Minuscule Representations

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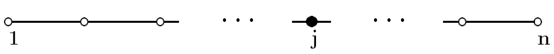
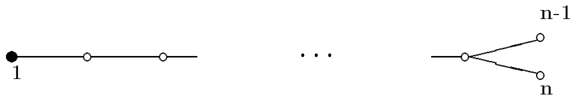
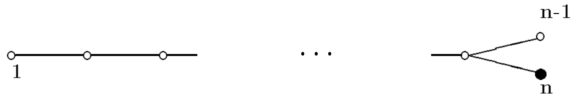
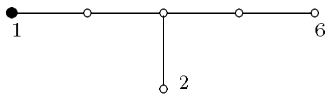
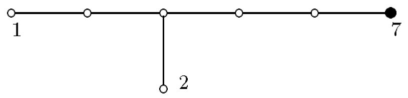
1 Introduction

Let G be a semisimple universal Chevalley group over an algebraically closed field F of arbitrary characteristic, W be its Weyl group, Ψ its root system and Π a fundamental set of roots in Ψ. In this paper we are concerned with the irreducible FG-modules corresponding to the *minuscule* fundamental weights (see Section 2 and note we include the trivial weight $\lambda = 0$). Such FG-modules are called minuscule. More particularly, we are interested in the restriction of minuscule modules to parabolic subgroups of G. It is well-known, see [6, Ch. VIII, §7.3], that when Ψ is indecomposable the non-trivial minuscule weights correspond to the following fundamental modules:

Type	Dynkin Diagram	Usual Name	Dimension
$A(n,j)$		j-th exterior power of natural module	$\binom{n+1}{j}$
$B(n,n)$		spin	2^n

$C(n,1)$		natural	$2n$
$D(n,1)$		natural	$2n$
$D(n,n)$		1/2-spin	2^{n-1}
$E(6,1)$		minimal	27
$E(7,7)$		minimal	56

Here and elsewhere we denote by $A(n,j)$, $B(n,n)$, etc. the highest weight module $V(\lambda_j)$ for A_n and $V(\lambda_n)$ for B_n, etc.. The cases $D(n,n-1)$ and $E(6,6)$ are not listed, as they are, respectively, algebraically conjugate to $D(n,n)$ and dual to $E(6,1)$.

When λ is a minuscule weight, [6] tells us that the weight spaces of $V(\lambda)$ form a single orbit under the action of W. That is the weight spaces are in one-to-one correspondence with the cosets W/W_J where W_J is the stabilizer in W of λ. In [13] Proctor exploits this correspondence. He proves [13, Proposition 3.1] that if an irreducible *Bruhat poset* is a *lattice*, then either W is of type G_2 or the poset is isomorphic to the poset induced on the W-orbit of a minuscule weight with respect to the usual ordering on weights. In another direction minuscule modules arise in [11] when char$F$ is odd, as the only modules (up to Frobenius twists and embeddings of the field into an algebraically closed field) on which both classes of root groups act quadratically. In a further direction, Hiller [9], is able to apply results about minuscule posets to obtain information about the *Schubert calculus* of G/P, where P is the stabilizer in G of a maximal weight space in a minuscule module.

Assume that W_J is the stabilizer in W of a minuscule weight λ. A first step in understanding restrictions of a minuscule module to parabolic subgroups in terms of the geometric structure of W/W_J is made in Theorem 2.4. This theorem tells us that the minuscule FG-module $V = V(\lambda)$ restricted to the semisimple part of a Levi complement, L'_K, behaves in the best possible way: it splits into a direct sum of minuscule FL'_K-modules. Moreover, Theorem 2.4 says that the direct summands are determined by the Ψ_K-dominant weights $w\lambda$ where w is from the set of minimal length (W_K, W_J) double coset representatives in W. So, in particular, there is a one-to-one relationship between (W_K, W_J) double cosets in W and L'_K-summands in $V|_{L'_K}$, the restriction of V to L'_K. Theorem 2.8 asserts that this situation is exactly the same for all fields. A partial converse of Theorem 2.4 is proved in Proposition 2.11.

In Section 3 we define a chamber system structure on W/W_J, indexed over Π, which we denote by $\mathcal{C}(W/W_J)$. This closely resembles the poset of the *weak Bruhat-Chevalley ordering* on W/W_J. But we keep the information about types of incidence which the Bruhat order forgets. One of the main reasons for considering this chamber system is that in $\mathcal{C}(W/W_J)$ the (W_K, W_J) double cosets in W are easily recognized as the K-connected components of $\mathcal{C}(W/W_J)$ for K a subset of Π. Because of Lemma 3.2, we can contract every K-connected component in $\mathcal{C}(W/W_J)$ to a point, so forming a new chamber system $\mathcal{C}_K(W/W_J)$ indexed over $\Pi \setminus K$. Our key result in Section 3 is Proposition 3.5, which implies that when W_J is the stabilizer of a minuscule weight and K a subset of Π, then the associated poset of $\mathcal{C}_K(W/W_J)$ is *ranked*, see Proposition 3.9. An interesting consequence of Proposition 3.5 is Corollary 3.7 which asserts that when K is maximal in Π then the Hasse-diagram of $\mathcal{C}_K(W/W_J)$ is a string. Notice that chambers in $\mathcal{C}_K(W/W_J)$ are in one-to-one correspondence with the (W_K, W_J) double cosets in W and thus with the L'_K-summands in the restriction of V to L'_K. In Proposition 3.10 we show that the poset structure on $\mathcal{C}_K(W/W_J)$ is equivalent to the poset $D_{K,J}$ with respect to the weak Bruhat-Chevalley ordering inherited from W, provided W_J is the stabilizer of a minuscule weight in W.

Given the geometrical results of Section 3, in Section 4 we prove our second main theorem, Theorem 4.3. This result describes the commutator series of a minuscule module $V = V(\lambda)$ with the unipotent radical, U_K, of an arbitrary standard parabolic subgroup, $P_K = L_K U_K$, in terms of the *depth* of chambers in $\mathcal{C}_K(W/W_J)$. That is, the subspace $[V, U_K; r]$ for $r \in \mathbb{Z}^+$ is the L_K-span of all the weight spaces $V_{w\lambda}$ where w is any representative of the (W_K, W_J) double cosets of depth at least r in $\mathcal{C}_K(W/W_J)$. Especially nice is the fact that Corollary 3.7 then implies that the restriction of a minuscule module to a maximal parabolic subgroup, P_K, of G is uniserial and the length of the commutator series is equal to the number of (W_K, W_J) double coset in W. This then leads to Corollary 4.7 which gives the length of the commutator series with a unipotent radical of an arbitrary standard parabolic subgroup. In Proposition 4.10 we show that the ranked structure of the poset $D_{K,J}$ corresponds to the L'_K-module structure of the successive quotients of the commutator series $[V, U_K; r]$.

At this stage our goal is almost reached. We know about the action of both $L_K = L'_K \times T$ and U_K and thus of P_K. It only remains to determine the isomorphism types of the summands in each commutator quotient $[V, U_K; r]/[V, U_K; r+1]$. In the exceptional cases this is done by inspection of the graph associated with $\mathcal{C}(W/W_J)$ which is presented in Appendix A. We deal with the classical cases in Section 5 and then only in the case when K is maximal. In Proposition 5.1 we produce an element g_K in W which is independent of the subset J associated with the minuscule module V and has the profitable property that its powers are distinct (W_K, W_J) double coset representatives in W and additionally, the cosets $g_K^n W_J$ contain the distinguished double coset representatives of minimal length mentioned earlier. Repeated application of g_K to λ then gives us the high weights for the successive quotients $[V, U_K; r]/[V, U_K; r+1]$. Finally, in Section 5 we apply our results to get an explicit description of the restriction of a minuscule FG-module to a subgroup of G of maximal rank.

Appendix A contains a method for constructing $\mathcal{C}(W/W_J)$ which is based upon Proposition 3.5. Though these structures are known via the work of Proctor [12] (see [8, Proposition 4.2] for an easy proof in the A_n case) as principal ideals in *Young lattices*, our construction has the advantage of carrying our labeling associated with the chamber system. We then go on to present some of the graphs associated with $\mathcal{C}(W/W_J)$. We conclude Appendix A with an example of a K-contraction of the minuscule chamber system related to E_6 which is neither *rank unimodal* nor *rank symmetric*. This shows that the K-contractions of minuscule posets are not as well-behaved as their non-contracted counterparts, which are both rank unimodal and rank symmetric [13, Theorem 10]. In Appendix B we prove Proposition 4.1.

As a general reference for algebraic groups we cite Borel's book [3] and for information on root systems we refer the reader to Bourbaki [5].

2 Restrictions of Minuscule Representations to Levi-Subgroups

The aim of this section is to prove Theorem 2.4 and to draw the reader's attention to some consequences of it. Let G be a semisimple universal Chevalley group over an algebraically closed field F of arbitrary characteristic. Let Ψ be the set of roots of G relative to a fixed maximal torus T in G with Π a set of fundamental roots in Ψ. Let $B \supset T$ be the Borel subgroup of G with respect to Π and U be the unipotent radical of B, thus $B = TU$. For a subset K in Π let Ψ_K denote the subsystem of roots in Ψ with base K and let $P_K = L_K U_K$ be the standard parabolic subgroup of G with respect to K, where U_K denotes the unipotent radical and L_K the Levi complement of P_K. Finally, we denote by W and W_K, respectively, the Weyl groups of G and L_K relative to T.

For $\lambda \in X(T)$ a dominant weight, $V_G(\lambda)$ is the irreducible FG-module of highest weight λ. If V is a rational FG-module and $\mu \in X(T)$ is a weight, then V_μ denotes the T-weight space of V relative μ.

A set of weights $Y \subseteq X(T)$ is called Ψ-*saturated* provided that for every $\mu \in Y$ and every root $\alpha \in \Psi$ the weight $\mu - t\alpha$ belongs to Y for every integer t between 0 and $\langle \mu, \alpha \rangle$ [6, Ch. VIII, §7.3]. So in particular, if Y is Ψ-saturated, then $s_\alpha \mu$ is in Y.

A dominant weight λ is called Ψ-*minuscule* provided there is no other dominant weight μ satisfying $\mu < \lambda$, where $\leq$ is the usual partial ordering on weights. In particular, we include the trivial case $\lambda = 0$. If λ is Ψ-minuscule, then $V_G(\lambda)$ is called a minuscule module for G.

In [6, Ch. VIII, §7.3] minuscule weights are defined only for indecomposable root systems. However, the following lemma gives a characterization of minuscule weights which holds for the more general situation of a decomposable root system.

