# Generalized one-point extensions

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## Introduction

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Suppose that there is an indecomposable projective left $A$-module $P$ satisfying $\dim_k \mathrm{End}_A(P) = 1$. We recall from [Dr2] that the full subcategory $K(A, P)$ of the category $A$-mod of all finitely generated left $A$-modules is just given by all modules $V \in A$-mod satisfying $\mathrm{Ext}^1_A(V, \mathrm{fac}\, P) = 0$. Of particular interest is the full subcategory $K_P(A, P)$ of $K(A, P)$ which consists only of those $V \in K(A, P)$ where $\mathrm{Hom}_A(P, X) \neq 0$ for each direct summand $X$ of $V$.

The importance of $K(A, P)$ comes from the fact that the fiber sum functor $F_P$ introduced in [Dr1] furnishes a full and dense functor from the subspace category $\check{\mathcal{U}}(K(A, P), \mathrm{Hom}_A(P, -))$ to the category $A$-mod. This shows immediately that $A$ is of finite representation type if and only if the vectorspace category $(K(A, P), \mathrm{Hom}_A(P, -))$ is of finite subspace type. Concerning tame representation type the situation was clarified in [Dr2]. Namely, if we write $P$ as $Ae$ for some primitive idempotent $e$, then the algebra $A$ is of tame representation type if and only if the factor algebra $A/AeA$ is of tame representation type, the vectorspace category $(K_P(A, P), \mathrm{Hom}_A(P, -))$ is of tame subspace type and for every $n \in \mathbb{N}$ there are up to isomorphism only finitely many indecomposable modules $X \in K_P(A, P)$ such that $\dim_k X \leq n$.

It is a well-known phenomenon in representation theory of algebras that finite or tame type is much easier to verify and to study for vectorspace categories rather than for algebras. Therefore one looks for reductions from module categories to subspace categories. [G-V], [Ri1] ,[Ri2] is a certainly very incomplete list of references where reductions of this kind were applied for tame type. The above characterization of tameness of an algebra $A$ via the vectorspace category $(K(A, P), \mathrm{Hom}_A(P, -))$ should give another efficient reduction tool for the presently very active reasearch on tame algebras, once this vectorspace category can actually be calculated. It is the major aim of this paper to investigate the structure of this vectorspace category for a big class of algebras $A$ and modules $P$.

To sketch our results, we recall from [Dr2] that the well-known one-point extension resp. coextension technique applied e.g. in [Ri1], [Ri2] can be considered as a special case of the fiber sum functor method if one chooses for $P$ a simple projective module. In this case the algebra $A$ has the form $A = \begin{pmatrix} B & 0 \\ R & k \end{pmatrix}$ where $R$ lies in the category mod-$B$ of finitely generated right $B$-modules. The point is that the vectorspace category $(K(A, P), \mathrm{Hom}_A(P, -))$ can almost be identified with $(B\text{-mod}, R \otimes_B -)$. More precisely one has to pass from $K(A, P)$ to the full subcategory $K(A, P) \setminus \mathrm{add}\, P$ of objects admitting no summand isomorphic to $P$ to arrive at isomorphic vectorspace categories. This is of course no essential loss of information.

As we want to deal with non-simple $P$, we need some more notations. Let us assume that $A$ is basic. Hence we can write $A$ as $k[\Delta]/I$ where $\Delta$ is a finite quiver and $I$ is an admissible ideal of the path algebra $k[\Delta]$ of the quiver $\Delta$ (see [Ga]). Denoting by $e(x)$ the primitive idempotents of $A$ induced by the vertices $x$ of $\Delta$, we obtain $P = Ae(s)$ for some $s \in \Delta_0$. We remark that we compose arrows from right to left.

It is convenient to introduce a transitive and reflexive relation $\leq_A$ on the vertex set $\Delta_0$ of $\Delta$ by defining $y \leq_A x$ provided there is a path in $\Delta$ from $x$ to $y$. Of course this relation will usually not be antisymmetric, but we require that the interval $[s,s]_A$ with respect to this relation consists only of the element $s$. Defining $e^s$ resp. $e_s$ as the sum of all $e(x)$ such that $x \not\geq_A s$ resp. $x \not\leq_A s$, we construct the algebras $A^s := e^s A e^s$ resp. $A_s := e_s A e_s$ which can be considered as factor algebras of $A/s := A/Ae(s)A$ in the natural way. We also need the idempotent $\bar{e}^s = e^s + e(s)$ and the corresponding algebra $\bar{A}^s = \bar{e}^s A \bar{e}^s$ which is not a factor of $A/s$ but still of $A$. Observe that the category $\bar{A}^s$-proj of projectives in $\bar{A}^s$-mod can be considered as a full subcategory of $A$-proj.

The vertex $s$ is called decomposing if every indecomposable module in $A/s$-mod lies either in $A^s$-mod or in $A_s$-mod. In this case the category $K(A,P)$ consists of the three parts $\bar{A}^s$-proj, $K^+$ and $K^-$ such that there are $R^+ \in A^s$-mod and $R^- \in$ mod-$A_s$ with the property that (up to some technicalities which we will discuss in section 2) the vectorspace categories $(K^+, (\mathrm{Hom}_A(P,-))$ resp. $(K^-, (\mathrm{Hom}_A(P,-))$ can be identified with $(A^s\text{-mod}, \mathrm{Hom}_{A^s}(R^+,-))$ resp. $(A_s\text{-mod}, R^- \otimes_{A_s} -)$. Hence we are in a situation similar pleasent as one-point coextension.

Actually if $P$ is simple projective or equivalently $s$ is a sink of $\Delta$, then $s$ is obviously decomposing. Moreover in this case $\bar{A}^s$-proj collapses to $\mathrm{add}\, P$ and $K^+$ is superfluous. This shows again that we are just generalizing the one-point coextension situation. As it turns out in section 3 that conversely a decomposition of $K(A,P)$ into three parts as above forces $s$ to be decomposing, we call algebras posessing a decomposing vertex $s$ also generalized one-point extensions by $s$.

Of course in applications the question arises how the parts $K^+$ and $K^-$ are related by homomorphisms. This problem is simplified by the observarion that one may path from the category $K_P(A,P)$ to the factor category $K_P(A,P)/I_P$ where $I_P$ denotes the kernel of the functor $\mathrm{Hom}_A(P,-)$ without changing the representation type. We show in section 2 that there are no non-zero maps from $K^-/I_P$ to $K^+/I_P$ and discuss the maps in the opposite direction in section 4 by investigating the quiver of $K_P(A,P)/I_P$.

In section 5 we will finish the construction of the quiver of $K_P(A,P)/I_P$ under certain finiteness and directedness conditions. In particular as an application of our access we are able to complete the description of hammocks of representation directed algebras presented in [Sc].

Section 6 is devoted to examples where our methods make it possible to understand the whole structure of $(K_P(A,P)/I_P, \mathrm{Hom}_A(P,-))$.

We finish the introduction by pointing out that until section 3 all our results are valid also for algebras over arbitrary fields or even artin algebras provided instead of requiring $\dim_k \mathrm{End}_A(P) = 1$ one demands that $\mathrm{End}_A(P)$ is a skew field $H$ and consequently replaces $k$ by $H$ at the appropriate places. In section 4 of course more care would have to be taken to replace quivers by species in the proper way and in section 5 we definitely need an algebraically closed base field. However, the formulations of our results are much shorter if we do not have to distinguish between $k$ and its finite skew field extensions. Moreover, presently we do not know applications if the base ring is not an algebraically closed field. Thus we refrained from starting out with the most general setup.

# 1 Preliminaries

1.1. Recall that we are considering a finite dimensonal algebra $A$ over an algebraically closed field $k$ which we write as $A = k[\Delta]/I$ where $\Delta$ is a finite quiver and $I$ is an admissible ideal of the path algebra $k[\Delta]$. The category of finitely generated left $A$-modules is denoted by $A$-mod. We compose morphisms in $A$-mod from left to right. Using the primitive idempotents $e(x)$ of $A$ induced by the vertices $x$ of $\Delta$, we define the indecomposable projective $A$-modules $P_x(A) := Ae(x)$ and the indecomposable injective $A$-modules $Q_x(A) := \mathrm{D}(e(x)A)$. The Auslander-Reiten translation $\tau_A = \mathrm{DTr}$ will be considered as a functor $A\text{-mod}/[A\text{-proj}] \to A\text{-mod}/[A\text{-inj}]$. It is an equivalence with inverse $\tau_A^- = \mathrm{TrD}$.

Note that for an arbitrary Krull-Schmidt $K$ category together with a full additive (i.e. closed under finite direct sums and direct summands) subcategory $L$ we denote by $[L]$ the ideal of $K$ consisting of the morphism factorizing through an object in $L$. To explain the above notations, we recall that the full subcategories $A$-proj resp. $A$-inj of projective resp. injective modules are additive subcategories of the Krull-Schmidt category $A$-mod.

If $L_1, L_2$ are full additive subcategories of a Krull-Schmidt category $K$, then $L_1 \setminus L_2$ is the full additive subcategory of $L_1$ consisting of the objects which admit no direct summand from $L_2$. The smallest full additive subcategory of $K$ containing $L_1$ and $L_2$ will be called $L_1 \vee L_2$.

For any occurring Krull-Schmidt category we choose one representative out of each isomorphism class of indecomposable objects and denote by $\mathrm{ind}\, K$ the induced full subcategory. In particular we abbreviate $\mathrm{ind}\,(A\text{-mod})$ to $A$-ind.

1.2. In the introduction we defined a reflexive and transitive relation $\leq_A$ on the vertex set $\Delta_0$ of $\Delta$ by setting $y \leq_A x$ if there is a path in $\Delta$ from $x$ to $y$. Using the idempotents $e(x)$ of $A$ induced by the points $x$ of the quiver $\Delta$, the relation $y \leq x$ can be characterized by the existence of a sequence $x_0, \ldots x_n$ in $\Delta_0$ such that $x_0 = y$, $x_n = x$ and $e(x_{i-1})Ae(x_i) \neq 0$ for all $i = 1, \ldots, n$. This does not make use of the representation of $A$ as $A = k[\Delta]/I$ but only refers to the decomposition $\sum_{x \in \Delta_0} e(x)$ of the unit element of $A$ into primitive idempotents.

For a subset $T$ of $\Delta_0$ we put $e(T) := \sum_{t \in T} e(t)$ and $A(T) := e(T)Ae(T)$. Clearly $A(T)$ is again a $k$-algebra and $\sum_{t \in T} e(t)$ is a decomposition of the unit elements $e(T)$ of $A(T)$ into primitive idempotents. For all $x, y \in T$ obviously $y \leq_A x$ follows from $y \leq_{A(T)} x$.

The canonical restriction functor $Res^A_{A(T)} : A\text{-mod} \to A(T)\text{-mod}$, $M \mapsto e(T)M$ has a left adjoint $L^A_{A(T)} := Ae(T) \otimes_{A(T)} -$ and a right adjoint $R^A_{A(T)} := \mathrm{Hom}_{A(T)}(e(T)A, -)$. Since $Res^A_{A(T)} L^A_{A(T)} \cong \mathrm{Id}_{A(T)\text{-mod}} \cong Res^A_{A(T)} R^A_{A(T)}$, the functors $L^A_{A(T)}$ and $R^A_{A(T)}$ are fully faithful. Moreover there exists a natural transformation $\varphi^A_{A(T)} : L^A_{A(T)} \to R^A_{A(T)}$ given by $(\varphi^A_{A(T)})_X : L^A_{A(T)}X \to R^A_{A(T)}X$, $ae(T) \otimes x \mapsto (e(T)a' \mapsto e(T)a'ae(T)x)$ for all $X \in A(T)$-mod.

