INFINITE PRODUCTS AND PARACONTRACTING MATRICES

L. Elsner

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131

33501 Bielefeld
FRG

1. Introduction.

In the investigation of chaotic iteration procedures for linear consistent systems matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [EKN], that if $A_1, \ldots, A_m$ are finitely many k–by–k complex matrices which are paracontracting with respect to the same norm, then for any sequence d_i, $1 \leq d_i \leq m$, $i = 1, 2, \ldots$ and any x_0 the sequence

$$x_{i+1} = A_{d_i} x_i \quad i = 1, 2, \ldots$$

is convergent. In particular $A^{(d)} = \lim_{i \to \infty} A_{d_i} \ldots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$.

Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [DL]. Following [DL], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i \to \infty} A_{d_i} A_{d_{i-1}} A_{d_1}$$

is continuous (which is equivalent to the set of fixed points of A_i being the same for all $1 \leq i \leq m$), a LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractedness is equivalent to the LCP–property.

2. Notations and known results.

Let $||\ ||$ denote a vector norm in $\mathbb{C}^k$. A k –by– k matrix P is **paracontracting** with respect to $||\ ||$, if for all x

$$Px \neq x \Leftrightarrow ||Px|| < ||x||.$$

We denote by $\mathcal{N}(||\ ||)$ the set of all k –by– k matrices paracontracting w.r.t. $||\ ||$.

For a bounded set $\Sigma = \Sigma_1$ of complex k–by–k matrices define $\Sigma_0 = \{I\}$ and for $n \geq 1$

$$\Sigma_n = \{M_1\ M_2 \dots M_n : M_i \in \Sigma\},$$

the set of all products of matrices in Σ of length n. Let $\Sigma = \{A_1, \dots, A_m\}$ be finite. For $d = (d_1, d_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$, i.e. $1 \leq d_i \leq m$ for $i \in \mathbb{N}$ define

$$A^{(d)} = \lim_{n \to \infty} A_{d_n} A_{d_{n-1}} \dots A_{d_1},$$

if the limit exists.
Σ is an LCP–set (left–convergent–product), if for all $d \in \{1, \dots, m\}^{\mathbb{N}}$ the limit $A^{(d)}$ exists. The function $d \to A^{(d)}$ mapping $\{1, \dots, m\}^{\mathbb{N}}$ into the space of k–by–k matrix is called the **limit function**.

We note in passing that in [DL] also the right–convergent–product property (RCP) was introduced. For convenience we restrict our considerations to the left convergence case.

Introducing in $\{1, \dots, m\}^{\mathbb{N}}$ the metric

$$\text{dist}\ (d, d') = m^{-r} \qquad r \ \text{ smallest index such that } \ d_r \neq d'_r$$

we define the concept of a **continuous limit function** in the standard way.

Σ is **product bounded**, if there exists $\Delta > 0$ such that

$$||A|| \leq \Delta \ \text{ for all } \ A \in \Sigma_n,\ n = 1, 2, \dots$$

Here $||\ ||$ denotes any matrix norm. Obviously this concept is independent of the norm. G. Schechtman has proved that LCP–sets are product bounded (see Theorem I of [BW]). We have the following statement.

Lemma 1. *For a set Σ of k–by–k matrices the following are equivalent:*

(1) Σ *is product bounded.*
(2) $\exists$ *vector norm* $||\ ||$ *such that* $||Ax|| \leq ||x||$ *for all* $A \in \Sigma,\ x \in \mathbb{C}^k$.
(3) $\exists$ *multiplicative matrix norm* $||\ ||$ *such that* $||A|| \leq 1$ *for all* $A \in \Sigma$.

Proof.
As (2) $\Longrightarrow$ (3) (the operator norm is multiplicative) and (3) $\Longrightarrow$ (1) are obvious, only (1) $\Longrightarrow$ (2) has to be shown.
For some vector norm ν define the norm

$$||x|| = \sup_{n \geq 0} \{ \sup_{A \in \Sigma_n} \nu(Ax)\}$$

which is finite by (1). Then $||Ax|| \leq ||x||$ for all $A \in \Sigma$. ■

Remark: This result could also be derived from [RS].

For a given matrix norm $||\ ||$ and bounded Σ let

$$\widehat{\rho}_n = \widehat{\rho}_n(\Sigma) = \max\{||A||, A \in \Sigma_n\}$$

and

$$\widehat{\rho} = \widehat{\rho}(\Sigma) = \lim_{n\to\infty} \widehat{\rho}_n^{1/n}.$$

$\widehat{\rho}$ is called the joint spectral radius of Σ. It has been introduced in [RS] for general bounded sets in a normed algebra. In [RS] and in [DL] the limit is replaced by lim sup, however, it is implicit in [DL], see there (3.12), that the limit exists.