Lemma 2.1. *Suppose that $Y \subseteq X(T)$ is Ψ-saturated and $\lambda \in Y$ is Ψ-dominant. Then the following are equivalent:*

1. *λ is Ψ-minuscule.*
2. *$Y = W \cdot \lambda$.*
3. *$\langle \mu, \alpha \rangle \in \{-1, 0, 1\}$ for each $\mu \in Y$ and each $\alpha \in \Psi$.*

Proof: The equivalence of 2.1.2 and 2.1.3 is part of [6, Ch. VIII, §7.3, Prop. 4], while the equivalence of 2.1.1 and 2.1.2 follows from [5, Ch. VI, §7.3, Exerc. 5(d)].

Lemma 2.2. *If λ is Ψ-minuscule, then the set of weights of $V_G(\lambda)$ forms a single orbit under the Weyl group W.*

Proof: This follows from Propositions 4 and 6 in [6, Ch. VIII, §7.3] and the fact that the Weyl module corresponding to λ is irreducible.

Restricting a weight μ in $X(T)$ of G to $T \cap L'_K$ gives rise to a weight for L'_K. We say that a weight μ in $X(T)$ is Ψ_K-dominant, if the restriction of μ to $T \cap L'_K$ is a Ψ_K-dominant weight. Likewise we speak of Ψ_K-minuscule weights and of Ψ_K-saturated sets of weights in $X(T)$. We maintain this language throughout the paper.

Finally, for subsets J and K of Π set

$$D_J = \{w \in W \mid w(\alpha) \in \Psi^+ \text{ for all } \alpha \in J\} \quad \text{and} \quad D_{K,J} = D_K^{-1} \cap D_J.$$

The following properties are well-known, see, for example, [7, 2.7].

Lemma 2.3. *Let J and K be subsets of Π. Then D_J is the set of representatives of left W_J-cosets of W of minimal length and $D_{K,J}$ is the set of minimal length (W_K, W_J) double coset representatives of W.*

Having established our notation we now present the main theorem of this section.

Theorem 2.4. *Suppose that $V = V(\lambda)$ is a minuscule FG-module, and let $P_J = N_G(V_\lambda)$ be the standard parabolic subgroup of G stabilizing the highest weight space V_λ. Let $P_K = L_K U_K$ be any standard parabolic subgroup of G. Then*

$$V(\lambda)|_{L'_K} = \bigoplus_{w \in D_{K,J}} V_{L'_K}(w \cdot \lambda),$$

where each $V_{L'_K}(w \cdot \lambda)$ is a minuscule L'_K-module and $\dim V_{L'_K}(w \cdot \lambda)$ equals the number of left W_J-cosets in $W_K w W_J$. In particular, $V(\lambda)|_{L'_K}$ is completely reducible and the number of L'_K-irreducible summands of $V(\lambda)|_{L'_K}$ is equal to the number of (W_K, W_J) double cosets in W.

Proof: Observe that $C_W(\lambda) = W_J$. Since λ is minuscule, $W \cdot \lambda = \cup_{w \in D_{K,J}} W_K \cdot w\lambda$ is the set of weights of $V(\lambda)$. Now for $w \in D_{K,J}$ and $\gamma \in \Psi_K^+$ we have $w^{-1} \in D_K$, so $w^{-1}\gamma \in \Psi^+$. Therefore, since λ is dominant, for each $\gamma \in \Psi_K^+$, $\langle w\lambda, \gamma \rangle = \langle \lambda, w^{-1}\gamma \rangle \geq 0$. Thus $w \cdot \lambda$ is Ψ_K-dominant. If $\gamma \in \Psi_K^+$ and $x \in W_K$, then, since λ is minuscule, $\langle xw\lambda, \gamma \rangle = \langle \lambda, (xw)^{-1}\gamma \rangle \in \{-1, 0, 1\}$. Whence, $W_K \cdot w\lambda$ is trivially Ψ_K-saturated. It follows by Lemma 2.1 that $w \cdot \lambda$ is Ψ_K-minuscule.

Further, for any $\alpha \in \Psi_K^+$ and $w \in D_{K,J}$ we have $\langle w\lambda + \alpha, \alpha \rangle \geq 2$, as $w\lambda$ is Ψ_K-dominant; and thus, as λ is Ψ-minuscule, the element $w\lambda + \alpha$ is not a weight of $V(\lambda)$ by Lemmas 2.1 and 2.2. Consequently, a non-trivial element in $V_{w\lambda}$ is fixed by the action of the root group X_α for each $\alpha \in \Psi_K^+$ and is therefore necessarily a maximal vector for L'_K. The following more general argument shows that this fact does not require λ to be minuscule.

Let $B_K = L_K \cap B$ be the Borel subgroup of L_K relative to Ψ_K^+. Observe that for $w \in D_K^{-1}$ and $\alpha \in \Psi_K^+$ we have $X_\alpha^w = X_{w^{-1}\alpha} \subseteq U$, since $w^{-1}\alpha \in \Psi^+$. Therefore, for each $w \in D_K^{-1}$, we have $B_K^w \subseteq B$ and thus

$$B_K \cdot V_{w\lambda} = B_K \cdot wV_\lambda = wB_K^w V_\lambda \subseteq wB \cdot V_\lambda = wV_\lambda = V_{w\lambda}.$$

Again we conclude that for each $w \in D_{K,J}$, a non-trivial element in $V_{w\lambda}$ is a maximal vector for L'_K (in fact for L_K). By [10, II.2.13] the submodule $\langle L'_K \cdot V_{w\lambda}\rangle$ is an epimorphic image of the corresponding irreducible Weyl module for L'_K, whence it is itself irreducible for L'_K, and since $w\lambda$ is Ψ_K-minuscule, $W_K \cdot w\lambda$ is its set of weights. It follows that $V(\lambda)|_{L'_K}$ decomposes into a direct sum of irreducible L'_K-modules.

Remark 2.5. The restriction of $V(\lambda)$ to L'_K provides knowledge of the restriction to L_K, as $L_K = L'_K \times T$ and T acts diagonally on any sum of weight spaces.

Example 2.6. Consider the 27-dimensional module $V(\lambda_1)$ for E_6. Let $P_K = P_J = N_G(V_{\lambda_1})$. It follows from Theorem 2.4 that $V(\lambda_1)|_{L'_K}$ decomposes as a direct sum of minuscule L'_K-modules. Using the associated graph for E_6 (see 6.6) and the information provided in Corollary 5.4, it is easily checked that $V(\lambda_1)|_{L'_K}$ decomposes as a direct sum of precisely a trivial, a natural, and a half-spin module for the D_5-Levi complement. Each of these is a minuscule module for D_5.

Remark 2.7. If $p = \mathrm{char} F$ is positive, then we may replace the minuscule weights λ in Theorem 2.4 by the weights $p^a\lambda$ ($a \in \mathbb{Z}^+$), and the statement of Theorem 2.4 also holds for these non-restricted weights. This follows, as $\dim V(p^a\lambda) = \dim V(\lambda)$ and the Frobenius endomorphism commutes with the restriction to a Levi complement.

The weights $p^a\lambda$ are no longer minuscule. However, the set of weights of the modules $V(p^a\lambda)$ still form a single orbit under the Weyl group W. This applies equally to the L'_K-constituents of $V(p^a\lambda)|_{L'_K}$. The equivalences of Lemma 2.1 fail for these non-restricted weights, because $W \cdot p^a\lambda$ is not Ψ-saturated.

Theorem 2.8. *Let F' be a subfield of F and assume that G is defined over F'. Then the statement of Theorem 2.4 still holds if G, L_K, etc. are replaced by their F'-rational points, $G(F')$, $L_K(F')$, etc. operating on the F-vector space $V(\lambda)$.*

Proof: If F' is finite, this follows from [15, Theorem 43] and if F' is infinite from a theorem of Borel and Tits [15, Theorem 42].

Corollary 2.9. *Assume the hypothesis of Theorem 2.8. Then*

$$C_{V(\lambda)}(U \cap L_K) = \langle V_{w\lambda} |\ w \in D_{K,J}\rangle.$$

In particular, $\dim_F C_{V(\lambda)}(U \cap L_K)$ *is equal to the number of* (W_K, W_J) *double cosets in* W.

Proof: This follows from Theorem 2.8 and [15, Theorem 39(d)].

We next point out a corollary to Theorem 2.8 which will form the basis of the proof of Theorem 4.3.

Corollary 2.10. *Assume the hypothesis of Theorem 2.8, and that $\alpha \in \Pi$. Then*

1. $[V(\lambda), X_\alpha, X_\alpha] = 0$;
2. $[V(\lambda), X_\alpha] = \langle V_{w\lambda} |\ w \in D_{\{\alpha\},J},\ s_\alpha w W_J \neq w W_J\rangle$; *and*

3. $\dim_F[V(\lambda), X_\alpha] = \dim_F V(\lambda)/C_{V(\lambda)}(X_\alpha)$ *is equal to the number of* (W_α, W_J) *double cosets in* W *which contain exactly two left* W_J*-cosets in* W.

Proof: Let $L'_\alpha = \langle X_\alpha, X_{-\alpha}\rangle \cong SL_2(F')$ be the derived subgroup of the Levi complement associated with the minimal parabolic subgroup P_α. Then by Theorem 2.8 the restriction $V(\lambda)|_{L'_\alpha}$ is a direct sum of 1 and 2-dimensional L'_α-modules. It follows immediately that $[V(\lambda), X_\alpha, X_\alpha] = 0$, so the first part holds. It follows also that there is a non-trivial commutator with X_α only on the 2-dimensional summands in $V(\lambda)|_{L'_\alpha}$. By Theorem 2.8 the 2-dimensional summands correspond to (W_α, W_J) double cosets in W which contain exactly two left W_J-cosets in W. This proves the remaining parts.

We close this section with a partial converse of Theorem 2.4.

Proposition 2.11. *Let* $\lambda \in X(T)$ *be a dominant weight and* $\emptyset \neq K \subset \Pi$. *Assume that there are roots of different length in* Ψ_K *whenever this is the case for* Ψ. *If each irreducible* L'_K*-composition factor of* $V(\lambda)|_{L'_K}$ *is a minuscule* L'_K*-module, then* $V(\lambda)$ *is a minuscule module for* G. *In particuar, this statement holds if there is only one root length in* Ψ.

Proof: Suppose that each irreducible L'_K-composition factor of $V(\lambda)|_{L'_K}$ is a minuscule L'_K-module. Then for every $w \in W$ the weight $w\lambda$ affords a weight of some minuscule L'_K-module occurring in $V(\lambda)|_{L'_K}$. Thus for each $\gamma \in \Psi_K$ we have $\langle w\lambda, \gamma\rangle \in \{-1, 0, 1\}$ by Lemmas 2.1 and 2.2. Now let α be an arbitrary root in Ψ. It follows from the hypothesis that there exists a root $\gamma \in \Psi_K$ and an element $w \in W$ such that $\alpha = w^{-1}\gamma$. Whence $\langle \lambda, \alpha\rangle = \langle \lambda, w^{-1}\gamma\rangle = \langle w\lambda, \gamma\rangle \in \{-1, 0, 1\}$, which implies that $W \cdot \lambda$ is Ψ-saturated. Thus λ is minuscule by Lemma 2.1.

