ON GAUSSIAN MEASURE OF BALLS IN $\mathbb{H}$

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ABSTRACT. Different estimates are derived for the density function of a random variable $|Y-a|^2$, where Y is a Gaussian random element in a Hilbert space H with zero mean, depending on possible various values of a. The estimates imply upper bounds for probabilities $P(|Y-a|>r)$ with the same behaviour with respect to a and r as the asymptotically sharp results of Linde (1991) obtained for the case $r\to\infty$. Lower bounds for $(P(|Y-a|>r)$ are also constructed.

1. Introduction and Notation.

Let H be a real separable Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $|x|=(x,x)^{1/2}$, $x\in H$. We denote by Y a H–valued Gaussian random element with zero mean and covariance operator V, i. e. the characteristic functional of Y has a form

$$E\exp\{i(x,Y)\}=\exp\{-(Vx,x)/2\},\quad x\in H,$$

where V is defined for all x,z from H by

$$(Vx,z)=E(Y,x)(Y,z).$$

Assume that the eigenvalues $\sigma_j^2, j=1,2,\dots$ of the covariance operator V satisfy

$$\sigma_1^2=\sigma_2^2=\cdots=\sigma_k^2>\sigma_{k+1}^2\geq\sigma_{k+2}^2\geq\cdots$$

for some $k\geq 1$ which is the multiplicity of σ_1^2 here and everywhere below. It is well known that there exists an orthonormal system $\{e_i\}_{i=1}^{\infty}$ in H such that $Ve_i=\sigma_i^2e_i$, or $Y=\sum_1^\infty Y_ie_i$, where $Y_i=(Y,e_i)$ are independent real Gaussian random variables with zero mean and variances σ_i^2. A density function of a random variable $|Y-a|^2$ will be denoted by $p(u,a)$, $u\geq 0$, $a\in H$.

Zolotarev (1961) proved the first sharp asymptotic result for $p(u,0)$

$$\lim_{u\to\infty}p(u,0)/f_k(u,0)=M(k),\tag{1}$$

where

$$M(k)=\prod_{j=k+1}^{\infty}(1-\sigma_j^2/\sigma_1^2)^{-1/2}$$

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and $f_k(u,0)$ is a density function of $\sum_1^k Y_i^2$, i. e.

$$f_k(u,0) = (2\sigma_1^2\Gamma(k/2))^{-1}(u/(2\sigma_1^2))^{k/2-1}\exp(-u/(2\sigma_1^2)),$$

where $\Gamma(\cdot)$ is gamma function. There are a lot of papers devoted to different generalizations of (1) on asymptotic behaviour of a Gaussian measure in a Hilbert space or in some Banach spaces (see e. g. Lifshits (1991) and Linde (1991) and bibliography in these papers). But in order to get generalizations of the law of the iterated logarithm to a Hilbert space (see e. g. Einmahl (1991)) or to study an accuracy of Gaussian approximation in infinite dimensional spaces (see e. g. survey by Bentkus et al. (1990)) one needs to know the behaviour of Gaussian measure of balls $B(a,r) = \{x \in H : |x-a| \le r\}$ not only for large but for all values of r.

One of the first results in this direction was obtained by Hoeffding (1964) who proved (cf. (1)) for all $u > 0$ and $k \ge 2$

$$p(u,0) \le M(k) f_k(u,0). \tag{2}$$

Later several upper estimates for $p(u,a)$ when $a \neq 0$ were derived in the papers on the central limit theorem in a Hilbert space (see the monographs by Paulauskas and Račkauskas (1989) and by Sazonov (1981) and the bibliography there and also Sazonov et al. (1988)). The discussion on these results see in Ulyanov (1991). However (2) does not follow from the above mentioned inequalities. In Ulyanov (1991) the inequality (2) was generalized in the following way: if $k \ge 2$ then for any $a \in H$

$$\begin{aligned} p(u,a) \le\ & M(k)(2\sigma_1^2\Gamma(k/2))^{-1} \\ & \times (u/(2\sigma_1^2))^{k/2-1}\exp(-(u^{1/2}-|a|)^2/(2\sigma_1^2)). \end{aligned} \tag{3}$$

