Non-linear combinatorial vectorspace categories

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Introduction

In the study of the representation theory of finite dimensional algebras besides module categories also other sorts of categories arise. These categories are frequently still Krull-Schmidt categories but fail to be abelian. One important class of these categories are the subspace categories of vectorspace categories which appeared classically when modules over one-point extension algebras were considered.

Let us first recall the relevant definitions. We fix an algebraically closed field k. A Krull-Schmidt category K (over k) is a k-additive category such that the morphism space $K(x,y)$ is finite dimensional for all objects x,y of K and the endomorphism algebra $K(x,x)$ is local for each indecomposable object x of K. We fix a set of representatives of the isomorphism classes of indecomposable objects in K and denote the induced full subcategory by $\operatorname{ind} K$.

A vectorspace category (over k) is a pair $\mathcal{K} = (K, |-|)$ consisting of a Krull-Schmidt category K and a k-linear functor $|-| : K \to k\text{-mod}$ where k-mod is the category of finite dimensional k-vectorspaces. Note that we compose morphisms in Krull-Schmidt categories usually from left to right whereas in k-mod we compose from right to left.

The subspace category $\check{\mathcal{U}}(\mathcal{K})$ of a vectorspace category $\mathcal{K} = (K, |-|)$ is again a Krull-Schmidt category whose objects are the triples $U = (U_0, U_\omega, \gamma_U)$ where U_0 is an object of K, U_ω is a finite dimensional k-vectorspace and $\gamma_U : U_\omega \to |U_0|$ is a k-linear map. For two objects U, V of $\check{\mathcal{U}}(\mathcal{K})$ the space of morphism form U to V is the set of pairs $f = (f_0, f_\omega)$ where $f_0 \in K(U_0, V_0)$ and $f_\omega \in \operatorname{Hom}_k(U_\omega, V_\omega)$ satisfying $|f_0|\gamma_U = \gamma_V f_\omega$. The morphisms in $\check{\mathcal{U}}(\mathcal{K})$ are composed componentwise.

The first aim in studying $\check{\mathcal{U}}(\mathcal{K})$ is to understand the indecomposable objects. A part of them can be constructed directly from K. Namely there is a canonical full embedding of K into $\check{\mathcal{U}}(\mathcal{K})$ defined by mapping an object x of K to the triple $(x, 0, 0)$. For general $\mathcal{K} = (K, |-|)$ there may be indecomposable objects x of K such that $|x| = 0$. It is an easy calculation that an indecomposable object U of $\check{\mathcal{U}}(\mathcal{K})$ such that there is an indecomposable direct summand x of U_0 with $|x| = 0$ has to be isomorphic to $(x, 0, 0)$. This shows that the full subcategory of $\check{\mathcal{U}}(\mathcal{K})$ supported by the indecomposables U with this property is equivalent to K. Therefore these objects of $\check{\mathcal{U}}(\mathcal{K})$ are uninteresting and we may suppose without loosing information that $\mathcal{K} = (K, |-|)$ is sincere i.e. $|x| \neq 0$ for all objects x of K.

If one is mainly interested in the indecomposables of a Krull-Schmidt category L one may always pass to a representation equivalent category by factoring out an ideal of L which does not contain an automorphism of x for any object x of L. Applying this to the subspace category $\mathcal{U}(\mathcal{K})$ of a sincere vectorspace category $\mathcal{K} = (K, |-|)$, we may factor out the ideal of all morphisms $f = (f_0, f_\omega)$ such that $|f_0| = 0$. The factor of $\mathcal{U}(\mathcal{K})$ by this ideal coincides with the subspace category of the vectorspace category $(K/\mathrm{Ker}\,|-|, |-|)$. But by construction the functor $|-|$ acts faithfully on the factor category $K/\mathrm{Ker}\,|-|$. So we have seen that for studying the indecomposables of $\mathcal{U}(\mathcal{K})$ we may always suppose that $\mathcal{K} = (K, |-|)$ is not only sincere but even faithful i.e. $|-|$ is a faithful functor.

If A is a one-point extension algebra, the category of finite dimensional A-modules can be identified with the subspace category of some vectorspace category $\mathcal{K} = (K, |-|)$ (see [Ri3]). A big part of research on algebras of finite representation type was based on the analysis of this situation in the case when A is even representation directed (see again [Ri3]). In this situation the occurring vectorspace categories $\mathcal{K} = (K, |-|)$ are always linear i.e. they are faithful and $\dim_k |x| = 1$ holds for each indecomposable object x of K.

The main advantage of linear vectorspace categories $\mathcal{K} = (K, |-|)$ is that one can reconstruct them completely from relatively few data. Namely one defines a partial ordered set $S = S(\mathcal{K})$ by choosing the objects of $\mathrm{ind}\, K$ as elements and putting $x \geq y$ if $K(x, y) \neq 0$. The point is that $\mathcal{K}$ is isomorphic to the vectorspace category $(\mathrm{add}\,(kS), can)$ where $\mathrm{add}\,(kS)$ is the additive closure of the category kS which has as objects the elements of S and between two objects x, y there is a 1-dimensional morphism space with basis element $b(x, y)$ if $x \geq y$ whereas otherwise the morphism space is 0. The composition in this category is defined by linear extending the rule $b(x, y)b(y, z) = b(x, z)$. The canonical functor *can* acts in the obvious way. This means that one can reconstruct a linear vectorspace category completely only from the knowledge when there exist non-zero morphisms between indecomposables.

Dealing with tame type, naturally vectorspace categories which are not linear occur. Nevertheless in certain situations one would like to have similar uniqueness results as in the linear case. Let us be more precise. In [Ri2] and [Ri3] certain tame faithful vectorspace categories $\mathcal{K} = (K, |-|)$ called patterns are considered which have the property that $\dim_k |x| \leq 2$ for all $x \in \mathrm{ind}\, K$. These vectorspace categories are visualized by the quiver of the category $\mathrm{ind}\, K$, which has as points just the objects of $\mathrm{ind}\, K$, and the number $\dim_k |x|$ attached to each of these points in the following way. The possible values for $\dim_k |x|$ are 1 and 2. The points where the value 2 is reached are presented as bold points. By our results in [Dr1] and [Dr2] we were led to vectorspace categories $\mathcal{K} = (K, |-|)$ where we could compute the quiver of $\mathrm{ind}\, K$, some of the imposed relations and the values of $\dim_k |x|$. The images we obtained resembled very much the ones corresponding to patterns. So we asked ourselves whether these data determine the vectorspace categories. Or in other words, is there a model vectorspace category to which all other vectorspace categories with the same data have to be isomorphic? Considering the non-isomorphic vectorspace categories of type $\mathbb{D}_{4\lambda}$ (compare [Ri3]) one can see that such a result cannot be true in complete generality.

In section 1 of this paper we will recall the relevant definitions on Krull-Schmidt categories and their quivers.

In section 2 we will introduce the concept of a pattern-like quiver which is a pair (Γ, T) consisting of a quiver Γ and a subset T of the vertex set of Γ satisfying certain properties. We will define a corresponding category $K_{(\Gamma,T)}$ as factor category of the k-linear path category $k\Gamma$ by an admissible ideal $J_{(\Gamma,T)}$. Furthermore, we will calculate $\dim_k K_{(\Gamma,T)}(x, y)$

for all objects x, y.

In section 3 we will use the results about $K_{(\Gamma,T)}$ to construct a faithful vectorspace category $(L_{(\Gamma,T)}, |-|_\omega)$ with the property that Γ is the quiver of $L_{(\Gamma,T)}$, $\dim_k|x| \leq 2$ for all vertices x and T is the just the set of points x satisfying $\dim_k|x| = 2$. Eventually, we will define pattern-like vectorspace categories $(K, |-|)$. Roughly speaking these are faithful vectorspace categories with the property that the quiver Γ of K together with the set of indecomposables with $\dim_k|x| = 2$ forms a pattern-like quiver (Γ, T). The main result of this paper will be that such a pattern-like vectorspace category has to be isomorphic to $(L_{(\Gamma,T)}, |-|_\omega)$. This shows that analogous to linear vectorspace categories the pattern-like vectorspace categories are uniquely determined by combinatorial data.