Remark 2.12. Observe that Proposition 2.11 is independent of the characteristic of F. Further, if the hypothesis on Ψ_K in Proposition 2.11 is not satisfied, then the converse of Theorem 2.4 does not hold. This is illustrated by the following example. For simplicity assume $\mathrm{char} F = 0$. Let G be of type B_n and let $K = \{\alpha_1, \ldots, \alpha_{n-1}\}$. Then L'_K is of type A_{n-1} and Ψ_K consists only of long roots. Consider the natural module $V(\lambda_1)$ for G. Then $V(\lambda_1)|_{L'_K}$ decomposes into a direct sum of three modules for L'_K, a trivial, a natural and its dual, each of which is a minuscule L'_K-module, while $V(\lambda_1)$ is not a minuscule FG-module.

3 The Chamber System $\mathcal{C}(W/W_J)$

In this section Ψ is a root system, Π is a set of fundamental roots of Ψ, $W = \langle s_\alpha |\ \alpha \in \Pi\rangle$ is the Coxeter group corresponding to Ψ, and w_0 is the element in W of maximal length. Our first lemma can be found in [2, Lemma 2.5].

Lemma 3.1. *Suppose that* $J \subseteq \Pi$, $\alpha \in \Pi$ *and assume that* $x \in D_J$. *Then either* $s_\alpha x \in D_J$ *or* $W_\alpha x W_J = xW_J$.

Proof: Suppose that $s_\alpha x \notin D_J$. Then there exists a simple root $\beta \in J$ such that $s_\alpha x(\beta) \in \Psi^-$. But as $x(\beta) \in \Psi^+$, it follows that $x(\beta) = \alpha$. Whence $s_\alpha x = xs_\beta$, and thus $s_\alpha xW_J = xs_\beta W_J = xW_J$.

Lemma 3.2. *Suppose that $K, J \subseteq \Pi$ and $\alpha \in \Pi$. Then $W_\alpha w W_J \subseteq W_K w W_J$, if, and only if, either $s_\alpha \in W_K$, or $s_\alpha w W_J = w W_J$.*

Proof: The 'if' part of the statement is clearly true. Thus we suppose that $W_\alpha w W_J \subseteq W_K w W_J$ with $s_\alpha w W_J \neq w W_J$. Then without loss of generality, we may assume that $w \in D_J$. So, by Lemma 3.1, $s_\alpha w \in D_J$. Since $W_\alpha w W_J \subseteq W_K w W_J$, we can find $x \in W_K$ such that $x w W_J = s_\alpha w W_J$. Choose such an x of minimal length. Then, by repeated application of Lemma 3.1 ($l(x)$-times), we find that $xw \in D_J$. We conclude, from Lemma 2.3, that $xw = s_\alpha w$, so $s_\alpha = x \in W_K$, as claimed.

Suppose that $J \subseteq \Pi$. We define a chamber system structure, indexed over Π, on the cosets W/W_J, as follows: $w_1 W_J \sim_\alpha w_2 W_J$ if and only if $s_\alpha w_1 W_J = w_2 W_J$ and $w_1 W_J \neq w_2 W_J$. We denote this chamber system by $\mathcal{C}(W/W_J)$. A gallery of length n in $\mathcal{C}(W/W_J)$ is a sequence of cosets $w_1 W_J \sim_{\alpha_1} w_2 W_J \ldots \sim_{\alpha_n} w_{n+1} W_J$. We say that two cosets are adjacent if there is a gallery of length one joining them. Observe that $\mathcal{C}(W/W_J)$ is connected. The *length of a coset*, wW_J, denoted by $l_c(wW_J)$, is the length of any shortest gallery joining W_J to wW_J. By Lemma 3.1, the length of a coset wW_J coincides with the length in the Coxeter group of the minimal representative of that coset. There is a unique coset of maximal length, $w_0 W_J$. Observe that the poset structure on $\mathcal{C}(W/W_J)$ defined by chains is precisely the (left-handed) weak Bruhat-Chevalley ordering on the set of cosets W/W_J.

The following lemma follows easily from Lemma 3.1.

Lemma 3.3. *Suppose that $J \subseteq \Pi$ and $xW_J \in \mathcal{C}(W/W_J)$ with $x \in D_J$.*

1. *If $x \neq 1$, then there exists $yW_J \in \mathcal{C}(W/W_J)$ with xW_J incident to yW_J and $l_c(yW_J) < l_c(xW_J)$.*
2. *If $xW_J \neq w_0 W_J$, then there exists $yW_J \in \mathcal{C}(W/W_J)$ with xW_J incident to yW_J and $l_c(yW_J) > l_c(xW_J)$.*

Let J be a subset of Π. We call the corresponding parabolic subgroup W_J of W minuscule provided that W_J is the stabilizer in W of a minuscule weight. Note this definition implies that Coxeter groups of type BC_n have two distinct proper minuscule parabolic subgroups; one arising from the minuscule weight of the B_n root system, the other from the minuscule weight of the C_n root system. Also observe that, in particular, W is itself a minuscule parabolic and that if Ψ is decomposable, then a minuscule parabolic of W need not be maximal.

Lemma 3.4. *Suppose that $\Pi = \{\alpha, \beta\}$ and $J \subseteq \Pi$ with W_J minuscule. Then W is of type $A_1 \times A_1$, A_2 or BC_2 and the structure of each $\mathcal{C}(W/W_J)$ is as presented in Figure 1.*

Proof: By [6, Ch. VIII, §7.3], W is of type $A_1 \times A_1$ and $J \in \{\emptyset, \{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$, or W is of type A_2 or BC_2 and $J \in \{\{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$. The presentations of each of the chamber systems given in Figure 1 are now easily calculated.

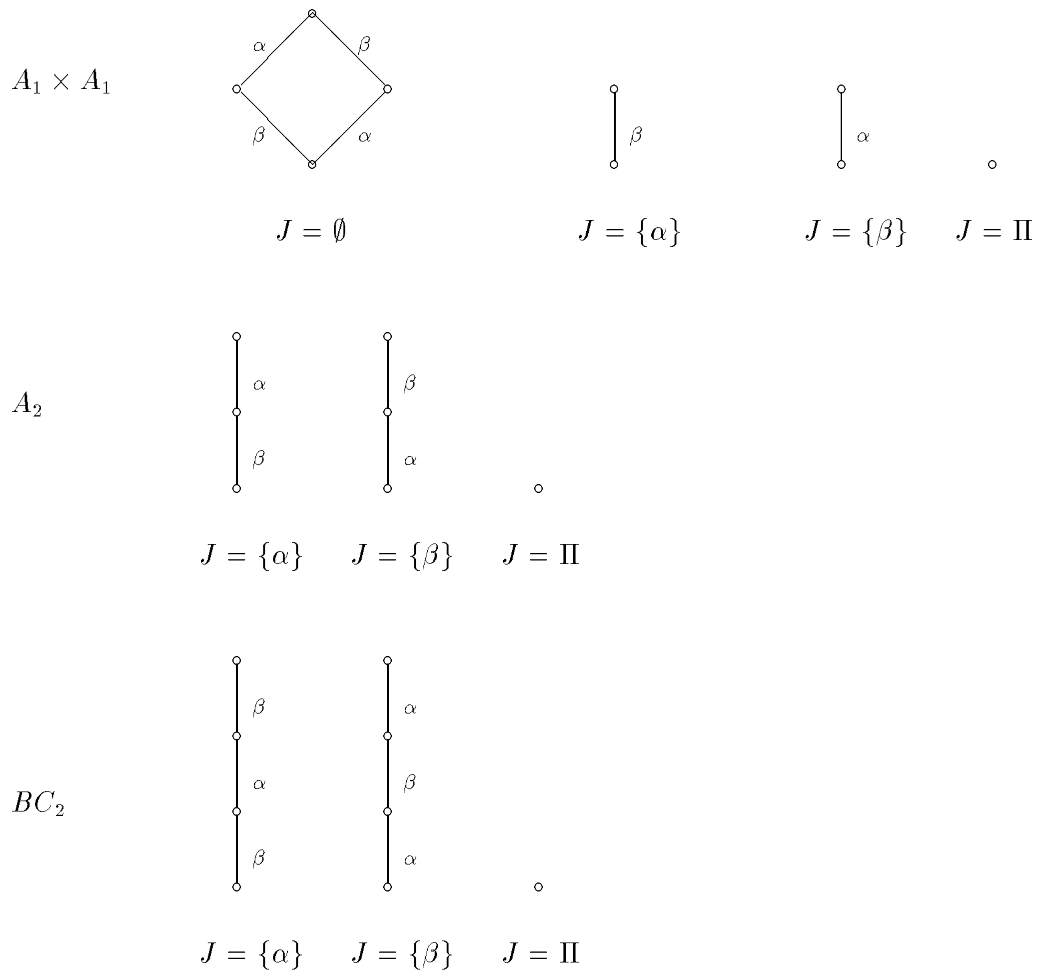


Figure 1

The following proposition is the key result of this section.

Proposition 3.5. *Suppose that $K, J \subseteq \Pi$ with W_J minuscule. Assume that $w \in D_J$, $\{d\} = W_K w W_J \cap D_{K,J}$, and write $w = xd$ where $x \in W_K$ and $l(x)$ is minimal. Then $l_c(s_\alpha w W_J) < l_c(wW_J)$ for some $\alpha \in \Pi \setminus K$ if, and only if, $l_c(s_\alpha dW_J) < l_c(dW_J)$ and s_α and x commute. Furthermore, in this case $W_K s_\alpha d W_J = W_K s_\alpha w W_J$.*

Proof: Clearly we may express w in terms of d and a suitable element $x \in W_K$ which is chosen to be of minimal length. Evidently, if $l_c(s_\alpha dW_J) < l_c(dW_J)$ with $\alpha \notin K$ and if s_α and x commute, then $l_c(s_\alpha wW_J) < l_c(wW_J)$ and $W_K s_\alpha dW_J = W_K s_\alpha wW_J$. So we assume that $l_c(s_\alpha wW_J) < l_c(wW_J)$ and aim to prove, by induction on $l(x)$, that $l_c(s_\alpha dW_J) < l_c(dW_J)$ and s_α and x commute. If $w = d$, then we are done. So assume that $w \neq d$. Then, there exists $\tau \in K$ such that $s_\tau wW_J \subseteq W_K wW_J$ and $l(s_\tau w) = l_c(s_\tau wW_J) < l_c(wW_J) = l(w)$. Now we consider the double coset $W_{\alpha\tau}wW_J$. By Theorem 2.4, this coset corresponds to a minuscule module for $L'_{\alpha\tau}$, so we find ourselves in the situation of Lemma 3.4 (and Figure 1). Thus, as both $l_c(s_\tau wW_J) < l_c(wW_J)$ and $l_c(s_\alpha wW_J) < l_c(wW_J)$, we infer that $W_{\alpha\tau}$ is of type $A_1 \times A_1$. Hence s_α and s_τ commute and $l_c(s_\alpha s_\tau wW_J) < l_c(s_\tau wW_J)$. Finally, by induction, $l_c(s_\alpha dW_J) < l_c(dW_J)$ and s_α and $s_\tau x$ commute; as s_τ and s_α also commute, we conclude that s_α and x commute.

