Domination Games on Infinite Graphs

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We consider two infinite games, played on a countable graph G given with an integer vertex labelling. One player seeks to construct a ray (a one-way infinite path) in G, so that the ray's labels dominate or elude domination by an integer sequence being constructed by another player. For each game, we give a structural characterization of the graphs on which one player or the other can win, providing explicit winning strategies. This answers a question of Deuber. [1]

1. Introduction

Let G be a countable graph, and let $\ell\colon V(G) \to \omega$ be an injective labelling of its vertices with natural numbers (thus distinct vertices have distinct labels). Consider the following game for two players, Adam and Eve. The two players move alternately, ω times, Eve having the first move. When Eve moves, she plays a natural number; when Adam moves, he plays a vertex of G. In the course of one game, Eve thus creates a sequence $e\colon \omega \to \omega$, $n \mapsto e_n$, of numbers, while Adam creates a sequence $a\colon \omega \to V(G)$, $n \mapsto a_n$, of vertices. More specifically, Adam tries to choose his vertices so as to define a *ray* (a one-way infinite path) in G, i.e. so that a_{n+1} is adjacent to a_n for every $n \in \omega$ and no vertices are repeated. *ray*

Let us say that a sequence $(a_n)_{n\in\omega}$ of natural numbers *dominates* another such sequence $(b_n)_{n\in\omega}$ if $a_n \geqslant b_n$ for all n greater than some $n_0 \in \omega$. In the *dominating game*, Adam wins if his function a defines a ray whose sequence $(\ell(a_n))$ of labels dominates Eve's sequence (e_n). Otherwise, Eve wins. In the *bounding game*, Adam merely tries to escape domination by Eve: he wins iff he constructs a ray whose labels exceed the corresponding terms of Eve's sequence again and again, i.e. such that (e_n) does not dominate $(\ell(a_n))$. Note that if Eve has won a particular instance of the bounding game, this same game would also go to her credit if viewed as an instance of the dominating game. Similarly, if Adam can win the dominating game on a particular graph, he can trivially also win the bounding game on that graph. *dominating game* *bounding game*

Let us look at a few examples. It is easy to see that Adam has a winning strategy for the dominating game on T_ω, the tree in which every vertex has countably infinite degree. Indeed, he can always find a new vertex a_{n+1} adjacent to his previous vertex a_n and such that $\ell(a_{n+1}) > e_{n+1}$. This, of course, also gives him a winning strategy for any graph that contains a copy of T_ω as a subgraph. T_ω

In Section 2, we shall prove that this simple example already exhausts Adam's resources for the dominating game. More precisely, we shall show that Adam has a winning strategy for the dominating game on G if and only if $T_\omega \subset G$, and that otherwise Eve has a winning strategy. The proof of this result will be fairly straightforward.

In contrast, the situation for the bounding game is rather more interesting. Again, Adam clearly has a winning strategy on T_ω. But more is true: since Adam only needs to beat Eve's sequence again and again, he will also win on any subdivision of T_ω, i.e. on any graph obtained from T_ω by replacing its edges with non-trivial paths whose interiors are pairwise disjoint. Any such graph will be called a $\mathrm{T}T_\omega$. $\mathrm{T}T_\omega$

FIGURE 1. The prototype bundle graph B

For similar reasons, Adam has a winning strategy on the graphs B and F shown in Figs. 1 and 2. Indeed, he simply constructs a ray from left to right, starting at the leftmost vertex (of B or F, respectively). Provided he never moves back towards the left on B or vertically up on F, he will again and again find himself at a vertex with an infinite choice of neighbours to the right: as all these neighbours carry different labels, he can choose one whose label exceeds the corresponding number just played by Eve. As before, Adam's winning strategies for B and F extend to subdivisions of these graphs; subdivisions of B will be called *bundle graphs*, subdivisions of F are *fan graphs*. *bundle graphs* *fan graphs*

FIGURE 2. The prototype fan graph F

In Section 3, we shall show that these three types of subgraph provide the unique discriminator between the graphs where Adam has a winning strategy and those where Eve can win. In other words, we shall prove that

Adam has a winning strategy for the bounding game on G if and only if G has a subgraph isomorphic to a $\mathrm{T}T_\omega$, a bundle graph or a fan graph, and that otherwise Eve has a winning strategy.