1 Quivers of Krull-Schmidt categories

1.1. We denote by rad_K the Jacobson radical of the Krull-Schmidt category K. For two indecomposable objects x, y of K the subspace $\mathrm{rad}_K(x,y)$ of $K(x,y)$ consists of all non-isomorphisms. As rad_K is an ideal of K, we may also consider its powers which are denoted by rad_K^n for all $n \in \mathbb{N}$. With this notation the quiver Γ_K of K is defined in the following way. The points of Γ_K are just the objects of $\mathrm{ind}\, K$ and for two points x, y there are exactly $\dim_k \mathrm{Irr}_K(x,y)$ arrows from x to y where $\mathrm{Irr}_K(x,y) = \mathrm{rad}_K(x,y)/\mathrm{rad}_K^2(x,y)$. Hence Γ_K is uniquely determined up to renaming the arrows, which justifies to talk about the quiver of K. Obviously between any two vertices of Γ_K there are at most finitely many arrows.

A morphism $f : x \to y$ in K is said to be a source map provided f is not a section, for every map $f' : x \to y'$ which is also not a section there exists $g : y \to y'$ such that $f' = fg$ and finally every endomorphism h of y satisfying $f = fh$ is an automorphism. The domain x of a source map f is always indecomposable. Source maps can be used to calculate the quiver of K. Namely, if $f : x \to y$ is a source map, then for each indecomposable object z of K the multiplicity of z as direct summand of y equals $\dim_k \mathrm{Irr}_K(x,z)$. The Krull-Schmidt category K is said to have source maps if for each $x \in \mathrm{ind}\, K$ there is a source map starting in x. Sink maps are introduced dually to source maps and consequently have the dual properties. If K has source and sink maps, then the quiver Γ_K is locally finite i.e. in each vertex x start and stop only finitely many arrows.

1.2. For a quiver Γ the path category W_Γ has the set Γ_0 of vertices of Γ as set of objects and for any two vertices x, y of Γ the set of morphisms from x to y is given by all directed paths from x to y in Γ. The composition is the concatenation of paths. Note that in order to obtain a category there has to be an identity 1_x for each point x of Γ which is considered as path of length 0.

The k-linear path category $k\Gamma$ associated with Γ has the same objects as W_Γ and the set of morphisms from x to y in W_Γ forms a k-basis of $k\Gamma(x,y)$. The composition is inherited from W_Γ in the natural way. The paths of positive length generate an ideal $k\Gamma^+$ of $k\Gamma$. Another ideal I of $k\Gamma$ is said to be preadmissible provided $I \subseteq (k\Gamma^+)^2$. It is called admissible if moreover $(k\Gamma/I)(x,y)$ is finite dimensional for all vertices x, y. In this case the additive closure K of the factor category $k\Gamma/I$ is a Krull-Schmidt category satisfying

$\operatorname{ind} K = k\Gamma/I$. If, additionally, the quiver Γ is locally finite, then this category K has source and sink maps induced by the arrows starting resp. ending in a given vertex.

We use the notation $x \rightsquigarrow y$ if there is a path from x to y in Γ or in other words a morphism from x to y in W_Γ. The quiver Γ is said to be interval finite if for any two points x, y of Γ the set of points z such that $x \rightsquigarrow z \rightsquigarrow y$ is finite. Note that finite quivers are always interval finite in our sense. In particular, there may still be infinitely many paths between two vertices. We will mostly deal with quivers Γ which are directed i.e. for each vertex x the trivial path 1_x is the only path from x to x. If Γ is directed and interval finite, then every preadmissble ideal I of $k\Gamma$ is automatically admissible.

A full subquiver Γ' of Γ is said to be convex provided that for each pair x, y of vertices of Γ' all vertices z of Γ satisfying $x \rightsquigarrow z \rightsquigarrow y$ belong to Γ' as well. Convex full subcategories Γ' have the pleasant property that $W_{\Gamma'}$ is just the full subcategory of W_Γ induced by Γ'_0 and consequently $k\Gamma'$ is the full subcategory of $k\Gamma$ induced by Γ'_0.

1.3. Let K be a Krull-Schmidt category. A k-linear functor $k\Gamma_K \to \operatorname{ind} K$ which is the identity on the objects is called presentation provided for any two indecomposables x, y the set $\{\gamma_1, \ldots, \gamma_r\}$ of arrows from x to y in Γ_K is mapped by F to a set of morphisms in $\operatorname{rad}_K(x, y)$ which induces a basis of $\operatorname{Irr}_K(x, y)$. This implies that $I_F = \operatorname{Ker} F$ is an admissible ideal of $k\Gamma_K$.

For $x, y \in \operatorname{ind} K$ we put $x \prec y$ if there is path in K from x to y i.e. a sequence $x_0, \ldots, x_n$ in $\operatorname{ind} K$ such that $\operatorname{rad}_K(x_{i-1}, x_i) \neq 0$ for all $i = 1, \ldots, n$ and $x_0 = x$, $x_n = y$. The category K is called interval finite if for any two $x, y \in \operatorname{ind} K$ there are only finitely many $z \in \operatorname{ind} K$ satisfying $x \prec z \prec y$. It is easy to see that every presentation F of an interval finite Krull-Schmidt category K is full. In this case the relations $\prec$ and $\rightsquigarrow$ coincide. Moreover $\operatorname{ind} K$ can be identified with $k\Gamma/I_F$.

In general it will be hard to check whether a given Krull-Schmidt category K is interval finite. A sufficient condition is that Γ_K is a directed, interval finite quiver, K has source and sink maps and the endomorphism algebra of each indecomposable object in K coincides with k.

1.4. Let us return to the category $\operatorname{add}(kS)$ associated with a partially orderd set S in the introduction. S is called interval finite if for any two elements x, y of S the interval $[y, x]$ of elements z satisfying $x \geq z \geq y$ is finite. For an interval finite poset the Hasse diagram Γ_S is defined by putting an arrow from x to y provided $[y, x]$ contains exactly two elements. We observe that $\Gamma_S = \Gamma_{\operatorname{add}(kS)}$ is interval finite and directed. Moreover all arrows $\gamma : x \to y$ in Γ_S are byepass free i.e. there is no further path in Γ_S from x to y. Obviously the relations $\leq$, $\rightsquigarrow$ and $\prec$ coincide.

By mapping every arrow $\gamma : x \to y$ in Γ_S to the corresponding base element $b(x, y)$ of $kS(x, y)$ we obtain a presentation F of kS such that the set of all differences $u - v$ generates $I_F(x, y)$ for all $x, y \in S$.

The canonical functor $can : \operatorname{add}(kS) \to k\text{-mod}$ can be made representable in the following way. We adjoin to S a unique maximal element ω to obtain an increased poset S^ω. The functor can is isomorphic to the restriction of $\operatorname{add}(kS^\omega)(\omega, -)$ to $\operatorname{add}(kS)$.

Our attempt to generalize the above description of a linear vectorspace category by representing the category by its quiver with relations and rewriting the functor as a representable functor led us to the definition and the study of pattern-like quivers and vectorspace categories presented in the following sections.

2 Pattern-like quivers

2.1. Suppose Γ is a quiver with set of vertices Γ_0 and set of arrows Γ_1. For each point $x \in \Gamma_0$ we denote by x^+ the set of all end points of arrows starting in x and by x^- the set of all start points of arrows ending in x. For a path $u = \alpha_1 \cdots \alpha_n$ in Γ und $x \in \Gamma_0$ we write $x \in u$ if x is start or end point of one of the arrows α_i. For two paths u, v in Γ we write $u|v$ provided there are paths w_1, w_2 such that $v = w_1 u w_2$.

A pair (Γ, T) consisting of a directed, interval finite, connected quiver Γ together with a subset $T = \{t_i : i \in I\}$ of its vertex set Γ_0 is called a *pattern-like quiver* if the following conditions are satisfied:

P1) All arrows of Γ are byepass free.