The following fromulas from [Sc] show how the Auslander-Reiten translations commute with the functors $L^A_{A(T)}$ and $R^A_{A(T)}$.

**Lemma.** *If $T \subseteq \Delta_0$, then $\tau_A^- R^A_{A(T)}X \cong L^A_{A(T)}\tau^-_{A(T)}X : A(T)$-mod/$[A(T)$-inj$] \to$ $A$-mod/$[A$-proj$]$ and $\tau_A L^A_{A(T)}X \cong R^A_{A(T)}\tau_{A(T)} : A(T) - mod/[A(T)$-proj$] \to$ $A$-mod/$[A$-inj$]$.*

1.3. For $U \subseteq \Delta_0$ we put $A/U := A/Ae(U)A$ and observe that $\sum_{t\in T}(e(t) + Ae(U)A)$ is a decomposition of the unit element of the factor algebra $A/U$ into primitive idempotents where we put $T := \Delta_0 \setminus U$. For all $x, y \in T$ clearly $y \leq_A x$ follows from $y \leq_{A/U} x$.

As $A/U$ is a factor algebra, we may consider $A/U$-mod as full subcategory of $A$-mod. The inclusion functor has the left adjoint $l^A_{A/U} := A/U \otimes_A -$ which is fully faithful. By $\rho^A_{A/U}$ we denote the corresponding natural epimorphism from the identity on $A$-mod to $l^A_{A/U}$.

We observe that the map $A(T) \to A/U$, $a \mapsto a + Ae(U)A$ is a surjective $k$-algebra homomorphism whose kernel is just $e(T)Ae(U)Ae(T)$. The subset $T \subseteq \Delta_0$ is called convex if it is convex with respect to the relation $\leq_A$. For convex $T$ we obtain $e(T)Ae(U)Ae(T) = 0$. Consequently we may identify $A(T)$ with $A/U$. Moreover $y \leq_{A(T)} x$ is equivalent zu $y \leq_A x$ for all $x, y \in T$.

Using the identification $A(T) = A/U$ for convex $T$, the functors $L^A_{A(T)}, R^A_{A(T)}$ and and the canonical embedding of $A/U$-mod into $A$-mod are related in the following way. On one hand we have a natural epimorphism $\pi^A_{A(T)}$ given by $(\pi^A_{A(T)})_X : L^A_{A(T)}X \to X$, $ae(T) \otimes x \mapsto ae(T)x$, and on the other hand a natural monomorphism $\iota^A_{A(T)}$ given by $(\iota^A_{A(T)})_X : X \to R^A_{A(T)}X$, $x \mapsto (e(T)a \mapsto e(T)ax)$. Obviously $\pi^A_{A(T)}\iota^A_{A(T)} = \varphi^A_{A(T)}$.

1.4. A subset $T \subseteq \Delta_0$ is called an ideal resp. coideal, provided $x \leq_A t$ (resp. $t \leq_A x$) implies $x \in T$ for all $t \in T$ und $x \in \Delta_0$. Ideals and coideals are convex. For an ideal $T$ the map $\pi^A_{A(T)}$ is an isomorphism, for a coideal $T$ the map $\iota^A_{A(T)}$ is an isomorphism. Using this together with the above lemma, we obtain the following assertion which can also be found in [Sc]. Note that the occurring functors are defined on factor categories according to lemma 1.2.

**Lemma.** *Suppose $T \subseteq \Delta_0$.*

*a) If $T$ is an ideal, then $\tau_A^- R^A_{A(T)}X \cong \tau^-_{A(T)}X$ and $\tau_A X \cong R^A_{A(T)}\tau_{A(T)}X$.*

*b) If $T$ is a coideal, then $\tau_A^- X \cong L^A_{A(T)}\tau^-_{A(T)}X$ and $\tau_A L^A_{A(T)}X \cong \tau_{A(T)}X$.*

1.5. Frequently we will consider intervals with respect to the relation $\leq_A$. We will also use intervals where the upper or lower bound is $\infty$. To give an example, for $x \in \Delta_0$ the interval $[x, \infty[_A$ denotes the set of all $y \in \Delta_0$ with $x \leq_A y$. Fixing now $s \in \Delta_0$, the set $\Delta_0 \setminus [s,\infty[_A$ is an ideal and the set $\Delta_0 \setminus ]\infty, s]_A$ is a coideal of $\Delta_0$ with respect to $\leq_A$. Consequently the set $\Delta_0 \setminus ([s,\infty_A[\cup]\infty,s]_A)$ is a convex subset of $\Delta_0$. Some of the following notations were already used in the introduction.

$$A^s := A(\Delta_0 \setminus [s,\infty[_A),\ e^s := e(\Delta_0 \setminus [s,\infty[_A)$$

$$A_s := A(\Delta_0 \setminus ]\infty,s]_A),\ e_s := e(\Delta_0 \setminus ]\infty,s]_A)$$

$$A_s^s := A(\Delta_0 \setminus ([s,\infty[_A \cup ]\infty,s]_A)),\ e_s^s := e(\Delta_0 \setminus ([s,\infty[_A \cup ]\infty,s]_A))$$

As $e(x)Ae(s)Ae(y) = 0$ holds for all $x, y \in \Delta_0 \setminus [s,\infty[$, we may identify $A^s$ with $(A/s)(\Delta_0 \setminus [s,\infty[_A)$. We make analogous identifications for $A_s$ and $A_s^s$.

Let us also consider the algebra $\overline{A}^s := A(\Delta_0 \setminus ]s,\infty[_A)$ induced by the idempotent $\bar{e}^s := e(\Delta_0 \setminus ]s,\infty[_A)$. The set $\Delta_0 \setminus ]s,\infty[_A$ is an ideal if and only if $[s,s]_A = \{s\}$. In this case $\operatorname{fac} P_s(A) \subseteq \overline{A}^s$-mod. Moreover, we observe $\overline{A}^s$-proj $\subseteq A$-proj and $l^A_{A/s} \cong l^{\overline{A}^s}_{A^s}$ : $\overline{A}^s$-mod $\to A^s$-mod.

# 2 Some parts of $K(A,P)$

2.1. As in the introduction we consider an indecomposable projective module $P \in A$-mod satisfying $\dim_k \mathrm{End}_A(P) = 1$. Clearly $P = P_s(A) = Ae(s)$ for some $s \in \Delta_0$. By [AS] the objects of the subcategory $K(A,P)$ can be characterized as those $V \in A$-mod such that $\mathrm{Hom}_A(P, \tau_A V) = 0$. Consequently $\tau_A^- : A/s\text{-mod}/[A\text{-inj}] \to K(A,P)/[A\text{-proj}]$ is an equivalence.

For every additive full subcategory $L$ of $K(A,P)$ we denote by $L_P$ the full subcategory of $L$ given by the objects $V$ which do not admit a direct summand $V'$ such that $\mathrm{Hom}_A(P,V') = 0$. In particular for $L = K(A,P)$ we obtain $K_P(A,P)$ as used in the introduction.

Let us recall some notations about vectorspace categories and their subspace categories. For abbreviation in the sequel we will write $K$ instead of $K(A,P)$ and $\check{\mathcal{U}}(K)$ instead of $\check{\mathcal{U}}(K, \mathrm{Hom}_A(P,-))$ if no confusion is possible. The subspace category $\check{\mathcal{U}}(K)$ has as objects the triples $X = (X_0, X_\omega, \gamma_X)$ where $X_0 \in K$, $X_\omega$ is a finite dimensional $k$-vectorspace and $\gamma_X \in \mathrm{Hom}_k(X_\omega, \mathrm{Hom}_A(P,X_0))$. For objects $X, X'$ the morphism space from $X$ to $X'$ consists of all pairs $(f_0, f_\omega)$ where $f_0 \in \mathrm{Hom}_A(X_0, X_0')$, $f_\omega : X_\omega \to X_\omega'$ is $k$-linear and $\gamma_{X'} f_\omega = \mathrm{Hom}_A(P, f_0)\gamma_X$. The composition is obvious.

As canonically $\mathrm{Hom}_k(X_\omega, \mathrm{Hom}_A(P,X_0)) \cong \mathrm{Hom}_A(P \otimes_k X_\omega, X_0)$, we will identify $\check{\mathcal{U}}(K)$ with the category $\mathrm{mat}(\operatorname{add} P, K)$ which has as objects all maps $\delta : P^m \to X_0$ where $P^m \in \operatorname{add} P$ and $X_0 \in K$. Morphisms from $\delta$ to $\delta'$ in $\mathrm{mat}(\operatorname{add} P, K)$ are just pairs $(h_\omega, h_0)$ where $h_\omega \in \mathrm{Hom}_A(P^m, P^{m'})$, $h_0 \in \mathrm{Hom}_A(X_0, X_0')$ and $h_\omega \delta' = \delta h_0$. The composition is again obvious.

Using the identification of $\check{\mathcal{U}}(K)$ and $\mathrm{mat}(\operatorname{add} P, K)$, the fiber sum functor $F_P$ maps an object of this category just to its cokernel which is a module in $A$-mod. It was proved in [Dr2] that $F_P$ is a full and dense functor. To describe the kernel of $F_P$, we recall that there are two inclusions of $K$ into $\check{\mathcal{U}}(K)$. The first one maps an object $V$ to the triple

$(V,0,0)$. The second one maps $V$ to $\overline{V} := (V, \mathrm{Hom}_A(P,V), \mathrm{id})$. Denoting by $K_0$ the full subcategory of objects $V$ of $K$ which lie also in fac $P$, it is shown in [Dr2] that the kernel of $F_P$ is just the ideal $[\mathrm{add}\,(0,k,0) \vee \overline{K_0}]$.

We derive that the composed functor $K \to A\text{-mod}\ V \mapsto F_P\overline{V}$ is full and has kernel $[K_0]$. On the other hand $F_P\overline{V} \cong l^A_{A/s}V$ for alle $V \in K$. This shows that $l^A_{A/s}$ induces a full and dense functor $K \to A/s\text{-mod}$ with kernel $[K_0]$.

2.2. **Proposition.** *For $K^-(A,P) := L^A_{A_s}(A_s\text{-mod})$ and $R^- := e(s)Ae_s$ the following assertions hold:*

*a)* $L^A_{A_s} : A_s\text{-mod} \to K^-(A,P)$ *is an equivalence.*

*b)* $\mathrm{Hom}_A(P,-)L^A_{A_s} \cong R^- \otimes_{A_s} - : A_s\text{-mod} \to k\text{-mod}$.

*c)* $K^-(A,P) \subseteq K(A,P)$.

*d)* $\tau_A L^A_{A_s} \cong \tau_{A_s} : A_s\text{-mod}/[A_s\text{-proj}] \to A/s\text{-mod}/[A\text{-inj}]$.

*e)* $l^A_{A/s}L^A_{A_s} \cong L^{A/s}_{A_s} : A_s\text{-mod} \to A/s\text{-mod}$.

*Proof:* $a)$ und $b)$ are obvious. For $c)$ and $d)$ we choose $X \in A_s\text{-ind}$ and observe that $\tau_A L^A_{A_s}X \cong \tau_{A_s}X$ holds by 1.4. To prove $e)$, we look at the exact sequence

$$0 \to Ae(s)Ae_s \longrightarrow Ae_s \longrightarrow (A/s)e_s \to 0$$

of $A$-$A_s$-bimodules induced by the canonical algebra homomorphism $A \to A/s$. This sequence in turn gives rise to the sequence

$$(Ae(s)Ae_s \otimes_{A_s} -) \longrightarrow L^A_{A_s} \longrightarrow ((A/s)e_s \otimes_{A_s} -) \to 0$$

of functors $A_s\text{-mod} \to A\text{-mod}$. Application of $l^A_{A/s}$ to this sequence yields an isomorphism $l^A_{A/s}L^A_{A_s} \cong L^{A/s}_{A_s}$ if we observe $A/s \otimes_A Ae(s)Ae_s = 0$ and $A/s \otimes_A (A/s)e_s \cong (A/s)e_s$.