Obviously, (3) coincides with (2) if $a = 0$, i. e. a equals the zero element in H. The Theorem in Richter and Ulyanov (1991) gave a hint that (3) can be improved under additional assumptions on a.
The aim of the present paper is to get these refinements of (3).
We give different upper bounds for $p(u,a)$ according to possible various values of a and k (see Theorem 1 and Remark after it).
In order to show how sharp the new results are, we construct upper and lower estimates for $P(|Y-a| > r)$ (see Theorem 2 and Corollary).
Our new estimates imply, in particular upper bounds for probabilities $P(|Y-a| > r)$ with the same behaviour with respect to a and r as the asymptotically sharp resutls of Linde (1991) obtained for the case $r \to \infty$ (cf. Theorem 3 and Remark 1 on p. 1274 in Linde (1991) and our Theorem 2).

2. Main Results.

Put for $a \in H$

$$\bar{a}_l = (a_1, \ldots, a_l),$$

where $a_i = (a, e_i), i = 1, 2, \ldots, l$.

Theorem 1. a) *If $k = 1$ then for any $a \in H$*

$$p(u,a) \le M(2)(2\sigma_1\sigma_2)^{-1}\exp(-(u^{1/2}-|a|)^2/(2\sigma_1^2)) \tag{4}$$

and

$$p(u,a) \le M(2)(2\sigma_1\sigma_2)^{-1}\exp\left(-(u^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2)\right) \times \exp\left(\frac{1}{2}\sum_3^\infty \frac{a_i^2}{\sigma_1^2-\sigma_i^2}\right). \tag{5}$$

b) *If $k \ge 2$ then for any $a \in H$*

$$p(u,a) \le M(k)\left(2\sigma_1^2\Gamma(k/2)\right)^{-1}\left(u/(2\sigma_1^2)\right)^{k/2-1} \times \exp\left(-(u^{1/2}-|a|)^2/(2\sigma_1^2)\right) \tag{6}$$

and

$$p(u,a) \le M(k)(2\sigma_1^2\Gamma(k/2))^{-1}(u/(2\sigma_1^2)^{k/2-1} \times \exp\left(-(u^{1/2}-|\bar{a}_k|)^2/(2\sigma_1^2)\right)\exp\left(\frac{1}{2}\sum_{k+1}^\infty \frac{a_i^2}{\sigma_1^2-\sigma_i^2}\right). \tag{7}$$

c) *If $k = 2$ and $|\bar{a}_2| > 0$ then*

$$p(u,a) \le c_1 M(3)(\sigma_1\sigma_3)^{-1}u^{1/4}|\bar{a}_2|^{-1/2} \times \exp\left(-(u^{1/2}-|a|)^2/(2\sigma_1^2)\right) \tag{8}$$

and

$$p(u,a) \le c_1 M(3)(\sigma_1\sigma_3)^{-1}u^{1/4}|\bar{a}_2|^{-1/2} \times \exp\left(-\left(u^{1/2}-|\bar{a}_3|\right)^2/\left(2\sigma_1^2\right)\right)\exp\left(\frac{1}{2}\sum_4^\infty \frac{a_i^2}{\sigma_1^2-\sigma_i^2}\right), \tag{9}$$

where c_1 is an absolute constant.

d) *If $k \ge 3$ and $|\bar{a}_k| > 0$ then*

$$p(u,a) \le c_2(k)M(k)\sigma_1^{-1}u^{(k-3)/4}|\bar{a}_k|^{-(k-1)/2} \times \exp\left(-(u^{1/2}-|a|)^2/(2\sigma_1^2)\right) \tag{10}$$

and

$$p(u,a) \le c_2(k)M(k)\sigma_1^{-1}u^{(k-3)/4}|\bar{a}_k|^{-(k-1)/2} \times \exp\left(-\left(u^{1/2}-|\bar{a}_k|\right)^2/(2\sigma_1^2)\right)\exp\left(\frac{1}{2}\sum_{k+1}^\infty \frac{a_i^2}{\sigma_1^2-\sigma_i^2}\right), \tag{11}$$

where $c_2(k)$ is a constant depending on k only.