Note that, for the domination game as for the bounding game, our characterizations depend only on the structure of G, and not on the particular labelling chosen.

For a fixed countable graph G with labelling ℓ, each game is Borel (meaning that the set of winning runs is Borel in the obvious product topology), and so it follows from Martin's theorem of Borel determinacy [5] that one player or the other must have a winning strategy. (In fact, each of our games is F_σ, so this follows already from Wolfe's theorem on F_σ determinacy [4].) However, we shall not need to rely on this result: we shall always be able to give explicit winning strategies.

Let us remark in passing that a typical winning strategy is unlikely to depend on the entire information encoded in the positions to which it assigns a next vertex or number.

In order to make the game set-up described above precise, we now briefly go through the (standard) definitions of the terms involved. The reader familiar with infinite games may wish to skip through to the start of the next section.

Let G be a fixed countable graph whose vertices are injectively labelled with natural numbers. For either game, a *position* of the game will be a pair of a sequence $v_0, \ldots, v_n$ of vertices of G ('those which Adam has played so far') and a sequence $k_0, \ldots, k_m$ of natural numbers ('those which Eve has played so far') such that either $n = m$ or $n = m - 1$ (depending on 'who is to move next'). A *strategy for Adam* is a function α which assigns to every position with $n = m - 1$ a vertex of G. A *strategy for Eve* is a function η which assigns to every position with $n = m$ a natural number. A *run* of the game is a pair of ω-sequences (a_n) and (e_n). Adam has *played* this run *according to* the strategy α if $a_n = \alpha(a_0, \ldots, a_{n-1} \mid e_0, \ldots, e_n)$ for every $n \in \omega$. Similarly, Eve has *played* this run *according to* the strategy η if $e_n = \eta(a_0, \ldots, a_{n-1} \mid e_0, \ldots, e_{n-1})$ for every $n \in \omega$.

position

strategy

run

A *winning strategy* (for either Adam or Eve) is a strategy such that every run on G played according to it is won.

winning strategy

2. The dominating game

We start by showing that, for the dominating game, Adam can force a win only if $G \supset T_\omega$.

Proposition 1. *Adam has a winning strategy for the dominating game on G if and only if $T_\omega \subset G$.*

Proof. As we saw in Section 1, Adam has an obvious winning strategy if $G \supset T_\omega$: he is easily able to beat Eve at every move. So it remains for us to show that if Adam has a winning strategy then $G \supset T_\omega$.

Let Adam's winning strategy be α. We claim that there exists a sequence $e_1, \ldots, e_k$ of natural numbers such that for every extension $e_1, \ldots, e_n$ $(n \geqslant k)$ we have $a_n \geqslant e_n$ when Eve plays $e_1, \ldots, e_n$ and Adam follows α. Indeed, if this were not the case then we could inductively construct an infinite sequence $e_1, e_2, \ldots$ such that if Eve plays this sequence and Adam follows α then $a_n < e_n$ for infinitely many n. However, this is impossible, as α is a winning strategy for Adam.

Now consider the subgraph H of G spanned by Adam's replies (following α) to all the sequences starting $e_1, \ldots, e_k$. By the choice of $e_1, \ldots, e_k$, it is clear that, for every such sequence $e_1, \ldots, e_n$ $(n \geqslant k)$, Adam must have an infinite choice of distinct replies to the sequences $e_1, \ldots, e_n, x$ as x varies over ω. Thus H is a (non-empty) graph in which every vertex has infinite degree. However, in any such graph it is easy to construct a T_ω inductively. $\square$

As we remarked earlier, Proposition 1 implies by Borel (or just F_σ) determinacy that Eve has a winning strategy for the dominating game on any graph not containing a T_ω. In the proof of the following extension of Proposition 1, we make such a winning strategy explicit.

