\le j}\frac{(-1)^\mu}{\binom{j}{\nu}}\binom{z-\mu}{\nu-\mu}\binom{z-\mu}{j-\nu}\binom{z-y}{\mu}\binom{z-(x+y)}{j-\nu}\binom{x}{\nu}\\
&= \sum_{0\le\nu\le j}\frac{1}{\binom{j}{\nu}}\binom{x}{\nu}\binom{z-\nu}{j-\nu}\binom{z-(x+y)}{j-\nu}\sum_{0\le\mu\le\nu}(-1)^\mu\binom{z-\mu}{\nu-\mu}\binom{z-y}{\mu}\\
&= \sum_{0\le\nu\le j}\frac{1}{\binom{j}{\nu}}\binom{x}{\nu}\binom{z-\nu}{j-\nu}\binom{z-(x+y)}{j-\nu}\binom{y}{\nu}
\end{aligned}$$

again via (9). So the Lemma is proven.

6 Applications

In this final paragraph we study the set of zeros of some special polynomials on $\mathbb{H}^{\nVdash}$. Put, for $\varphi\in P_n(\mathbb{H}^{\nVdash})$

$$Z(\varphi)=\{P\in\mathbb{H}^{\nVdash}\,|\varphi(\mathbb{P})=\nvdash\}.$$

We denote by $\bar\varphi$ the polynomial with values $\bar\varphi(P)=\overline{\varphi(P)}$, so, if

$$\varphi=\sum a_{klm}z^k\bar z^l t^m$$

then

$$\bar\varphi=\sum \bar a_{lkm}z^k\bar z^l t^m.$$

If φ and $\bar\varphi$ are linearly dependent over $\mathbb{C}$ then, in order to study $Z(\varphi)$ one may assume with no loss of generality $\varphi=\bar\varphi$.

Let first $\varphi=a+bz+c\bar z+dt\in P_1(\mathbb{H}^{\nVdash})$ not in $P_0(\mathbb{H}^{\nVdash})$, and let, in addition, $\varphi=\bar\varphi$, so $a,d\in\mathbb{R}$, $=\not\succsim$ It is then elementary to check that $Z(\varphi)$, qualitatively, depends on $ad-b\bar b$ in the following way: If $ad-b\bar b>0$ then $Z(\varphi)$ is empty; if $ad-b\bar b=0$ then, again, $Z(\varphi)=\emptyset$, but $h_1(\varphi)$ has exactly one zero (on $\mathbb{C}$), namely $-\frac{\bar b}{d}$. (The coefficient d cannot be 0, otherwise would be $b=0$ and φ constant). If $ad-b\bar b<0$ then $Z(\varphi)$ is a hyperbolic plane which will be recognized from the set of zeros of $h_1(\varphi)$ on $P^1\mathbb{C}$: This is the circle with center $-\frac{\bar b}{d}$ and radius equal to $\sqrt{\frac{b\bar b-ad}{d^2}}$ if $d\neq 0$, and the straight line

$$Re(bz)=-\frac{1}{2}a$$

in case $d = 0$.

If $\varphi \in P_1(\mathbb{H}^3)$ and its complex conjugate are linearly independent over $\mathbb{C}$, then

$$Z(\varphi) = Z(\varphi) \cap Z(\bar{\varphi}) = Z(\varphi + \bar{\varphi}) \cap Z(i(\varphi - \bar{\varphi})),$$

so, if not empty, will be the intersection of two hyperbolic planes, so a hyperbolic line.