Notice that for $K \subseteq \Pi$, the union in W of the cosets wW_J corresponding to a K-connected component in $\mathcal{C}(W/W_J)$ is a (W_K, W_J) double coset of W. Because of Lemma 3.2, we may contract any K-connected component of $\mathcal{C}(W/W_J)$ to a point, without forming loops in the resulting graph. We call the chamber system over $\Pi \backslash K$, constructed in this way, the K-contraction of $\mathcal{C}(W/W_J)$, and we denote it by $\mathcal{C}_K(W/W_J)$. Hence, every point of $\mathcal{C}_K(W/W_J)$ represents a (W_K, W_J) double coset of W. We set $c_0 = W_K w_0 W_J$ and $c_1 = W_K 1 W_J$. A gallery in $\mathcal{C}_K(W/W_J)$, $e_1 \sim_{\alpha_1} e_2 \sim_{\alpha_2} \dots \sim_{\alpha_{n-1}} e_n$, where each $\alpha_i \in \Pi \backslash K$, is *strictly descending*, if, for each $1 \leq i < n$, there exist $wW_J \in W_K wW_J = e_i$ with $l_c(s_{\alpha_i} wW_J) < l_c(wW_J)$. If $c \in \mathcal{C}_K(W/W_J)$, then the *$K$-depth* of c, $\mathrm{dep}_K(c)$, is the length of a shortest strictly descending gallery from c_0 to c.

Corollary 3.6. *Suppose that $K, J \subseteq \Pi$ with W_J minuscule. If $c \in \mathcal{C}_K(W/W_J)$ with $c \neq c_1$, then there exist $\alpha \in \Pi \setminus K$ and $e \in \mathcal{C}_K(W/W_J)$ such that $c \sim_\alpha e$ is strictly descending; moreover, for any given α in $\Pi \setminus K$, there is at most one such e.*

Proof: Suppose that $c = W_K dW_J$, where $d \in D_{K,J}$ and, by assumption $d \neq 1$. Then, by Proposition 3.5, if $e \in \mathcal{C}_K(W/W_J)$ and $c \sim_\alpha e$ is strictly descending, then $e = W_K s_\alpha dW_J$, whence the uniqueness part of the statement holds. The existence part follows from Lemma 3.3.1 as $d \neq 1$.

Corollary 3.7. *Let $K, J \subseteq \Pi$ with W_J minuscule and K maximal. Then $\mathcal{C}_K(W/W_J)$ is a string.*

Proof: Let α be the element of Π not in K. Then, by Corollary 3.6, given $c \in \mathcal{C}_K(W/W_J)$ with $c \neq c_1$, there exists a unique $e \in \mathcal{C}_K(W/W_J)$ with $\mathrm{dep}_K(e) = \mathrm{dep}_K(c) + 1$. As there is a unique maximal element in $\mathcal{C}_K(W/W_J)$ we are done.

Lemma 3.8. *Assume that $K, J \subseteq \Pi$ with W_J minuscule and suppose that $c \in \mathcal{C}_K(W/W_J)$. Then for any $\alpha \in \Pi \setminus K$ the number of occurrences of an α-incidence in any strictly descending gallery from c_0 to c is independent of the gallery chosen. In particular, any two strictly descending galleries from c_0 to c are of the same length.*

Proof: In the case that K is maximal this is just Corollary 3.7. Thus we assume that K is not maximal. Suppose that $c \in \mathcal{C}_K(W/W_J)$. Let $X = (c_0, x_1, \dots, x_n = c)$ and $Y = (c_0, y_1, \dots, y_m = c)$ be two strictly descending galleries from c_0 to c. Choose $\alpha \in \Pi \setminus K$ and enlarge K to a maximal subset K_α of Π not containing α. Then, in passing from $\mathcal{C}_K(W/W_J)$ to $\mathcal{C}_{K_\alpha}(W/W_J)$, Lemma 3.2, Proposition 3.5 and Corollary 3.7 imply that X and Y contract to the same gallery. In particular, there are the same number of occurrences of α in both X and Y, which is our first claim. Finally, as the choice of $\alpha \in \Pi \setminus K$ was arbitrary, it follows that the galleries X and Y have equal length.

For $c = W_K w W_J \in \mathcal{C}_K(W/W_J)$ we define the *K-level* of c by $\mathrm{lev}_K(c) = \mathrm{dep}_K(c_1) - \mathrm{dep}_K(c)$, where, as above, $c_1 = W_K 1 W_J$. In other words, the K-level of $c \in \mathcal{C}_K(W/W_J)$ is the length of a shortest gallery from c_1 to c in $\mathcal{C}_K(W/W_J)$. This is in a sense dual to the K-depth introduced earlier.

We conclude this section with two results concerning the associated poset structure of $\mathcal{C}_K(W/W_J)$ defined by ascending galleries. The importance of Proposition 3.5 and the meaning of the definition of K-level is revealed in the following propositions.

Proposition 3.9. *Assume that $K, J \subseteq \Pi$ with W_J minuscule. Then the poset associated with the chamber system $\mathcal{C}_K(W/W_J)$ is ranked by K-level.*

Proof: This follows readily from the definition of K-level above, Corollary 3.8, and the definition of a ranked poset (for instance see [9, Ch. V, §1]).

Finally, we compare the poset structure on $\mathcal{C}_K(W/W_J)$ to the weak Bruhat-Chevalley ordering on $D_{K,J}$ inherited from W.

Proposition 3.10. *Let $J, K \subseteq \Pi$ with W_J minuscule. Then the posets $D_{K,J}$, endowed with the weak Bruhat-Chevalley order induced from W, and $\mathcal{C}_K(W/W_J)$ are equivalent. In particular, the poset $D_{K,J}$ is ranked and in both posets any two maximal chains have the same length $\mathrm{lev}_K(c_0) = \mathrm{dep}_K(c_1)$.*

Proof: Granting the equivalence of the posets, the remaining statements follow from Proposition 3.9. Let $\preceq$ denote the weak ordering on W. For $d_1, d_2 \in D_{K,J}$ let $e_1 = W_K d_1 W_J$ and $e_2 = W_K d_2 W_J$. We need to show that $d_1 \preceq d_2$, if and only if, there is a chain in $\mathcal{C}_K(W/W_J)$ from e_1 to e_2.

Suppose that $d_1 \prec d_2$ in the weak order on W. Then any chain in $D_{K,J}$ from d_2 to d_1 can be reduced to a strictly descending gallery in $\mathcal{C}_K(W/W_J)$ from e_2 to e_1.

Conversely, suppose that there is a strictly descending chain in $\mathcal{C}_K(W/W_J)$ from e_2 to e_1; in particular, $e_1 \neq e_2$. Then, without loss we may assume that e_1 and e_2 are adjacent. It follows from the definition of adjacency in $\mathcal{C}_K(W/W_J)$ that there exist elements w_1 and w_2 in D_J and a root $\alpha \in \Pi \setminus K$ satisfying $e_1 = W_K w_1 W_J$, $e_2 = W_K w_2 W_J$, and $e_1 = W_K s_\alpha w_2 W_J$. Now Proposition 3.5 implies that $e_1 = W_K s_\alpha d_2 W_J$. By [7, Prop. 2.7.5] there exist elements $x \in W_K$ and $y \in W_J$ such that $s_\alpha d_2 = x d_1 y$ and additionally $l(x d_1 y) = l(x) + l(d_1) + l(y)$. In particular, $l(x d_1) = l(x) + l(d_1)$, which implies that $d_1 \preceq x d_1$. Moreover, since $s_\alpha d_2 W_J \neq d_2 W_J$, we have $s_\alpha d_2 \in D_J$ by Lemma 3.1. Then, since $s_\alpha d_2 W_J = x d_1 W_J$, we infer that $l(x d_1) \geq l(s_\alpha d_2) = l(x d_1) + l(y)$. Therefore, $y = 1$ and $s_\alpha d_2 = x d_1$. Finally, we conclude $d_1 \preceq x d_1 = s_\alpha d_2 \prec d_2$; the last inequality holds, as $l(s_\alpha d_2) < l(d_2)$ by Proposition 3.5 and $s_\alpha(s_\alpha d_2) = d_2$. Whence, $d_1 \prec d_2$, as desired. This completes the proof of the proposition.

Remark 3.11. The hypothesis in Propositions 3.5, 3.9, and 3.10 that W_J is minuscule is crucial. That the assertions do in general not hold when W_J is not minuscule, can be seen, for example, by considering the A_2 system with $J = \emptyset$ and K singleton or the F_4 system with $J = \{\alpha_2, \alpha_3, \alpha_4\}$ maximal in Π and $K = \{\alpha_1, \alpha_3, \alpha_4\}$ [14, 2.5.10].

4 Commutator Series in Minuscule Modules

Let G be a semisimple Chevalley group defined over a field F and let $P_K = L_K U_K$ be a parabolic subgroup of G ($K \subset \Pi$). Throughout this section we assume that G does not admit simple components of type G_2. This is not a great loss, as the only minuscule module for G_2 is the trivial one. In the next proposition we use the following notation. If A is a collection of subgroups in G, then $\langle A\rangle^{L_K}$ denotes the smallest L_K-invariant subgroup of G containing the subgroup $\langle A\rangle$ generated by the elements of A.

Proposition 4.1. *Let G be a semisimple Chevalley group defined over a field F and let $P_K = L_K U_K$ be a parabolic subgroup of G. In case G admits simple components of type B_n, C_n, or F_4 and char$F = 2$, assume that $|F| \geq 4$. Then*

$$U_K = \langle X_\alpha | \alpha \in \Pi \setminus K\rangle^{L_K}.$$

In particular, the statement holds when F is algebraically closed.

We postpone the proof of Proposition 4.1 until Appendix B as to not interrupt the present development.

Remark 4.2. The restriction $|F| \geq 4$ in 4.1 can not be removed, for the result is not true if F only has two elements, as is shown by this example: a Sylow 2-subgroup of $Sp_4(2)$ is isomorphic to the direct product of a Dihedral group of order 8 and a cyclic group of order 2 and thus can be generated by three but not by two elements.