Remark. Always we give estimates for $p(u,a)$ with two type of dependence on a. It is explained by the fact that the right-hand sides of inequalities depend on many parameters and one form of the estimate gives sharper result than the other form in some cases and vice versa. Let us compare for instance (4) and (5). Take $a = (0,0,1,0,0\dots)$ and $\sigma_1^2 = 1$. Then (4) can be rewritten in the form

$$p(u,a) \le M(2)(2\sigma_2)^{-1}\exp(-\frac{u}{2})\exp\left(\frac{2u^{1/2}-1}{2}\right). \tag{12}$$

At the same time we have from (5)

$$p(u,a) \le M(2)(2\sigma_2)^{-1}\exp(-\frac{u}{2})\exp\left(\frac{1}{2(1-\sigma_2^2)}\right). \tag{13}$$

It follows from (12) and (13) that for any fixed u we can choose σ_2 so close to 1 that the right-hand side of (13) will be much larger that the right-hand side of (12). Vice versa, for any fixed value of σ_2^2 it is possible to take u so large that (13) will be much sharper than (12).

Theorem 2 *a) Let r and a be such that $r/2 > |\bar a_k| > 1/r$. Then for all $k \geqslant 3$ there exist constants $c_3(k)$ and $c_4(k)$ depending only on k such that*

$$\begin{aligned} c_3(k) &\le P(|Y-a|>r)\exp\{(r-|\bar a_k|)^2/(2\sigma_1^2)\}\times\sigma_1 r^{-(k-3)/2}|\bar a_k|^{(k-1)/2} \\ &\le c_4(k)M(k)\min\{\exp(\frac12\sum_{k+1}^{\infty}a_i^2/(\sigma_1^2-\sigma_i^2)),\exp(\frac{1}{2\sigma_1^2}((r-|\bar a_k|)^2-(r-|a|)^2))\}. \end{aligned} \tag{14}$$

b) Let r and a be such that $r \ge 1$ and $|\bar a_k| \le 1/r$. Then for all $k \ge 2$ there exists constants $c_5(k)$ and $c_6(k)$ depending only on k such that

$$\begin{aligned} c_5(k) &\le P(|Y-a|>r)\exp\{(r-|\bar a_k|)^2/(2\sigma_1^2)\}\times\sigma_1^{2-k}r^{k-2} \\ &\le c_6(k)M(k)\min\{(\frac12\sum_{k+1}^{\infty}a_i^2/(\sigma_1^2-\sigma_i^2)),\exp(\frac{1}{2\sigma_1^2}((r-|\bar a_k|)^2-(r-|a|)^2))\}. \end{aligned} \tag{15}$$

Corollary. *a) Let r and a satisfy conditions of Theorem 2 a). Then for all $k \ge 1$*

$$c_3(k) \le P(|\bar Y_k-\bar a_k|>r)\exp\{(r-|\bar a_k|)^2/(2\sigma_1^2)\}\times\sigma_1 r^{-(k-3)/2}|\bar a_k|^{(k-1)/2} \le c_4(k). \tag{16}$$

b) Let r and a satisfy conditions of Theorem 2 b). Then for all $k \ge 1$

$$c_5(k) \le P(|\bar Y_k-\bar a_k|>r)\exp\{(r-|\bar a_k|)^2/(2\sigma_1^2)\}\times\sigma_1^{2-k}r^{k-2} \le c_6(k). \tag{17}$$

Remark. Inequalities (16) and (17) give two–sided estimates for a distribution function of noncentral chi–square distribution in terms of “simple” functions.

The proofs of the Theorems and Corollary are based on a series of lemmas which will be proved in the next section.

3. Auxiliary lemmas.

Lemma 1. *Let $b > -1$ and $a > 0$. Then*

$$I \equiv \int_0^1 (1-z)^{-1/2} z^b \exp(-az) dz \le a^{-(b+1)}(\sqrt{2}\Gamma(b+1) + B(b+1; 1/2)(b+1)^{b+1}), \tag{18}$$

where $\Gamma(\cdot)$ and $B(\cdot;\cdot)$ are gamma and beta functions respectively.