Next, if $\varphi = \bar{\varphi} \in P_n(\mathbb{H}^3)$ then $Z(\varphi)$ to contain a given hyperbolic line is equivalent with a system of $n+1$ linear conditions on φ: Let

$$\varphi(z.\bar{z}, t) = \sum_{k+l+m \leq n} a_{klm} z^k \bar{z}^l t^m \in P_n(\mathbb{H}^3)$$

and the line in question, without loss of generality, be given as the axis $Z(\varphi_0)$, where $\varphi_0(z, \bar{z}, t) = z$. We have

$$a_{lkm} = \bar{a}_{klm},$$

especially $a_{kkm} \in \mathbb{R}$. Trivially, $Z(\varphi) \supset Z(\varphi_0)$ means

$$\varphi(0, 0, t) = 0$$

identically, so

$$a_{00m} = 0$$

for $0 \leq m \leq n$. This is a system of $n+1$ conditions on the coefficients of φ. This fact can also be expressed the following way: In order $Z(\varphi)$ to contain a hyperbolic line it is nescessary and sufficient for $Z(\varphi)$ to contain $n+1$ pairwise distinct points of this line. This leads to the following observation:

Proposition. Let $L_j, j \bmod (n+2)$, be $n+2$ hyperbolic lines in $\mathbb{H}^3$ forming an $(n+2)$–gon; in other words, $L_j \cap L_{j+1}$ consisting of exactly one point. Then there is a $\varphi = \bar{\varphi} \in im h_n = h_n(P_n(\mathbb{C})) \subset \mathbb{P}_n(\mathbb{H}^3)$ such that $Z(\varphi)$ contains all lines L_j, $j = 1, \cdots, n+2$. So $Z(\varphi)$, in some sense, is, partly, a membrane spanned by the polygon.

The proof is based on an easy counting of dimensions: Requiring $Z(\varphi)$ to contain each line L_j means to require only n points of each line to lie on $Z(\varphi)$. This, altogether, is a system of $n(n+2)$ linear homogeneous conditions on $\varphi = \bar{\varphi}$ which, as an element of $h_n(P_n(\mathbb{C}))$ ranges over a real vector space of dimension $(n+1)^2 = n(n+2)+1$.

For the purpose of the next proposition we make use of the fact that hyperbolic lines in $\mathbb{H}^3$ and involutions in $PSL_2(\mathbb{C})$ are in one–one correspondance via assigning to each $L \in SL_2(\mathbb{C})$ such that $L^2 = -\mathbb{E}$ its set of fixed points in $\mathbb{H}^3$. Let, in this sense,

$$L_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in SL_2(\mathbb{C}), \quad j = 1, 2, 3,$$

be a tripel of arbitary lines in $\mathbb{H}^3$. Put

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

as elements of $\mathbb{C}^{\nVdash}$. $\bar{a}, \bar{b}, \bar{c}$ will denote a, b, c barred componentwise, respectively. We define seven polynomials $q_j, 0 \le j \le 6$, in dependence of the entries of L_1, L_2, L_3:

$$\begin{aligned}
q_0 &= -i \cdot det(a+\bar{a}, b, \bar{b}) \\
q_1 &= 2i \cdot det(a, \bar{a}, \bar{b}) + i \cdot det(b, \bar{b}, c) \\
q_2 &= i \cdot det(a+\bar{a}, \bar{b}, \bar{c}) - i \cdot det(a+\bar{a}, b, c) \\
q_3 &= i \cdot det(a-\bar{a}, \bar{b}, c) \\
q_4 &= i \cdot det(a-\bar{a}, b, c) + i \cdot det(a-\bar{a}, \bar{b}, \bar{c}) \\
q_5 &= 2i \cdot det(a, \bar{a}, c) + i \cdot det(\bar{b}, c, \bar{c}) \\
q_6 &= i \cdot det(a+\bar{a}, c, \bar{c}).
\end{aligned}$$

q_0, q_2, q_4, and q_6 are $\mathbb{R}$–valued. Let $\varphi = \bar{\varphi} \in P_2(\mathbb{H}^{\nVdash})$ be defined by

$$\varphi_{L_1,L_2,L_3}(z, \bar{z}, t) = q_0 + q_1 z + \bar{q}_1 \bar{z} + q_2 t + q_3 z^2 + \bar{q}_3 \bar{z}^2 + q_4 z\bar{z} + q_5 zt + \bar{q}_5 \bar{z} t + q_6 t^2.$$

Proposition. With the above notation we have: $Z(\varphi)$ contains L_1, L_2, and L_3.