We give a characterization of $\widehat{\rho}(\Sigma)$, which can be found essentially in [RS].

Lemma 2. *For any bounded set Σ of k–by–k matrices*

$$\widehat{\rho}(\Sigma) = \inf_{\nu \text{ operatornorm}} \sup_{A\in\Sigma} \nu(A).$$

Proof.
For any $\epsilon > 0$ the set

$$\Sigma_\epsilon = \{\frac{1}{\widehat{\rho}+\epsilon} A,\ A \in \Sigma\}$$

is product bounded, as for any $B \in (\Sigma_\epsilon)_n$

$$||B|| \leq \frac{1}{(\widehat{\rho}+\epsilon)^n} \widehat{\rho}_n \to 0 \quad \text{as} \quad n \to \infty.$$

Hence by Lemma 1, (2) there exists a norm ν_ϵ such that

$$\nu_\epsilon(\frac{A}{\widehat{\rho}+\epsilon} x) \leq \nu_\epsilon(x) \quad \text{for all} \quad A \in \Sigma,\ x \in \mathbb{C}^k.$$

and therefore

$$\nu_\epsilon(Ax) \leq (\widehat{\rho}(\Sigma)+\epsilon)\nu_\epsilon(x) \quad \text{for all} \quad A \in \Sigma,\ x \in \mathbb{C}^k.$$

■

In the following the subspaces

$$M_i = \{x : A_i x = x\} = N(I - A_i) \qquad i = 1, \ldots, m$$

play a fundamental role. If Σ has the LCP–property, then in particular $\lim_{n\to\infty} A_i^n$ exist, and hence

$$\mathbb{C}^k = N(I - A_i) \oplus R(I - A_i) \qquad i = 1, \ldots, m.$$

The same holds if $A_i \in \mathcal{N}(||\ ||)$ for some norm, see [EKN] and [NN]. This is not surprising in view of the following result, which is just a restatement of the Theorem in [EKN].

Theorem 1. *Let $\Sigma \subset \mathcal{N}(||\ ||)$ for some vector norm $||\ ||$, Σ finite. Then Σ has the LCP–property.*

3. Main result.

It is tempting to conjecture that the reverse statement of Theorem 1 also holds, namely that if Σ is an LCP-set, then there exists a vector norm $||\ ||$ such that $\Sigma \subset \mathcal{N}(||\ ||)$. We were unable to decide this question.

However, if Σ is an LCP-set with a continuous limit function, then this is true. More precisely, the following holds:

Theorem 2. *Let $\Sigma = \{A_1, \dots, A_m\}$ be a finite set of k–by–k matrices and $M_i = N(I - A_i)$, $i = 1, \dots, m$. Then the following are equivalent:*

(1) There exists a vector norm $||\ ||$ in $\mathbb{C}^k$ such that $\Sigma \subset \mathcal{N}(||\ ||)$ and for $i, j = 1, \dots, m$

$$M_i = M_j.$$

(2) Σ has the LCP–property and for $i, j = 1, \dots, m$

$$M_i = M_j.$$

(3) Σ has the LCP–property with continuous limit function.

Proof.
We will show (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).
(1) $\Longrightarrow$ (2) is an immediate consequence of Theorem 1. To prove (2) $\Longrightarrow$ (3), we are going to show that

(x) $$||A^{(d)} - A^{(d')}|| \leq (2 + \Delta)||A_{(r)} - A^{(d)}||$$

where $||\ ||$ is a fixed operator norm, $(d), (d') \in \{1, \dots, m\}^{\mathbb{N}}$, $d_i = d'_i$ for $i \leq r$ and Δ the bound in the definition of product boundedness. Here we use the fact that by [BW] Σ is product bounded. Also we use the notation

$$A_{(r)} = A_{d_r} A_{d_{r-1}} \dots A_{d_1},\ A'_{(s)} = A_{d'_s} \dots A_{d'_1}.$$

Let $M_0 = N(I - A_i)$, $i = 1, \dots, m$ the common pointwise invariant subspace of the $A'_i s$. If $i \in \{1, \dots, m\}$ occurs infinitely often in the sequence $d_1, d_2, \dots$, then by the usual reasoning

$$A_i A^{(d)} = A^{(d)}$$

and hence all columns of $A^{(d)}$ are in M_0. Hence $A_j A^{(d)} = A^{(d)}$ for all $A_j \in \Sigma$. This implies the relation

$$A'_{(r+s)} - A_{(r)} = (A_{d'_{r+s}} \dots A_{d'_{r+1}} - I)(A_{(r)} - A^{(d)}) \qquad s > 0$$