Proof. Let us fix any $\gamma : 0 < \gamma < 1$,and write

$$I = I_1 + I_2, \tag{19}$$

where

$$I_1 = \int_0^\gamma (1-z)^{-1/2} z^b \exp(-az) dz$$

and

$$I_2 = I - I_1.$$

We have

$$I_1 \le (1-\gamma)^{-1/2} \int_0^\gamma z^b \exp(-az) dz \le (1-\gamma)^{-1/2} a^{-(b+1)} \Gamma(b+1) \tag{20}$$

and

$$I_2 \le \exp(-a\gamma) \int_\gamma^1 (1-z)^{-1/2} z^b dz \le a^{-(b+1)} [(b+1)/(\gamma e)]^{b+1} B(b+1; 1/2). \tag{21}$$

Taking $\gamma = 1/2$ in (20) and (21) we get (18) from (19).

Lemma 2.

Let $Y_1, \dots, Y_k$ are independent gaussian random variables with mean zero and variance 1. We denote by $f_k(u,a)$ the density function of

$$|\bar{Y}_k - \bar{a}_k|^2 = (Y_1 - a_1)^2 + \dots + (Y_k - a_k)^2.$$

Then for all $|\bar{a}_k| = (a_1^2 + \dots + a_k^2)^{1/2} > 0$ and $k \ge 1$

$$f_k(u,a) \le c(k) u^{(k-3)/4} |\bar{a}_k|^{-(k-1)/2} \exp(-(u^{1/2} - |\bar{a}_k|)^2/2), \tag{22}$$

where $c(k) = \pi^{-1/2} + (\frac{k-1}{2})^{(k-1)/2}/\Gamma(k/2)$.

Remark 1. The estimate of the type (22) was derived earlier by my student Digailova under additional assumptions on u and $\bar{a}_k$ using another arguments.

Remark 2. In other words (22) gives an upper estimate for the density $f_k(u,a)$ of noncentral chi–square distribution in terms of "simple" functions. It is known that (see e.g. vol. 6 of Encyclopedia of Mathematics (1990))

$$f_k(u,a) = 2^{-k/2}(\Gamma(1/2))^{-1}\exp\{-(u+|\bar{a}_k|^2)/2\}u^{(k-2)/2}\sum_{r=0}^{\infty}\frac{|\bar{a}_k|^{2r}u^r\Gamma(r+1/2)}{(2r)!\Gamma(r+k/2)}$$

.

Proof. If $k=1$ then (22) follows from the fact that the density function of $(Y_1-a_1)^2$ equals
$(2\pi u)^{-1/2}\cdot 2^{-1}\cdot[\exp\{-(u^{1/2}-a_1)^2/2\}+\exp\{-(u^{1/2}+a_1)^2/2\}]$.
In the sequel $k\geq 2$.
Note that $f_k(u,a)$coincides with the density function of $Y_1^2+\cdots+Y_{k-1}^2+(Y_k-|\bar{a}_k|)^2$.
We recall that $f_{k-1}(u,0)$ stands for the density function of $Y_1^2+\cdots+Y_{k-1}^2$ and

$$f_{k-1}(u,0) = 2^{-(k-1)/2}(\Gamma((k-1)/2))^{-1}u^{(k-1)/2-1}\exp(-u/2). \tag{23}$$

let $g_k(u,|\bar{a}_k|)$ denote the density function of $(Y_k-|\bar{a}_k|)^2$. It is easy to see that

$$g_k(u,|\bar{a}_k|) \leq (2\pi u)^{-1/2}\exp(-(u^{1/2}-|\bar{a}_k|)^2/2). \tag{24}$$

Put $c_1(k) = (2\pi)^{-1/2}2^{-(k-1)/2}(\Gamma((k-1)/2))^{-1}$. It follows from (23) and (24) that

$$\begin{aligned} f_k(u,a) &= \int_0^u g_k(u-v,|\bar{a}_k|)f_{k-1}(v,0)dv \\ &\leq c_1(k)\exp(-\frac{u+|\bar{a}|^2}{2})\int_0^u(u-v)^{-1/2}v^{(k-3)/2}\exp(|\bar{a}_k|(u-v)^{1/2})dv \\ &= c_1(k)u^{(k-2)/2}\exp(-\frac{u+|\bar{a}_k|^2}{2}) \\ &\times\int_0^1(1-z)^{-1/2}z^{(k-3)/2}\exp(|\bar{a}_k|u^{1/2}(1-z)^{1/2})dz. \end{aligned} \tag{25}$$

Since for all $z: 0\leq z\leq 1$,

$$(1-z)^{1/2}-1\leq -z/2,$$

from Lemma 1 and (25) we obtain (22).