P2) For all $x \in \Gamma_0$ and $i \in I$ either $x \rightsquigarrow t_i$ or $t_i \rightsquigarrow x$ holds.

P3) For all $i \in I$ there are vertices $\tau t_i, p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, q_1^{(i)}, q_2^{(i)}, q_3^{(i)}, \tau^- t_i$ in $\Gamma_0 \setminus T$ with the properties $(\tau t_i)^+ = \{p_1^{(i)}, p_2^{(i)}, p_3^{(i)}\} = t_i^-$ and $t_i^+ = \{q_1^{(i)}, q_2^{(i)}, q_3^{(i)}\} = (\tau^- t_i)^-$.

Let us note that by P1) all arrows in Γ are simple i.e. between two vertices there is at most one arrow. Conditions P1) and P2) imply that $(p_\mu^{(i)})^+ = \{t_i\} = (q_\mu^{(i)})^-$ for all $i \in I$ and $\mu = 1, 2, 3$. Moreover, P2) shows that if $i \in I$, $x, y \in \Gamma_0$ and $x \rightsquigarrow t_i \rightsquigarrow y$, then $t_i \in u$ for all $u \in W_\Gamma(x, y)$. Finally, the assumption that Γ is interval finite together with P2) enables us to choose the index set I as an interval in $\mathbb{Z}$. This interval carries the usual order and we may suppose that $i \leq j$ is equivalent to $t_i \rightsquigarrow t_j$ for all $i, j \in I$.

In order to give names to certain arrows for given $i \in I$ we consider the full subquiver of Γ induced by the set $\{\tau t_i, p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, t_i, q_1^{(i)}, q_2^{(i)}, q_3^{(i)}, \tau^- t_i\}$. We call this subquiver the double mesh corresponding to t_i.

$$\begin{tikzcd}
 & p_1^{(i)} \arrow[dr, "\beta_1^{(i)}"] & & q_1^{(i)} \arrow[dr, "\delta_1^{(i)}"] & \\
\tau t_i \arrow[ur, "\alpha_1^{(i)}"] \arrow[r, "\alpha_2^{(i)}"] \arrow[dr, "\alpha_3^{(i)}"'] & p_2^{(i)} \arrow[r, "\beta_2^{(i)}"] & t_i \arrow[ur, "\gamma_1^{(i)}"] \arrow[r, "\gamma_2^{(i)}"] \arrow[dr, "\gamma_3^{(i)}"'] & q_2^{(i)} \arrow[r, "\delta_2^{(i)}"] & \tau^- t_i \\
 & p_3^{(i)} \arrow[ur, "\beta_3^{(i)}"'] & & q_3^{(i)} \arrow[ur, "\delta_3^{(i)}"'] &
\end{tikzcd}$$

Obviously the double meshes are convex subquivers of Γ. We omit the superscript $-^{(i)}$ if no confusion is possible.

2.2. To define an admissible idael of $k\Gamma$ which is suitable for our purposes, we need some preparations. Let $C := \{(\mu, \nu) \in \{1, 2, 3\}^2 : \mu \neq \nu\}$ and $\mathcal{S}_3$ the symmetric group on the set $\{1, 2, 3\}$. The group $\mathcal{S}_3$ acts on C by $s \cdot (\mu, \nu) := (s(\mu), s(\nu))$.

This action is faithful and transitive. Thus for $(\mu,\nu),(\mu',\nu') \in C$ there is exactly one $s \in \mathcal{S}_3$ such that $s\cdot(\mu,\nu) = (\mu',\nu')$. We define $\sigma((\mu,\nu),(\mu',\nu')) := \operatorname{sign} s$ obtaining the equations $\sigma((\mu,\nu),(\mu',\nu')) = \sigma((\mu',\nu'),(\mu,\nu))$, $(\sigma((\mu,\nu),(\mu',\nu')))^2 = 1$ and $\sigma((\mu,\nu),(\mu',\nu'))\sigma((\mu',\nu'),(\mu'',\nu'')) = \sigma((\mu,\nu),(\mu'',\nu''))$ for all $(\mu,\nu),(\mu',\nu'),(\mu'',\nu'') \in C$.

Suppose that (Γ,T) is a pattern-like quiver and $x,y \in \Gamma_0\setminus T$. We denote by $W''_\Gamma(x,y)$ the subset of $W_\Gamma(x,y)$ given by all u with the property that there are $i\in I$ and $\mu\in\{1,2,3\}$ with $\beta^{(i)}_\mu\gamma^{(i)}_\mu | u$. Furthermore we put $W'_\Gamma(x,y) := W_\Gamma(x,y)\setminus W''_\Gamma(x,y)$.

As Γ is interval finite, the set $\{l\in I : x \rightsquigarrow t_l \rightsquigarrow y\}$ is a finite (possibly empty) subinterval of I and consequently of the shape $[i,j]$ for some $i,j\in\mathbb{Z}$. P4) shows that we can write each $u \in W_\Gamma(x,y)$ as $u = u^{(i-1)}\prod_{l=i}^{j}\beta^{(l)}_{\mu(l,u)}\gamma^{(l)}_{\nu(l,u)}u^{(l)}$. Representing some $u\in W'_\Gamma(x,y)$ in this form, we obtain $(\mu(l,u),\nu(l,u))\in C$ for all $l=i,\dots,j$. Hence for all $u,v\in W'_\Gamma(x,y)$ the number $\sigma(u,v) := \prod_{l=i}^{j}\sigma((\mu(l,u),\nu(l,u)),(\mu(l,v),\nu(l,v)))\in k$ is well-defined and the above formulas immediately lead to the equations $\sigma(u,v)=\sigma(v,u)$, $(\sigma(u,v))^2=1$ and $\sigma(u,v)\sigma(v,w)=\sigma(u,w)$ for all $u,v,w\in W'_\Gamma(x,y)$.

For $x,y\in\Gamma_0$ we define subsets $R'(x,y)$ and $R''(x,y)$ of $k\Gamma(x,y)$ in the following way:

$$R'(x,y) := \begin{cases} \{u-\sigma(u,v)v : u,v\in W'_\Gamma(x,y)\} & \text{if } x,y\in\Gamma_0\setminus T \\ \emptyset & \text{else}\end{cases}$$

$$R''(x,y) := \begin{cases} \{\sum_{\mu=1}^{3}\alpha^{(i)}_\mu\beta^{(i)}_\mu\} & \text{if there exists } i\in I \text{ such that } x=\tau t_i, y=t_i \\ \{\sum_{\mu=1}^{3}\gamma^{(i)}_\mu\delta^{(i)}_\mu\} & \text{if there exists } i\in I \text{ such that } x=t_i, y=\tau^- t_i \\ \{\beta^{(i)}_\mu\gamma^{(i)}_\mu\} & \text{if there exist } i \text{ and } \mu \text{ such that } x=p^{(i)}_\mu \text{ and } y=q^{(i)}_\mu \\ \emptyset & \text{else}\end{cases}$$

We denote by $J_{(\Gamma,T)}$ the ideal of $k\Gamma$ generated by R' and R''. From P1) follows that $J_{(\Gamma,T)}$ is a preadmissible hence an admissible ideal of $k\Gamma$. Finally we put $K_{(\Gamma,T)} := k\Gamma/J_{(\Gamma,T)}$ and $\bar u := u + J_{(\Gamma,T)}$ for all paths u in W_Γ.