Defining $R^-_t := e(t)Ae_s$ for all $t \in ]\infty, s]_A$, we obtain a decomposition of the right $A_s$-module $(1-e_s)Ae_s$ as direct sum $\oplus_{t\in]\infty,s]}R^-_t$. Note that $R^- = R^-_s$. For all $t \in ]\infty, s]_A$ there exists a well-known natural isomorphism $e(t)L^A_{A_s} = R^-_t \otimes_{A_s} - \cong \mathrm{D}^2(R^-_t \otimes_{A_s} -) \cong \mathrm{D}\mathrm{Hom}_{A_s}(R^-_t, \mathrm{D}-)$.

2.3. From now on we suppose $[s,s]_A = \{s\}$.

**Lemma.** *$K(\overline{A}^s, P)$ is the full subcategory of all $V \in K(A,P)$ satisfying $l^A_{A/s}V \in A^s\text{-mod}$. In particular $K_0(A,P) \subseteq K(\overline{A}^s, P)$.*

*Proof:* First we consider an indecomposable module $X \in K(\overline{A}^s, P)$. 1.4 yields $e(s)\tau_A X \cong e(s)R^A_{\overline{A}^s}\tau_{\overline{A}^s}X \cong e(s)\tau_{\overline{A}^s}X = 0$ and therefore $X \in K(A,P)$. Evidently $l^A_{A/s}X \in A^s\text{-mod}$.

Conversely, let us choose $V \in K(A,P)$ indecomposable with $l^A_{A/s} V \in A^s$-mod. Using $l^A_{A/s}V \cong F_P\overline{V}$, we obtain a short exact sequence $0 \to U \longrightarrow V \longrightarrow F_P\overline{V} \to 0$ with $U \in \operatorname{fac} P \subseteq \overline{A}^s$-mod. Hence $V$ has to lie in $\overline{A}^s$-mod. Applying 1.4, we get $e(s)\tau_{\overline{A}^s}V \cong e(s)R^A_{\overline{A}^s}\tau_{\overline{A}^s}V \cong e(s)\tau_A V = 0$ and thus $V \in K(\overline{A}^s, P)$.

2.4. Still supposing $[s,s]_A = \{s\}$, we want to get more precise informations about $K(\overline{A}^s, P)$. We introduce the left $A^s$-module $R^+_t := e^s Ae(t)$ for all $t \in [s, \infty[_A$ and in particular $R^+ := R^+_s$. Recalling furthermore the notation $I_P$ for the ideal $\operatorname{Ker}\operatorname{Hom}_A(P,-)$ of $A$-mod, we make the following observations:
*a*) The homomorphisms in $A^s$-mod lying in $[\overline{A}^s$-inj$]$ form an ideal of the category $A^s$-mod which is contained in $[A^s$-inj$]$ and in $\operatorname{Ker}\operatorname{Hom}_{A^s}(R^+,-)$.
*b*) Suppose $X \in A^s$-ind. Then $X \in \overline{A}^s$-inj (i.e. $\operatorname{End}_{A^s}(X,X) \subseteq [\overline{A}^s$-inj$]$) is equivalent to $X \in A^s$-inj and $\operatorname{Hom}_{A^s}(R^+,X) = 0$.
c) $[\overline{A}^s$-proj$](M,N) \subseteq I_P(M,N)$ for all $M,N \in \overline{A}^s$-ind such that $M \not\cong P$.

We are now in the position to reduce the description of $K(\overline{A}^s, P)$ to $A^s$-mod.

**Proposition.** *Suppose* $[s,s]_A = \{s\}$. *For* $K^+(A,P) := \tau^-_{\overline{A}^s}(A^s\text{-mod})$ *and* $R^+ := e^sAe(s)$ *the following assertions hold:*

*a)* $\tau^-_{\overline{A}^s} : A^s\text{-mod}/[\overline{A}^s\text{-inj}] \to K^+(A,P)/[\overline{A}^s\text{-proj}]$ *is an equivalence.*

*b)* $\operatorname{Hom}_A(P,-)\tau^-_{\overline{A}^s} \cong \operatorname{Hom}_{A^s}(R^+,-) : A^s\text{-mod}/[\overline{A}^s\text{-inj}] \to k\text{-mod}$.

*c)* $K(\overline{A}^s, P) = K^+(A,P) \vee \overline{A}^s$-proj.

*d)* $\tau_A\tau^-_{\overline{A}^s} \cong R^A_{\overline{A}^s} : A^s\text{-mod}/[\overline{A}^s\text{-inj}] \to A/s\text{-mod}/[A\text{-inj}]$.

*e)* $l^{\overline{A}^s}_{A^s}\tau^-_{\overline{A}^s} \cong \tau^-_{A^s}\pi : A^s\text{-mod}/[\overline{A}^s\text{-inj}] \to A^s\text{-mod}/[A^s\text{-proj}]$ *where* $\pi : A^s\text{-mod}/[\overline{A}^s\text{-inj}] \to A^s\text{-mod}/[A^s\text{-inj}]$ *is the canonical projection functor.*

*Proof:* *a*) is obvious. For *b*) we start by showing the isomorphism $\operatorname{Ext}^1_{\overline{A}^s}(Q_s(\overline{A}^s),-) \cong \operatorname{Hom}_{A^s}(R^+,-) : A^s\text{-mod} \to k\text{-mod}$. Namely the exact sequence $0 \to R^+ \longrightarrow P \longrightarrow Q_s(\overline{A}^s) \to 0$ yields the following exact sequence of functors whose start and end term are 0.

$$\operatorname{Hom}_{\overline{A}^s}(P,-) \to \operatorname{Hom}_{A^s}(R^+,-) \to \operatorname{Ext}^1_{\overline{A}^s}(Q_s(\overline{A}^s),-) \to \operatorname{Ext}^1_{\overline{A}^s}(P,-)$$

Now we recall the formulas $\operatorname{Ext}^1_{\overline{A}^s}(Q_s(\overline{A}^s),-) \cong \mathrm{D}\underline{\operatorname{Hom}}_{\overline{A}^s}(\tau^-_{\overline{A}^s}-, Q_s(\overline{A}^s)) : \overline{A}^s\text{-mod}/[\overline{A}^s\text{-inj}] \to k\text{-mod}$ and $\operatorname{Hom}_{\overline{A}^s}(P,-) \cong \mathrm{D}\operatorname{Hom}_{\overline{A}^s}(-, Q_s(\overline{A}^s)) : \overline{A}^s\text{-mod} \to k\text{-mod}$. The composition of all these natural isomorphisms gives our claim.

To prove *c*) we first consider $X \in A^s$-ind with $X \notin \overline{A}^s$-inj and calculate $e(s)\tau_{\overline{A}^s}\tau^-_{\overline{A}^s}X \cong e(s)X = 0$. This shows $K^+(A,P) \subseteq K(\overline{A}^s,P)$. If conversely $V$ is an indecomposable module in $K(\overline{A}^s,P)$ which is not projective, then we know $V \cong \tau^-_{\overline{A}^s}\tau_{\overline{A}^s}V$. On the other hand $e(s)\tau_{\overline{A}^s}V = 0$ shows that $\tau_{\overline{A}^s}V$ is an indecomposable $A^s$-module which is not $\overline{A}^s$-injective.

From 1.4 follows $\tau_A \tau^-_{\overline{A}^s} \cong R^A_{\overline{A}^s} \tau_{\overline{A}^s} \tau^-_{\overline{A}^s} \cong R^A_{\overline{A}^s}$ which implies $d$).

For $e$) at first we prove $l^{\overline{A}^s}_{A^s} \mathrm{Tr}_{\overline{A}^s} \cong \mathrm{Tr}_{A^s} \tilde{\pi}$ : mod-$A^s$/D([$\overline{A}^s$-inj]) $\to$ $A^s$-mod/[$A^s$-proj] using the notation $\tilde{\pi} := \mathrm{D}\pi\mathrm{D}$ : mod-$A^s$/D([$\overline{A}^s$-inj]) $\to$ mod-$A^s$/[proj-$A^s$]. Since $A^s = \overline{A}^s/s$, there is an exact sequence $0 \to \overline{A}^s e(s) \overline{A}^s \longrightarrow \overline{A}^s \longrightarrow A^s \to 0$ which observing $\overline{A}^s e(s) \overline{A}^s e^s = 0$ and $A^s = A^s e^s$ leads to an isomorphism $- \otimes_{\overline{A}^s} A^s \cong Res^{\overline{A}^s}_{A^s}$ : $\overline{A}^s$-mod $\to$ $A^s$-mod. If for given $X \in$ ind-$A^s$ we choose a minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow X \to 0$ of $X$ in mod-$\overline{A}^s$, then $P_1 \otimes_{\overline{A}^s} A^s \longrightarrow P_0 \otimes_{\overline{A}^s} A^s \longrightarrow X \otimes_{\overline{A}^s} A^s \to 0$ is a minimal projective presentation of $X \otimes_{\overline{A}^s} A^s \cong X$ in mod-$A^s$. In the exact sequences $l^{\overline{A}^s}_{A^s} \mathrm{Hom}_{\overline{A}^s}(P_0, \overline{A}^s) \to l^{\overline{A}^s}_{A^s} \mathrm{Hom}_{\overline{A}^s}(P_1, \overline{A}^s) \to l^{\overline{A}^s}_{A^s} \mathrm{Tr}_{\overline{A}^s} X \to 0$ and $\mathrm{Hom}_{A^s}(P_0 \otimes_{\overline{A}^s} A^s, A^s) \to \mathrm{Hom}_{A^s}(P_1 \otimes_{\overline{A}^s} A^s, A^s) \to \mathrm{Tr}_{A^s} X \to 0$ the first two terms are related by the natural isomorphisms $\mathrm{Hom}_{A^s}(P_i \otimes_{\overline{A}^s} A^s, A^s) \cong \mathrm{Hom}_{\overline{A}^s}(P_i, \mathrm{Hom}_{A^s}(A^s, A^s)) \cong \mathrm{Hom}_{\overline{A}^s}(P_i, A^s) \cong A^s \otimes_{\overline{A}^s} \mathrm{Hom}_{\overline{A}^s}(P_i, \overline{A}^s) = l^{\overline{A}^s}_{A^s} \mathrm{Hom}_{\overline{A}^s}(P_i, \overline{A}^s)$. Consequently we obtain a natural isomorphism $l^{\overline{A}^s}_{A^s} \mathrm{Tr}_{\overline{A}^s} X \to \mathrm{Tr}_{A^s} X$ for the cokernels. We derive easily $l^{\overline{A}^s}_{A^s} \tau^-_{\overline{A}^s} = l^{\overline{A}^s}_{A^s} \mathrm{Tr}_{\overline{A}^s} \mathrm{D} \cong \mathrm{Tr}_{A^s} \tilde{\pi} \mathrm{D} = \mathrm{Tr}_{A^s} \mathrm{D} \pi = \tau^-_{A^s} \pi$.