Theorem 2. *If $T_\omega \subset G$ then Adam has a winning strategy for the dominating game on G. Otherwise, Eve has a winning strategy.*

Proof. We assume that G contains no T_ω, and construct a winning strategy for Eve. We start by recursively defining a rank function ρ on some or all of the vertices of G. For each ordinal α, give rank $\alpha = \rho(v)$ to all vertices v such that all but finitely many neighbours w of v have rank $\rho(w) < \alpha$. If any vertex remains unranked, then it has infinitely many unranked neighbours, and so the unranked vertices span a (non-empty) graph in which every vertex has infinite degree. We may then construct a $T_\omega \subset G$ from these vertices inductively. Thus, since $G \not\supset T_\omega$ by assumption, ρ gets defined for every vertex of G.

We may now choose a winning strategy for Eve as follows. Let Eve's first move be arbitrary (say 0). Later, if Adam's last chosen vertex is v, let Eve play the number

$$1 + \max\{\, \ell(w) \mid w \text{ is a neighbour of } v \text{ and } \rho(w) \geqslant \rho(v) \,\};$$

note that, by definition of $\rho(v)$, this maximum is taken over just a finite set.

Now consider a run of the game which Eve has played according to this strategy. If Adam fails to construct a ray, then Eve wins by definition. So assume that Adam does indeed construct a ray $R \subset G$. Since there is no infinite descending sequence of ordinals, R has infinitely many vertices each of whose rank is at most that of its successor on R. But Eve beats Adam on all these successors, so Adam's sequence $(\ell(a_n))$ fails to dominate Eve's sequence (e_n). Thus Eve's strategy is indeed a winning strategy. □

3. The bounding game

We now turn our attention to the bounding game. It turns out that the key notion here is that of a propagating set of paths [2], which we now describe. Call a non-empty set $\mathcal{P}$ of finite (directed) paths *propagating* if every path in $\mathcal{P}$ has infinitely many extensions in $\mathcal{P}$ of some common length. (A path $v_0 \dots v_n$ is an *extension* of a path $u_0 \dots u_m$ if $m \leqslant n$ and $v_i = u_i$ for all $i \leqslant m$.) Equivalently, $\mathcal{P}$ is propagating if and only if every $P \in \mathcal{P}$ has an extension Q such that $\mathcal{P}$ contains infinitely many one-vertex extensions of Q. Note that the paths from left to right in a bundle graph and the paths from left to right and down in a fan graph form propagating sets: indeed, this was precisely the property we used in Section 1 to show that Adam can win the bounding game on F and on B.

The following proposition shows that the concept of a propagating set of paths captures precisely the structural properties of a graph that enable Adam to win the bounding game—again, independently of the choice of the graph's labelling.

Proposition 3. *Adam has a winning strategy for the bounding game on G if and only if G contains a propagating set of paths.*

Proof. We have seen that Adam has a winning strategy if G contains a propagating set of paths. So, for the converse, suppose Adam has a winning strategy α for the bounding game on G. We claim that the collection of plays $a_1, \dots, a_k$ in which Adam follows α defines a propagating set of paths.

Indeed, suppose to the contrary that some such path $a_1 \dots a_k$ has only finitely many extensions of each length n. Let Eve play a sequence forcing Adam to play $a_1, \dots, a_k$ (remember, Adam is following α). Then, at each $n > k$, when Adam has played $a_1, \dots, a_n$, let Eve play a number greater than the maximum of the finite set of labels of possible next moves for Adam when he follows α. Eve clearly wins this run of the bounding game, a contradiction. □

We now turn to winning strategies for Eve.

Theorem 4. *If G contains a propagating set of paths then Adam has a winning strategy for the bounding game on G. Otherwise, Eve has a winning strategy.*

Proof. We assume that G contains no propagating set of paths, and construct a winning strategy for Eve.

