THE GENUS OF THE BARNES-WALL LATTICE

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1 Introduction

In this paper, we combine the general concept of the root system of a Euclidean lattice as studied in [B-S] with the "main lemma" Proposition 1 of [Ve] to obtain the full classification of all 2-elementary totally even 16-dimensional lattices of determinant 2^8. It turns out that this genus of lattices consists of 24 isometry classes. Like in the case of 24-dimensional, even unimodular lattices, 23 of them are **reflective** in the sense that the root system has maximal rank 16. We derive a list of 'possible' root systems, which are subject to two essential restrictions. It turns out that for each root system in our list, there exists a unique lattice.

Acknowledgement. Most of this work was done during a stay of the second author at the Sonderforschungsbereich 343 of the Faculty of Mathematics, University of Bielefeld, in November and December 1992. B. Venkov thanks this institution for its hospitality. R. Scharlau' s thanks go to the Max-Planck-Institut for Mathematik, Bonn, for a short invitation in May 1993 which allowed him to finish this work. Both authors are indebted to H.-G. Quebbemann for his interest and advice.

2 Results

By 'Euclidean lattice', or 'lattice' for short, we mean as usual a $\mathbb{Z}$-lattice of full rank in a rational vector space with a positive definite scalar product (x, y). The lattice is called integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$, and ***p*-elementary**, for some prime number p, if $pL^\# \subseteq L$. Here $L^\# = \{y \in \mathbb{Q}L \mid (x, y) \in \mathbb{Z} \text{ for all } x \in L\}$ denotes the dual lattice of L.

By a left upper index $\alpha \in \mathbb{Q}^+$, we denote scaling: ${}^{\alpha}L$ equals L as a lattice, but the form is multiplied by α. If L is p-elementary, then ${}^{p}(L^\#)$ is again (integral and) p-elementary. In this paper, we specialize to the case $p = 2$, and we write ${}^{2}(L^\#) = L^\natural$ for short. We say that L is **totally even** if both L and $L^\natural$ are even, i.e. $(x, x) \in 2\mathbb{Z}$ for all $x \in L$ and $(y, y) \in \mathbb{Z}$ for all $y \in L^\#$.

For a given dimension n and determinant $\det L = |L^{\#}/L| = 2^r$, all 2-elementary totally even lattices form one genus. They exist if and only if either $r \in \{0, n\}$ and $n \equiv 0(8)$, or $0 < r < n$, $r \equiv 0(2)$, $n \equiv 0(4)$. If $n = 4m$ and $r = 2m$, the mapping $L \to L^{\natural}$ is an involution on (the isometry classes of) this genus. A $\natural$-invariant representative is $mD_4 = D_4 \perp .... \perp D_4$, where D_4 as usual denotes the D_4-root lattice. Another, prominent, lattice in the genus of $4D_4$ is the Barnes-Wall lattice $BW = BW_{16}$ which has no vector of norm 2 and again satisfies $BW \cong BW^{\natural}$.

From [B-S] we recall the definition

$$\mathsf{R}(L) := \{v \in L \mid v \,\text{primitive}, \, 2(v, L)/(v, v) \subseteq \mathbb{Z}\}$$

of the **root system** of an arbitrary lattice L. The quadratic form is part of the structure of such root systems R. Therefore, scaling ${}^{\alpha}\mathsf{R}$ makes sense, and the irreducible ones are ${}^{\alpha}\mathsf{A}_l, {}^{\beta}\mathsf{B}_l, {}^{\gamma}\mathsf{C}_l, \ldots$. The lattice generated by a root system A, B, C, $\ldots$, R is denoted by $A, B, C, \ldots$, R. Specializing again to 2-elementary lattices, we observe, that

$$\mathsf{R}(L) = \{v \in L \mid (v, v) = 2\} \;\dot{\cup}\; \{v \in 2L^{\#} \mid (v, v) = 4\} =: \mathsf{R}_{sh} \dot{\cup}\, \mathsf{R}_{lo}$$

('short' and 'long' roots). Thus, the possible irreducible components of $\mathsf{R}(L)$ are A_l, ${}^2\mathsf{A}_l$, ${}^2\mathsf{B}_l$, C_l, D_l, ${}^2\mathsf{D}_l$, F_4, E_6, ${}^2\mathsf{E}_6$, E_7, ${}^2\mathsf{E}_7$, E_8, ${}^2\mathsf{E}_8$.