2.5. We define the set $\check{s} := \{t \in \Delta_0 : e(t)Ae(s) \neq 0, t \neq s\}$ and observe that $\check{s} = \{t \in \Delta_0 \setminus [s, \infty[_A : \mathrm{Hom}_{A^s}(R^+, Q_t(A^s)) \neq 0\}$. For each $t \in \check{s}$ we choose a $k$-basis $a_1, \ldots, a_r$ of $e(t)Ae(s)$ and denote by $C(t)$ the cokernel of the induced map $(a_1, \ldots, a_r) : P_t(A) \to P_s(A)^r$. It is easy to calculate that $\tau^-_{\overline{A}^s} Q_t(A^s) = C(t)$. Thus the above proposition immediately yields:

**Corollary.** *If* $[s, s]_A = \{s\}$, *then* $\mathrm{ind}\,(K_0(A, P) \setminus \mathrm{add}\, P) = \{C(t) : t \in \check{s}\}$. *In particular the set* $\mathrm{ind}\, K^0(A, P)$ *is finite.*

2.6. **Lemma.** *If* $[s, s]_A = \{s\}$ *then* $\mathrm{Hom}_A(K^-(A, P), K(\overline{A}^s, P)) \subseteq I_P$.

*Proof:* Let $X \in K(\overline{A}^s, P)$, $Y \in A_s$-mod und $f \in \mathrm{Hom}_A(L^A_{A_s} Y, X)$. The equality $e(s) L^A_{A_s} Y = e(s) A e_s \otimes_{A_s} Y = e(s)(\mathrm{Rad}\, A) A e_s \otimes_{A_s} Y = e(s)(\mathrm{Rad}\, A) L^A_{A_s} Y$ shows $f(e(s) L^A_{A_s} Y) \subseteq e(s)(\mathrm{Rad}\, A) X = e(s)(\mathrm{Rad}\, A) e(\Delta_0 \backslash ]s, \infty[_A) X = 0$.

# 3 Decomposing points

3.1. As before we suppose $P = P_s(A) = Ae(s)$ for some $s \in \Delta_0$ and use the abbreviation $K = K(A, P)$. The vertex $s$ is called decomposing provided $[s, s]_A = \{s\}$ and $A/s$-mod $=$ $A^s$-mod $\vee$ $A_s$-mod. In this case for each $Y \in A_s$-mod $\setminus$ $A^s_s$-mod the natural epimorphism $(\pi^{A/s}_{A_s})_Y : L^{A/s}_{A_s} Y \to Y$ (see 1.3) is an isomorphism. Analogously for all $X \in A^s$-mod $\setminus$ $A^s_s$-mod the natural monomorphism $(\iota^A_{\overline{A}^s})_X : X \to R^A_{\overline{A}^s} X$ is an isomorphism.

Another observation will be helpful. By $R^- e^s_s = 0$ we obtain $Y \notin A^s_s$-mod for all $Y \in A_s$-ind satisfying $R^- \otimes_{A_s} Y \neq 0$. Dually because of $e^s_s R^+ = 0$ always $X \notin A^s_s$-mod for all $X \in A^s$-ind such that $\mathrm{Hom}_{A^s}(R^+, X) \neq 0$.

**Theorem.** *Suppose* $[s, s]_A = \{s\}$. *The point* $s$ *is decomposing if and only if* $K =$

$\overline{A}^s$-proj $\vee K^+ \vee K^-$.

*Proof:* For one implication by 2.3 we only have to show that any indecomposable object $V$ in $K \setminus K(\overline{A}^s, P)$ lies in $K^-$. From 2.3 we know $l^A_{A/s}V \notin A^s$-mod. Since $A^s_s$-mod $\subseteq A^s$-mod, we derive $l^A_{A/s}V \in \operatorname{ind}(A_s\text{-mod} \setminus A^s_s\text{-mod})$. Using 2.2.$d$ and the above remarks, we get $l^A_{A/s}L^A_{A_s}l^A_{A/s}V \cong L^{A/s}_{A_s}l^A_{A/s}V \cong l^A_{A/s}V$. As neither $V$ nor $L^A_{A_s}l^A_{A/s}V$ belong to $K_0(A,s)$, by 2.1 we obtain $V \cong L^A_{A_s}l^A_{A/s}V$.

For the converse implication we start out with $N \in A/s$-ind and by 2.1 find $V \in \operatorname{ind} K$ with $l^A_{A/s}V \cong N$. If $N$ does not lie in $A^s$-mod, then by 2.3 the module $V$ cannot belong to $\overline{A}^s$-proj $\vee K^+$. Hence $V = L^A_{A_s}X$ for some $X \in A_s$-ind and therefore $N \cong l^A_{A/s}L^A_{A_s}X \cong L^{A/s}_{A_s}X$. The indecomposable module $X$ cannot lie in $A^s$-mod since otherwise $N$ would also belong to this subcategory. By the above remarks $(\pi^{A/s}_{A_s})_X$ has to be an isomorphism and thus $N \cong X$.

3.2. We want to provide some examples and always suppose $[s,s]_A = \{s\}$.

$a$) Let $C_s$ resp. $C^s$ be the union of all connected components $C$ of the quiver of $A/s$ such that $C \cap ]\infty, s[_A \neq \emptyset$ resp. $C \cap ]s, \infty[_A \neq \emptyset$. If $C_s \cap C^s = \emptyset$, then $s$ is obviously decomposing. Particular cases of this are the following:

$i$) If $\Delta$ is a tree, then every $s \in \Delta_0$ is decomposing.

$ii$) If $s$ is a sink or source of $\Delta$, then $s$ is decomposing. Let us briefly comment the case that $s$ is a sink i.e. $A$ is a one-point coextension with coextension vertex $s$ and consequently $P$ is a simple projective module. One checks immediately that $K = \overline{A}^s\text{-proj} \vee K^- = \operatorname{add} P \vee K^-$. Hence we reproved that we are just generalizing the one-point coextension technique. From our point of view the essential feature of the one-point coextension situation is that the computation of $K$ can be reduced to a module category over a smaller algebra. Our above theorem shows that decomposing points share this property whereas for general $s$ it is not clear how to control $K$. Therefore, if $s$ is a decomposing point, then we also denote $A$ as generalized one-point extension by $s$.

$b$) If $A$ is representation directed, then every $s \in \Delta_0$ is decomposing. Namely, if there would exist an indecomposable $A/s$-module $X \notin A^s\text{-mod} \vee A_s\text{-mod}$, then the support of this module would be a non-convex subset of $\Delta_0$ contradicting [Bo, 2.3].

3.3. By the preceding results for a decomposing point $s$ we are able to construct the objects of $K$ but we are still far from controlling the morphisms. On the other hand our original aim was to find out the representation type of the subspace category $\check{\mathcal{U}}(K_P, \operatorname{Hom}_A(P,-))$ which is obviously representation equivalent to $\check{\mathcal{U}}(K_P/I_P, \operatorname{Hom}_A(P,-))$. For the factor category $K_P/I_P$ lemma 2.6 provides some informations which we now want to prepare for further use.

Obviously $\operatorname{Hom}_A(V,P)/I_P = 0$ holds for each $V \in \operatorname{ind} K$ such that $V \ncong P$. As $(\overline{A}^s\text{-proj})_P = \operatorname{add} P$ by 3.1 we obtain:

**Proposition.** *If $s$ is decomposing, then $K_P = \operatorname{add} P \vee K^+_P \vee K^-_P$. This union is disjoint in the sense that any two of these categories do not have common indecomposable* *objects. Moreover* $\mathrm{Hom}_A(K_P^-, \mathrm{add}\, P \vee K_P^+)/I_P = 0$ *and* $\mathrm{Hom}_A(K_P^+ \vee K_P^-, \mathrm{add}\, P)/I_P = 0$.

3.4. We finish this section by a lemma which will be helpful in the next section.

**Lemma.** *Suppose $s$ is decomposing.*
*a) If $Q \in A^s$-inj is indecomposable such that $Q \notin A^s_s$-mod, then $Q \in A/s$-inj. If $\Sigma = 0 \to X \longrightarrow Y \longrightarrow Z \to 0$ is an Auslander-Reiten sequence in $A^s$-mod such that $X \notin A^s_s$-mod, then $\Sigma$ is an Auslander-Reiten sequence in $A/s$-mod.*
*b) If $P \in A_s$-proj is indecomposable such that $P \notin A^s_s$-mod, then $P \in A/s$-proj. If $\Sigma = 0 \to X \longrightarrow Y \longrightarrow Z \to 0$ is an Auslander-Reiten sequence in $A_s$-mod such that $Z \notin A^s_s$-mod, then $\Sigma$ is an Auslander-Reiten sequence in $A/s$-mod.*

*Proof:* By duality we only prove $a$). Only the second part is not immediately clear. By 1.4 and 3.1 we obtain $\tau_{A/s}Z \cong \tau_{A/s}\tau^-_{A^s}X \cong \tau_{A/s}\tau^-_{A/s}R^{A/s}_{A^s}X \cong R^{A/s}_{A^s}X \cong X$. Hence the Auslander-Reiten sequence in $A/s$-mod with end term $Z$ is actually a sequence in $A^s$-mod.

# 4 The quiver of $K'_P(A,P)/I_P$

4.1. A kind of skeleton for the morphism spaces in a Krull-Schmidt category $K$ is given by the so-called quiver $\Gamma_K$ of $K$. Recall that a quiver $\Gamma$ is given by a set of vertices $\Gamma_0$ and a set of arrows $\Gamma_1$ such that each arrow has a unique start and a unique end vertex. The quiver $\Gamma_K$ has as vertices the objects of $\mathrm{ind}\, K$. For all $X, Y \in \mathrm{ind}\, K$ there are exactly $d := \dim_k \mathrm{Irr}_K(X,Y)$ arrows $\alpha_1, \ldots, \alpha_d$ from $X$ nach $Y$ where $\mathrm{Irr}_K(X,Y) := \mathrm{rad}_K(X,Y)/\mathrm{rad}^2_K(X,Y)$. Note that the radical $\mathrm{rad}_K(X,Y)$ of $K$ for indecomposable objects $X, Y$ is just the space of all non-isomorphisms. $\Gamma_K$ is only well-defined up to an isomorphism which is the identity on $\mathrm{ind}\, K$.

A morphism $f : X \to Y$ in $K$ is called left resp. right almost split if $f$ is not a section resp. retraction and for every non-section $f' : X \to Y'$ resp. non-retraction $f' : X' \to Y$ there exists $h : Y \to Y'$ with $f' = fh$ resp. there exists $h : Y' \to Y$ with $f' = hf$. The domain of a left almost split map and the range of a right almost split map is always indecomposable. $f$ is said to be left resp. right minimal if every endomorphism $g$ of $X$ resp. $Y$ satisfying $fg = g$ resp. $gf = f$ is an automorphism. Note that we are composing morphisms in $K$ from left to right. Morphisms which are left minimal and left almost split resp. right minimal and right almost split are sometimes called source resp. sink maps (see [Ri2]).

Source and sink maps need not exist but if they do one can use them to describe the quiver $\Gamma_K$. Namely for $X, Y \in \mathrm{ind}\, K$ the number $\dim_k \mathrm{Irr}_K(X,Y)$ is exactly the multiplicity of $Y$ as direct summand of the range of the source map starting in $X$ resp. the multiplicity of $X$ as direct summand of the domain of the sink map ending in $Y$.

For constructing a source map starting in an indecomposable object $X$ of $K$ it sufffices to find a left almost split map $f : X \to Y$. Namely, if we write $Y = \oplus_{i=1}^n Y_i^{r_i}$ where the $Y_i$ are indecomposable and mutually non-isomorphic, then $\mathrm{rad}_K(X,Y_i)/f\mathrm{Rad}\,K(Y,Y)K(Y,Y_i)$ is a finite dimensional vectorspace over $k \cong K(Y_i,Y_i)/\mathrm{Rad}\,K(Y_i,Y_i)$ for all $i = 1,\dots,n$. More precisely, the corresponding components $f_i^{(1)},\dots,f_i^{(r_i)}$ of $f$ build generating sets of these spaces. Choosing a $k$-bases in all these generating sets yields a source map $f' : X \to Y'$ such that there is a retraction $r : Y \to Y'$ satisfying $f' = fr$. Note, moreover, that $f\mathrm{Rad}\,K(Y,Y)K(Y,Y_i) = \mathrm{rad}_K^2(X,Y_i)$ for all $i = 1,\dots,n$. Clearly all these considerations are valid for sink maps up to duality.