Recall that a non-empty set $\mathcal{P}$ of finite paths in G is propagating if

$$\text{for every } P \in \mathcal{P} \text{ there exists an } n \in \omega \text{ such that } \mathcal{P} \text{ contains infinitely many extensions of } P \text{ all of length } n. \tag{$*$}$$

Starting with the set $\mathcal{P}_0$ of all finite paths in G, let us force property $(*)$ on to this set by recursively deleting any paths P that violate $(*)$. More precisely, let us define for each ordinal $\alpha > 0$ a set $\mathcal{P}_\alpha$, as follows. If α is a successor ordinal, $\alpha = \beta + 1$ say, let

$$\mathcal{P}_\alpha = \mathcal{P}_\beta \setminus \{ P \in \mathcal{P}_\beta \mid P \text{ violates } (*) \text{ for } \mathcal{P} = \mathcal{P}_\beta \} ;$$

thus, $\mathcal{P}_\alpha$ is obtained from $\mathcal{P}_\beta$ by deleting from it any path P such that, for each $n \in \omega$, $\mathcal{P}_\beta$ contains only finitely many extensions of P of length n. If α is a limit ordinal, let $\mathcal{P}_\alpha = \bigcap_{\beta<\alpha} \mathcal{P}_\beta$.

Choose γ large enough that $\mathcal{P}_{\gamma+1} = \mathcal{P}_\gamma$ (remember, we have defined $\mathcal{P}_\alpha$ for arbitrarily large α), and set $\mathcal{P}^* = \mathcal{P}_\gamma$. Clearly, $\mathcal{P} = \mathcal{P}^*$ satisfies $(*)$. By assumption, however, G contains no propagating set of paths; therefore $\mathcal{P}^*$ must be empty.

We may now define a strategy for Eve as follows. Consider any position where Eve is next to move. If the vertices $v_0, \dots, v_n$ which Adam has played so far do not form a path (in this order), let Eve's move be arbitrary (say 0). If they do form a path, P say, let α be minimal such that $P \notin \mathcal{P}_\alpha$; as $\mathcal{P}^* = \emptyset$, this α certainly exists. Moreover, α is a successor ordinal (note that $\alpha > 0$, because $P \in \mathcal{P}_0$), say $\alpha = \beta + 1$. Then $P \in \mathcal{P}_\beta$, but P fails to satisfy $(*)$ for $\mathcal{P} = \mathcal{P}_\beta$. In particular, P has only finitely many extensions in $\mathcal{P}_\beta$ by just one vertex, and we may define as Eve's next move the number

$$1 + \max \{ \ell(w) \mid v_0 \dots v_n w \text{ is a path in } \mathcal{P}_\beta \} .$$

Let us now show that this is a winning strategy for Eve. Consider any run of the bounding game which Eve has played according to this strategy. Let $v_0, v_1, \dots$ be the vertices chosen by Adam. If they fail to form a ray in G, then Eve has won by definition. So suppose that $v_0 v_1 \dots$ is a ray. Let α be minimal such that $P = v_0 \dots v_k \notin \mathcal{P}_\alpha$ for some $k \in \omega$. As before, $\alpha = \beta + 1$ for some β, so P is in $\mathcal{P}_\beta$ but fails to satisfy $(*)$ for $\mathcal{P} = \mathcal{P}_\beta$. But then the same is true for every path $P' = v_0 \dots v_n$ with $n \geqslant k$: by the minimality of α, we have $P' \in \mathcal{P}_\beta$, and P' cannot have infinitely many extensions in $\mathcal{P}_\beta$ of any common length, because these would also be extensions of P. Hence, all the paths $v_0 \dots v_n$ with $n \geqslant k$ were deleted at the same time, when $\mathcal{P}_\alpha$ was formed from $\mathcal{P}_\beta$. Thus in each case, the number played by Eve as e_{n+1} is at least one greater than the label of $w = v_{n+1}$, and so Eve beats Adam in every move from the k'th position onwards. $\square$

Theorem 4 presents us with an obvious question: which graphs contain a propagating set of paths?

A graph G is called *bounded* if for every labelling $\ell\colon V(G) \to \omega$ of its vertices there exists an ω-sequence (d_n) of natural numbers which dominates all the sequences $(\ell(v_n))$ for rays $v_0v_1\ldots$ in G. Otherwise, the graph G is *unbounded*. Thus to prove from the definition of boundedness that a certain graph G is bounded is like playing the part of Eve in the bounding game, but with a handicap: a labelling of G is given to us, and we have to produce a sequence (d_n) which dominates every ray in G according to this labelling; we are not allowed however (as Eve is) to let the definition of the later values of d_n depend on the initial segments of the ray we are trying to dominate. Similarly, a proof that G is unbounded is like playing the role of Adam, but with an additional advantage: we may now choose even the beginning of our ray in the full knowledge of the sequence we are trying to elude.