By "Niemeier lattice" we mean a 24-dimensional even unimodular lattice. A full classification of this genus of lattices into 24 isometry classes has been given by Niemeier in [Nie], and was simplified in [Ve]. The purpose of this note is to annouce and partially prove a classification result for the genus of $4D_4$ which is completely analogous to the classification of Niemeier lattices in the form presented in [Ve]. Incidently there are, again, precisely 24 isometry classes. In the following, h denotes the Coxeter number of an irreducible root system R. It is characterized by the following formula:

$$(1) \qquad \sum_{v \in \mathsf{R}} \frac{(v, x)^2}{(v, v)} = h \cdot (x, x) \text{ for all } x \in \mathbb{Q}\mathsf{R}.$$

See e.g. [Bou], Chap.V, §6.2, p. 121. By $\mathcal{G}$ we denote the genus consisting of all integral, 2-elementary, totally even lattices of dimension 16 and determinant 2^8.

Theorem 1 *Let L be a lattice in $\mathcal{G}$ and $\mathsf{R} = \mathsf{R}(L)$ its root system. Then the following properties hold:*

a) $\mathsf{R} = \emptyset$ *or* rank $\mathsf{R} = 16$.

b) If $\mathsf{R} \neq \emptyset$, *then*

(i) the Coxeter number h is the same for all irreducible components of R,

(ii) $|\mathsf{R}_{sh}| = |\mathsf{R}_{lo}|$,

(iii) The determinant $\det \mathsf{R}$ *is a square.*

Theorem 2 *Let* R *be a* 16*-dimensional root system consisting of roots of norm* 2 *and* 4 *only. If properties (i), (ii) and (iii) of Theorem 1, part b) hold, then there exists a lattice* L *in* $\mathcal{G}$ *with* $\mathsf{R}(L) = \mathsf{R}$*. For a given* R*, the lattice* L *is determined uniquely up to isometry.*

The explicit classification of $\mathcal{G}$ follows immediately from Theorems 1 and 2 and the following elementary proposition which is obtained simply by inspecting all 16-dimensional combinations of irreducible roots systems with fixed Coxeter number $h = 2, 3, 4, \ldots, 16, 18, 30$.

Proposition 1 *The complete list of all root systems in dimension* 16*, consisting of roots only of norm* 2 *and* 4*, and satisfying properties (i) and (ii) is the following:*

$8\mathsf{A}_1\,8\,{}^2\mathsf{A}_1$	$4\mathsf{A}_2\,4\,{}^2\mathsf{A}_2$	$8\mathsf{C}_2$	$2\mathsf{A}_3\,2\,{}^2\mathsf{A}_3\,2\mathsf{C}_2$	$2\mathsf{A}_4\,2\,{}^2\mathsf{A}_4$	$2\mathsf{D}_4\,2\,{}^2\mathsf{D}_4$
$\mathsf{D}_4\,4\,{}^2\mathsf{B}_3$	${}^2\mathsf{D}_4\,4\mathsf{C}_3$	$\mathsf{A}_5\,{}^2\mathsf{A}_5\,\mathsf{C}_3\,{}^2\mathsf{B}_3$	$2\mathsf{C}_4\,2\,{}^2\mathsf{B}_4$	$\mathsf{A}_7\,{}^2\mathsf{D}_5\,{}^2\mathsf{B}_4$	${}^2\mathsf{A}_7\,\mathsf{D}_5\,\mathsf{C}_4$
$\mathsf{A}_8\,{}^2\mathsf{A}_8$	$\mathsf{D}_6\,2\,{}^2\mathsf{B}_5$	${}^2\mathsf{D}_6\,2\mathsf{C}_5$	$4\mathsf{F}_4$	$\mathsf{C}_6\,{}^2\mathsf{B}_6\,\mathsf{F}_4$	$\mathsf{E}_6\,{}^2\mathsf{E}_6\,\mathsf{F}_4$
$\mathsf{D}_8\,{}^2\mathsf{D}_8$	$\mathsf{C}_8\,{}^2\mathsf{B}_8$	$\mathsf{C}_9\,{}^2\mathsf{E}_7$	${}^2\mathsf{B}_9\,\mathsf{E}_7$	$\mathsf{E}_8\,{}^2\mathsf{E}_8$	$\mathsf{A}_3\,{}^2\mathsf{A}_3\,5\mathsf{C}_2$.

We observe the the additional property $|\mathsf{R}_{sh}| = 8h$ holds for all reflective lattices in Proposition 1. This is readily derived from properties (i) and (ii).

Now recall that the Barnes-Wall lattice is the unique lattice in $\mathcal{G}$ with minimum 4 (for a proof, see [Que], Theorem 4), and observe that only the root system $\mathsf{A}_3\,{}^2\mathsf{A}_3\,5\mathsf{C}_2$ in the proposition above has a non-square determinant. The next theorem combines the results stated so far.