4.2. We need some basic results about the relation of a Krull-Schmidt category $K$ and its factor category by some ideal $I$. Let us denote by $\pi : K \to K/I$ the canonical projection functor. Furthermore, we put $K_I := \mathrm{add}\,\{X \in \mathrm{ind}\,K : \mathrm{id}_X \notin I(X,X)\}$.

Sink and source maps in $K$ are transformed to sink and source maps in $K/I$ by $\pi$. For all $X,Y \in \mathrm{ind}\,K_I$ the functor $\pi$ induces a $k$-epimorphism $\mathrm{Irr}_K(X,Y) \to \mathrm{Irr}_{K/I}(X,Y)$ with kernel $(I(X,Y) + \mathrm{rad}_K^2(X,Y))/\mathrm{rad}_K^2(X,Y)$. Moreover $\mathrm{Irr}_{K/I}(X,Y) = \mathrm{Irr}_{K_I/I}(X,Y)$ holds.

If $I = [L]$ for some full additive subcategory $L$ of $K$, then $K_I = K \setminus L$ and $I(X,Y) \subseteq \mathrm{rad}_K^2(X,Y)$ for all $X,Y \in K \setminus L$. If $\mathrm{ind}\,L = \{Z_1,\dots,Z_m\}$ is finite, then source and sink maps in $K/[L]$ can be lifted to $K$. More precisely, suppose that $f : X \to Y$ is a morphism in $K$ with $X,Y \in K \setminus L$ such that $f + [L]$ is a source map in $K/[L]$. We put $r_i := \dim_k K(X,Z_i)$ for all $i = 1,\dots,m$ and define a morphism $h : X \to Z := \oplus_{i=1}^m Z_i^{r_i}$ whose components are just $k$-bases of the respective spaces $K(X,Z_i)$. Then $(f,h) : X \to Y \oplus Z$ is a left almost split map in $K$ which can be reduced to a sink map as outlined above.

4.3. Now we come back to studying the category $K(A,P)$ defined as in the preceding sections. Let $K'(A,P) := K(A,P) \setminus \mathrm{add}\,P$. The subspace category $\check{\mathcal{U}}(K(A,P),\mathrm{Hom}_A(P,-))$ has up to isomorphism only two indecomposanle objects more than $\check{\mathcal{U}}(K'(A,P),\mathrm{Hom}_A(P,-))$ namely $(P,0,0)$ and $(P,\mathrm{End}_A(P),\mathrm{id})$. Hence concerning representation type we can as well study the vectorspace category $(K',\mathrm{Hom}_A(P,-))$ resp. $(K'_P/I_P,\mathrm{Hom}_A(P,-))$. In the sequel we will again abbreviate $K(A,P)$ to $K$ and consequently $K'(A,P)$ to $K'$.

As $\mathrm{Hom}_A(X,P_s(A)) \subseteq I_P$ for $X \in K'$, we obtain $\mathrm{rad}_{K'/I_P}(X,Y) = \mathrm{rad}_{K/I_P}(X,Y)$ and in particular $\mathrm{Irr}_{K/I_P}(X,Y) = \mathrm{Irr}_{K'/I_P}(X,Y)$ for all $X,Y \in K$ such that $X \in K'$.

**Lemma.** *Suppose $[s,s]_A = \{s\}$. For each $X \in \mathrm{ind}\,K'$ there is a source map in $K$ starting in $X$ and a sink map in $K$ ending in $X$.*

*Proof:* By 2.1 the functor $l^A_{A/s} : K \to A/s\text{-mod}$ is full and dense with kernel $[K_0]$. It is well-known that $A/s$-mod has source and sink maps. Since by 2.5 $\mathrm{ind}\,K_0$ is finite, these lift to source and sink maps in $K$.

By 2.3 and 2.4 the intersection of $\mathrm{ind}\,K_0$ and $\mathrm{ind}\,(A\text{-proj})$ is just $\{P\}$. Hence the equivalence $\tau_A^- : A/s\text{-mod}/[A\text{-inj}] \to K/[A\text{-proj}]$ may be used in a similar fashion to complete the proof.

**Corollary.** *For each indecomposable object $X$ in the category $K'/I_P$ there is a source map in $K'/I_P$ starting in $X$ and a sink map in $K'/I_P$ ending in $X$. In particular, the* *quiver of $K'/I_P$ is locally finite.*

4.4. As $]\infty, s]_A$ is an ideal of $\Delta_0$, by $e(]\infty, s]_A)L^A_{A_s}$ we obtain a subfunctor of the functor $L^A_{A_s}$. Denoting by $\kappa$ the canonical inclusion, for each $Y \in A_s$-mod there is an exact sequence $0 \to e(]\infty, s]_A)L^A_{A_s}Y \xrightarrow{\kappa_Y} L^A_{A_s}Y \xrightarrow{(\pi^A_{A_s})_Y} Y \to 0$.

Dually $[s, \infty[_A$ is a coideal of $\Delta_0$ yielding the factor functor $e([s, \infty[_A)R^A_{A^s}$ of the functor $R^A_{A^s}$. Denoting by $\lambda$ the canonical epimorphism, for each $X \in A^s$-mod there is an exact sequence $0 \to X \xrightarrow{(\iota^A_{A^s})_X} R^A_{A^s}X \xrightarrow{\lambda_X} X \to 0$.

**Theorem.** *Suppose $s$ is decomposing. Let $V, W \in \operatorname{ind} K'_P$ such that $r(V, W) \neq 0$ where by definition $r(V, W) := \dim_k \mathrm{Irr}_{K'/I_P}(V, W)$. Then the following holds:*

*a) If $V = L^A_{A_s}X \in K^-_P$ with $X \in A_s$-ind, then $W = L^A_{A_s}Y \in K^-_P$ for some $Y \in A_s$-ind and $r(V, W) = \dim_k(\mathrm{rad}_{A_s\text{-mod}}/(\mathrm{rad}^2_{A_s\text{-mod}} + J^-))(X, Y)$ where $J^-$ is the kernel of the functor $(R^- \otimes_{A_s} -)$.*

*b) If $W = \tau^-_{\overline{A}^s}Y \in K^+_P$ with $Y \in A^s$-ind, then $V = \tau^-_{\overline{A}^s}X \in K^+_P$ for some $X \in A^s$-ind and $r(V, W) = \dim_k(\mathrm{rad}_{A^s\text{-mod}}/(\mathrm{rad}^2_{A^s\text{-mod}} + J^+))(X, Y)$ where $J^+$ is the kernel of the functor $\mathrm{Hom}_{A^s}(R^+, -)$.*

*c) If $V = \tau^-_{\overline{A}^s}X \in K^+_P \setminus K_0$ with $X \in A^s$-ind and $W = L^A_{A_s}Y \in K^-_P$ with $Y \in A_s$-ind, then $X \notin A^s$-inj, $\tau^-_{A^s}X \in A^s_s$-mod and $r(V, W) = \dim_k(\mathrm{rad}_{A_s\text{-mod}}/(\mathrm{rad}^2_{A_s\text{-mod}} + J_0))(\tau^-_{A^s}X, Y)$ where $J_0$ is the kernel of the functor $e(]\infty, s]_A)L^A_{A_s}$.*

*d) If $V = \tau^-_{\overline{A}^s}X \in K^+_P$ with $X \in A^s$-ind and $W = L^A_{A_s}Y \in K^-_P \setminus A$-proj with $Y \in A_s$-ind, then $Y \notin A_s$-proj, $\tau_{A_s}Y \in A^s_s$-mod and $r(V, W) = \dim_k(\mathrm{rad}_{A^s\text{-mod}}/(\mathrm{rad}^2_{A^s\text{-mod}} + J^0))(X, \tau_{A_s}Y)$ where $J^0$ is the kernel of the functor $e([s, \infty[_A)R^A_{A^s}$.*

*Proof:* By proposition 3.3 parts $a$) and $b$) are immediate consequences of 2.2 and 2.4.

To establish $c$), we derive $X \notin A^s$-inj from 2.4. By 4.2 and 4.3 we know $\mathrm{Irr}_K(\tau^-_{\overline{A}^s}X, L^A_{A^s}Y) \neq 0$, which using 2.1 leads to $\mathrm{Irr}_{A/s\text{-mod}}(l^A_{A/s}\tau^-_{\overline{A}^s}X, l^A_{A/s}L^A_{A_s}Y) \neq 0$. By 2.2 and 2.4 we can rewrite the involved modules as $l^A_{A/s}\tau^-_{\overline{A}^s}X = \tau^-_{A^s}X$ and $l^A_{A/s}L^A_{A_s}Y = Y$. Now lemma 3.4 implies $\tau^-_{A^s}X \in A^s$-mod $\cap\, A_s$-mod $= A^s_s$-mod.

In order to show the remainder of assertion $c$) we at first want to prove that $l^A_{A/s}I_P(\tau^-_{\overline{A}^s}X, L^A_{A^s}Y) = J_0(\tau^-_{A^s}X, Y)$. For abbreviation we put $Z := \tau^-_{A^s}X$. As $l^A_{A/s} : K \to A/s$-mod is a full functor with kernel $[K_0]$ it suffices to pick $f \in \mathrm{Hom}_A(\tau^-_{\overline{A}^s}X, L^A_{A^s}Y)$ and to show that $f \in I_P + [K_0]$ is equivalent to $l^A_{A/s}f \in J_0$. Since $e(]\infty, s]_A)L^A_{A_s}Z \subseteq (\mathrm{Rad}\, A)L^A_{A_s}Z$, we see that $l^A_{A/s}f \in J_0$ is equivalent to $e(]\infty, s]_A)L^A_{A_s}l^A_{A/s}f$ factorizing through $\kappa_Z$. This in turn is equivalent to $l^A_{A/s}f$ factorizing through $(\pi^A_{A_s})_Y = (\rho^A_{A/s})_{L^A_{A_s}Y}$. Let us show in the sequel that this happens if and only if $f \in I_P + [K_0]$.

If $l^A_{A/s}f = h(\pi^A_{A_s})_Y$ for some $h \in \mathrm{Hom}_A(Z, L^A_{A_s}Y)$, then $l^A_{A/s}(f - (\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X}h) = l^A_{A/s}f - (l^A_{A/s}(\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X})(l^A_{A/s}h) = l^A_{A/s}f - l^A_{A/s}h = l^A_{A/s}f - h(\pi^A_{A_s})_Y = 0$. As $(\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X}h \in I_P$, we arrive at $f \in I_P + [K_0]$. Conversely, if $f = h' + h'' \in I_P + [K_0]$, then there exists

$h \in \mathrm{Hom}_A(Z, L^A_{A_s}Y)$ such that $h' = (\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X} h$. Hence $l^A_{A/s}f = (l^A_{A/s}(\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X})(l^A_{A/s}h) = h(\pi^A_{A_s})_Y$.