Lemma 3.

For all $k\geq 1$

$$f_k(u,a) \leq (2\Gamma(k/2))^{-1}(u/2)^{k/2-1}\exp(-(u^{1/2}-|\bar{a}_k|)^2/2). \tag{26}$$

Proof. The estimate (26) is proved in Ulyanov (1991).

Remark. Comparing (22) and (26) we see that (22) gives sharper estimate than (26) when $k\geq 2$ and $(2\Gamma(k/2))^{-1}(u/2)^{k/2-1} > c(k)u^{(k-3)/4}|\bar{a}_k|^{-(k-1)/2}$, or

$$u^{1/2}|\bar{a}_k| > c_7(k), \tag{27}$$

where $c_7(k) = [2^{k/2}\Gamma(k/2)c(k)]^{2/(k-1)} \geq 1$. Below we give two different lower estimates for
$P(|\bar{Y}_k - \bar{a}_k| > r)$ depending on whether (27) is satisfied for $r = u^{1/2}$ or not.

Lemma 4.

Let r and $\bar{a}_k$ be such that $r/2 > |\bar{a}_k| > 1/r$ then for $k \geq 1$

$$P(|\bar{Y}_k - \bar{a}_k| > r) \geq c_8(k) r^{(k-3)/2} |\bar{a}_k|^{-(k-1)/2} \exp\{-(r - |\bar{a}_k|)^2/2\}, \tag{28}$$

where $c_8(k)$ is a constant depending only on k.

Proof (cf. the Theorem and its proof in Richter and Ulyanov (1991)).
For $k = 1$ the inequality (28) is trivial.
Let $k \geq 2$.
Applying Richter's representation formula (see Richter (1985)) for the standard Gaussian measure in $\mathbb{R}^k$ we get

$$P(|\bar{Y}_k - \bar{a}_k| > r) = c_9(k) \exp\{-(r - |\bar{a}_k|)^2/2\} \int_0^\infty ((r - |\bar{a}_k|)^2 + 2c)^{k/2-1} e^{-c} F(B, 2c) dc, \tag{29}$$

where

$$F(B, 2c) = F_B(((r - |\bar{a}_k|)^2 + 2c)^{1/2}),$$

$$F_B(l) = W_l(B \cap S(l)), \quad l \geq 0,$$

$$B = \{\bar{x}_k \in \mathbb{R}^k : |\bar{x}_k - \bar{a}_k| > r\},$$

W_l is the uniform probability distribution over $S(l)$,

$$S(l) = \{\bar{x}_k \in \mathbb{R}^k : |\bar{x}_k| = l\}$$

and

$$c_9(k) = 2\pi^{k/2}/\Gamma(k/2)$$

is the surface area of the unit sphere in $\mathbb{R}^k$. It was remarked above that without loss of generality we can assume that

$$\bar{a}_k = (0, \ldots, 0, |\bar{a}_k|).$$

Therefore, the function $F(B, 2c)$ in (29) describes the surface area part of the sphere with center 0 and radius $((r - |\bar{a}_k|)^2 + 2c)^{1/2}$ which belongs to the set

$$\{\bar{x}_k \in \mathbb{R}^k : |\bar{x}_{k-1}|^2 + (x_k - |\bar{a}_k|)^2 \geq r^2\}$$

We have

$$F(B, 2c) \geq 1/2 \tag{30}$$

if $(r - |\bar{a}_k|)^2 + 2c > r^2 - |\bar{a}_k|^2$, or, equivalently,

$$|\bar{a}_k|^2 - r|\bar{a}_k| + c \geq 0. \tag{31}$$

To evaluate $F(B, 2c)$ in the opposite case

$$|\bar{a}_k|^2 - r|\bar{a}_k| + c < 0, \tag{32}$$

we shall solve the system of equations

$$|\bar{x}_{k-1}|^2 + x_k^2 = (r - |\bar{a}_k|)^2 + 2c,$$

$$|\bar{x}_{k-1}|^2 + (x_k - |\bar{a}_k|)^2 = r^2.$$

We get

$$|\bar{x}_{k-1}|^2 = \frac{c}{|\bar{a}_k|}(2r - \frac{c}{|\bar{a}_k|}).$$

Therefore, if (32) is satisfied then

$$F(B, 2c) \geq c_{10}(k)[\frac{c}{|\bar{a}_k|}(2r - \frac{c}{|\bar{a}_k|})((r - |\bar{a}_k|)^2 + 2c)^{-1}]^{(k-1)/2} \tag{33}$$