In the forthcoming theorem and its corollaries we define the K-depth of $w \in W$ as $\mathrm{dep}_K(w) = \mathrm{dep}_K(W_K w W_J)$. Also, similarly to above, if A is a set of subspaces of an FG-module V, then $\langle A\rangle^{L_K}$ denotes the smallest L_K-invariant F-subspace of V containing the F-subspace of V spanned by A.

Theorem 4.3. *Suppose that $V = V(\lambda)$ is a minuscule FG-module, and let $P_J = N_G(V_\lambda)$. Further, let $P_K = L_K U_K$ be any standard parabolic subgroup of G. Then for $r \in \mathbb{Z}^+$*

$$[V, U_K; r] = \langle V_{w\lambda}|\ w \in D_{K,J},\ dep_K(w) \geq r\ \rangle^{L_K} = \langle W_K V_{w\lambda}|\ w \in D_{K,J},\ dep_K(w) \geq r\ \rangle.$$

Proof: For this proof we represent the 1-spaces of V that correspond to weight spaces by the cosets wW_J where $w \in D_J$. We also define for each $m \in \mathbb{Z}^+$, $Y_m = \langle W_K w W_J |\ \mathrm{dep}_K(w) \geq m\rangle \subseteq V$. Notice that, by Theorem 2.8, Y_m is an L'_K-invariant subspace of V which is also invariant under T, whence invariant under L_K.

Suppose now that wW_J is any weight space in V. Then, by Corollary 2.10.2, for $\alpha \in \Pi$, $[wW_J, X_\alpha] \neq 0$ if and only if, $[wW_J, X_\alpha] = s_\alpha w W_J$, with $s_\alpha w \in D_{\{\alpha\},J}$ and $s_\alpha w W_J \neq w W_J$, which, using the notation of Section 3 and Lemma 3.1, is if and only if, $l_c(wW_J) > l_c(s_\alpha w W_J)$. Now, for $wW_J \subseteq W_K d W_J$, where $d \in D_{K,J}$, and $\alpha \in \Pi \setminus K$, it follows from Proposition 3.5 that $l_c(wW_J) > l_c(s_\alpha w W_J)$ if, and only if $l_c(dW_J) > l_c(s_\alpha d W_J)$, and in this case $W_K s_\alpha d W_J = W_K s_\alpha w W_J \supseteq [wW_J, X_\alpha]$. Therefore, for $wW_J \subseteq W_K d W_J$, we have $0 \neq [wW_J, X_\alpha]$, if and only if, $\mathrm{dep}_K(W_K s_\alpha w W_J) = \mathrm{dep}_K(W_K d W_J) + 1$, and, if this is the case, then $[W_K d W_J, X_\alpha] \subseteq W_K s_\alpha w W_J$.

Whence, we have proven that for all $\alpha \in \Pi \setminus K$, $[W_K dW_J, X_\alpha] \subseteq W_K s_\alpha w W_J$, where $\mathrm{dep}_K(W_K s_\alpha w W_J) = \mathrm{dep}_K(W_K w W_J) + 1$. It now follows immediately that for each $m \in \mathbb{Z}^+$,

$$[Y_m, \langle X_\alpha | \alpha \in \Pi \setminus K \rangle] \subseteq Y_{m+1}.$$

Thus, as each Y_m, $m \in \mathbb{Z}^+$, is an L_K-invariant subspace of V, Proposition 4.1 implies that

$$[Y_m, U_K] = [Y_m, \langle X_\alpha | \alpha \in \Pi \setminus K \rangle^{L_K}] \subseteq Y_{m+1}.$$

In particular, as $V = Y_0$, we have $[V, U_K; m] \subseteq Y_m$, for each $m \in \mathbb{Z}^+$, and this is half of what we wanted to prove. On the other hand, if $W_K w W_J$ is any coset of depth m, then there exists a strictly descending gallery in $\mathcal{C}_K(W/W_J)$,

$$c_0 = W_K w_0 W_J \sim_{\alpha_1} W_K w_1 W_J \ldots \sim_{\alpha_{m-1}} W_K w_{m-1} W_J \sim_{\alpha_m} W_K w_m W_J = W_K w W_J,$$

joining c_0 to $W_K w W_J$. Now, $0 \neq [W_K w_0 W_J, X_{\alpha_1}] \subseteq W_K w_1 W_J$, whence

$$W_K w_1 W_J = [W_K w_0 W_J, X_{\alpha_1}]^{W_K} \subseteq [W_K w_0 W_J, U_K].$$

By iterating this procedure it follows at once that $W_K w W_J \subseteq [V, U_K; m]$ and therefore, $Y_m \subseteq [V, U_K; m]$ for each $m \in \mathbb{Z}^+$. This completes the proof of the theorem.

Putting Theorems 2.8 and 4.3 together we obtain the following two corollaries:

Corollary 4.4. *Assume the hypothesis of Theorem 4.3. Then for $r \in \mathbb{Z}^+$*

$$\dim_F [V(\lambda), U_K; r] = \sum_{\substack{w \in D_{K,J} \\ dep_K(w) \geq r}} \dim_F V_{L'_K}(w \cdot \lambda).$$

Corollary 4.5. *Assume the hypothesis of Theorem 4.3. Then for $r \in \mathbb{Z}^+$*

$$[V(\lambda), U_K; r]/[V(\lambda), U_K; r+1] \cong \bigoplus_{\substack{w \in D_{K,J} \\ dep_K(w) = r}} V_{L'_K}(w \cdot \lambda).$$

Corollary 4.6. *Assume the hypothesis of Theorem 4.3. If K is maximal in Π, then, for each $1 \leq r \leq dep_K(1)$, we have*

$$[V(\lambda), U_K; r]/[V(\lambda), U_K; r+1] \cong V_{L'_K}(w \cdot \lambda)$$

which is an irreducible L'_K-module of highest weight $w \cdot \lambda$, where $dep_K(w) = r$. In particular, $V(\lambda)|_{P_K}$ is a uniserial P_K-module.

Proof: Combine Corollaries 4.5 and 3.7.

For any FG-module, V, we define $m(V, K) = m(V, U_K) = \min\{r \in \mathbb{Z} \mid [V, U_K; r] = 0\}$. When K is a maximal subset of Π and V is a minuscule G-module, then, by Theorem 2.4 and Corollary 4.6

$$m(V, K) = \mathrm{dep}_K(1) + 1 = |D_{K,J}|.$$

Now by Lemma 3.8, if K is any subset of Π, then

$$\mathrm{dep}_K(1) = \sum_{\substack{\Pi \supseteq M \supseteq K \\ M \text{ max. in } \Pi}} \mathrm{dep}_M(1).$$

Whence we have:

Corollary 4.7. *Assume the hypothesis of Theorem 4.3. Then*

$$m(V(\lambda), K) = 1 + \sum_{\substack{\Pi \supsetneq M \supseteq K \\ M \text{ max. in } \Pi}} dep_M(1) = 1 + \sum_{\substack{\Pi \supsetneq M \supseteq K \\ M \text{ max. in } \Pi}} (|D_{M,J}| - 1).$$

In the following table we present the length of the commutator series of U_K, K maximal in Π, on each of the classical minuscule modules $V = V(\lambda)$. We use the standard notation for the minuscule modules and set $\{k\} = \Pi \setminus K$.

Type of V	$m(V,K) = \lvert D_{K,J} \rvert$	Condition
$A(n,j)$	$k+1$	$k \le j$
	$j+1$	$j \le k \le n-j+1$
	$n-k+2$	$k \ge n-j+1$
$B(n,n)$	$k+1$	
$C(n,1)$	3	$k \le n-1$
	2	$k = n$
$D(n,1)$	3	$k \le n-2$
	2	$k = n-1,\ n$
$D(n,n)$	$k+1$	$k \le n-2$
	$(n+2)/2$	n even, $k = n$
	$n/2$	n even, $k = n-1$
	$(n+1)/2$	n odd, $k = n-1,\ n$

Table 1

Remark 4.8. In Table 1 we present the information for $A(n,j)$ only for $2j \le n+1$. The remaining cases can be obtained by duality under a graph automorphism; in particular, $A(n,j)$ is dual to $A(n,n-j)$. The same applies to the modules $D(n,n-1)$ and $D(n,n)$.

With the aid of Table 1 we can sum over all the maximal subsets of Π and use Corollary 4.7 to get

Corollary 4.9. *Assume that $V = V(\lambda)$ is a minuscule FG-module.*

1. *If $V = A(n,j)$, then $m(V,U) = j(n-j+1)+1$.*
2. *If $V = B(n,n)$, then $m(V,U) = (n^2+n+2)/2$.*
3. *If $V = C(n,1)$, then $m(V,U) = 2n$.*
4. *If $V = D(n,1)$, then $m(V,U) = 2n-1$.*
5. *If $V = D(n,n)$ or $D(n,n-1)$, then $m(V,U) = (n^2-n+2)/2$.*

Our last result in this section describes a connection between the L'_K-module structure of the successive quotients in the commutator series $[V(\lambda), U_K; r]$ for $r \in \mathbb{Z}^+$ and the weak Bruhat-Chevalley ordering on $D_{K,J}$ inherited from W. By Proposition 3.10 the posets $D_{K,J}$ and $\mathcal{C}_K(W/W_J)$ are equivalent and ranked provided W_J is minuscule. Thus we may speak of the K-depth or the K-level of elements in $D_{K,J}$. The following is a direct consequence of Proposition 3.10 and Corollary 4.5.

Proposition 4.10. *Suppose that $V = V(\lambda)$ is a minuscule FG-module with $P_J = N_G(V_\lambda)$. Further, let $P_K = L_K U_K$ be any standard parabolic subgroup of G. Then*

1. *The number of irreducible L'_K-summands in $[V, U_K; r]/[V, U_K; r+1]$ equals the number of elements in $D_{K,J}$ of K-depth r, or equivalently, of K-level $lev_K(c_0) - r$ for $r \in \mathbb{Z}^+$.*
2. *The number of non-trivial quotients in V as in part 1., which is $m(V, K) - 1$, equals the number of K-levels that occur in $D_{K,J}$, which equals $dep_K(c_1) = lev_K(c_0)$.*

We conclude this section by remarking that, since the dual of a minuscule module is also minuscule, we have results analogous to Theorem 4.3 and its corollaries for centralizer series.

5 The Weyl Group Elements g_K

In this section we continue the notation of Sections 3 and 4. Specifically, we recall the notion of K-level in $\mathcal{C}_K(W/W_J)$ from Section 3. Additionally, if $K \subseteq \Pi$, then w_K^0 denotes the element of maximal length in W_K. Throughout this section K is a maximal subset of Π, $\{k\} = \Pi \setminus K$ and $K' = K \cap k^\perp$ is the set of simple roots in K that are orthogonal to the root k.