Proof. It requires a straightforward but lengthy computation to show the invariance of the statement of the proposition with respect to hyperbolic transformations: If $S \in SL_2(\mathbb{C})$, then

$$\varphi_{SL_1S^{-1}, SL_2S^{-1}, SL_3S^{-1}}|_2 S = \varphi_{L_1,L_2,L_3}.$$

Here, again, one may restrict oneself to matrices $S = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We omit the details. So we may assume $L_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ which geometrically, is the r–axis in $\mathbb{H}^{\nVdash}$. We compute

$$\begin{aligned}
q_0 &= -i \cdot det \begin{pmatrix} a_1+\bar{a}_1 & b_1 & \bar{b}_1 \\ a_2+\bar{a}_2 & b_2 & \bar{b}_2 \\ 0 & 0 & 0 \end{pmatrix} = 0, \\
q_2 &= i \cdot det \begin{pmatrix} a_1+\bar{a}_1 & \bar{b}_1 & \bar{b}_2 \\ a_2+\bar{a}_2 & \bar{c}_1 & \bar{c}_2 \\ 0 & 0 & 0 \end{pmatrix} - i \cdot det \begin{pmatrix} a_1+\bar{a}_1 & b_1 & c_1 \\ a_2+\bar{a}_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{pmatrix} = 0, \\
q_6 &= i \cdot det \begin{pmatrix} a_1+\bar{a}_1 & c_1 & \bar{c}_1 \\ a_2+\bar{a}_2 & c_2 & \bar{c}_2 \\ 0 & 0 & 0 \end{pmatrix} = 0
\end{aligned}$$

which shows that $L_3 \subset Z(\varphi)$. So the proposition is proven.

Proposition. Using the notation of the previous proposition, let L_1, L_2, L_3 be three coplanar hyperbolic lines which are not contained in one bundle; in other words which neither do possess a common point in $\mathbb{H}^{\nVdash}$ or on its boundary nor a common perpendicular. Let $\varphi = \bar{\varphi} \in P_1(\mathbb{H}^{\nVdash})$ be such that $Z(\varphi)$ equals the hyperbolic plane spanned by L_1, L_2 and L_3. Then $\varphi^2 = q\varphi_{L_1,L_2,L_3}$ with some nonzero $q \in \mathbb{R}$.

Proof. We may assume L_1, L_2, L_3 being contained in the plane $z = \bar{z}$. This means

$$\bar{L}_j = -L_j$$

for $j = 1, 2, 3$. We compute, in this situation,

$$\begin{aligned}
q_0 &= 0 \\
q_1 &= 0 \\
q_2 &= 0 \\
q_3 &= i \cdot det \begin{pmatrix} 2a_1 & -b_1 & c_1 \\ 2a_2 & -b_2 & c_2 \\ 2a_3 & -b_3 & c_3 \end{pmatrix} = -2i \cdot det(a, b, c) \\
q_3 &= i \cdot det \begin{pmatrix} 2a_1 & b_1 & c_1 \\ 2a_2 & b_2 & c_2 \\ 2a_3 & b_3 & c_3 \end{pmatrix} + i \cdot det \begin{pmatrix} 2a_1 & -b_1 & -c_1 \\ 2a_2 & -b_2 & -c_2 \\ 2a_3 & -b_3 & -c_3 \end{pmatrix} = 4i \cdot det(a, b, c) \\
q_5 &= 0 \\
q_6 &= 0.
\end{aligned}$$

So

$$\varphi_{L_1, L_2, L_3}(z, \bar{z}, t) = -2idet(a, b, c)(z - \bar{z})^2$$

The factor $-2idet(a, b, c)$ will be a real number not vanishing under the geometric assumptions of the proposition.

We finish with a few remarks on a class of hyperbolic surfaces which may be useful when fundamental domains of lattices in $SL_2(\mathbb{C})$ have to be constructed, or,more generally, in the context of creation of three dimensional hyperbolic manifolds.