for some constant $c_{10}(k)$ depending only on k. It follows from the assumption of Lemma 4 on r and $|\bar{a}_k|$ that

$$r|\bar{a}_k| - |\bar{a}_k|^2 > r|\bar{a}_k|/2 > 1/2.$$

Hence, (29), (32) and (33) imply

$$\begin{aligned} P(|\bar{Y}_k - \bar{a}_k| > r) &\geq c_{11}(k) \exp\{-(r - |\bar{a}_k|)^2/2\} \\ &\int_0^{r|\bar{a}_k|/2} [\frac{c}{|\bar{a}_k|}(2r - \frac{c}{|\bar{a}_k|})]^{(k-1)/2}((r - |\bar{a}_k|)^2 + 2c)^{-1/2} e^{-c} dc \\ &\geq c_{11}(k) \exp\{-(r - |\bar{a}_k|)^2/2\} r^{(k-1)/2} |\bar{a}_k|^{-(k-1)/2} \\ &\int_0^{1/2} c^{(k-1)/2}((r - |\bar{a}_k|)^2 + 2c)^{-1/2} e^{-c} dc \\ &\geq c_8(k) r^{(k-3)/2} |\bar{a}_k|^{-(k-1)/2} \exp\{-(r - |\bar{a}_k|)^2/2\}, \end{aligned}$$

that is Lemma 4 is proved.

Lemma 5.

Let $r > 1$ and $\bar{a}_k$ be such that $|\bar{a}_k| \leq c_{12}(k)/r$ for some constant $c_{12}(k)$. Then for $k \geq 1$

$$P(|\bar{Y}_k - \bar{a}_k| > r) \geq c_{13}(c_{12}, k) r^{k-2} \exp\{-(r - |\bar{a}_k|)^2/2\}, \tag{34}$$

where $c_{13}(c_{12}, k)$ is a constant depending on k and $c_{12}(k)$.

Proof. Note that

$$r|\bar{a}_k| - |\bar{a}_k|^2 < r|\bar{a}_k| \leq c_{12}(k).$$

Therefore, it follows from (29) – (31) that

$$\begin{aligned} P(|\bar{Y}_k - \bar{a}_k| > r) &\geq c_9(k) \exp\{-(r - |\bar{a}_k|)^2/2\} \int_{c_{12}(k)}^{\infty} ((r - |\bar{a}_k|)^2 + 2c)^{k/2-1} e^{-c}/2 dc \\ &\geq c_{13}(c_2, k) r^{k-2} \exp\{-(r - |\bar{a}_k|)^2/2\}, \end{aligned}$$

which proves Lemma 5.

Lemma 6.

We have

$$p(u,a) \le \overline{\lim_m} f_m(u,a), \tag{35}$$

where $f_m(u,a)$ is a density function of $|\bar{Y}_m - \bar{a}_m|^2 = (Y_1-a_1)^2 + \cdots + (Y_m - a_m)^2$.

Proof (cf. the proof of the Theorem in Ulyanov (1991)). Fix $a \in M$ and put $p(u) = p(u,a)$ and $f_m(u) = f_m(u,a)$. Now take an arbitary $\epsilon > 0$. Obviously there exists $M_1 = M_1(\epsilon)$ such that for all $m \ge M_1$

$$\sum_{i=m+1}^{\infty} a_i^2 < \epsilon^2.$$

Therefore, since for any $m \ge 1$

$$\sum_{i=m+1}^{\infty} (Y_i - a_i)^2 \le 2(\sum_{m+1}^{\infty} Y_i^2 + \sum_{m+1}^{\infty} a_i^2),$$

we have for $m \ge M_1$

$$\begin{aligned} P(u-\epsilon < |Y-a|^2 < u+\epsilon) &\le P(u - \epsilon - 2\epsilon^2 - 2\sum_{m+1}^{\infty} Y_i^2 < \sum_{i=1}^{m}(Y_i-a_i)^2 < u+\epsilon) \\ &\le P(\sum_{m+1}^{\infty} Y_i^2 \ge \epsilon^2) + P(u - \epsilon - 4\epsilon^2 \\ &< \sum_{i=1}^{m}(Y_i - a_i)^2 < u + \epsilon). \end{aligned} \tag{36}$$