To complete the proof of $c$), we will show that $l^A_{A/s}\mathrm{rad}^2_K(\tau^-_{\overline{A}^s}X, L^A_{A_s}X)$ equals $\mathrm{rad}^2_{A_s\text{-mod}}(\tau^-_{A^s}X, Y)$. Obviously the second space is contained in the first one. For the converse inclusion we pick $f \in \mathrm{rad}^2_K(\tau^-_{\overline{A}^s}X, L^A_{A_s}X)$ and observe that $l^A_{A/s}f = gh$ where $g, h$ are homomorphisms in $A/s$-mod such that $g$ is not a section and $h$ is not a retraction. Let $h : U \to Y$. Since $Y \in A_s$-mod, we can factorize $h$ as $h = (\rho^{A/s}_{A_s})_U l^{A/s}_{A_s}h$. Decomposing $U = \oplus^n_{i=1}U_i$ into indecomposable direct summands, the induced components $h_i$ of $h$ give rise to components $l^{A/s}_{A_s}h_i$ of $l^{A/s}_{A_s}h$. If $U_i \in A_s$-mod, then $h_i = l^{A/s}_{A_s}h_i$ is not an isomorphism. If $U_i \notin A_s$-mod, then $U_i \in A^s$-mod and consequently $l^{A/s}_{A_s}U_i \in A^s_s$-mod. Thus $l^{A/s}_{A_s}h_i$ is not a retraction. Altogether this shows that $l^{A/s}_{A_s}h$ is not a retraction as well. Obviously $g(\rho^{A/s}_{A_s})_U$ is not a section such that $l^A_{A/s}f$ lies in $\mathrm{rad}^2_{A_s\text{-mod}}(\tau^-_{A^s}X, Y)$

The proof of $d$) is more or less similar. $Y \notin A_s$-proj is obvious. We obtain $\mathrm{Irr}_{A/s\text{-mod}}(\tau_A\tau^-_{\overline{A}^s}X, \tau_A L^A_{A_s}Y) \neq 0$ and using 2.2 and 2.4 rewrite the involved modules as $\tau_A\tau^-_{\overline{A}^s}X = R^A_{\overline{A}^s}X = X$ and $\tau_A L^A_{A_s}Y = \tau_{A_s}Y$. Since $X \notin A^s_s$-mod, by 3.4 we get $\tau_{A_s}Y \in A^s_s$-mod.

Observing that $\tau_A$ maps $(\rho^A_{A/s})_{\tau^-_{\overline{A}^s}X} + [A\text{-proj}]$ to $(\iota^A_{A^s})_X + [A\text{-inj}]$, we see that $\tau_A(I_P(\tau^-_{\overline{A}^s}X, L^A_{A_s}Y) + [A\text{-proj}])$ is just the set of all $f + [A\text{-inj}]$ where $f \in \mathrm{Hom}_A(X, \tau_{A_s}Y)$ is a homomorphism factorizing through $(\iota^A_{A^s})_X$. Using dual arguments as for $c$), this factorization property for $f$ is equivalent to $f \in J^0$.

Now we remark that, since $X \in A^s$-mod, the injective hull of $X$ in $A$-mod actually consists of modules $R^A_{A^s}Q$ where $Q \in A^s$-inj. This shows that every homomorphism in $[A\text{-inj}](X, \tau_{A_s}Y)$ factorizes through $(\iota^A_{A^s})_X$. This fact together with a dual argumentation as at the end of the proof of $c$) completes the prove of $d$).

4.5. Using the remark at the end of 2.2, we see that the calculation of the ideals $J^+, J^-, J_0, J^0$ can always be reduced to the calculation of the kernel of a functor $\mathrm{Hom}_B(R, -)$ for a suitable algebra $B$ and $R \in B$-mod. As the quiver of $B$-mod is just the Auslander-Reiten quiver of $B$, there are effective methods for its computation. The fact that $\mathrm{Hom}_B(R, -)$ maps every Auslander-Reiten sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ where $Z$ is not a summand of $R$ to an exact sequence of $k$-spaces enables us to control the function $\dim_k\mathrm{Hom}_B(R, -)$ on the indecomposable $B$-modules. Thus we can find the incecomposable objects of the factor category $B$-mod/$\mathrm{Ker}\,\mathrm{Hom}_B(R, -)$. Unfortunately it may happen that $\mathrm{Ker}\,\mathrm{Hom}_B(R, -)(X, Y) \not\subseteq \mathrm{rad}^2_{B\text{-mod}}$ although $X, Y$ are indecomposable $B$-modules such that $\mathrm{Hom}_B(R, X) \neq 0 \neq \mathrm{Hom}_B(R, Y)$. In other words, in order to pass from the full subquiver of the Auslander-Reiten quiver of $B$ induced by the indecomposables with $\mathrm{Hom}_B(R, X) \neq 0$ to the quiver of $B$-mod/$\mathrm{Ker}\,\mathrm{Hom}_B(R, -)$ one usually has to cancel some arrows. An example where this actually happens can be found in [Ri1, p. 238]. A sufficient condition on $R$ to avoid this effect is given in the following lemma.

**Lemma.** *Let $B$ be a $k$-algebra and $R \in B$-mod satisfying $\mathrm{Ext}^1_B(R, \mathrm{fac}\, R) = 0$. If we denote by $J$ the ideal $\mathrm{Ker}\,\mathrm{Hom}_B(R, -)$ of $B$-mod and consider $X, Y \in B$-ind such that $\mathrm{Hom}_B(R, X) \neq 0 \neq \mathrm{Hom}_B(R, Y)$, then $J(X, Y) \subseteq \mathrm{rad}^2_B(X, Y)$.*

*Proof:* By defining $t(Z)$ as the sum of all images of homomorphisms $h \in \mathrm{Hom}_B(R, Z)$ for every $Z \in B$-mod, we obtain a radical of the category $B$-mod. Eeach $f \in I(X, Y)$ factorizes through $X/t(X)$ and this module satisfies $\mathrm{Hom}_B(R, X/t(X)) = 0$.

# 5 Relevant pairs

5.1. We continue to consider a decomposing vertex $s$ and use the abbreviations $K = K(A, P)$ and $K' = K'(A, P)$ introduced in the preceding sections. Applying theorem 4.4, we can read off nearly all arrows of the quiver $\Gamma = \Gamma_{K'_P/I_P}$ from the Auslander-Reiten quivers of $A_s$ and $A^s$. The finite set $\mathcal{D}$ of all pairs $(V, W) \in \mathrm{ind}\, K_P^+ \times \mathrm{ind}\, K_P^-$ where $V \in K_0$ and $W \in A$-proj is exactly the set of all pairs $(V, W)$ of vertices of $\Gamma$ for which we do not yet have any information about $\dim_k \mathrm{Irr}_{K'_P/I_P}(V, W)$. We will call a subset $\mathcal{R} \subseteq \mathcal{D}$ a set of relevant pairs provided $\mathrm{rad}_{K'_P/I_P}(V, W) = \mathrm{rad}^2_{K'_P/I_P}(V, W)$ for all $(V, W) \in \mathcal{D} \setminus \mathcal{R}$. Of course $\mathcal{D}$ itself is a set of relevant pairs, but usually one can find much smaller sets of this kind using trivial considerations.

For a given set $\mathcal{R}$ of relevant pairs let us introduce a new quiver $\Gamma_{\mathcal{R}}$ with vertex set $\mathrm{ind}\, K_P$ by defining the number of arrows from $V$ to $W$ for $(V, W) \notin \mathcal{D}$ as prescribed by theorem 4.4 and putting $\dim_k \mathrm{Hom}_A(V, W)$ arrows from $V$ to $W$ if $(V, W) \in \mathcal{R}$. The quiver $\Gamma$ may be considered as a subquiver of $\Gamma_{\mathcal{R}}$ such that there can only exist strictly more arrows from $V$ to $W$ in $\Gamma_{\mathcal{R}}$ than in $\Gamma$ provided $(V, W) \in \mathcal{R}$.

In general we do not know an easy recipe to reduce $\Gamma_{\mathcal{R}}$ to $\Gamma$. Nevertheless, in applications it happens frequently (see also the examples in the next section) that one only has to deal with a linear subvectorspace category to erase the superfluous arrows from $\Gamma_{\mathcal{R}}$. Subsequently we will discuss this in more detail.

5.2. Let us first establish some more facts about arbitrary Krull-Schmidt categories $K$. We introduce two relations on $\mathrm{ind}\, K$. Namely, we put $X \rightsquigarrow Y$ provided there is a path from $X$ to $Y$ in $\Gamma_K$ i.e. there is a sequence $X_0, \ldots, X_n$ of objects in $\mathrm{ind}\, K$ such that there is an arrow in $\Gamma_K$ from $X_{i-1}$ to $X_i$ for all $i = 1, \ldots, n$ and $X_0 = X$, $X_n = Y$. The relation $X \prec Y$ is defined very similar. Here we demand that there is a path from $X$ to $Y$ in $K$ i.e. there is a sequence $X_0, \ldots, X_n$ of objects in $\mathrm{ind}\, K$ such that $\mathrm{rad}_K(X_{i-1}, X_i) \neq 0$ for all $i = 1, \ldots, n$ and $X_0 = X$, $X_n = Y$. Of course $X \rightsquigarrow Y$ implies $X \prec Y$ but not vice versa. A path $X_0, \ldots, X_n$ in $K$ is called a cycle if $n \geq 1$ and $X_0 = X_n$. An object $X \in \mathrm{ind}\, K$ is said to be directing provided there does not exist any cycle starting in $X$.

A map $\varrho$ from the set of arrows of $\Gamma_K$ to the set of morphisms of $\mathrm{ind}\, K$ is said to be a presentation of $K$ provided for each pair $(X, Y)$ of objects of $\mathrm{ind}\, K$ the set of arrows from $X$ to $Y$ in $\Gamma_K$ is mapped to a set of morphism in $\mathrm{rad}_K(X, Y)$ which induces a basis of the $k$-space $\mathrm{Irr}_K(X, Y)$. For a presentation $\varrho$ the smallest full subcategory of $K$ containing the images $\varrho(\alpha)$ of all arrows $\alpha$ of $\Gamma_K$ is denoted by $K_\varrho$.

**Lemma.** *Suppose $K$ is a Krull-Schmidt category and $T$ is a finite subset of* $\mathrm{ind}\, K$.

*a) If $T$ is convex w.r.t. $\prec$, then $K(X, Y) = K_\varrho(X, Y)$ for all $X, Y \in T$ and all presentations $\varrho$ of $K$.*

b) *Suppose $T$ is convex w.r.t. $\rightsquigarrow$, each $X \in T$ is a directing object of $K$ and for each path $X = X_0, \ldots, X_n = Y$ in $\Gamma_K$ with $n \geq 1$ and $X, Y \in T$ there is either a source map $X \to X^+$ in $K$ such that all indecomposable direct summands of $X^+$ lie in $T$ or there is sink map $Y^- \to Y$ in $K$ such that all indecomposable direct summands of $Y^-$ lie in $T$. If the full subquiver of $\Gamma_K$ induced by $T$ is directed (i.e. has neither loops nor oriented cycles), then $T$ is convex w.r.t. $\prec$.*

*Proof:* Let $L$ be the full additive subcategory of $K$ generated by $T$. The convexity of $T$ shows $\mathrm{rad}_K^n(X,Y) = \mathrm{rad}_L^n(X,Y)$ for all $X, Y \in T$ and all $n \in \mathbb{N}$. Hence for the proof of $a$) we may suppose $L = K$. But for Krull-Schmidt categories $K$ where $\mathrm{ind}\, K$ is finite assertion $a$) is well-known.