Proposition 5.1. *Suppose that J, $K \subseteq \Pi$ with W_J minuscule and K maximal. Define*

$$g_K = s_k w_{K'}^0 w_K^0 = w_{K'}^0 s_k w_K^0.$$

Then

1. *$\{g_K^i \mid 0 \leq i < |D_{K,J}|\}$ is a complete set of (W_K, W_J) double coset representatives in W. In particular, $W = W_K \langle g_K \rangle W_J$.*
2. *$\{dW_J \mid d \in D_{K,J}\} = \{g_K^i W_J \mid 0 \leq i < |D_{K,J}|\}$.*
3. *If $d \in D_{K,J}$ with $dW_J = g_K^i W_J$, then $lev_K(W_K dW_J) = i$.*

Proof: Clearly, 5.1.2 implies 5.1.1. We prove 5.1.2 and 5.1.3 by induction on i. Certainly, $g_K^0 W_J = W_J$ and $\mathrm{lev}_K(W_K 1 W_J) = 0$. Assume that $g_K^i W_J = dW_J$ where $\mathrm{lev}_K(W_K dW_J) = i$. Suppose additionally that $i < |D_{K,J}| - 1$. This means that $W_K dW_J \neq W_K w_0 W_J$ and so, by Corollary 3.7, there is a unique element $c \in \mathcal{C}_K(W/W_J)$ with $\mathrm{lev}_K(c) = \mathrm{lev}_K(W_K dW_J) + 1$. Now we apply g_K in its constituent parts. First, $w_K^0 dW_J \subseteq W_K dW_J$, and, as $d \in D_K^{-1}$, $l(w_K^0 d) = l(w_K^0) + l(d)$ ([7, Prop. 2.3.3]). Hence we conclude that $w_K^0 dW_J$ has maximal coset-length among all the left W_J-cosets in $W_K dW_J$. Therefore, as $w_K^0 dW_J \neq w_0 W_J$, Lemma 3.3 implies $l_c(s_k w_K^0 dW_J) > l_c(w_K^0 dW_J)$ and $c = W_K s_k w_K^0 dW_J$. Set $\{d_1\} = c \cap D_{K,J}$. Now, by Proposition 3.5, $s_k w_K^0 dW_J \subseteq W_{K'} d_1 W_J$, and moreover, by construction, $l_c(s_k w_K^0 dW_J)$ is maximal in $W_{K'} s_k w_K^0 dW_J$. But, then $w_{K'}^0 d_1 W_J = s_k w_K^0 dW_J$ and so we have $d_1 W_J = w_{K'}^0 s_k w_K^0 dW_J = g_K dW_J = g_K^{i+1} W_J$, as desired.

Remark 5.2. The element g_K defined in Proposition 5.1 is independent of the choice of the minuscule parabolic subgroup W_J of W.

The next result is a reformulation of Theorem 2.4 and Corollary 4.6 in terms of the elements g_K.

Corollary 5.3. *Suppose that $V = V(\lambda)$ is a minuscule FG-module and set $P_J = N_G(V_\lambda)$. Let $P_K = L_K U_K$ be any maximal standard parabolic subgroup in G, set $m = m(V, K)$ and let g_K be as in Proposition 5.1. Then*

1. $V(\lambda)|_{L'_K} = \bigoplus_{i=1}^m V_{L'_K}(g_K^i \cdot \lambda)$.
2. $[V(\lambda), U_K; r-1]/[V(\lambda), U_K; r] \cong V_{L'_K}(g_K^{m-r} \cdot \lambda)$ *for* $1 \leq r \leq m$.

Proof: This follows from Theorem 2.4, Corollaries 4.6, 4.7, and Proposition 5.1.

Our next lemma presents a more usable realization of g_K in the case when W is of classical type.

Lemma 5.4. *Suppose that J, $K \subseteq \Pi$ with W_J minuscule and K maximal with $\{k\} = \Pi \setminus K$.*

1. *If W is of type A_n, then* $g_K = s_k s_{k-1} \cdots s_1 s_{k+1} \cdots s_n$.
2. *If W is of type BC_n, then* $g_K = s_k s_{k-1} \cdots s_1 s_{k+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{k+1}$.
3. *Suppose W is of type D_n.*
   (a) *If $K \supseteq \{n-2, n-1, n\}$, then* $g_K = s_k s_{k-1} \cdots s_1 s_{k+1} \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_{k+1}$.
   (b) *If $n-2 \notin K$, then* $g_K = s_{n-2} s_{n-3} \cdots s_1 s_{n-1} s_n$.
   (c) *If $n-1 \notin K$, then* $g_K = s_{n-1} s_{n-2} \cdots s_1 s_n s_{n-2} \cdots s_2$.
   (d) *If $n \notin K$, then* $g_K = s_n s_{n-2} \cdots s_1 s_{n-1} s_{n-2} \cdots s_2$.

Proof: Suppose that W is of type A_n and let K be a maximal subset of Π. Then K divides the Dynkin-diagram of W into two connected pieces with associated fundamental systems denoted by K_1 and K_2 (one of which may be empty). Now $w_{K'}^0 w_K^0 = w_{K_1'}^0 w_{K_1}^0 w_{K_2'}^0 w_{K_2}^0$. So to describe g_K we only need to describe the elements $w_{K_1'}^0 w_{K_1}^0$ and $w_{K_2'}^0 w_{K_2}^0$. Note that $d = w_{K_1'}^0 w_{K_1}^0$ is the minimal right coset representative of $W_{K_1'} w_K^0$ relative to W_{K_1}. Then, as K_1' is a minuscule subset of K_1, the element d of W_{K_1} is as presented in 6.1 of Appendix A of this paper, $w_{K_1'}^0 w_{K_1}^0 = s_{k-1} \cdots s_1$. Likewise, we find $w_{K_2'}^0 w_{K_2}^0 = s_{k+1} \cdots s_n$. This proves the first part. The BC_n and D_n cases follow similarly using the descriptions of $\mathcal{C}(W/W_J)$ given in 6.1, 6.4 and 6.5 in Appendix A. The only difference being the nodes $n-1$ and n when the type is D_n. In which case one has to describe the element of maximal length in $D_{K_1'}$ (relative to K_1) when K_1' does not contain the second node of a Dynkin-diagram of type A_{n-1}. However, this element can also be read directly from the information provided in 6.2 of Appendix A, and thus we obtain the descriptions of g_K given in parts 3(c) and 3(d).

In the next lemma $\{\lambda_i | \ 1 \leq i \leq n\}$ denotes the set of fundamental weights associated with Π. We also define $\lambda_0 = \lambda_{n+1} = 0$.

Lemma 5.5. *Suppose that $V = V(\lambda)$ is a minuscule G-module with $P_J = N_G(V_\lambda)$. Let P_K be any maximal standard parabolic subgroup in G. Set $\{k\} = \Pi \setminus K$. Then for $1 \leq i < |D_{K,J}|$ the following hold:*

1. *If G is of type A_n and if $j \leq k$, then $g_K^i \cdot \lambda_j = \lambda_{j-i} - \lambda_k + \lambda_{k+i}$.*
2. *Suppose that G is of type B_n.*
   - (a) *If $k \neq n$, then $g_K^i \cdot \lambda_n = \lambda_{k-i} - \lambda_k + \lambda_n$.*
   - (b) *If $k = n$, then $g_K^i \cdot \lambda_n = \lambda_{k-i} - \lambda_n$.*
3. *Suppose that G is of type C_n.*
   - (a) *If $n \in K$, then $g_K \cdot \lambda_1 = \lambda_{k+1} - \lambda_k$ and $g_K^2 \cdot \lambda_1 = \lambda_{k-1} - \lambda_k$.*
   - (b) *If $n \notin K$, then $g_K \cdot \lambda_1 = \lambda_{n-1} - \lambda_n$.*
4. *Suppose that G is of type D_n and $\lambda = \lambda_1$.*
   - (a) *If $K \supset \{n-2, n-1, n\}$, then $g_K \cdot \lambda_1 = \lambda_{k+1} - \lambda_k$ and $g_K^2 \cdot \lambda_1 = \lambda_{k-1} - \lambda_k$.*
   - (b) *If $n-2 \notin K$, then $g_K \cdot \lambda_1 = \lambda_n + \lambda_{n-1} - \lambda_{n-2}$ and $g_K^2 \cdot \lambda_1 = \lambda_{n-3} - \lambda_{n-2}$.*
   - (c) *If $n-1 \notin K$, then $g_K \cdot \lambda_1 = \lambda_n - \lambda_{n-1}$.*
   - (d) *If $n \notin K$, then $g_K \cdot \lambda_1 = \lambda_{n-1} - \lambda_n$.*
5. *Suppose that G is of type D_n and $\lambda = \lambda_n$.*
   - (a) *If $K \supset \{n-1, n\}$, then $g_K^i \cdot \lambda_n = \lambda_{k-i} - \lambda_k + \lambda_{n-1}$, i odd, $g_K^i \cdot \lambda_n = \lambda_{k-i} - \lambda_k + \lambda_n$, i even.*
   - (b) *If $n-1 \notin K$, then $g_K^i \cdot \lambda_n = \lambda_{n-(2i+1)} - \lambda_{n-1}$.*
   - (c) *If $n \notin K$, then $g_K^i \cdot \lambda_n = \lambda_{n-2i} - \lambda_n$.*

Proof: Use the description of g_K given in Lemma 5.4.

The information provided in Lemma 5.5 in conjunction with Corollary 5.3 gives us a precise description of the commutator series of the unipotent radical of a maximal parabolic subgroup on a minuscule FG-module.

Example 5.6. *Suppose that G has type B_7 and $V = B(7,7)$ is the spin module for G. Assume that $K = \Pi \setminus \{\alpha_4\}$. Then L'_K is of type $A_3 \times B_3$ and*

$$V|_{P_K} = B(3,3)/(A(3,3) \otimes B(3,3))/(A(3,2) \otimes B(3,3))/(A(3,1) \otimes B(3,3))/B(3,3).$$

For $K \subset \Pi$ maximal define $\tilde{K} = K \cup \{-\rho\}$, where $\rho = \sum n_\beta(\rho)\beta$ (the sum over all $\beta \in \Pi$) is the highest root in Ψ. Then $\tilde{K}$ is the base for a subsystem of roots of Ψ (which may be decomposable) of the same rank as Ψ. The corresponding subsystem subgroup of G, denoted by $G_{\tilde{K}}$, is a so-called maximal-rank subgroup of G, see [4]. The next result provides the analogue of Theorem 2.4 for the maximal-rank subgroups of G. More precisely, it forms the iterative base from which the restrictions of a minuscule FG-module to any maximal-rank subgroup can be determined (*loc. cit.*).