and hence $||A'_{(r+s)} - A_{(r)}|| \leq (1 + \Delta)||A_{(r)} - A^{(d)}||$. Taking $s \to \infty$, we get

$$||A^{(d')} - A_{(r)}|| \leq (1 + \Delta)||A_{(r)} - A^{(d)}||,$$

from which (x) follows. This implies continuity: Given $\epsilon > 0$, as $A_{(r)} \to A^{(d)}$, there exists r_0 such that

$$||A_{(r_0)} - A^{(d)}|| \leq (2+\Delta)^{-1}\epsilon.$$

Now, if (d') is such that

$$\text{dist}\,(d,d') \leq m^{-r_0-1}$$

then $d_i = d'_i$ for $i \leq r_0$ and hence by (x)

$$||A^{(d')} - A^{(d)}|| \leq (2+\Delta)||A_{(r_0)} - A^{(d)}|| \leq \epsilon.$$

Finally we show (3) $\Longrightarrow$ (1).

Assume that (3) holds. By Theorem 4.2 in [DL] the subspaces M_i are the same for $i = 1, \ldots, m$. By a similarity transformation, i.e.

$$\Sigma \to S^{-1}\Sigma S = \{S^{-1}A_iS : \ i = 1, \ldots, m\}$$

which does not change the properties involved, we can assume that M_i is spanned by the first r unit vectors $e_1, \ldots, e_r$ so that for $i = 1, \ldots, m$

$$A_i = \begin{pmatrix} I_r & C_i \\ 0 & \widetilde{A}_i \end{pmatrix}.$$

Obviously $\widetilde{\Sigma} = \{\widetilde{A}_1, \ldots, \widetilde{A}_m\}$ has the LCP–property also and its limit function is identically zero. Otherwise if $\widetilde{A}^{(d)} \neq 0$, for some $d \in \{1, \ldots, m\}^{\mathbb{N}}$ we would have $\widetilde{A}_r\widetilde{A}^{(d)} = \widetilde{A}^{(d)}$ for at least one r and $\widetilde{A}_r$ would have 1 as an eigenvalue. This contradicts our assumptions. But then, from Theorem 4.1 in [DL], it follows that $\widehat{\rho}(\widetilde{\Sigma}) < 1$. We select some q in $(\widehat{\rho}(\widetilde{\Sigma}), 1)$. By Lemma 2 we find a norm $||\ ||$ on $\mathbb{C}^{k-r}$ such that

$$||\widetilde{A}_i x|| \leq q||x|| \ \text{ for all } x \in \mathbb{C}^{k-r} \ \text{ and all } \ i = 1, \ldots, m.$$

Denoting by $||\ ||_2$ the Euclidean norm in $\mathbb{C}^r$, we introduce for positive K the following vector norm in $\mathbb{C}^k$:

$$\mu_K(x) = \mu_K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ||x_1||_2 + K||x_2||.$$

Then, as

$$\mu_K(A_ix) = \mu_K\begin{pmatrix} x_1 + C_ix_2 \\ \widetilde{A}_ix_2 \end{pmatrix} = ||x_1 + C_ix_2||_2 + K||\widetilde{A}_ix_2||$$

$$\leq ||x_1||_2 + \left(\frac{||C_i||}{K} + q\right)K||x_2||$$

where $||C_i|| = \max\left\{\frac{||C_ix||_2}{||x||}, x \in \mathbb{C}^{k-r}\right\}$, we have that for K large enough, $x_2 \neq 0$ (which is equivalent to $A_ix \neq x$) implies $\mu_K(A_ix) < \mu_K(x)$. Hence $\Sigma \subset \mathcal{N}(\mu_K)$. ■

This paper was partially written during a stay at the University of Calgary. The author thanks the members of the Department of Mathematics, in particular P. Lancaster and P. Binding for their hospitality and support.

References

[BW] Marc A. Berger and Yang Wang, *Bounded Semigroups of Matrices*, Linear Algebra Appl. **166** (1992), 21–27.

[DL] I. Daubechies and J.C. Lagarias, *Sets of Matrices All Infinite Products of Which Converge*, Linear Algebra Appl. **161** (1992), 227–263.

[EKN] L. Elsner, I. Koltracht and M. Neumann, *On the Convergence of Asynchronous Paracontractions with Applications to Tomographic Reconstruction from Incomplete Data*, Linear Algebra Appl. **130** (1990), 65–82.

[NN] S. Nelson and M. Neumann, *Generalization of the Projection Method with Applications to SOR Method for Hermitian Positive Semidefinite Linear Systems*, Numer. Math. **51** (1987), 123–141.

[RS] G.–C. Rota and W.G. Strang, *A Note on the Joint Spectral Radius*, Indagationes **22** (1960), 379–381.

14. September 1993