2.3. **Lemma.** *Suppose (Γ,T) is a pattern-like quiver and $x,y\in\Gamma_0\setminus T$. We denote by U'' the subspace of $k\Gamma(x,y)$ generated by the $u\in W''_\Gamma(x,y)$ and by U' the subspace of $k\Gamma(x,y)$ generated by the elements $u-\sigma(u,v)v$ of $R'(x,y)$. Then the following holds:*
a) $\dim_k U'' = \operatorname{card}(W''_\Gamma(x,y))$.
b) If $W'_\Gamma(x,y)\neq\emptyset$ then $\dim_k U' = \operatorname{card}(W'_\Gamma(x,y))-1$.
c) $J_{(\Gamma,T)}(x,y) = U'\oplus U''$.
In particular this shows $\dim_k K_{(\Gamma,T)}(x,y) = \begin{cases} 1 & \text{if } W'_\Gamma(x,y)\neq\emptyset \\ 0 & \text{else.}\end{cases}$

Proof: a) is obvious. For the proof of b) we fix $u\in W'_\Gamma(x,y)$ and want to show that the differences $u-\sigma(u,v)v$ with $v\in W'_\Gamma(x,y), v\neq u$ form a k-basis of U'. As the linear independence is apparent, we only have to guarantee that each difference $v-\sigma(v,w)w$ with $v,w\in W'_\Gamma(x,y)$ and $v\neq u\neq w$ is a linear combination of our proposed base vectors. This follows from $v-\sigma(v,w)w = -\sigma(u,v)(u-\sigma(u,v)v)+\sigma(u,v)(u-\sigma(u,w)w)$.

Concerning c) of course $U'\cap U''=0$ and "$\supseteq$" are clear. To show "$\subseteq$", we choose $r\in R'(x',y')\cup R''(x',y')$ and $u\in W_\Gamma(x,x'), v\in W_\Gamma(y',y)$. For $r=\beta_\mu\gamma_\mu$ one gets immediately

$urv \in U''$. So without loss of generality we may suppose $u \in W'_\Gamma(x,x')$, $v \in W'_\Gamma(y',y)$. If $r = u' - \sigma(u',v')v' \in R'(x',y')$, then $uu'v, uv'v \in W'_\Gamma(x,y)$ and $\sigma(uu'v, uv'v) = \sigma(u',v')$. Hence $urv = uu'v - \sigma(uu'v, uv'v)uv'v \in U'$.

For $r = \sum_{\mu=1}^{3} \alpha_\mu \beta_\mu$ after possibly rearranging the indices, we assume $v = \gamma_3 w$. Denoting by s the transposition in $\mathcal{S}_3$ which permutes 1 and 2, we observe $s \cdot (1,3) = (2,3)$. Consequently $\sigma(u\alpha_1\beta_1\gamma_3 w, u\alpha_2\beta_2\gamma_3 w) = \operatorname{sign} s = -1$. Therefore we obtain $urv = (u\alpha_1\beta_1\gamma_3 w - \sigma(u\alpha_1\beta_1\gamma_3 w, u\alpha_2\beta_2\gamma_3 w)u\alpha_2\beta_2\gamma_3 w) + u\alpha_3\beta_3\gamma_3 w \in U' + U''$. The case $r = \sum_{\mu=1}^{3} \gamma_\mu \delta_\mu$ is treated analogously.

2.4. **Lemma.** *Let (Γ, T) be a pattern-like quiver. We put $K := K_{(\Gamma,T)}$ and fix $i \in I$. Then for each $x \in \Gamma_0$ such that $x \neq t_i$ the following holds:*
a) The sequence

$$K(x, \tau t_i) \xrightarrow{K(x,a)} K(x, \oplus_{\mu=1}^{3} p_\mu^{(i)}) \xrightarrow{K(x,b)} K(x, t_i) \to 0$$

where $a = (\bar\alpha_1^{(i)}, \bar\alpha_2^{(i)}, \bar\alpha_3^{(i)})$ and $b = (\bar\beta_1^{(i)}, \bar\beta_2^{(i)}, \bar\beta_3^{(i)})^T$ is exact.
b) The map $K(x,a)$ is not injective if and only if $x = t_j \in T$, $W'_\Gamma(\tau^- t_j, \tau t_i) = \emptyset$ and $W'_\Gamma(q_\mu^{(j)}, \tau t_i) \neq \emptyset \neq W'_\Gamma(\tau^- t_j, p_\mu^{(i)}) \neq \emptyset$ for all $\mu = 1,2,3$.
c) If $K(x,a)$ is not injective, then $\dim_k \operatorname{Ker} K(x,a) = 1$.

Proof: By P4) and the definition of $J_{(\Gamma,T)}$ the sequence in a) is exact.

For b) we first want to show that $K(x,a)$ is injective provided the presented condition fails. In case $x \notin T$ by 2.3 we may suppose that $W'_\Gamma(x, \tau t_i)$ is non-empty and choose a path u in this set. From 2.3 we derive that $\bar u$ is a k-basis of $K(x, \tau t_i)$ and for all $\mu = 1,2,3$ the class $\bar u \bar\alpha_\mu$ is a k-basis of $K(x, p_\mu)$.

For $x = t_j \in T$ we define $M := \{\mu = 1,2,3 : W'_\Gamma(q_\mu^{(j)}, \tau t_i) \neq \emptyset\}$. Considering an arbitrary $\bar f \in \operatorname{Ker} K(x,a)$, by 2.3 we can write f as $f = \sum_{\mu \in M} \lambda_\mu \gamma_\mu^{(j)} u_\mu$ where $\lambda_\mu \in k$ and $u_\mu \in W'_\Gamma(q_\mu^{(j)}, \tau t_i)$. If $M \neq \{1,2,3\}$, from $\bar\beta_\nu^{(j)} \bar f \in \operatorname{Ker} K(p_\nu^{(j)}, a)$ for $\nu = 1,2,3$ we obtain that all λ_μ are 0 by applying 2.3.

If $M = \{1,2,3\}$ but there exists $u \in W'_\Gamma(\tau^- t_j, \tau t_i)$, then modulo $J_{(\Gamma,T)}$ the morphism f coincides with $\sum_{\mu=1}^{2} \gamma_\mu^{(j)}(\lambda_\mu u_\mu - \lambda_3 \sigma(u_3, \delta_3^{(j)} u)\delta_\mu^{(j)} u)$. Now by 2.3 modulo $J_{(\Gamma,T)}$ we may rewrite $\lambda_\mu u_\mu - \lambda_3 \sigma(u_3, \delta_3^{(j)} u)\delta_\mu^{(j)} u$ as $\lambda'_\mu u_\mu$ where $\lambda'_\mu \in k$. As above we use $\bar\beta_\nu^{(j)} \bar f \in \operatorname{Ker} K(p_\nu^{(j)}, a)$ for $\nu = 1,2$ to arrive at $\lambda'_1 = \lambda'_2 = 0$.

Supposing now that also $W'_\Gamma(\tau^- t_j, \tau t_i) = \emptyset$, we obtain that without loss of generality $W'_\Gamma(\tau^- t_j, p_3^{(i)}) = \emptyset$. Since by the dual of part a) of this lemma the sequence

$$K(\tau^- t_j, p_3^{(i)}) \longrightarrow K(\oplus_{\mu=1}^{3} q_\mu^{(j)}, p_3^{(i)}) \longrightarrow K(t_j, p_3^{(i)}) \to 0$$

is exact, from $0 = \bar f \bar\alpha_3^{(i)}$ we get $(\lambda_1 \bar u_1 \bar\alpha_3^{(i)}, \lambda_2 \bar u_2 \bar\alpha_3^{(i)}, \lambda_3 \bar u_3 \bar\alpha_3^{(i)}) = (\bar\delta_1^{(j)}, \bar\delta_2^{(j)}, \bar\delta_1^{(j)})\bar v$. Now $W'_\Gamma(\tau^- t_j, \tau t_i) = \emptyset$ implies $\bar v = 0$ and consequently $\lambda_\mu = 0$ for $\mu = 1,2,3$.