Confirming a long-standing conjecture of Halin, it was proved in [2] that a countable graph is bounded if and only if it contains no $\mathrm{T}T_\omega$, no bundle graph, and no fan graph. Now, it is easy to see that any graph containing a propagating set of paths must be unbounded, and we have seen that any $\mathrm{T}T_\omega$, bundle graph or fan graph does contain a propagating set of paths. We thus have the following classification theorem for bounded graphs.

Theorem 5. [2] The *following statements are equivalent for countable graphs G:*

(i) *G is bounded;*

(ii) *G has no subgraph isomorphic a $\mathrm{T}T_\omega$, a bundle graph, or a fan graph;*

(iii) *G does not contain a propagating set of paths.*

Putting Theorems 4 and 5 together, we obtain our desired structural characterization for the bounding game.

Theorem 6. *If G has a subgraph isomorphic to a $\mathrm{T}T_\omega$, a bundle graph or a fan graph, then Adam has a winning strategy for the bounding game on G. Otherwise, Eve has a winning strategy.* □

Let us once more point out the fact that, as a corollary of the above characterization theorems, the outcome of the domination game or the bounding game does not depend on 'how fast' the labelling ℓ 'grows': it depends only on the structure of the graph G on which the game is played. As long as G is countable (which we have so far assumed; but cf. Section 4), one can also see this directly. Indeed, the bounding and the dominating game on G with labels are then equivalent to the following game on G itself: at each move Eve chooses a finite set E_n of vertices, while Adam tries to construct a ray $a_0a_1\ldots$ in such a way that $a_n \notin E_n$ eventually (for the dominating game) or infinitely often (for the bounding game).

4. Concluding remarks

The graphs G we considered in this paper were all countable. However, the reader may have noticed that we never make full use of the fact that the labellings ℓ are injective: all we actually use is that every vertex of infinite degree has an infinite set of neighbours with distinct labels. With this version of a labelling, one could extend the domination and bounding games to uncountable graphs. Alternatively, we might leave it to Adam to choose a labelling before Eve makes her first move. It is not difficult to show that these two generalizations are in fact equivalent.

With these adaptations, all our results extend to uncountable graphs. Theorems 2 and 4 remain true, because the countability of G is not used in their proofs. Theorem 6 remains true, because the equivalence of (ii) and (iii) in Theorem 5 still holds in the uncountable case: note that any graph with a propagating set of paths must contain a countable such graph.

We remark, however, that the equivalence of (i) and (ii) in Theorem 5 does not extend to uncountable graphs. A simple example of an unbounded graph not containing a $\mathrm{T}T_\omega$, a bundle graph or a fan graph is the disjoint union of 2^ω rays. In [2], it is proved (assuming CH) that a graph is bounded if and only if it does not contain a $\mathrm{T}T_\omega$, a bundle graph, a fan graph or the disjoint union of 2^ω rays.

Finally, let us remark that there is also a notion of a dominating graph: G is called *dominating* if there exists a labelling $\ell\colon V(G) \to \omega$ of its vertices such that every ω-sequence of integers is dominated by the labelling along some ray in G. Viewed from the perspective of our games, the dominating graphs differ from the bounded graphs in an interesting way: the structural distinction between the graphs that are dominating and those that are not does not run parallel, even in the countable case, to the distinction between those graphs on which Adam wins the dominating game and those where Eve wins. Thus, the dominating graphs are not merely those that contain a T_ω: in the countable case, a graph is dominating if and only if it contains a *uniform* subdivision of T_ω (one where at each branch vertex the incident edges are subdivided a bounded number of times), while in the uncountable case there is a similar (yet surprisingly different) characterization. The interested reader is referred to [3], where the dominating graphs are classified.

Acknowledgement. *We thank the referee for several thoughtful comments and suggestions.*

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