From Lemma 3 in Sazonov et al (1988) we get

$$\begin{aligned} P(\sum_{m+1}^{\infty} Y_i^2 \ge \epsilon^2) &\le 2\exp\{-\epsilon^2/(2E\sum_{m+1}^{\infty} Y_i^2)\} \\ &= 2\exp\{-\epsilon^2/(2\sum_{m+1}^{\infty}\sigma_i^2)\}. \end{aligned}$$

Hence, there exists $M_2 = M_2(\epsilon)$ such that for all $m \ge M_2$

$$P(\sum_{m+1}^{\infty} Y_i^2 \ge \epsilon^2) \le \epsilon^2.$$

Then for any $m \ge M_1 + M_2$ the inequality (36) implies

$$P(u-\epsilon < |Y-a|^2 < u+\epsilon) \le \epsilon^2 + (2\epsilon + 4\epsilon^2) \sup_{y \in T(\epsilon,u)} f_m(u), \tag{37}$$

where $T(\epsilon,u) = \{y \in \mathbb{R}^1 : u - \epsilon - 4\epsilon^2 < y < u + \epsilon\}$. Moreover, it is easy to show that we have for $m \ge 4$ and any $\delta > 0$

$$f_m(u+\delta) \le f_m(u) + c(m,\sigma_4,a)\delta^{1/8}, \tag{38}$$

where $c(m,\sigma_4,a)$ is a constant depending on m, a, and σ_4. Dividing the both sides of (37) by 2ϵ and using (38) we get (35).

4. Proofs of Theorems and Corollary.

Proof of Theorem 1. Estimates (4) and (6) are proved in Ulyanov (1991).
Let us prove (5). In Ulyanov (1991) it is proved that for the density function $f_2(u,a)$ of $|\bar{Y}_2-\bar{a}_2|^2$ we have

$$f_2(u,a)\le(2\sigma_1\sigma_2)^{-1}\exp(-(u^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2)). \tag{39}$$

Let $g_3(u,a)$ be the density function of $(Y_3-a_3)^2$. Then

$$\begin{aligned}
f_3(u,a)&=\int_0^u f_2(u-v,a)g_3(v,a)dv\\
&\le(2\sigma_1\sigma_2)^{-1}\int_0^u\exp(-((u-v)^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2))g_3(v,a)dv\\
&=(2\sigma_1\sigma_2)^{-1}\exp(-(u^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2))\int_0^u(2\pi v)^{-1/2}\sigma_3^{-1}2^{-1}\exp(-\frac{a_3^2}{2\sigma_3^2}(1-\frac{\sigma_1^2}{\sigma_1^2-\sigma_3^2}))\\
&\times[\exp(-\frac{1}{2\sigma_1^2\sigma_3^2}(v(\sigma_1^2-\sigma_3^2)+\sigma_1^2a_3/(\sigma_1^2-\sigma_3^2)^{1/2})^2)+\\
&\exp(-\frac{1}{2\sigma_1^2\sigma_3^2}(v(\sigma_1^2-\sigma_3^2)-\sigma_1^2a_3/(\sigma_1^2-\sigma_3^2)^{1/2})^2)]dv\\
&\le(2\sigma_1\sigma_2)^{-1}\sigma_1/(\sigma_1^2-\sigma_3^2)^{1/2}\exp(-(u^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2))\exp(a_3^2/(2(\sigma_1^2-\sigma_3^2))).
\end{aligned}$$

Similar arguments and mathematical induction imply for any $m\ge 3$

$$f_m(u,a)\le(2\sigma_1\sigma_2)^{-1}\prod_3^m(1-\sigma_j^2/\sigma_1^2)^{-1/2}\exp(-(u^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2))\exp(\frac{1}{2}\sum_3^m a_i^2/(\sigma_1^2-\sigma_j)^2). \tag{40}$$