In the sequel we want to prove that $K(x,a)$ is not injective provided our condition is satisfied. Thus we may choose elements $u_\mu \in W'_\Gamma(q_\mu^{(j)}, \tau t_i)$ and $v_\mu \in W'_\Gamma(\tau^- t_j, p_\mu^{(i)})$. More precisely, we will show that $\operatorname{Ker} K(x,a)$ is the k-space generated by $\bar f$ where $f =$

$\sum_{\mu=1}^{3}\sigma(u_\mu\alpha_1^{(i)},\delta_\mu^{(j)}v_1)\gamma_\mu^{(j)}u_\mu$. Obviously $\bar f\bar\alpha_1^{(i)}=0$. To see that also $\bar f\bar\alpha_2^{(i)}=0$, for each $\mu=1,2,3$ we calculate

$$\sigma(u_\mu\alpha_1^{(i)},\delta_\mu^{(j)}v_1)=\sigma(u_\mu\alpha_1^{(i)}\beta_1^{(i)}\gamma_3^{(i)},\delta_\mu^{(j)}v_1\beta_1^{(i)}\gamma_3^{(i)})=$$

$$=\sigma(u_\mu\alpha_1^{(i)}\beta_1^{(i)}\gamma_3^{(i)},\delta_\mu^{(j)}v_2\beta_2^{(i)}\gamma_3^{(i)})\sigma(\delta_\mu^{(j)}v_2\beta_2^{(i)}\gamma_3^{(i)},\delta_\mu^{(j)}v_1\beta_1^{(i)}\gamma_3^{(i)})=$$

$$=(-1)\sigma(u_\mu\alpha_2^{(i)}\beta_2^{(i)}\gamma_3^{(i)},\delta_\mu^{(j)}v_2\beta_2^{(i)}\gamma_3^{(i)})\sigma(\delta_\mu^{(j)}v_2\beta_2^{(i)}\gamma_3^{(i)},\delta_\mu^{(j)}v_1\beta_1^{(i)}\gamma_3^{(i)})=$$

$$=\sigma(u_\mu\alpha_2^{(i)},\delta_\mu^{(j)}v_2)(-1)\sigma(v_2\beta_2^{(i)}\gamma_3^{(i)},v_1\beta_1^{(i)}\gamma_3^{(i)})$$

and observe that the factor $(-1)\sigma(v_2\beta_2^{(i)}\gamma_3^{(i)},v_1\beta_1^{(i)}\gamma_3^{(i)})$ is independent of μ.

If conversely an element $\bar g\in\operatorname{Ker}K(x,a)$ is given where $g=\sum_{\mu=1}^{3}\lambda_\mu\gamma_\mu^{(j)}u_\mu$, by subtracting from $0=\bar g\bar\alpha_3^{(i)}$ the class of $\lambda_3\sigma(u_3\alpha_1^{(i)},\delta_3^{(j)}v_1)\sum_{\mu=1}^{3}\gamma_\mu^{(j)}\delta_\mu^{(j)}v_1$ and multiplying the result with $\bar\beta_1^{(j)}$ resp. $\bar\beta_2^{(j)}$ we get $\lambda_\mu=\lambda_3\sigma(u_\mu\alpha_1^{(i)},\delta_\mu^{(j)}v_1)\sigma(u_3\alpha_1^{(i)},\delta_3^{(j)}v_1)$ for $\mu=1,2$.

The proof of a) and also b) is completed by showing that $\bar f\neq 0$. But actually $\bar f=0$, again using the dual of a), would imply $\sigma(u_\mu\alpha_1^{(i)},\delta_\mu^{(j)}v_1)\bar u_\mu=\bar\delta_\mu^{(j)}\bar v$ for $\mu=1,2,3$ and some $\bar v\in K(\tau^-t_j,\tau t_i)$ which has to be 0 by $W'_\Gamma(\tau^-t_j,\tau t_i)=\emptyset$. This would be a contradiction.

By a *total source* of Γ we understand a vertex ω such that $\omega\rightsquigarrow x$ for every vertex $x\in\Gamma_0$. The two preceding results allow to compute $\dim_k K_{(\Gamma,T)}(x,y)$ for arbitrary $x,y\in\Gamma_0$. In particular for a total source ω using $x\rightsquigarrow\tau t_i$ for all $i\in I$ we obtain:

Corollary. *If (Γ,T) is a pattern-like quiver, ω is a total source of Γ and $x\in\Gamma_0$, then* $\dim_k K_{(\Gamma,T)}(\omega,x)=\begin{cases}2 & \text{if } x\in T\\ 1 & \text{else.}\end{cases}$

2.5. For a pattern-like quiver (Γ,T) such that Γ has a total source ω we consider the vectorspace category $(K_{(\Gamma,T)},K_{(\Gamma,T)}(\omega,-))$. Using the corollary this vectorspace category is sincere but in general it is not faithful. So we pass to the factor category $L_{(\Gamma,T)}:=K_{(\Gamma,T)}/H$ where H is the ideal of K given by the morphisms mapped to 0 by the functor $K_{(\Gamma,T)}(\omega,-)$. By construction $L_{(\Gamma,T)}(\omega,-)$ acts faithfully on $L_{(\Gamma,T)}$. Our next aim is to analyze the dimensions of the morphism spaces $L_{(\Gamma,T)}(x,y)$. In particular this will show that H is induced from an admissible ideal of $k\Gamma$ which implies that Γ is still the quiver of $L_{(\Gamma,T)}$.

Lemma. *Let (Γ,T) be a pattern-like quiver such that Γ has a total source ω. Suppose $x,y\in\Gamma_0$. If we put $\Delta(x,y):=(\dim_k K_{(\Gamma,T)}(\omega,x))\cdot(\dim_k K_{(\Gamma,T)}(\omega,y))$, then* $\dim_k L_{(\Gamma,T)}(x,y)=\min\{\dim_k K_{(\Gamma,T)}(x,y),\Delta(x,y)\}$.

Proof: Of course we may suppose $x\neq y$. We put $K:=K_{(\Gamma,T)}$, $L:=L_{(\Gamma,T)}$ and $m:=\min\{\dim_k K_{(\Gamma,T)}(x,y),\Delta(x,y)\}$. As $L(x,y)$ is an epimorphic image of $K(x,y)$ and the map $L(\omega,-):L(x,y)\to\operatorname{Hom}_k(L(\omega,x),L(\omega,y))$ is injective, we know $\dim_k L(x,y)\leq m$. So we have to prove $\dim_k L(x,y)\geq m$. This inequality follows from the following claim: There is a set of m k-linear independent elements of $K(x,y)$ such that the subspace generated by these elements intersects $H(x,y)$ only in 0. Clearly we may suppose $m>0$.

Let us first deal with the case $x, y \notin T$. This means $m = \dim_k K(x,y) = 1$. In other words, we will establish that $K(\omega, -)$ acts faithfully on $K(x,y)$. Actually, by 2.3 there exists $u \in W'_\Gamma(x,y)$ such that $\bar{u}$ is a k-basis of $K(x,y)$. We choose $v \in W'_\Gamma(\omega, x)$ and derive $vu \in W'_\Gamma(x,y)$ implying $K(\omega,\bar{u})(\bar{v}) = \bar{v}\bar{u} \neq 0$.

If $x \notin T$ but $y = t_i \in T$, then $1 \leq m = \dim_k K(x,y) \leq 2$. Hence we also will show that $K(\omega,-)$ acts faithfully on $K(x,y)$. Let us suppose $m = 2$ since $m = 1$ is even easier. By definiton of K after possibly renumbering the indices we know that the space $K(x,y)$ has a k-basis of the form $\bar{u}_1\bar{\beta}_1^{(i)}, \bar{u}_2\bar{\beta}_2^{(i)}$ where $u_i \in W'_\Gamma(x, p_\mu^{(i)})$. By multiplying an arbitrary linear combination a of these vectors with $\bar{\gamma}_1^{(i)}$ and $\bar{\gamma}_2^{(i)}$ we see that $a \in H$ implies $a = 0$. The case $x \in T$ but $y \notin T$ is handled dually.