Proposition. Let $L_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, L_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix}$ elements of $SL_2(\mathbb{C})$, regarded as hyperbolic lines in $\mathbb{H}^{\nvDash}$. Let

$$\begin{aligned}
g_1(z) &= c_1 z^2 - 2a_1 z - b_1, \\
g_2(z) &= c_2 z^2 - 2a_2 z - b_2
\end{aligned}$$

denote the polynomials whose roots are the “feet” of L_1 and L_2 on $\partial\mathbb{H}^{\nvDash}$, respectively, or, equivalently, the fixed points of L_1 and L_2 as Möbius transformations. Call

$$f(z, \bar{z}) = g_1(z)\overline{g_1(z)} - g_2(z)\overline{g_2(z)}$$

as an element of $P_2(\mathbb{C})$, and

$$\varphi = \eta_2(f) \in P_2(\mathbb{H}^{\nvDash}).$$

Then

$$Z(\varphi) = \{P \in \mathbb{H}^{\nvDash} | \mathbb{P} \eqsim\eth\sim \shortparallel\approxeq\eth\lessdot \sqsupset\sim\approx\eth\ltimes \mho\ddots\rtimes\gtrdot \mathbb{L}_{\nVdash} \eth\ltimes \mathbb{L}_{\nvDash}\}.$$

Proof. The distance of two points

$$\begin{pmatrix} \xi_1 & \zeta_1 \\ \bar{\zeta}_1 & \eta_1 \end{pmatrix}, \begin{pmatrix} \xi_2 & \zeta_2 \\ \bar{\zeta}_2 & \eta_2 \end{pmatrix} \in \mathcal{H}_2^+$$

is measured by

$$-\frac{\zeta_1\bar{\zeta}_2+\bar{\zeta}_1\zeta_2-\xi_1\eta_2-\eta_1\xi_2}{\sqrt{\xi_1\eta_1-\zeta_1\bar{\zeta}_1}\sqrt{\xi_2\eta_2-\zeta_2\bar{\zeta}_2}}$$

which, in terms of Poincaré model coordinates equals

$$\frac{|z_1-z_2|^2+r_1^2+r_2^2}{r_1r_2}. \tag{11}$$

Let $L=\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL_2(\mathbb{C})$. Then the distance of a point $(z,r)\in \mathbb{H}^{\nvDash}$ from the line L is half the distance between $(z.r)$ and $L(z,r)$ which, by formula (11), is measured by

$$\begin{aligned}
d_L(z,r) &= \frac{\left|\frac{(az+b)\overline{(cz-a)}+a\bar{c}r^2}{j(L;z,r)}-z\right|^2+\frac{r^2}{j^2(L;z,r)}+r^2}{\frac{r^2}{j(L;z,r)}} \\
&= \frac{1}{r^2}(|cz^2-2az-b|^2+2|cz-a|^2r^2+|c|^2r^4) \\
&= \frac{1}{r^2}\eta_2(g)(z,r)
\end{aligned}$$

where $g(z,\bar{z})=|cz^2-2az-b|^2$ as an element of $P_2(\mathbb{C})$. This proves the proposition.

We, qualitatively, describe the surface consisting of those points with equal distance from the two lines L_1 and L_2 in $\mathbb{H}^{\nvDash}$ by looking at its "trace" on the boundary S^2 of $\mathbb{H}^{\nvDash}$. Making use of the equivariance of η_2 we may assume L_1 and L_2 to be in special positions: Let, with no loss of generality,

$$\begin{aligned}
L_1 &= \frac{i}{1-\omega}\begin{pmatrix} 1+\omega & -2\omega \\ 2 & -(1+\omega) \end{pmatrix}, \\
L_2 &= \frac{i}{1-\omega}\begin{pmatrix} -(1+\omega) & -2\omega \\ 2 & 1-\omega \end{pmatrix},
\end{aligned}$$

where $\omega\in\mathbb{C}, \omega\neq \nvDash$. So L_1 is the hyperbolic line with endpoints 1 and ω, L_2 the line with endpoints -1 and $-\omega$ on $\mathbb{C}$. Coplanarity of L_1 and L_2 is equivalent to