Theorem 3 *The class number of the genus* $\mathcal{G}$ *of* $4D_4$ *is equal to* 24*. Representatives of* $\mathcal{G}$ *are the Barnes-Wall lattice and* 23 *lattices having the first* 23 *root systems listed in Proposition 1.*

The orthogonal group $O(L)$ of a reflective lattice L (i.e. rank $\mathsf{R}(L) = \dim L$) is of the form $O(L) = W(L) \rtimes A(L)$, where $W(L)$ is the Weyl group of (the root system of) L, and $A(L)$ can be identified with a subgroup of the outer automorphism group (group of diagram automorphisms) of the root system. In the following table, we list all our lattices together with the Coxeter number h and the order of $O(L)$.

lattice	h	root system	order of $O(L)$
$L_1 = BW$	–	empty	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$
L_2	2	$8\mathsf{A}_1\, 8\,{}^2\mathsf{A}_1$	$2^{25} \cdot 3 \cdot 7$
L_3	3	$4\mathsf{A}_2\, 4\,{}^2\mathsf{A}_2$	$2^{14} \cdot 3^9$
L_4	4	$8\mathsf{C}_2$	$2^{30} \cdot 3 \cdot 7$
L_5	4	$2\mathsf{A}_3\, 2\,{}^2\mathsf{A}_3\, 2\mathsf{C}_2$	$2^{22} \cdot 3^4$
L_6	5	$2\mathsf{A}_4\, 2\,{}^2\mathsf{A}_4$	$2^{15} \cdot 3^4 \cdot 5^4$
L_7	6	$2\mathsf{D}_4\, 2\,{}^2\mathsf{D}_4$	$2^{27} \cdot 3^5$
L_8	6	$\mathsf{D}_4\, 4\,{}^2\mathsf{B}_3$	$2^{25} \cdot 3^6$
$L_9 = L_8^\natural$	6	${}^2\mathsf{D}_4\, 4\mathsf{C}_3$	$2^{25} \cdot 3^6$
L_{10}	6	$\mathsf{A}_5\, {}^2\mathsf{A}_5\, \mathsf{C}_3\, {}^2\mathsf{B}_3$	$2^{17} \cdot 3^6 \cdot 5^2$
L_{11}	8	$2\mathsf{C}_4\, 2\,{}^2\mathsf{B}_4$	$2^{30} \cdot 3^4$
L_{12}	8	$\mathsf{A}_7\, {}^2\mathsf{D}_5\, {}^2\mathsf{B}_4$	$2^{22} \cdot 3^4 \cdot 5^2 \cdot 7$
$L_{13} = L_{12}^\natural$	8	${}^2\mathsf{A}_7\, \mathsf{D}_5\, \mathsf{C}_4$	$2^{22} \cdot 3^4 \cdot 5^2 \cdot 7$
L_{14}	9	$\mathsf{A}_8\, {}^2\mathsf{A}_8$	$2^{15} \cdot 3^8 \cdot 5^2 \cdot 7^2$
L_{15}	10	$\mathsf{D}_6\, 2\,{}^2\mathsf{B}_5$	$2^{26} \cdot 3^4 \cdot 5^3$
$L_{16} = L_{15}^\natural$	10	${}^2\mathsf{D}_6\, 2\mathsf{C}_5$	$2^{26} \cdot 3^4 \cdot 5^3$
L_{17}	12	$4\mathsf{F}_4$	$2^{31} \cdot 3^9$
L_{18}	12	$\mathsf{C}_6\, {}^2\mathsf{B}_6\, \mathsf{F}_4$	$2^{27} \cdot 3^6 \cdot 5^2$
L_{19}	12	$\mathsf{E}_6\, {}^2\mathsf{E}_6\, \mathsf{F}_4$	$2^{22} \cdot 3^{10} \cdot 5^2$
L_{20}	14	$\mathsf{D}_8\, {}^2\mathsf{D}_8$	$2^{28} \cdot 3^4 \cdot 5^2 \cdot 7^2$
L_{21}	16	$\mathsf{C}_8\, {}^2\mathsf{B}_8$	$2^{30} \cdot 3^4 \cdot 5^2 \cdot 7^2$
L_{22}	18	$\mathsf{C}_9\, {}^2\mathsf{E}_7$	$2^{26} \cdot 3^8 \cdot 5^2 \cdot 7^2$
$L_{23} = L_{22}^\natural$	18	${}^2\mathsf{B}_9\, \mathsf{E}_7$	$2^{26} \cdot 3^8 \cdot 5^2 \cdot 7^2$
L_{24}	30	$\mathsf{E}_8\, {}^2\mathsf{E}_8$	$2^{28} \cdot 3^{10} \cdot 5^4 \cdot 7^2$