For $x = t_j$, $y = t_i$ because of $\tau^- t_j \rightsquigarrow t_i$ and $t_j \rightsquigarrow \tau t_i$ by a suitable renumbering of the indices we obtain the existence of some $u \in W'_\Gamma(\tau^- t_j, p_3^{(i)})$ and some $v \in W'_\Gamma(q_3^{(j)}, \tau t_i)$. We denote by M the subspace of $K(x,y)$ generated by the classes of $\gamma_\mu^{(j)}\delta_\mu^{(j)}u\beta_3^{(i)}$ and $\gamma_3^{(j)}v\alpha_\mu^{(i)}\beta_\mu^{(i)}$ where $\mu = 1,2,3$. By the definition of K the classes of the paths $\gamma_1^{(j)}\delta_1^{(j)}u\beta_3^{(i)}$, $\gamma_2^{(j)}\delta_2^{(j)}u\beta_3^{(i)}$ and $\gamma_3^{(j)}v\alpha_1^{(i)}\beta_1^{(i)}$ generate M. Only the trivial linear combination of these classes lies in H, which can be seen by multiplying such a combination with suitable $\bar{\beta}_\mu^{(j)}$ resp. $\bar{\gamma}_\mu^{(i)}$. Hence $m \geq 3$ and for $m = 3$ we are done. If $m = 4$, there has to be a path $\gamma_\mu^{(j)}w\beta_\nu^{(i)}$ whose class does not lie in M. Consequently $\mu \neq 3 \neq \nu$. Multiplication of an element of the space generated by M and $\bar{\gamma}_\mu^{(j)}\bar{w}\bar{\beta}_\nu^{(i)}$ with $\bar{\beta}_3^{(j)}$ from the left and $\bar{\gamma}_3^{(i)}$ from the right shows that also this subspace has only trivial intersection with H.

Corollary. *If* $\dim_k K_{(\Gamma,T)}(x,y) \leq 4$ *for all* $x, y \in T$, *then* $L_{(\Gamma,T)} = K_{(\Gamma,T)}$.

3 Pattern-like vectorspace categories

3.1. Let us call a pattern-like quiver (Γ, T) *bounded above* by a finite set of sources $Z = \{z_1, \ldots, z_m\}$ provided for each $x \in \Gamma_0$ there exists $z_j \in Z$ satisfying $z_j \rightsquigarrow x$. If (Γ, T) is bounded above and $I \neq \emptyset$ then I has a unique minimal element i_0 and $z \rightsquigarrow t_{i_0}$ for all $z \in Z$.

We increase a pattern-like quiver (Γ, T) bounded above by Z by adding a vertex ω to Γ_0 and an arrow $\varepsilon_z : \omega \to z$ for each $z \in Z$. The obtained quiver will be denoted by Γ^e. Obviously (Γ^e, T) is again a pattern-like quiver. The vertex ω is a total source of Γ^e. Since Γ is a convex subquiver of Γ^e, we see that $K_{(\Gamma,T)}$ is a full subcategory of $K_{(\Gamma^e,T)}$. Hence we may consider the vectorspace category $(K_{(\Gamma,T)}, K_{(\Gamma^e,T)}(\omega,-))$ which by 2.4 satisfies $\dim_k K_{(\Gamma^e,T)}(\omega, t_i) = 2$ for all $i \in I$ and $\dim_k K_{(\Gamma^e,T)}(\omega, x) = 1$ for all $x \in \Gamma_0 \setminus T$.

3.2. Let now (Γ, T) be a general pattern-like quiver. Recall that an antichain Z w.r.t. $\rightsquigarrow$ is just a non-empty subset of Γ_0 such that $x \not\rightsquigarrow y$ for all $x \neq y \in Z$. For such an antichain Z we denote by Γ_Z the full subquiver of Γ supported by all $x \in \Gamma_0$ satisfying $z \rightsquigarrow x$ for some $z \in Z$. This is a convex subquiver of Γ.

If Z is an antichain, by P2) we know that there is $i_0 \in I$ such that $z \rightsquigarrow t_{i_0}$ but, if $i_0 - 1 \in I$, then $z \not\rightsquigarrow t_{i_0-1}$ for each $z \in Z$. Putting $T_Z := T \cap (\Gamma_Z)_0$, it is easy to see that (Γ_Z, T_Z) is again a pattern-like quiver if and only if Z is finite and there is $z \in Z$ satisfying $z \rightsquigarrow \tau t_{i_0}$. Obviously this implies $p_\mu^{(i_0)} \notin Z$ for all $\mu = 1,2,3$. If (Γ_Z, T_Z) is pattern-like, then it is actually bounded above by Z.

We need to choose a set $\mathcal{Z}$ of finite antichains Z w.r.t. $\rightsquigarrow$ and certain paths connecting the elements of $\Omega := \cup_{Z \in \mathcal{Z}} Z$. For this purpose we distinguish two cases.

If the interval I has a minimal element i_0, then we define $\mathcal{Z}$ to be the set of all finite antichains Z which are subsets of the set of all $x \in \Gamma_0$ such that $x = \tau t_{i_0}$ or $\tau t_{i_0} \not\rightsquigarrow x$ having the additional property that there exists $z \in Z$ such that $z \rightsquigarrow \tau t_{i_0}$. For $x, y \in \Omega$ satisfying $x \rightsquigarrow y$ we choose an arbitrary $u_{x,y} \in W'_\Gamma(x,y)$.

In the case that I does not have a minimal element, we define $\mathcal{Z}$ to be the set of all 1-element sets $\{\tau t_i\}$ for all $i \in I$. Now for each $i \in I$ we choose $v_i \in W'_\Gamma(\tau t_{i-1}, \tau t_i)$ and put $u_{\tau t_j, \tau t_i} := \prod_{\mu=j+1}^{i} v_\mu$ for all $j, i \in I$ satisfying $j \leq i$.

3.3. Given a pattern-like quiver (Γ, T) and the set $\mathcal{Z}$ of antichains chosen above, we extend Γ by adding a vertex ω and an arrow $\varepsilon_x : \omega \to x$ for each $x \in \Omega$. Denoting the extended quiver by Γ^ω we observe that Γ is a convex subquiver. Moreover for each $Z \in \mathcal{Z}$ the quiver Γ_Z^e is a subquiver of Γ^ω which usually will not be full.

We call $J^\omega_{(\Gamma,T)}$ the ideal of $k\Gamma^\omega$ generated by the union of all the $J_{(\Gamma_Z^e, T)}$ where $Z \in \mathcal{Z}$ and the differences $\varepsilon_x u_{x,y} - \varepsilon_y$ for all $x, y \in \Omega$ satisfying $x \rightsquigarrow y$. In addition, we define $K^\omega_{(\Gamma,T)}$ to be the factor of $k\Gamma^\omega$ by the ideal $J^\omega_{(\Gamma,T)}$. By construction for each $Z \in \mathcal{Z}$ the inclusion of Γ_Z^e into Γ^ω induces a functor $\iota_Z : K_{(\Gamma_Z^e, T_Z)} \to K^\omega_{(\Gamma,T)}$.

Lemma. *For each $Z \in \mathcal{Z}$ the functor $\iota_Z : K_{(\Gamma_Z^e, T_Z)} \to K^\omega_{(\Gamma,T)}$ is fully faithful.*

Proof: Let us denote by Γ'_Z the full subquiver of Γ^ω induced by $(\Gamma_Z^e)_0$ which is obviously convex. Hence the corresponding full subcategory K'_Z of $K^\omega_{(\Gamma,T)}$ is the factor of $k\Gamma'_Z$ by the induced ideal. By sending all arrows ε_x such that $x \in \Omega$ but $x \notin Z$ to $\varepsilon_z u_{z,x}$ for some $z \in Z$ such that $z \rightsquigarrow x$ and mapping all other arrows and all the vertices identically to themselves, we obtain a functor $K'_Z \to K_{(\Gamma_Z^e, T_Z)}$ which is a retraction for $\iota_Z : K_{(\Gamma_Z^e, T_Z)} \to K'_Z$. Since ι_Z is obviously full, the proof is complete.