By the directedness condition we know that the set $T$ is partially ordered by the relation $\rightsquigarrow$. We want to show by induction on the cardinality $c(X,Y)$ of the interval between $X, Y \in T$ w.r.t. $\rightsquigarrow$ that every indecomposable object appearing on a path in $K$ from $X$ to $Y$ belongs to $T$. If $c(X,Y) = 1$, then $X = Y$ and this follows from the fact that $X$ is directing. For $c(X,Y) > 1$ let us consider a path $X_0, \ldots, X_n$ in $K$ from $X$ to $Y$ such that $n \geq 1$. Without loss of generality we may suppose that there is a source map $f : X \to X^+$ such that $X^+ \in L$. By factorizing a non-zero element of $\mathrm{rad}_K(X_0, X_1)$ through $f$ we find a direct summand $Z$ of $X^+$ with the property $K(Z, X_1) \neq 0$. Since $c(Z,Y) < c(X,Y)$, we obtain $X_1, \ldots, X_{n-1} \in T$.

5.3. Let us come back to the calculation of the quiver $\Gamma$ of the category $K'_P/I_P$.

**Proposition.** *Let $\mathcal{R}$ be a set of relevant pairs for a decomposing vertex $s$ and $T$ a finite subset of $\mathrm{ind}\, K'_P$. Suppose that $T$ is a convex set w.r.t. the relation $\rightsquigarrow$ defined by the quiver $\Gamma_{\mathcal{R}}$ satisfying the following properties:*

*i)* $\dim_k \mathrm{Hom}_A(P, X) = 1$ *for all $X \in T$.*

*ii) For each path $X = X_0, \ldots, X_n = Y$ in $\Gamma_{\mathcal{R}}$ with $n \geq 1$ and $X, Y \in T$ either the set of all direct successors of $X$ or the set of all direct predecessors of $Y$ in $\Gamma_{\mathcal{R}}$ lies entirely in $T$.*

*iii) $X$ is a directing object of $K'_P/I_P$ for all $X \in T$.*

*Then the subquiver $\Psi$ of $\Gamma_{\mathcal{R}}$ induced by $T$ is directed and there is at most one arrow beween any two vertices of $\Psi$. Denoting this arrow by $\alpha(V,W)$ for a pair $(V,W) \in \mathcal{R}$ such that $V, W \in T$, the inequality $\mathrm{Irr}_{K'_P/I_P}(V,W) \neq 0$ holds if and only if $\alpha(V,W)$ is the only path form $V$ to $W$ in $\Psi$.*

*Proof:* Condition $i$) imlies that every non-zero morphism between two objects in $T$ is mapped by $\mathrm{Hom}_A(P, -)$ to an isomorphism between two 1-dimensional $k$-spaces. Therefore $\Psi$ must be directed. The same argument also shows that there is at most one arrow in $\Psi$ from $X$ to $Y$ provided $(X,Y) \notin \mathcal{R}$. and $\dim_k \mathrm{Hom}_A(V,W) \leq 1$ for all $(V,W) \in \mathcal{R}$.

Now we can apply lemma 5.2 to $T$ using an arbitrary presentation $\varrho$ of $K'_P/I_P$. Let $(V,W) \in \mathcal{R}$. Obviously $\mathrm{Irr}_{K'_P/I_P}(V,W) \neq 0$ provided $\alpha(V,W)$ is the only path from $V$ to $W$. Conversely, if there is another path from $V$ to $W$ in $\Psi$ then this has to be of length bigger than 1 giving rise to a non-zero map $h$ in $\mathrm{rad}^2_{K'_P/I_P}(V,W)$. Any other map in $K'_P/I_P(V,W)$ has to be a skalar multiple of $h$ which implies $\mathrm{Irr}_{K'_P/I_P}(V,W) = 0$.

We have already seen in the preceding sections how to check the conditions *i)* and *ii)*. Condition *iii)* follows if we start out from directing objects in $A^s$-mod resp. $A_s$-mod. More precisely, from section 2 and 3.1 we obtain the following lemma.

**Lemma.** *Suppose $s$ is a decomposing point.*

*a) If $X \in \mathrm{ind}\, A^s$ is a directing $A^s$-module such that $\mathrm{Hom}_{A^s}(R^+, X) \neq 0$, then $\tau_{\overline{A}^s} X$ is a directing object of $K'_P/I_P$.*

*b) If $Y \in \mathrm{ind}\, A_s$ is a directing $A_s$-module such that $R^- \otimes_{A_s} Y \neq 0$, then $L^A_{A_s} Y$ is a directing object of $K'_P/I_P$.*

We want to add some remarks about condition *i)* in the above proposition. Denoting by $L$ the additive subcategory of $K'_P/I_P$ generated by $T$ the vectorspace category $(L, \mathrm{Hom}_A(P,-))$ has the properties that $\mathrm{Hom}_A(P,-)$ acts faithful on $L$ and each indecomposable object is mapped to a 1-dimensional $k$-space. Vectorspace categories with this properties are said to be linear (see [Ri2]). Up to isomorphism they are of the shape $(\mathrm{add}\, k\mathcal{S}, can)$ where $\mathcal{S}$ is a partially ordered set considered as category and $\mathrm{add}\, k\mathcal{S}$ is the additivization of the $k$-linear hull of this category. Moreover *can* is the canonical functor mapping the elements to the space $k$ and the non-zero morphisms to $\mathrm{id}_k$. Since $\mathrm{ind}\, L$ is finite the Hasse diagram of $\mathcal{S}$ is just the quiver of $L$.

5.4. As indicated in the introduction, we are now able to give the construction of $K'_P/I_P$ for arbitrary $s$ provided that $A$ is a representation directed algebra i.e. all indecomposables in $A$-mod are directing (see [Ri2]). This completes the description of $K'_P/I_P$ obtained in [Sc] for this case. Namely, the access via hammock functions used in this paper did not allow to find the arrows of $\Gamma$ going from objects in $K^+_P$ to objects in $K^-_P$.

We recall from 3.2 that for a representation directed algebra all vertices $s$ are decomposing. It is well-known (and easy to see) that the vectorspace category $(K'_P/I_P, \mathrm{Hom}_A(P,-))$ is linear and $\mathrm{ind}\, K'_P$ is finite since by [Ri2] the algebra $A$ is of finite representation type. It follows that $\dim_k e(y)Ae(x) \leq 1$ for all $x, y \in \Delta_0$. Hence $C(t) = Ae(s)/Ae(t)Ae(s)$ for all $t \in \check{s}$ and consequently $\mathrm{Hom}_A(C(t), Ae(u)) \neq 0$ if and only if $e(s)Ae(u) \neq 0$ but $e(t)Ae(s)Ae(u) = 0$ where $u \in \Delta_0$.

**Corollary.** *Suppose $A = k[\Delta/I]$ is a representation directed algebra and $s$ is a vertex of $\Delta$. Denote by $\Gamma^+$ the full subquiver of the Auslander-Reiten quiver $\Gamma^s$ of $A^s$ supported by all modules $X \in \mathrm{ind}\, A^s$ such that $\mathrm{Hom}_A(R^+, X) \neq 0$ and by $\Gamma^-$ the full subquiver of the Auslander-Reiten quiver $\Gamma_s$ of $A_s$ supported by all modules $Y \in \mathrm{ind}\, A_s$ such that $R^- \otimes_{A_s} X \neq 0$. Add a single arrow from the vertex $X$ of $\Gamma^+$ to the vertex $Y$ of $\Gamma^-$ in the following cases:*

*i) $X$ is not injective, $\tau^-_{A^s}X \in A^s_s$-ind, there is an arrow from $\tau^-_{A^s}X$ to $Y$ in $\Gamma_s$ and there exists $t \in ]\infty, s]_A$ such that $\mathrm{Hom}_{A_s}(R^-_t, \mathrm{D}\tau^-_{A^s}X) \neq 0 \neq \mathrm{Hom}_{A_s}(R^-_t, \mathrm{D}Y)$.*

*ii) $X$ is injective, $Y$ is not projective, $\tau_{A_s}Y \in A^s_s$-ind, there is an arrow from $X$ to $\tau_{A_s}Y$ in $\Gamma^s$ and there exists $t \in [s, \infty[_A$ such that $\mathrm{Hom}_{A^s}(R^+_t, X) \neq 0 \neq \mathrm{Hom}_{A^s}(R^+_t, \tau_{A_s}Y)$.*

*iii) $X = Q_t(A^s)$ is injective, $Y = P_u(A_s)$ is projective and $e(t)Ae(s)Ae(u) = 0$.*

*Cancel from the obtained glueing $\Gamma'$ of the quivers $\Gamma^+$ and $\Gamma^-$ all arrows $\alpha$ starting in a injective $X \in \Gamma^+$ and ending in a projective $Y \in \Gamma^-$ such that there is a path from $X$ to $Y$ in $\Gamma'$ different from $\alpha$. The quiver $\Gamma$ constructed in this way is the quiver of the category $K'_P/I_P$ w.r.t. $P = Ae(s)$. Since the vectorspace category $(K'_P/I_P, \mathrm{Hom}_A(P,-))$ is linear, it is isomorphic to $(\mathrm{add}\, k\mathcal{S}, can)$ where $\mathcal{S}$ is a finite partially ordered set with Hasse diagram $\Gamma$.*

*Proof:* We apply the preceding proposition with $\mathcal{R} = \mathcal{D}$. Thus we only have to ensure that the ideals $J^+, J^-, J_0, J^0$ are contained in the square of the radical of the respective module categories. By 4.5 it suffices to prove $\mathrm{Ext}^1_B(R, \mathrm{fac}\, R) = 0$ or equivalently $\mathrm{Hom}_B(R, \tau_B R) = 0$ for the involved modules $R^+_t$ resp. $R^-_t$ over the algebra $B = A^s$ resp. $B = A_s$. Since $A$ is representation directed, the algebras $A(T)$ are representation directed for all subsets $T \subseteq \Delta_0$. Any of the modules $R$ we have to examine can be interpreted as the radical of an indecomposable projective module over an algebra of the shape $A(T)$ where $T = (\Delta_0 \setminus [s, \infty[_A) \cup \{t\}$ resp. $T = (\Delta_0 \setminus ]\infty, s]_A) \cup \{t\}$. By [HR] these radicals have to be directing $B$-modules. But the definition of (possibly decomposable) directing modules in [HR] immediately implies $\mathrm{Hom}_B(R, \tau_B R) = 0$.

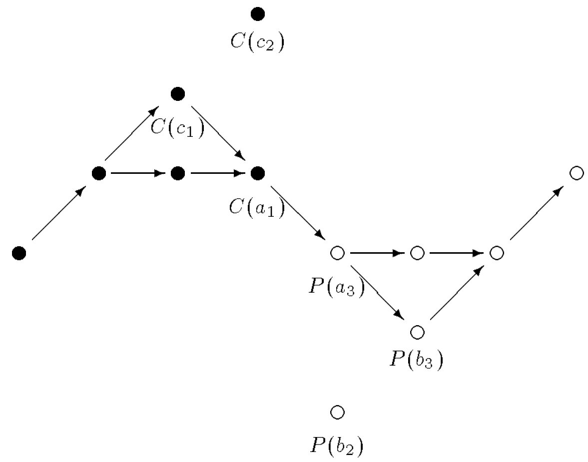
# 6 Examples

In all the examples in this section we specify an algebra $A = k[\Delta]/I$ by presenting its quiver $\Delta$ and a set of generators for the ideal $I$. After choosing a vertex $s$ and therefore a projective module $P = Ae(s)$, we display the quiver $\Gamma$ of $K'_P/I_P$. We mark the vertices $V$ of this quiver belonging to $K^+$ resp. $K^-$ by $r$ times the symbol $\bullet$ resp. $\circ$ where $r = \dim_k \mathrm{Hom}_A(P, V)$. Moreover we distinguish the indecomposables in $K_0$ and $A$-proj by writing $C(t)$ resp. $P(t)$ below the corresponding vertex.