Using the orders of the orthogonal groups, one can readily confirm our enumeration by the mass formula. The mass of $\mathcal{G}$ is equal to

$$M(\mathcal{G}) = \frac{691}{2^{30} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} \cdot \frac{1}{2} \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 43 \cdot 127.$$

The first factor is the mass of even unimodular 16-dimensional lattices which can be looked up for instance in [Se], Chap. V, §2. The second factor is a correction factor coming from the determinant times the quotient of the 2-adic densities of the genera in question. It is readily derived e.g. from [Pf], Satz 1, Hilfssatz 8. The mass can also be obtained from [C-S2]. In the notation of that paper, our genus reads $1_{II}^{8+}2_{II}^{8+}$.

3 Some proofs

If L is any integral lattice, we denote by $T(L) := L^\#/L$ its discriminant group. It carries a $\mathbb{Q}/\mathbb{Z}$-valued symmetric bilinear form, the so-called discriminant (bilinear) form $(x+L, y+L) := (x,y)+\mathbb{Z}$, where $x, y \in L^{\#}$. If L is even, we also have the **discriminant quadratic form** $q(y+L) = \frac{1}{2}(y,y)+\mathbb{Z}$. The integral even over-lattices M of L correspond bijectively to the subgroups $\mathcal{M}$ $(:= M/L)$ of $T(L)$ which are totally isotropic with respect to q. The discriminant group (and form) of M can be identified with (the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by q_L on) $\mathcal{M}^{\perp}/\mathcal{M}$. Here, $\mathcal{M}^{\perp}$ denotes the othogonal complement of $\mathcal{M}$ with respect to q_L (or rather its associated bilinear form). The following proposition is an immediate consequence of the remarks made so far.

Proposition 2 *Let L_1, L_2 be even, integral lattices, and*

$$\varphi : (T(L_1), q_1) \to (T(L_2), -q_2)$$

be an isometry. Then

$$L_1 \times_{\varphi} L_2 := \{(x_1, x_2) \in L_1^{\#} \perp L_2^{\#} \mid \varphi \bar{x}_1 = \bar{x}_2\}$$

is an integral, even, unimodular lattice. Conversely, any even unimodular over-lattice of $L_1 \perp L_2$ can be obtained in this way.

Indeed, $\mathcal{L} := \{(\bar{x}, \varphi\bar{x}) \mid \bar{x} \in T(L_1)\} \subseteq T(L_1 \perp L_2)$ is obviously totally isotropic, and for reasons of group orders, it must coincide with $\mathcal{L}^{\perp}$. The converse is also easy.

We shall use Proposition 2 in the following special case: $L_1 = L$ is an arbitrary lattice in the genus $\mathcal{G}$, and $L_2 = {}^2E_8$. Then, φ as above actually exists (take for instance $L = E_8 \perp {}^2E_8$). We shall, by the way, not use the fact that the discriminant form determines the genus (see e.g. [Nik]), but only the trivial fact that it depends only on the genus. In our particular case, the "glueing map" φ is essentially unique (i.e., two possible φ's can be transformed into each other by an automorphism of $L_1 \times L_2$ preserving the components). This is a trivial consequence of the well-known fact that the orthogonal group $O(E_8)$ maps surjectively onto the orthogonal group of the $\mathbb{F}_2$-valued quadratic form $\frac{1}{2}(x,x)+2\mathbb{Z}$ on $E_8/2E_8$. Thus, we have a mapping $L \longmapsto \widetilde{L} := L \times_{\varphi} {}^2E_8$ from $\mathcal{G}$ to the Niemeier lattices which is well defined on isometry classes.

Proof of Theorem 1. We begin with the equality $|\mathsf{R}_{sh}| = |\mathsf{R}_{lo}|$. It is readily derived from the basic observation that the long roots are in one-to-one correspondence with the short roots of $L^{\#}$, taking into account the following facts: the theta series ϑ_L and $\vartheta_{L^{\natural}}$ are modular forms for $\Gamma_0(2)$ of weight 8; the corresponding space of cusp forms is one-dimensional (see e.g. [Ra] Chap. 1.5, 7.1); the transformation formula for theta series gives a linear transformation mapping coefficients of ϑ_L, onto those of $\vartheta_{L^{\natural}}$. (Using a slightly more refined argument, Quebbemann shows that $\vartheta_L = \vartheta_{L^{\natural}}$ for any L in $\mathcal{G}$.)