3.4. Before we can prove our main theorem, we will provide some informations about the structure of the double meshes of a pattern-like quiver (Γ, T) induced by the relations R''. For this reason let us denote by Δ the quiver presented below and by J' the ideal of $k\Delta$ generated by $\sum_{\mu=1}^3 \alpha_\mu \beta_\mu$, $\sum_{\mu=1}^3 \gamma_\mu \delta_\mu$ and $\beta_\mu \gamma_\mu$ for all $\mu = 1,2,3$. For any two $(\mu,\nu), (\mu',\nu') \in C$ then $\alpha_\mu \beta_\mu \gamma_\nu \delta_\nu - \sigma((\mu,\nu),(\mu',\nu')) \alpha_{\mu'} \beta_{\mu'} \gamma_{\nu'} \delta_{\nu'} \in J'$ holds. Namely let $(\mu',\nu') = s \cdot (\mu,\nu)$. As $\mathcal{S}_3$ is generated by the transpositions, without loss of generality for s we may choose a fixed transposition and calculate for all 6 elements $(\mu,\nu) \in C$ that $\alpha_\mu \beta_\mu \gamma_\nu \delta_\nu + \alpha_{\mu'} \beta_{\mu'} \gamma_{\nu'} \delta_{\nu'}$ lies in J'.

$$\begin{tikzcd}
& p_1 \arrow[dr, "\beta_1"] & & q_1 \arrow[dr, "\delta_1"] & \\
\tau t \arrow[ur, "\alpha_1"] \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_3"'] & p_2 \arrow[r, "\beta_2"] & t \arrow[ur, "\gamma_1"] \arrow[r, "\gamma_2"] \arrow[dr, "\gamma_3"'] & q_2 \arrow[r, "\delta_2"] & \tau^- t \\
& p_3 \arrow[ur, "\beta_3"'] & & q_3 \arrow[ur, "\delta_3"'] &
\end{tikzcd}$$

3.5. We call a faithful vectorspace category $(K, |-|)$ such that $\dim_k |x| \leq 2$ for all $x \in \operatorname{ind} K$ *pattern-like* provided the following conditions are satisfied:

i) If Γ is the quiver of K and $T := \{x \in \Gamma_0 : \dim_k |x| = 2\}$, then (Γ, T) is a pattern-like quiver.

ii) There is a full presentation $F : k\Gamma \to \operatorname{ind} K$ such that $R'' \subseteq I_F$.

iii) For each $i \in I$ the images of the k-linear maps $|F\beta_\mu^{(i)}|$, $\mu = 1, 2, 3$ are pairwise different subspaces of $|t_i|$.

We say that (Γ, T) is the pattern-like quiver associated with $(K, |-|)$. Note that $\sum_{\mu=1}^{3} \operatorname{Im} |F\beta_\mu^{(i)}| = |t_i|$ for each $t_i \in T$. Moreover for each $i \in I$ and $\mu = 1, 2, 3$ the image of $|F\beta_\mu^{(i)}|$ coincides with the kernel of $|F\gamma_\mu^{(i)}|$.

Theorem. *Let $(K, |-|)$ be a pattern-like vectorspace category with associated pattern-like quiver (Γ, T). Then there is a full functor $\Phi : K_{(\Gamma,T)} \to \operatorname{ind} K$, which is the identity on Γ_0, and a natural isomorphism $\varphi : |-|_\omega \to |-|\Phi$ where $|-|_\omega := K^\omega_{(\Gamma,T)}$.*

Proof: Using the full presentation F, we clearly may suppose $K = k\Gamma/J$ where $J = I_F$ is a preadmissible ideal containing R''. We write $\tilde{u}$ for the element of K induced by the path u in W_Γ. For each $x \in \Gamma_0 \setminus T$ we choose a k-basis c_x of $|x|$ and denote by Γ_1' the set of all arrows $\alpha : x \to y$ in Γ_1 such that $x, y \in \Gamma_0 \setminus T$. Since J is preadmissible and $|-|$ is faithful, for every $\alpha \in \Gamma_1'$ the linear map $|\tilde{\alpha}|$ is an isomorphism. Hence there exists $0 \neq \lambda_\alpha \in k$ such that $|\tilde{\alpha}|(c_x) = \lambda_\alpha c_y$.

No we substitute all $\alpha \in \Gamma_1'$ by $\lambda_\alpha^{-1}\alpha$. More precisely this means that we define an equivalence $\widetilde{\psi} : k\Gamma \to k\Gamma$ by fixing the vertices and mapping the arrows $\alpha : x \to y$ in the following way:

$$\widetilde{\psi}(\alpha) := \begin{cases} \lambda_\alpha^{-1}\alpha & \text{if } \alpha \in \Gamma_1' \\ \lambda_{\alpha_\mu}^{(i)}\alpha & \text{if } \alpha = \beta_\mu^{(i)} \\ \lambda_{\delta_\mu}^{(i)}\alpha & \text{if } \alpha = \gamma_\mu^{(i)} \end{cases}$$

The ideal $J' := \widetilde{\psi}^{-1}(J)$ of $k\Gamma$ is still preadmissible and contains R''. Denoting by $\psi : k\Gamma/J' \to k\Gamma/J$ the equivalence induced by $\widetilde{\psi}$, the vectorspace category $(k\Gamma/J', |-|\psi)$ is isomorphic to $(K, |-|)$. Therefore without loss of generality we can write again J for J' and $|-|$ for $|-|\psi$ gaining the additional property $|\tilde{\alpha}|(c_x) = c_y$ for all $\alpha : x \to y$ in Γ_1'.

Using the definiton of pattern-like vectorspace categories, we observe that for every $t_i \in T$ and every $(\mu,\nu) \in C$ there is $0 \neq \lambda^{(i)}_{(\mu,\nu)}$ such that $|\tilde{\beta}^{(i)}_\mu \tilde{\gamma}^{(i)}_\nu|(c_{p_\mu}) = \lambda^{(i)}_{(\mu,\nu)} c_{q_\nu}$. Now 3.4 implies $\lambda^{(i)}_{(\mu,\nu)} = \sigma((\mu,\nu),(\mu',\nu'))\lambda^{(i)}_{(\mu',\nu')}$ for all $(\mu,\nu),(\mu',\nu') \in C$. This shows that $R' \subseteq J$ and thus $J_{(\Gamma,T)} \subseteq J$.

If the interval I has a minimal element, we may procede. Otherwise we have to make another substitution. Namely for all $i \in I$ we choose $\lambda_{i-1} \in k$ such that by substituting the arrows $\gamma^{(i-1)}_\mu$ by $\lambda_{i-1}\gamma^{(i-1)}_\mu$ we obtain $|\tilde{v}_i| c_{\tau t_{i-1}} = c_{\tau t_i}$ and consequently $|\tilde{u}_{\tau t_j, \tau t_i}|(c_{\tau t_j}) = c_{\tau t_i}$.

Now we extend J to an ideal J^ω of $k\Gamma^\omega$ in the obvious way such that $J^\omega_{(\Gamma,T)} \subseteq J^\omega$. Using the convexity of Γ in Γ^ω we may identify K with the full subcategory of $K^\omega := k\Gamma^\omega / J^\omega$ induced by Γ_0. Because of our substitutions we may extend $|-|$ to a faithful functor $|-| : K^\omega \to k\text{-mod}$ by putting $|\omega| = k$ and $|\bar{\varepsilon}_z|(c_\omega) = c_z$ for all $z \in \Omega$ where $c_\omega = 1 \in k$.

We consider the canonical projection $\pi : K^\omega_{(\Gamma,T)} = k\Gamma^\omega / J^\omega_{(\Gamma,T)} \to k\Gamma^\omega / J^\omega$ and observe that by $\varphi_x : |x|_\omega \to |x|$, $a \mapsto |\pi(a)|(c_\omega)$ we obtain a natural homomorphism $\varphi : |-|_\omega \to |-|\pi$ which is actually the one obtained from $c_\omega \in |\omega|$ by using the Yoneda lemma.