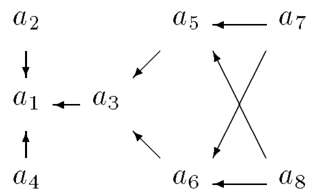
6.1. As first example we consider the algebra $A := k[\Delta]/I$ where $\Delta$ is the following tree.

$$\begin{tikzcd}
 & b_1 \arrow[d] & b_2 \arrow[d] & b_3 \arrow[d] & \\
a_0 & a_1 \arrow[l] \arrow[d] & a_2 \arrow[l] \arrow[d] & a_3 \arrow[l] \arrow[d] & a_4 \arrow[l] \\
 & c_1 & c_2 & c_3 &
\end{tikzcd}$$

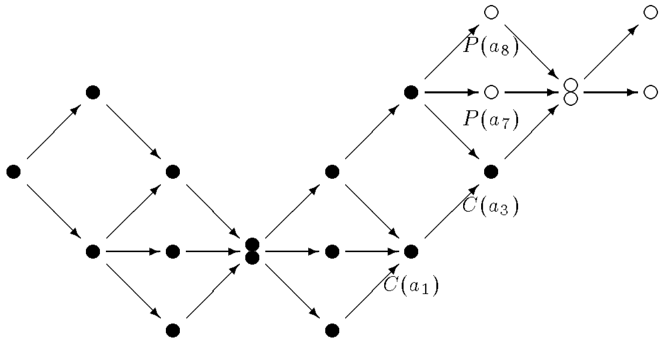
Let $I$ be the ideal generated by the unique paths joining the vertices $a_{i-1}$ and $a_{i+1}$ for $i = 1, 2, 3$. Since $\Delta$ is a tree, each vertex and in particular $s = a_2$ is decomposing. The algebras $A^s$ and $A_s \cong (A^s)^{op}$ are representation directed. Hence there Auslander-Reiten quivers are easy to calculate. The vectorspace category $(K'_P/I_P, \mathrm{Hom}_A(P,-))$ turns out to be linear with the following quiver $\Gamma$. This vectorspace category is tame but not of polynomial growth (see [Na], [Za]).



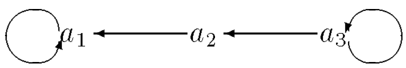

6.2. Let $A = k[\Delta]/I$ with the following quiver $\Delta$ and the ideal $I$ generated by all commutativity relations.




The algebra $A/a_6$ is representation directed which makes it easy to verify that the vertex $s = a_6$ is decomposing. As in the preceding example one has no problem to calculate $\Gamma$ using our results of the preceding sections. The occurring vectorspace category is not linear but it is easy to see that it is just a convex subcategory of the tame vectorspace category $(\widetilde{D}_n, {}^{n-2}_{n-2})$ appearing in [Ri1].




6.3. In the definition of decomposing points in section 3 we demanded $[s,s]_A = \{s\}$ but not that the whole quiver $\Delta$ should be directed. We want to illustrate by a very trivial example that this is actually not necessary.



Let us consider the above quiver $\Delta$ and the ideal $I$ of $k[\Delta]$ generated by all paths of length 2. The vertex $s = a_2$ is decomposing and the algebras $A^s$, $A_s$ both are isomorphic to $k[T]/(T^2)$. The modules $R^+$ resp. $R^-$ are both the unique $k[T]/(T^2)$-module of $k$-dimension 1. The vectorspace category $K'_P/I_P$ is again linear and its quiver $\Gamma$ has the following shape.

$$\begin{tikzcd} \bullet \arrow[r] & \underset{C(a_1)}{\bullet} \arrow[r] & \underset{P(a_3)}{\circ} \arrow[r] & \circ \end{tikzcd}$$

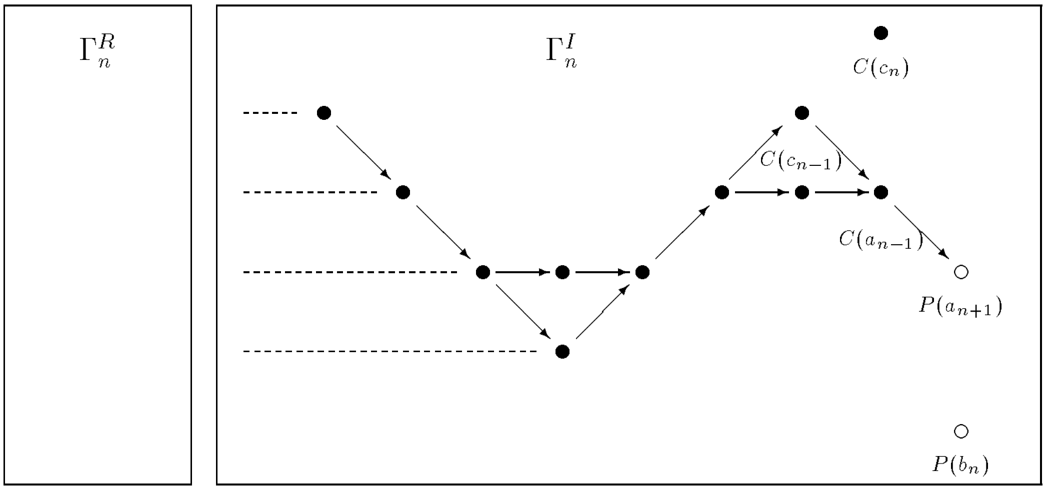
6.4. In our final example we want to illustrate how one can prove the tameness of a whole class of algebras by combining the results of this paper and our characterization of tamenes given in [Dr2]. This characterization goes as follows. Fixing a vertex $s$ such that $\dim_k \mathrm{End}_A(P) = 1$ where $P = Ae(s)$, the algebra $A$ is tame if and only if the factor algebra $A/s$ is tame, the vectorspace category $(K'_P(A,P)/I_P, \mathrm{Hom}_A(P,-))$ is subspace tame and for every $n \in \mathbb{N}$ there are up to isomorphism only finitely many indecomposable modules $X \in K'_P(A,P)$ such that $\dim_k X \leq n$.

For each $n \in \mathbb{N}$ let the algebra $A_n = k[\Delta_n]/I_n$ be given by the following tree $\Delta_n$.

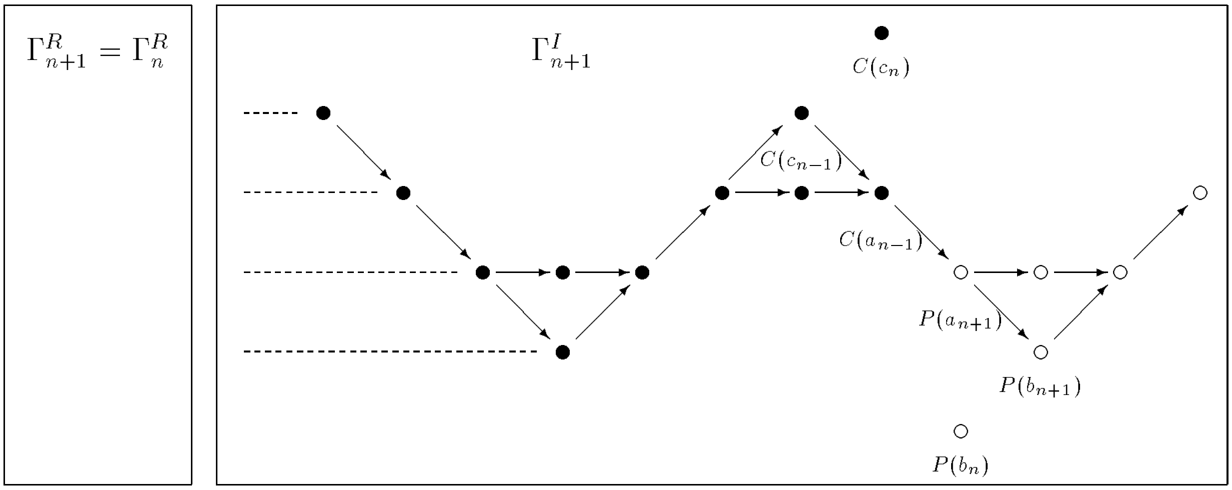
$$\begin{tikzcd} & b_1 \arrow[d] & b_2 \arrow[d] & \cdots & b_{n-1} \arrow[d] & b_n \arrow[d] & \\ a_0 & a_1 \arrow[l] & a_2 \arrow[l] & \cdots & a_{n-1} & a_n \arrow[l] & a_{n+1} \arrow[l] \\ & c_1 \arrow[from=u] & c_2 \arrow[from=u] & \cdots & c_{n-1} \arrow[from=u] & c_n \arrow[from=u] & \end{tikzcd}$$

The generators of $I_n$ will be the unique paths joining the points $a_{i-1}$ and $a_{i+1}$ for all $i = 1, \ldots, n$. Thus $A_3$ is just our first example in this section which shows that $A_n$ is tame for $n = 1, 2, 3$.

We want to apply induction to prove that $A_n$ is tame for all $n \in \mathbb{N}$. Supposing as induction hypothesis that $A_n$ is tame, we use the above tameness characterization to dervive that the vectorspace category $(K'_P(A_n, P)/I_P, \mathrm{Hom}_{A_n}(P,-))$ corresponding to $s = a_n$ is subspace tame. We use our results of this paper to calculate the quiver $\Gamma_n$ of $K'_P(A_n, P)/I_P$. It turns out that this quiver is the disjoint union of two connected subquivers $\Gamma_n^R$ and $\Gamma_n^I$. Namely $\Gamma_n^I$ is a connected component of $\Gamma_n$ obtained by glueing the preinjective component of the Auslander-Reiten quiver of $(A_n)^s$ and the Auslander-Reiten quiver of the algebra $(A_n)_s \cong k^{n+1}$ using 5.3. Let us denote by $G_n^R$ resp. $G_n^I$ the induced full additive subcategories of $K'_P(A_n, P)/I_P$. We do not know much about $\Gamma_n^R$ or $G_n^R$. The point is that we will not need this knowledge. We display $\Gamma_n^I$ as in the previous examples.




Now we calculate $(K'_P(A_{n+1},P)/I_P, \mathrm{Hom}(P,-))$ again with respect to $s = a_n$. The quiver $\Gamma_{n+1}$ of $K'_P(A_{n+1},P)/I_P$ is also the disjoint union of two connected subquivers $\Gamma^R_{n+1}$ and $\Gamma^I_{n+1}$ where $\Gamma^I_{n+1}$ is constructed analogously to $\Gamma^I_n$ by gluing the preinjective component of the Auslander-Reiten quiver of $(A_{n+1})^s$ and the Auslander-Reiten quiver of the representation directed algebra $(A_{n+1})_s$ using 5.3. Denoting by $G^R_{n+1}$ resp. $G^I_{n+1}$ the induced full additive subcategories of $K'_P(A_{n+1},P)/I_P$, from $(A_n)^s = (A_{n+1})^s$ we obtain that $G^R_{n+1} = G^R_n$. Moreover $\Gamma^I_{n+1} \cong \Gamma^I_n$ and consequently $G^I_{n+1} \cong G^I_n$ by 5.2. Using the existence of sink maps in $K'_P(A_{n+1},P)/I_P$ (see 4.3), we derive that each finite full subvectorspace category of $(K'_P(A_{n+1},P)/I_P, \mathrm{Hom}_{A_{n+1}}(P,-))$ is isomorphic to a finite full subvectorspace category of $(K'_P(A_n,P)/I_P, \mathrm{Hom}_{A_n}(P,-))$. This means that also $(K'_P(A_{n+1},P)/I_P, \mathrm{Hom}_{A_{n+1}}(P,-))$ is subspace tame. Since obviously all other conditions of our tameness criterion are satisfied, we conclude that the algebra $A_{n+1}$ is tame.



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