$$trL_1L_2=2\left(1+\frac{8\omega}{(\omega-1)^2}\right)$$

being real, which is the case if and only if $\omega\in\mathbb{R}$ or $|\omega|=1$. L_1 and L_2 will intersect if and only if $\omega<0$. We compute

$$\begin{aligned}
\frac{1-\omega}{2i}g_1(z) &= z^2-(\omega+1)z+\omega \\
\frac{1-\omega}{2i}g_2(z) &= z^2+(\omega+1)z+\omega.
\end{aligned}$$

So the surface in question hits the boundary of $\mathbb{H}^{\nvDash}$ along the curve

$$(\bar{\omega}+1)z^2\bar{z}+(\omega+1)z\bar{z}^2+\bar{\omega}(\omega+1)z+\omega(\bar{\omega}+1)\bar{z}=0.$$

This is, looking at $\mathbb{C}$ as a real euclidian plane, the real part of an elliptic curve, at least for ω generic, with an asymptotic line

$$(\bar{\omega}+1)z+(\omega+1)\bar{z}=0. \tag{12}$$

This curve will be a line on $\mathbb{C}$ if and only if L_1 and L_2 are coplanar, and then this line will equal the one described by equation (12). So in this case the corresponding surface will be a plane.

We define a subset $M \subset \mathbb{H}^3$ to be tubelike if there is a hyperbolic line L in $\mathbb{H}^3$ with the following property: For each $P_0 \in M$ the set of all points $P \in \mathbb{H}^3$ on the perpendicular $L^\perp(P_0)$ between P_0 and its foot $L \cap L^\perp(P_0)$ on L, including the foot itself, belongs to M. For each line L which serves this purpose we say: M is tubelike with respect to L.

Proposition. Let L_1, L_2 be two nonintersecting hyperbolic lines in $\mathbb{H}^3$, and let F_1 be the set of all points in $\mathbb{H}^3$ whose distance from L_1, is at most their distance from L_2. Then F_1 is tubelike with respect to L_1. Consequently, the surface F^0 consisting of all points with equal distance from L_1 and L_2 divides $\mathbb{H}^3$ into two connected components.

Proof. We normalize the geometric situation to

$$L_1=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, L_2=\frac{i}{\omega-1}\begin{pmatrix} \omega+1 & -2\omega \\ 2 & -(\omega+1) \end{pmatrix}:$$

The fixed points of L_1 on $\partial\mathbb{H}^3$ are specialized to 0 and infinity, the corresponding ones of L_2 to 1 and ω where $\omega \in \mathbb{C}$ is different from 1 and nonnegative. We compute, following the notation of the previous proposition:

$$\begin{aligned} g_1(z) &= -2iz \\ g_2(z) &= \frac{2i}{\omega-1}(z^2-(\omega+1)z+\omega). \end{aligned}$$

So

$$-\frac{1}{4}|\omega-1|^2 f(z,\bar{z}) = |z^2-(\omega+1)z+\omega|^2-|\omega-1|^2|z|^2,$$

and

$$\begin{aligned} &\eta_2\left(-\frac{|\omega-1|^2}{4}f\right)(z,r) \\ = \;& |z^2-(\omega+1)z+\omega|^2-|\omega-1|^2|z|^2+\frac{1}{2}\left(|2z-(\omega+1)|^2-|\omega-1|^2\right)r^2+r^4. \end{aligned}$$