In order to show that φ is an isomorphism, it suffices to establish that each map φ_x is surjective. Since we have chosen $\mathcal{Z}$ in a way such that for each $x \in \Gamma_0$ there exists $Z \in \mathcal{Z}$ satisfying $x \in (\Gamma_Z)_0$, by lemma 3.3 we may suppose $\Gamma^\omega = \Gamma^e_Z$. Thus we may apply induction on the number $\operatorname{prd}(x)$ of all $y \in \Gamma^\omega_0$ such that $y \rightsquigarrow x$. Of course $\operatorname{prd}(x) = 1$ just means $x = \omega$. In this case φ_ω is an epimorphism by $\varphi_\omega(\bar{1}_\omega) = 1 \in k$. If $\operatorname{prd}(x) > 1$, then $x^- \neq \emptyset$ and $\operatorname{prd}(y) < \operatorname{prd}(x)$ for all $y \in x^-$. Thus all the φ_y are surjective by induction. For $x \notin T$ we choose an arbitrary $y \in x^-$ and observe that $|\tilde{\alpha}|$ has to be surjective where α is the arrow from y to x in Γ because $\dim_k |x| = 1$ and $|-|$ is faithful. The equation $|\tilde{\alpha}|\varphi_y = \varphi_x |\bar{\alpha}|_\omega$ yields the assertion. If $x = t_i$, the surjectivity of φ_x follows from $|x| = \sum_{\mu=1}^3 \operatorname{Im}(|\tilde{\beta}_\mu| \varphi_{p_\mu}) = \sum_{\mu=1}^3 \operatorname{Im}(\varphi_x |\bar{\beta}_\mu|_\omega) \subseteq \operatorname{Im}\varphi_x$.

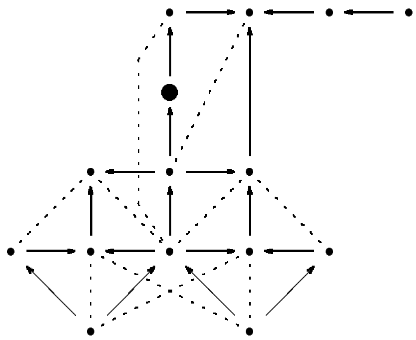
3.6. Using 2.5 and 3.3, we can check easily whether $|-|_\omega$ acts faithfully on $K_{(\Gamma,T)}$. In any case 2.5 shows that the kernel H_ω of $|-|_\omega$ is induced by an admissible ideal of $k\Gamma$. Putting $L_{(\Gamma,T)} := K_{(\Gamma,T)}/H_\omega$ and denoting the induced functor on this factor category again by $|-|_\omega$, the above theorem yields:

Corollary. *Let $(K, |-|)$ be a pattern-like vectorspace category with associated pattern-like quiver (Γ, T). Then there is an equivalence $\Phi : L_{(\Gamma,T)} \to \operatorname{ind} K$, which is the identity on the objects, and an isomorphism $\varphi : |-|_\omega \longrightarrow |-|\Phi$.*

Remark. It happens in practice that all axioms of a pattern-like vectorspace category are satisfied with the only exception that the quiver Γ of K is merely a convex, connected full subquiver of some quiver Γ' such that (Γ', T) is pattern-like. This only means that I has a minimal element j_0 such that τt_{j_0} and possibly some of the $p^{(j_0)}_\mu$ are missing or I has a maximal element i_0 such that $\tau^- t_{i_0}$ and possibly some of the $q^{(i_0)}$ are missing. In this case one may add to Γ the missing vertices and corresponding arrows to obtain a pattern-like quiver $(\hat{\Gamma}, T)$. Imposing the induced relations from K and the necessary relations for the added arrows, one constructs a category $\hat{K}$ which has K as full convex subcategory. Then one extends the functor $|-|$ from K to $\hat{K}$ in the obvious way to arrive at an actually pattern-like vectorspace category to which the above results can be applied.

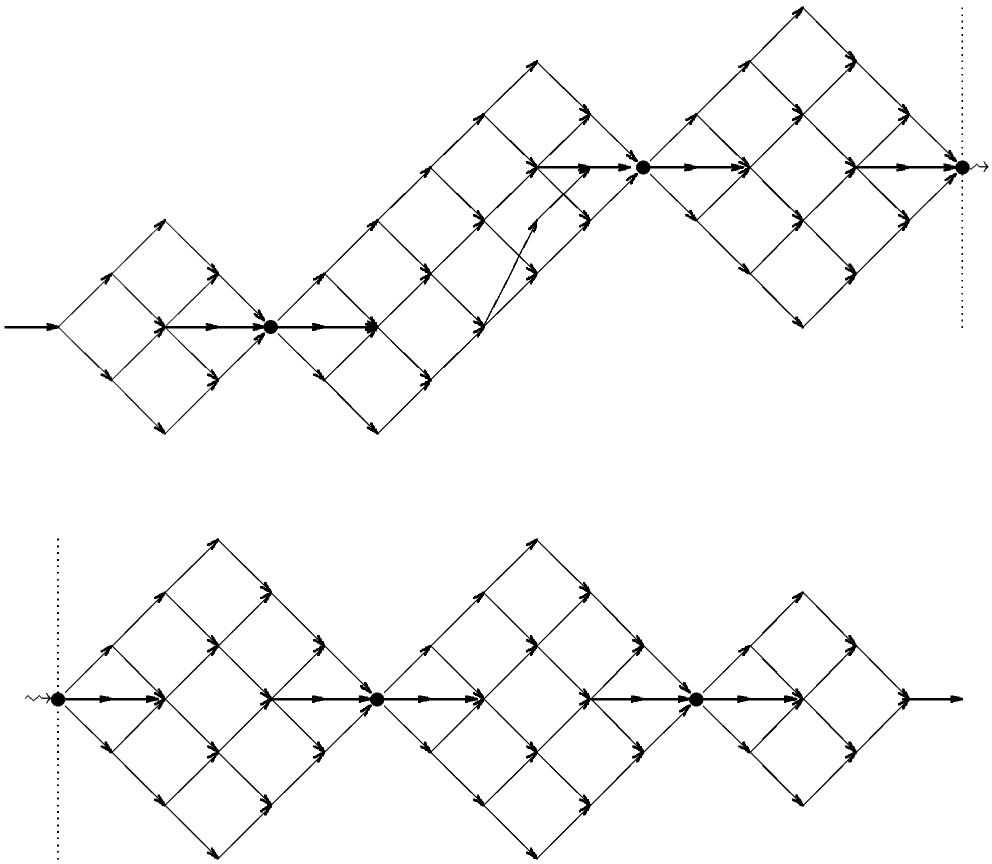
3.7. We want to finish this paper by an showing an example that pattern-like vectorspace categories which are not themselves patterns (see [Ri2]) in fact occur in 'nature'. Recall from [Dr1] that many informations about the representation theory of a finite dimensional k-algebra A are encoded in the vectorspace category $(K(A,P), \mathrm{Hom}_A(P,-))$ where P is an indecomposable projective A-module such that $\mathrm{End}_A(P) = k$ and $K(A,P)$ is the full subcategory of all finite dimensional A-modules V satisfying $\mathrm{Ext}^1_A(V, \mathrm{fac}\, P) = 0$. In particular finite resp. tame representation type of A is characterized by finite resp. tame subspace type of $(K(A,P), \mathrm{Hom}_A(P,-))$.

Let us denote by $(K, |-|)$ the faithful vectorspace category associated with $(K(A,P), \mathrm{Hom}_A(P,-))$ as explained in the introduction. In [Dr2] we developed a technique to calculate the quiver Γ of K together with some relations and in addition the dimensions of the spaces $|x|$ for the vertices x of Γ. We apply this to the algebra A given by the following quiver with relations. This example is borrowed from [PT] where it is shown that this algebra is iterated tubular and consequently of tame representation type.



The faithful vectorspace category $(K, |-|)$ obtained from the indecomposable projective module belonging to the distinguished vertex is actually pattern-like with the associated pattern-like quiver (Γ, T) displayed below. The set T is represented as the set of bold points.

Using the list of tubular vectorspace categories from [Ri3] and adapting the concept of iterated tubular extension to vectorspace categories one can construct an iterated tubular vectorspace category $(K', |-|')$ which is pattern-like with the same associated pattern-like quiver (Γ, T). Iterated tubular vectorspace categories will be studied in the forthcoming paper [Ba]. In particular, by construction these vectorspace categories are of tame subspace type. On the other hand by corollary 3.6 the vectorspace categories $(K, |-|)$ and $(K', |-|')$ are isomorphic. This means that we not only found a natural example for the occurrance of pattern-like vectorspace categories but also gave a new proof for the tameness of A.



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