Now assume that $\mathsf{R} = \mathsf{R}(L) \neq \emptyset$. Then also $\mathsf{R}_{sh} \neq \emptyset$, and thus $\widetilde{\mathsf{R}} := \mathsf{R}(\widetilde{L}) \neq \emptyset$. Let us describe $\widetilde{\mathsf{R}} = \{v \in \widetilde{L} \mid (v,v) = 2\}$ explicitly in terms of L. The vectors $\tilde{v}$ in $\widetilde{L}$ of norm 2 are of two kinds: either $\tilde{v} \in L$, or $\tilde{v}$ is of the form $\tilde{v} = (v_1 + v_2), v_1 \in L^{\#}, v_2 \in \frac{1}{2}{}^2E_8, (v_1, v_1) = (v_2, v_2) = 1, \varphi\bar{v}_1 = \bar{v}_2$.

We now have to recall the following well-known property of the E_8-root-lattice: for any element in $E_8/2E_8$ which is anisotropic with respect to the quadratic form $\frac{1}{2}(x,x)+2\mathbb{Z}$, there exists a representative in E_8 of norm $\frac{1}{2}(x,x)=1$. It is unique up to sign. It follows that the roots of $\widetilde{L}$ of the second kind are of the shape $v_1 \pm v_2$, where $2v_1$ runs through the long roots of L, and $v_2 = \varphi v_1$ is a function of v_1 (and a root of E_8). We see that

(2) $\quad |\widetilde{\mathsf{R}}| = |\mathsf{R}_{sh}| + 2|\mathsf{R}_{lo}| = 3|\mathsf{R}_{sh}|.$

Now we use the following formula from [Ve], Proposition 1:

(3) $\quad \displaystyle\sum_{v\in\widetilde{\mathsf{R}}}(v,x)^2 = \frac{1}{12}(x,x)|\widetilde{\mathsf{R}}| \text{ for all } x \in \widetilde{L}.$

The above description of $\widetilde{\mathsf{R}}$ gives

(4) $\quad \displaystyle\sum_{v\in\widetilde{\mathsf{R}}}(v,x)^2 = \sum_{v\in\mathsf{R}_{sh}}(v,x)^2 + 2\sum_{v\in\mathsf{R}_{lo}}(\frac{v}{2},x)^2 = 2\sum_{v\in\mathsf{R}}\frac{(v,x)^2}{(v,v)} \text{ for all } x \in \widetilde{L}.$

Now specify an irreducible component R_i of R and choose x non-zero in R_i. Substituting (4) into (3) and comparing with (1) gives

$$h_i = \frac{1}{24}|\widetilde{\mathsf{R}}|$$

for the Coxeter number h_i of R_i. This proves (i). (In view of (2), it also proves the additional property $|\mathsf{R}_{sh}| = 8h$.) $\square$

We observe that the Coxeter number h is equal to the Coxeter number of any component of $\widetilde{\mathsf{R}}$.

Some cases of Theorem 2.

<u>The lattice L with $\mathsf{R}(L) = 2\mathsf{C}_2\, 2\mathsf{A}_3\, 2\,{}^2\!\mathsf{A}_3$.</u>

The root lattice is $L_0 = 4\mathsf{A}_1\, 2\mathsf{A}_3\, 2\,{}^2\!\mathsf{A}_3$. Its discriminant group is of the form

$$T(L_0) = \langle g_1,\ldots,g_4; g_5,g_6; \tfrac{1}{2}g_7, \tfrac{1}{2}g_8\rangle \cong Z_2^4 \times Z_4^2 \times Z_8^2,$$

where $g_1,\ldots,g_4$ are generators of the four copies of $T(\mathsf{A}_1)$, and $g_5,\ldots,g_8$ are generators of the four copies of $T(\mathsf{A}_3)$, and $Z_m := \mathbb{Z}/m\mathbb{Z}$. By a general result of [B-S], the desired code $\mathcal{L}$ is contained in the subgroup of $T(L_0)$ which is obtained by forgetting the scaling (observe that ${}^{\alpha}M^{\#} = \frac{1}{\alpha}M^{\#}$, and hence $T(M) \subseteq T({}^{\alpha}M)$, for any M and α):

$$\mathcal{L} \subseteq T = \langle g_1,\ldots,g_4; g_5,g_6; g_7,g_8\rangle \cong Z_2^4 \times Z_4^2 \times Z_4^2.$$

Since $\det L_0 = 2^{18}$, we look for an $\mathcal{L}$ with $|\mathcal{L}| = 2^5$. The discriminant form on T is the diagonal form

$$diag\langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{4}, \tfrac{3}{4}; \tfrac{3}{2}, \tfrac{3}{2}\rangle,$$

where the entries are actually the values $\ell(g_i)$ of the **length function** on T which is defined as

$$\begin{aligned}&\ell : T(L_0) \to \mathbb{Q}\\ &\ell(x+L_0) = \min\{(y,y) \mid y \in x+L_0\} \quad \text{where } x \in L_0^{\#}.\end{aligned}$$