The points $(z,r)\in\mathbb{H}^3$ which are closer to L_1 than to L_2 are characterized by

$$\eta_2\left(-\frac{|\omega-1|^2}{4}f\right)(z,r)>0.$$

We study the behaviour of this function along an arbitary line perpendicular to L_1:

$$(z,r)=(t\cdot p, |p|\sqrt{1-t^2})$$

with some fixed $p \in \mathbb{C}, \imath \neq \nvdash$, and $-1 < t < 1$. It is easily computed that

$$\begin{aligned}
\varphi(t) &:= \eta_2\left(-\frac{|\omega-1|^2}{4}f\right)\left(pt, |p|\sqrt{1-t^2}\right) \\
&= (p+\bar{p})(\bar{\omega}p+\omega\bar{p})t^2 - \left((\omega\bar{p}+p)(\bar{\omega}+|p|^2)+(\bar{\omega}p+\bar{p})(\omega+|p|^2)\right)t + |\omega+|p|^2|^2
\end{aligned}$$

is a polynomial in t of degree at most 2. $t=0$ corresponds to a point on L_1. We have $\varphi(0) = |\omega+|p|^2|^2 > 0$ as L_1 and L_2 do not intersect. We prove:

$$\{t \in \mathbb{R} \,|\, -\nVdash < \approx < \nVdash, \varphi(\approx) > \nvdash\}$$

is connected which will then imply the statement of the proposition. But this fact is obvious except in the case of the leading coefficient of φ being positive. So let

$$sgn(p+\bar{p}) = sgn(\bar{\omega}p+\omega\bar{p}) \neq 0.$$

$\varphi(t)$ attains its minimum at

$$\begin{aligned}
t_0 &= \frac{(\omega\bar{p}+p)(\bar{\omega}+|p|^2)+(\bar{\omega}p+p)(\omega+|p|^2)}{2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})} \\
&= \frac{(\bar{\omega}p+\omega\bar{p})(1+|p|^2)+(p+\bar{p})(|\omega|^2+|p|^2)}{2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})}.
\end{aligned}$$

This number has the same *sign* as $p+\bar{p}$ and $\bar{\omega}p+\omega\bar{p}$. We compute

$$\begin{aligned}
t_0 - 1 &= \frac{(\bar{\omega}p+\omega\bar{p})(1+|p|^2)+(p+\bar{p})(|\omega|+|p|^2)-2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})}{2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})} \\
&= \frac{(\bar{\omega}p+\omega\bar{p})|p-1|^2+(p+\bar{p})|\omega-p|^2}{2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})}
\end{aligned}$$

and similarly

$$t_0 + 1 = \frac{(\bar{\omega}p+\omega\bar{p})|p+1|^2+(p+\bar{p})|\omega+p|^2}{2(p+\bar{p})(\bar{\omega}p+\omega\bar{p})}$$

So, if $t_0 > 0$ then even $t_0 > 1$, and, similarly, if $t_0 < 0$, then $t_0 < -1$. This shows that φ cannot have two different or one double zero in the intervall $-1 < t < 1$. So the proposition is proven.

We finish by the remark that the last proposition, at least in theory, allows to modify Dirichlet's construction of fundamental domains of lattices in $PSL_2(\mathbb{C})$. The intersection of a family of subsets of $\mathbb{H}^{\nVdash}$ all of them tubelike with respect to the same hyperbolic line will be tubelike again. So, if $\Gamma \subset PSL_2(\mathbb{C})$ is a lattice and L a hyperbolic line with the property $S(L) \cap L = \emptyset$ or $S(L) = L$ *for all* $S \in \Gamma$ then

$$\bigcap_{S \in \Gamma \setminus Stab(L)} F_S$$

factored by the stabilizer of L in Γ will be a fundamental domain of Γ. (Here F_S means the set of all points which are closer to L than to $S(L)$). As a line L in this context the axis of some element of Γ may serve which seems, in some respect, to be a more natural choice than the choice of some arbitary base point in Dirichlet's construction. On the other hand the advantages of this approach in practice are not clear, yet.

7 References

[1] L. Carlitz: The Saalschützian Theorems, The Fibonacci Quarterly, vol. 14, nr. 1, 1976

[2] W. Fenchel: Elementary Geometry in Hyperbolic Space, W. de Gruyter 1989

[3] D. Foata: Une démonstration combinatoire de l'identité de Pfaff–Saalschütz, C.R. Acad. Sc. Paris, t. 297, Serie I

[4] L. Saalschütz: Eine Summationsformel, Z. für Math. u. Phys. 35, 1890