Theorem 5.7. *Suppose that $V = V(\lambda)$ is a non-trivial minuscule FG-module with $P_J = N_G(V_\lambda)$. Let $G_{\tilde{K}}$ be the maximal-rank subgroup of G corresponding to $\tilde{K} = K \cup \{-\rho\}$, where K is maximal in Π. Let $\{\alpha\} = \Pi \setminus K$, $m = m(V,K)$, $r = n_\alpha(\rho)$, and let g_K be as in Proposition 5.1. Then*

$$V(\lambda)|_{G_{\tilde{K}}} = \bigoplus_{i=1}^{r} V_{G_{\tilde{K}}}(g_K^{m-i} \cdot \lambda),$$

where each direct summand is a minuscule $G_{\tilde{K}}$-module. In particular, the number of $G_{\tilde{K}}$-summands in $V|_{G_{\tilde{K}}}$ is equal to the number of $(W_{\tilde{K}}, W_J)$ double cosets in W is equal to $n_\alpha(\rho)$.

Proof: We argue as in the proof of Theorem 2.4 that any $\Psi_{\tilde{K}}$-dominant weight μ is automatically $\Psi_{\tilde{K}}$-minuscule and any non-zero vector in a weight space corresponding to μ is a maximal vector for $G_{\tilde{K}}$. Because there is a unique $\Psi_{\tilde{K}}$-dominant weight in each $W_{\tilde{K}}$-orbit on $W \cdot \lambda$, arguing again as in the proof of Theorem 2.4 we conclude that $V|_{G_{\tilde{K}}}$ decomposes as a direct sum of minuscule $G_{\tilde{K}}$-modules and the number of summands is precisely the number of $(W_{\tilde{K}}, W_J)$ double cosets in W. Thus, to complete the proof of the theorem, it only remains to find the $\Psi_{\tilde{K}}$-dominant weights in $W \cdot \lambda$. Clearly, these weights are, in particular, Ψ_K-dominant. Whence any maximal vector for $G_{\tilde{K}}$ is also a maximal vector for L'_K. Therefore, Theorem 2.4 and Proposition 5.1 imply that the candidates for $\Psi_{\tilde{K}}$-dominant weights lie in the set $\mathcal{S} = \{g_K^i \cdot \lambda \mid 0 \leq i < |D_{K,J}|\}$. In the two cases when G is exceptional, our claims are easily checked by using the diagramatic presentations of $\mathcal{C}(W/W_J)$ given in 6.6 of Appendix A. In the classical cases we simply use Lemma 5.5 and verify that the weights described in the statement of the theorem indeed have inner product 0 or 1 with $-\rho$ and are thus $\Psi_{\tilde{K}}$-dominant. Furthermore, it is straightforward to see that all the other weights in $\mathcal{S}$ have negative inner product with $-\rho$ and are thus not $\Psi_{\tilde{K}}$-dominant.

Once again we remark that the isomorphism type of the summands in Theorem 5.7 can be read from the information provided in Lemma 5.5.

Example 5.8. *Suppose that G is of type B_8 and $V = B(8,8)$. Let $\tilde{K} = (\Pi \setminus \{\alpha_5\}) \cup \{-\rho\}$. Then $G_{\tilde{K}}$ is of type $D_5 \times B_3$ and*

$$V|_{G_{\tilde{K}}} \cong (D(5,-\rho) \otimes B(3,3)) \oplus (D(5,1) \otimes B(3,3)).$$

6 Appendix A: The Incidence Graphs Associated with $\mathcal{C}(W/W_J)$

In this appendix Π is a set of fundamental roots of a finite indecomposable root system, $W = \langle s_\alpha | \alpha \in \Pi \rangle$ is the corresponding Coxeter group and W_J is a minuscule parabolic subgroup of W (see Section 3). We will give an iterative method by which the incidence graph of the chamber system $\mathcal{C}(W/W_J)$ can be constructed (when W is classical). We remark that the graphs that we present are related to the *Hasse diagrams* of the corresponding Bruhat-Chevalley posets W/W_J. Thus the incidence graphs (and the poset structure) of the $\mathcal{C}(W/W_J)$ minus the labels are known as certain principal ideals in the *Young Lattice* (see [13, 9, 8]); however, once again, it is important in our situation that the graphs carry the incidence information that we need in order to produce K-contractions.

Assume that W is classical. Then, using Table 1, we observe that in each case there is a maximal subset $K \subset \Pi$ for which $|D_{K,J}| = 2$. More specifically,

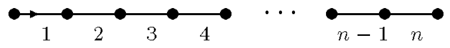
1. if W is of type A_n, then for all minuscule W_J, $K = \Pi \setminus \{1\}$ (or $\Pi \setminus \{n\}$);
2. if W is of type BC_n and $J = \Pi \setminus \{n\}$, then $K = \Pi \setminus \{1\}$;
3. if W is of type BC_n and $J = \Pi \setminus \{1\}$, then $K = \Pi \setminus \{n\}$;
4. if W is of type D_n and $J = \Pi \setminus \{1\}$, then $K = \Pi \setminus \{n\}$ (or $\Pi \setminus \{n-1\}$);
5. if W is of type D_n and $J = \Pi \setminus \{n-1\}$ or $\Pi \setminus \{n\}$, then $K = \Pi \setminus \{1\}$.

Assume then that W is classical and K is chosen from the above list. Then digramatically $\mathcal{C}_K(W/W_J)$ is a string with two vertices, $W_K 1 W_J$ and $W_K w_0 W_J$ and a single bond with the label k ($\{k\} = \Pi \setminus K$). Thus, before $\mathcal{C}(W/W_J)$ was contracted there were precisely two K-connected components, $\mathcal{C}_1$ and $\mathcal{C}_2$, which were joined by a number of edges each labeled with a k. Now, by Theorem 2.4, $\mathcal{C}_1$ and $\mathcal{C}_2$ are also both minuscule chamber systems. Hence we have $\mathcal{C}_1 = \mathcal{C}(W_K/W_{J_1})$ and $\mathcal{C}_2 = \mathcal{C}(W_K/W_{K\cap J})$ respectively, where J_1 is a subset of K determined by the information presented in Lemma 5.5 with W_{J_1} a minuscule parabolic subgroup of W_K. We suppose, by induction, that we know the graphs of both $\mathcal{C}_1$ and $\mathcal{C}_2$. So to construct $\mathcal{C}(W/W_J)$, it is simply a matter of *glueing* $\mathcal{C}_1$ on top of $\mathcal{C}_2$. This is achieved by using Proposition 3.5: let d_1 be the minimal element of $\mathcal{C}_1$, and let m_2 be the maximal element of $\mathcal{C}_2$. Then Proposition 3.5 tells us that $\mathcal{C}_2$ is glued (with k-bonds) to $\mathcal{C}_1$ precisely on the connected K'-component of $\mathcal{C}_1$ containing d_1 and the connected K'-component of $\mathcal{C}_2$ containing m_2 ($K' = K \cap k^\perp$). We illustrate the method by building the incidence graphs of the chamber systems associated with the minuscule modules $A(n,1)$, $A(n,2)$ and and some of the incidence graphs associated with $D(n,n)$.

6.1. The Incidence Graphs related to $A(n,1)$.

Let $K = \Pi \setminus \{1\}$. Then L'_K is of type A_{n-1}. By Lemma 5.5, $\mathcal{C}_1$ is the graph of $A(n-1,1)$ (on the set $\{2,3,\ldots,n\}$) and $\mathcal{C}_2$ is the graph associated with a trivial A_{n-1}-module, (i.e. a single point). We have to add an s_1-bond. As there is only a single point in $\mathcal{C}_2$, we can add only one bond. Thus we have:

$A(n,1)$



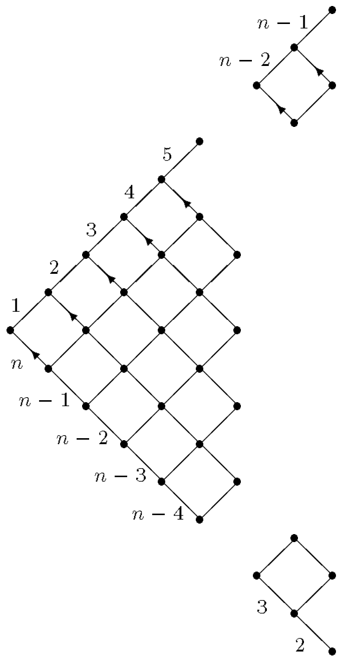
Now the elements of D_J and D_J^{-1} (where W_J is the minuscule parabolic associated with $A(n,1)$) can now be read from the incidence graph. Hence we have

$$D_J = \{s_1, s_1s_2, \ldots, s_1s_2\cdots s_n\} \quad \text{and} \quad D_J^{-1} = \{s_1, s_2s_1, \ldots, s_ns_{n-1}\cdots s_2s_1\}.$$

6.2. The Incidence Graphs related to $A(n,2)$.

This time we choose $K = \Pi \setminus \{n\}$ and L'_K is of type A_{n-1}. Using Lemma 5.5, $\mathcal{C}_1$ is the graph associated with $A(n-1,1)$ and $\mathcal{C}_2$ is the graph associated with $A(n-1,2)$. Thus we get:

$A(n,2)$

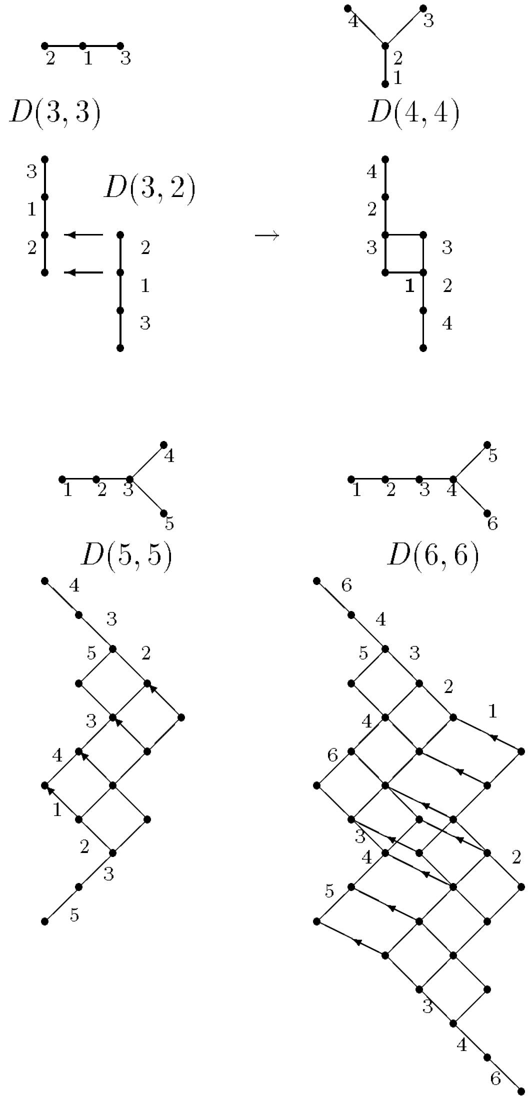


The minimum representative in w_0W_J is $s_{n-1}s_{n-2}\cdots s_1s_ns_{n-1}\cdots s_3s_2$.