We shall also use the values

$$\ell(2g_5) = \ell(2g_6) = 1, \quad \ell(2g_7) = \ell(2g_8) = 2.$$

We now observe the following conditions on $\mathcal{L}$, where $L = L(\mathcal{L})$ denotes the inverse image of $\mathcal{L}$ in $L_0^{\#}$:

(5) the length function ℓ takes integral, even values on $\mathcal{L}$, and $\ell(e) > 2$ for all $e \in \mathcal{L}\backslash\{0\}$.

This is the condition that L should be even, and all vectors of norm 2 in L should be contained in L_0. Since $\frac{1}{2}\ell \bmod \mathbb{Z}$ gives back the discriminant quadratic form q, the previous general condition that $\mathcal{L}$ should be totally isotropic with respect to q, is included in condition (5).

(6) the transpositions (12) and (34) preserve $\mathcal{L}$, but (13), (14), (23) do not.

This comes from the fact that we want the root system $4\mathsf{A}_1 =: \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\}$ to be enlarged to $2\mathsf{C}_2 = 4\mathsf{A}_1 \cup \{\pm a_1 \pm a_2, \pm a_3 \pm a_4\}$.

If we furthermore arrange $\mathcal{L}$ in such a way that L becomes 2-elementary and 2-even, it follows from Theorem 1 b) (ii) that L not only has no "new roots" of norm 2, but also no "new roots" of norm 4. In this connection observe that $L^{\#}/L \cong \mathcal{L}^{\perp}/\mathcal{L}$. For the actual calculation of $L^{\#}$, split L_0 as

$$L_0 = L_{0,1} \perp L_{0,2} \perp L_{0,3} = 4\mathsf{A}_1 \perp 2\mathsf{A}_3 \perp 2\,{}^2\!\mathsf{A}_3.$$

Compute $\mathcal{L}^{\perp}$ with respect to the bilinear form

$$diag\langle \tfrac{1}{2}, \ldots, \tfrac{1}{2}; \tfrac{3}{4}, \tfrac{3}{4}; \tfrac{3}{4}, \tfrac{3}{4} \rangle$$

on T. Then

$$L^{\#} = \{y_1 + y_2 + \tfrac{1}{2}y_3 \mid y_i \in \mathcal{L}_{0,i}^{\#},\ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \in \mathcal{L}^{\perp}\},$$

and L is 2-elementary if and only if $2\bar{y}_1 + 2\bar{y}_2 + \bar{y}_3 \in \mathcal{L}$ for all $\bar{y} = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \in \mathcal{L}^{\perp}$. In particular, $pr_3(\mathcal{L}^{\perp}) \subseteq \mathcal{L}$ (projection onto the third factor).

Having collected all these conditions, it is now relatively easy to see that the following $\mathcal{L}$ is a solution to our problem, and is unique up to outer automorphisms of the root systems $2\mathsf{C}_2\,2\mathsf{A}_3\,2\,{}^2\!\mathsf{A}_3$

$$\begin{aligned}
\mathcal{L} &= \left\langle e_1, e_2, e_3 \right\rangle, \quad \text{where} \\
e_1 &= g_1 + g_2 + g_5 + g_6 + g_7 \\
e_2 &= g_3 + g_4 + g_5 - g_6 + g_8 \\
e_3 &= 2g_5 + g_7 - g_8.
\end{aligned}$$

We finally calculate the orthogonal group $O(L) = W(L) \rtimes A(L)$. Recall that $A(L)$ can be identified with either the subgroup of $\operatorname{Aut} T(\mathsf{R}(L))$ preserving $\mathcal{L}$, or a certain subgroup of the diagram automorphisms of $\mathsf{R}(L)$. The latter description shows that

$$A(L) \hookrightarrow Z_2 \times Di_8 \times Di_8,$$

where Di_8 is the dihedral group of order 8. We represent the automorphisms of $T \cong Z_2^2 \times Z_4^2 \times Z_4^2$ by triples of 2×2-matrices; then the diagram automorphisms correspond to the monomial matrices with entries ± 1. We use the notation

$$\sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and if π is a matix in $\langle \sigma_0, \tau_0 \rangle \cong Di_8$, then π_i, $(i = 1, 2, 3)$ denotes the corresponding element acting on the i-th factor of T. Direct verification shows that the following elements preserve $\mathcal{L}$:

$$\sigma = \mathrm{id}_1 \sigma_2 \tau_3, \quad \tau = \sigma_1 \tau_2 \sigma_3, \quad \rho = \sigma_1 (-\mathrm{id})_2 \mathrm{id}_3.$$