From Lemma 6 and (40) we get (5). To prove (7) we have to use Lemma 3 instead of (39). In the same manner using Lemma 2 we prove (10) and (11).
Now we prove (8) and (9). From Lemma 2 we have

$$\begin{aligned}
f_3(u,a)&=\int_0^u f_2(u-v,a)g_3(v,a)dv\\
&\le c\sigma_1^{-1}|\bar{a}_2|^{-1/2}\sigma_3^{-1}\\
&\int_0^u(u-v)^{-1/4}v^{-1/2}\exp(-((u-v)^{1/2}-|\bar{a}_2|)^2/(2\sigma_1^2)-(v^{1/2}-|a_3|)^2/(2\sigma_3^2))dv\\
&\le c\sigma_1^{-1}\sigma_3^{-1}|\bar{a}_2|^{-1/2}\exp(-(u-|\bar{a}_3|)^2/(2\sigma_1^2))\int_0^u(u-v)^{-1/4}v^{-1/2}dv\\
&\le c_1(\sigma_1\sigma_3)^{-1}u^{1/4}|\bar{a}_2|^{-1/2}\exp(-(u-|\bar{a}_3|)^2/(2\sigma_1^2)).
\end{aligned}$$

This estimate, Lemma 6 and analysis similar to that in the proof of (40) yield (8) and (9).
Thus Theorem 1 is proved.

Proof of Theorem 2 and Corollary. The left–hand sides of (16) and (17) follow immediately from Lemmas 4 and 5 respectively. Note that the density function of

$|\bar{Y}_k - \bar{a}_k|$ equals $2uf_k(u^2, a)$. Then Lemmas 2 and 3 imply the right–hand sides of (16) and (17) respectively.
In order to prove left–hand sides of (14) and (15) it is enough to note that
$P(|Y - a| > r) \geq P(|\bar{Y}_k - \bar{a}_k| > r)$.
The right–hand sides of (14) and (15) follow from (10), (11) and (6), (7) respectively.

References.

Bentkus V., Götze F., Paulauskas V. and Račkaukas A., *The accuracy of Gaussian approximation in Banach spaces*, SFB 343 "Diskrete Strukturen in der Mathematik", Preprint 90 – 100 (1990), Universität Bielefeld.

Einmahl U., *On the almost sure behavior of sums of iid random variables in Hilbert space*, Ann. Probab. 19 (1991), 1227–1263.

Encyclopedia of Mathematics (Transl. of the Soviet "Mathematical Encyclopedia"), Kluwer Academic Publishers (1990)

Hoeffding W., *On a theorem of V.M. Zolotarev*, Theory Probab. Appl. 9 (1964), 89–91.

Lifshits M.A., *Gaussian large deviations of a smooth seminorm*, submitted to Notes of scientific Seminars of Steklov Mathem. Inst., Leningrad Branch (1991).

Linde W., *Gaussian measure of large balls in l_p*, Ann. Probab. 19 (1991), 1264–1279.

Paulauskas V. and Račkauskas A., *Approximation Theory in the Central Limit Theorem. Exact Results in Banach spaces*, Kluwer Academic Publishers (1989), 1–156.

Richter W.–D., *Laplace–Gauss integrals, Gaussian measure asymptotic behaviour and probabilities of moderate deviations*, Z. Anal. Anw., 4 (1985), 257–267.

Richter W.–D. and Ulyanov V.V., *Two–sided estimates for the Gaussian measure of the complement of a ball in Hilbert space*, Theory Probab. Appl. 36 (1991), 837–838.

Sazonov V.V., *Normal approximation. Some recent advances*, Lect. Notes Math. 879, (1981), 1–105.

Sazonov V.V., Ulyanov V.V. and Zalesskii B.A., *Normal approximation in Hilbert space*, Theory Probab. Appl. 33 (1988), 207–227, 473–487.

Ulyanov V.V., *Non–uniform estimates of the density of the norm of a Gaussian vector in Hilbert space.* In: New Trends in Probability and Statistics. V. Sazonov and T. Shervashidze (Eds.), VSP–Mokslas, Vilnius (1991), 255–263.

Zolotarev V.M., *Concerning a certain probability problem*, Theory Probab. Appl. 6 (1961), 201–204.