6.3. Examples of the Incidence Graphs related to $B(n-1,n-1)$ and $D(n,n)$.

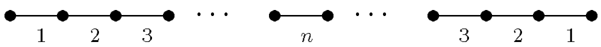
We construct examples of the incidence graph of the chamber system related to $D(n,n)$. The incidence graphs for $B(n-1,n-1)$ can then be obtained by changing all the s_n labels to s_{n-1} (so we get twice as many s_{n-1} labels) and the incidence graphs for $D(n,n-1)$ are gotten by interchanging the labels n and $n-1$.

We take $K = \Pi \setminus \{1\}$. Then L'_K is of type D_{n-1}. We deduce, from Lemma 5.5, that the graph associated with $D(n,n)$ consists of the graph of $D(n-1,n-1)$ glued on top of the graph associated with $D(n-1,n-2)$ over the graph associated with $D(n-2,n-2)$. Observe that this glueing process demands a relabeling of the bonds.



6.4. The Incidence Graphs related to $C(n,1)$.

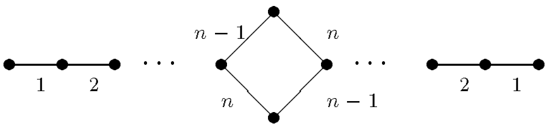
$C(n,1)$



The minimal representative of w_0W_J is $s_1s_2\cdots s_n\cdots s_2s_1$.

6.5. The Incidence Graphs Related to $D(n,1)$.

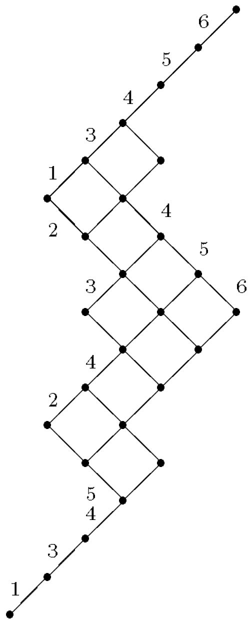
$D(n,1)$



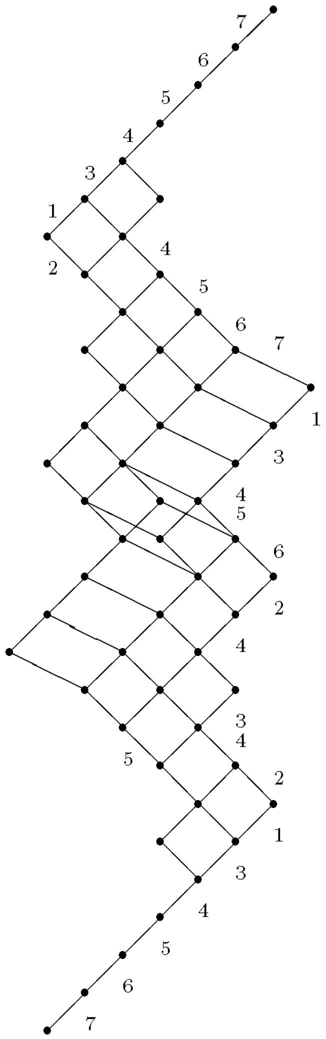
The minimal representative of $w_0 W_J$ is $s_1 s_2 \cdots s_{n-2} s_n s_{n-1} s_{n-2} \cdots s_2 s_1$.

6.6. The Incidence Graphs related to $E(6,1)$ and $E(7,7)$. Extracted from [16].

$E(6,1)$



$E(7,7)$



As a final point we mention a result of Proctor which asserts that minuscule posets are *rank symmetric* and *rank unimodal* [12, Theorem 10]. The same, however, is not true for their K-contracted cousins as the following example illustrates: let Ψ be of type E_6, $J = \Pi \setminus \{\alpha_1\}$, and $K = \{\alpha_6\}$. Using the graph associated with $\mathcal{C}(W/W_J)$ for E_6 above, it is easily seen that the resulting K-contraction is neither rank symmetric nor rank unimodal.

7 Appendix B: A proof of Proposition 4.1

Assume the same notation as in Section 4 and also that G does not have simple constituents of type G_2. This assumption on G is not strictly necessary but it greatly simplifies the proof of Proposition 4.1. However, the statement also holds for G_2 under some mild restrictions. We indicate the result for G_2 in Remark 7.4 below.

Let $K \subset \Pi$ and $P_K = L_K U_K$ be the corresponding parabolic subgroup of G. For a root $\beta = \sum_{\alpha \in \Pi} n_\alpha(\beta)\alpha$ we define the *K-height* of β, as $\sum_{\alpha \in \Pi \setminus K} n_\alpha(\beta) \in \mathbb{Z}$ and the *K-shape* of β, as $\sum_{\alpha \in \Pi \setminus K} n_\alpha(\beta)\alpha$ in the $\mathbb{Z}$-span of $\Pi \setminus K$ in $\mathbb{Z}\Psi$. The following result appears as Lemma 2.1 in [1].

Lemma 7.1. *If β and γ are roots in $\Psi^+ \setminus \Psi_K^+$ of the same length and K-shape, then they are W_K-conjugate.*

Lemma 7.2. *If $\alpha \in \Pi \setminus K$ is short and $\beta \in \Psi^+$ has K-shape α, then β is also short.*

Proof: We employ an argument from the proof of Lemma 2.1 in [1]. We may assume that β is chosen of minimal height with this property within its W_K-orbit. Then $\langle \alpha, \gamma \rangle \leq 0$ and $\langle \beta, \gamma \rangle \leq 0$ for each $\gamma \in K$. Since the K-shape of β is α, the $\mathbb{Z}$-span of $\{\alpha\} \cup K$ contains β. This implies that $\langle \beta, \alpha \rangle > 0$, as otherwise $\{\alpha, \beta\} \cup K$ is a set of mutually obtuse positive roots and hence linearly independent, which is absurd. Further, for j a non-negative integer, $\beta - j\alpha$ is a root only for $j = 0$ or $j = 1$. Since $\langle \beta, \alpha \rangle > 0$, the root $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ is different from β. Thus we must have $\langle \beta, \alpha \rangle = 1$. Assuming that α is short, this implies that α and β are of the same length.

Proof of Proposition 4.1: Let $Y = \langle X_\alpha | \alpha \in \Pi \setminus K \rangle^{L_K}$. Clearly, $Y \leq U_K$. So we must show $U_K \leq Y$. We first prove that $X_\beta \leq Y$ for any root β of K-height 1.

Let $\alpha \in \Pi \setminus K$ and $\beta \in \Psi^+ \setminus \Psi_K^+$ of K-shape α. If both α and β are of the same length, then they are W_K-conjugate by Lemma 7.1, and therefore $X_\beta \leq Y$.

Now suppose that β is of K-shape α but of different length than α. It follows from Lemma 7.2 that α is long and β is short. Again, we may assume that β is chosen of minimal height in its W_K-orbit and, as in the proof of Lemma 7.2, we conclude that $\langle \beta, \alpha \rangle > 0$. Then α and β span a subsystem of roots of type B_2. Further, $\gamma = \beta - \alpha \in \Psi_K^+$ and $\{\alpha, \gamma\}$ is a base of that B_2 subsystem with $\alpha + \gamma = \beta$ and $s_\gamma(\alpha) = \alpha + 2\gamma$. The structure constants in the commutator relations relative to simple roots in a system of type B_2 are $+1$ or -1 [15, Lemma 33]. Thus elements of $X_{\alpha+\gamma} = X_\beta$ occur as factors in the commutator of elements of X_α and X_γ. Since $X_{\alpha+2\gamma} = X_\alpha^{s_\gamma}$ is a subgroup of Y, so is X_β.

Thus for each root β of K-shape $\alpha \in \Pi \setminus K$ the root subgroup X_β is in Y. As $\alpha \in \Pi \setminus K$ was arbitrary, $X_\beta \leq Y$ for each root β of K-height 1.

Next assume that β is a root of K-height 2. Then $\beta = \gamma + \delta$ for some roots γ and δ of K-height 1. If these three roots are all of the same length, then X_β is contained in the commutator of X_γ and X_δ.

If both γ and δ are short and β is long, then γ and δ span an $A_1 \times A_1$ subsystem of a B_2 subsystem in Ψ. In particular, $\gamma - \delta \in \Psi_K$ and $\{\gamma - \delta, \delta\}$ is a base of this B_2. As before, it follows that $X_\beta \leq Y$.

Finally, we consider the case when γ is long and both δ and β are short. Then $\{\gamma, \delta\}$ is a base for a B_2 subsystem in $\Psi^+ \setminus \Psi_K^+$. We argue as follows to show that $X_\beta = X_{\gamma+\delta} \leq Y$ provided $|F| \geq 4$ in case $\mathrm{char} F = 2$. We have $[X_\gamma(s), X_\delta(t)] = X_\beta(\pm st) X_{\gamma+2\delta}(\pm st^2)$. Thus

$$[X_\gamma(s), X_\delta(t)] \cdot [X_\gamma(-t), X_\delta(s)] = X_{\gamma+2\delta}(\mp st(s-t)).$$

Whence $X_{\gamma+2\delta} \leq Y$ and from that $X_\beta \leq Y$, provided $s \neq t$. This fails precisely when $|F| = 2$ and in that case the result is not true, as shown in Remark 4.2.

It now follows by iteration on K-height that each root subgroup relative to any root in $\Psi^+ \setminus \Psi_K^+$ is in Y. This completes the verification of Proposition 4.1.

Remark 7.3. It follows from the proof of Proposition 4.1 above that the restriction on the field, $|F| \geq 4$, is only required in case there is an embedding of the unipotent radical of a Borel subgroup of a group of type B_2 into the unipotent subgroup U_K.

Remark 7.4. Let G be of type G_2 and $\{\alpha, \beta\}$ a base of Ψ with α long. Using the commutator relations for G_2 [15, p. 151], one can show that Proposition 4.1 holds for G_2, if and only if, $|F| \geq 4$ in case $\mathrm{char} F = 2$ and $K = \{\beta\}$ or $K = \emptyset$. In particular, there are no restrictions on the size of the field F if $\mathrm{char} F = 3$ and Proposition 4.1 holds for G_2 if F is algebraically closed.

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