They clearly generate a group isomorphic to $Di_8 \times Z_2$. In order to show that this is all of $A = A(L)$, it suffices to show that the projection p_{12} onto the first two components of $Z_2 \times Di_8 \times Di_8$ is injective. Its kernel $H = \{(\mathrm{id}, \mathrm{id}, -)\} \cap A$ maps under p_3 onto a normal subgroup of $p_3 A = Di_8$. Assuming H nontrivial, we have $(\mathrm{id}, \mathrm{id}, -\mathrm{id}) =: \pi \in H$. This is not the case, since $\pi e_1 - e_1 \notin \mathcal{L}$.

The lattice L with $\mathsf{R}(L) = \mathsf{A}_7\,{}^2\mathsf{D}_5\,{}^2\mathsf{B}_4$

We start with the necessary information about the discriminant group of A_7. It is cyclic of order 8; in the standard coordinates $\mathsf{A}_7 = \{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid \sum x_i = 0\}$ it consists of the residue classes

$$\begin{aligned}
g_1 &\equiv \tfrac{1}{8}(-7,1,1,1,1,1,1,1), & 2g_1 &\equiv \tfrac{1}{8}(-6,-6,2,2,2,2,2,2), \\
3g_1 &\equiv \tfrac{1}{8}(-5,-5,-5,3,3,3,3,3), & 4g_1 &\equiv \tfrac{1}{8}(-4,-4,-4,-4,4,4,4,4)
\end{aligned}$$

and their negatives. These vectors are in fact shortest coset representatives, thus the length function has the values

$$\ell(g_1) = \tfrac{7}{8},\ \ell(2g_1) = \tfrac{3}{2},\ \ell(3g_1) = \tfrac{15}{8},\ \ell(4g_1) = 2.$$

The generator g_1 is by the way equal to the fundamental highest weight usually called $\overline{\omega}_1$; in a root basis, it equals $\frac{1}{8}(7,6,5,4,3,2,1)$. Writing our complete root lattice as $L_0 = \mathsf{A}_7\, {}^2\mathsf{D}_5\, 4\mathsf{A}_1$, its discriminant group is

$$T(L_0) = \langle g_1; \tfrac{1}{2}g_2; g_3, g_4, g_5, g_6\rangle \cong Z_8 \times Z_8 \times Z_2^4,$$

(where g_2 is the canonical generator of $T(\mathsf{D}_5) \cong Z_4$). As above, the desired group $\mathcal{L}$ is contained in

$$T := \langle g_1; g_2; g_3, g_4, g_5, g_6\rangle \cong Z_8 \times Z_4 \times Z_2^4.$$

Notice that $\ell(g_2) = \frac{5}{2}$, $\ell(2g_2) = 2$. Since $\det L_0 = 2^{14}$, we want $|\mathcal{L}| = 2^3$. Since we want $4\mathsf{A}_1$ to be enlarged to ${}^2\mathsf{B}_4$, the group $\mathcal{L}$ must be invariant under arbitrary permutations of g_3, g_4, g_5, g_6. Having in mind this fact and the fundamental restriction $\ell(g) \equiv 0 \bmod 2$, $\ell(g) > 2$ for all $g \in \mathcal{L}\backslash\{0\}$, we easily arrive at the group

$$\mathcal{L} = \langle 2g_1 + g_2,\ 2g_2 + g_3 + g_4 + g_5 + g_6\rangle.$$

Arguing as above, one sees that $T(L)$ consists of all $\{y_1g_1 + y_2\frac{g_2}{2} + y_3g_3 + \ldots + y_6g_6\}$, where $y_1 - y_2 \equiv 0 \bmod 4$ and $y_2 + y_3 + \ldots + y_6 \equiv 0 \bmod 2$. It readily follows that $(v,v) \in \mathbb{Z}$ for all $v \in L^{\#}$, and $2L^{\#} \subseteq L$. Thus L is 2-elementary and 2-even.

The determination of the orthogonal group is trivial in this case, since $W(L)$ is only of index 4 in $\operatorname{Aut}\mathsf{R}(L)$, and $-\mathrm{id} \notin W(L)$. We must have

$$O(L) = \langle -\mathrm{id}\rangle \times W(L).$$

Indeed $O(L)$ cannot be all of $\operatorname{Aut}\mathsf{R}(L)$, since in this case ${}^2\mathsf{D}_5$ would enlarge to ${}^2\mathsf{B}_5$.

4 Further results

A remarkable property of the genus $\mathcal{G}$ is that, for any possible root system, the corresponding lattice does exist, and is unique (Theorem 2). This phenomenon is not completely understood. This is so even in the widely studied case of the genus of 24-dimensional unimodular lattices. In view of the fact that the proof of Theorem 2 goes case by case, we find it necessary to present, in a continuation of this paper, the explicit contruction of each lattice, and a sketch of each uniqueness proof. We shall also present a different proof of the existence (and uniqueness) of each lattice L in $\mathcal{G}$ making further use of the associated Niemeier lattice $\widetilde{L} \supset L$. If we define an involution σ on $\mathbb{Q}\widetilde{L}$ by $\sigma_{|L} = \mathrm{id}$, $\sigma_{|L^\perp} = -\mathrm{id}$, then σ is an isometry of $\widetilde{L}$. We shall describe all (conjugacy classes of) involutions on Niemeier lattices such that the fixed lattice $\widetilde{L}^\sigma$ is 16-dimensional, 2-elementary, totally even, and $(\widetilde{L}^\sigma)^\perp$ contains no roots.

We shall also list all other 2-elementary totally even lattices of dimensions $n \leq 16$ and determinant 2^{2k}, $k \leq n/4$. It is known from the work of Esselmann [Es], though not explicitly stated there, that all these genera except for the one treated here, are totally reflective (i.e., consist of reflective lattices only).

For $n \leq 12$, the lattices can be easily obtained in an ad hoc way, starting from the smallest possible k ($= 0$ or 1), without paying much attention to the possible root systems. The class numbers $h(n,k)$ are:

$$h(4,1) = h(8,0) = h(8,1) = h(8,2) = 1$$

$$h(12,1) = 2 \quad h(12,2) = 2 \quad h(12,3) = 3.$$

In the last case, the root systems are $3\,\mathsf{F}_4$, ${}^2\mathsf{B}_6\,\mathsf{C}_6$, $\mathsf{E}_6\,{}^2\mathsf{E}_6$.

Of course, this classification can also be derived as an immediate corollary of Theorem 3, simply by adding an orthogonal summand D_4 (with root system F_4). It is, by the way, a general fact that a root system ${}^{\alpha}\mathsf{F}_4$ in an arbitrary lattice always splits off; see [B-S]. This allows us to reduce the existence and uniqueness proof for those root systems in Proposition 1 containing an F_4-component to the 12-dimensional case.

Results completely analogous to Theorems 1 and 2 hold for the 12-dimensional genus of $6A_2$, or of the Coxeter-Todd lattice. The number of short roots equals $6h$ (in the reflective case), the root systems are $6\mathsf{A}_1\,6\,{}^3\mathsf{A}_1$, $3\mathsf{A}_2\,3\,{}^3\mathsf{A}_2$, $2\mathsf{A}_3\,2\,{}^3\mathsf{A}_3$, $\mathsf{D}_4\,{}^3\mathsf{D}_4\,2\mathsf{G}_2$, $6\mathsf{G}_2$, $\mathsf{A}_5\,{}^3\mathsf{A}_5\,\mathsf{G}_2$, $\mathsf{A}_6\,{}^3\mathsf{A}_6$, $\mathsf{D}_6\,{}^3\mathsf{D}_6$, $\mathsf{E}_6\,{}^3\mathsf{E}_6$, the class number is 10.

We unify and extend our results by treating the other genera of 'extremal modular' lattices considered by Quebbemann in [Que]. In that paper, a lattice L is called **modular** (of level p) if it is isometric to its p-scaled dual: $L \cong {}^p(L^\#)$. Such a lattice is in particular p-elementary of determinant $p^{n/2}$.

Theorem 4 *Let p be one of the prime numbers $1,2,3,5,7,11,23$ (i.e., $p+1 \mid 48$), let the dimension be equal to $n = \frac{48}{1+p} = 24,16,12,8,6,4,2$. For each of these pairs (p,n), there exist a unique p-elementary even lattice of determinant at most $p^{n/2}$ and with minimum 4. All other lattices in the respective genera are reflective. For $p \geq 3$, all these lattices are modular.*

The 8-dimensional lattice with determinant 5^4 and minimum 4 is the lattice $Q_8(1)$ described in [C-S1]. The 6-dimensional lattice with determinant 7^3 and minimum 4 is the lattice $Q_6(1) = P_6$ first found by Barnes. The class numbers for $p = 5,7,11,23$ are $5,3,3,2$, respectively. All 'possible' root systems (i.e., $|\mathsf{R}_{sh}| = |\mathsf{R}_{lo}|$) do occur.

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