1 Introduction

The general framework of a decision model is a set of situations together with a set of strategies belonging to these situations. In this paper we will only be interested in stationary strategies, which means independent of when we arrive at a certain situation we will always make the same decision. The aim is to minimize or maximize a certain cost function. The problem is that the outcome of certain decisions is random. In spite of this, we ask, if we can propose a certain subset of strategies as optimal with respect to the cost function.

For example, consider a mathematician having two different ways to prove a hypothesis e.g. say a direct proof and a proof by contradiction. After each hour of work he has to decide, whether to continue the work on the first or on the second type of proof.

Of course the mathematician does not know in advance, whether the next hour of work will be successful in the sense that he will be "closer" to the proof of the hypothesis afterwards. Anyhow, he will be looking for a fastest strategy to prove the hypothesis or to see that both approaches are unsuccessful.

More precisely we will consider random walks of several particles on a connected graph, that means each unit of time we are allowed to activate exactly one particle which, then describes one step of a random walk. Moreover, we are given a certain criterion concerning when the process stops.

The example with the mathematician can be described in terms of a 2 particles nearest-neighbor random walk on a line with two absorbing barriers. We ask for the fastest strategy to answer the question, whether at least one of the particles will be absorbed e.g. at the left barrier or both of them will be absorbed at the right barrier.

In the following we will also talk about best-case strategies when we mean fastest strategies and analogously worst-case strategies, when we mean slowest strategies. Note, that this terminology does not coincide with the "best-case" and "worst-case" terminology used in complexity theory.

Related questions have been studied from a more theoretical point of view for the meeting time of 2 particle random walks on arbitrary connected graphs in [Ald **91**], [CTW **92**] and [TW **93**], where an upper bound for the minimal running time has been established. Worst-case meeting times have been studied in [CTW **92**]. For a survey on random walks on graphs see the rich bibliography of Aldous [Ald **89**].

The paper is organized as follows:

In Section 2 we introduce the basic models we are going to work with and the central questions which are valid for the following sections.

Section 3 contains a best-case and worst-case analysis of the model with 2 particles and with two absorbing barriers.

Section 4 is one way to generalize Section 3 in the way that we now consider 3 particles on a line and ask for the fastest strategy to answer where the majority of the particles will be absorbed.

A second generalization of Section 3 is given in Section 5 where we are given s particles. We ask for the best-case strategy to answer the question, whether one of them will be absorbed at the left or all of them will be absorbed at the right barrier.

In Section 6 we compare the best-case and worst-case strategies for the models above.

Section 7 contains a further type of decision model with random walks. Instead of several particles on a line we consider the case of 3 and 4 particles on the n-circle C_n, where a collision of 2 particles leads to the end of the process.

In Section 8 we pose some open problems related to decision models with random walks.

2 A Basic Model

In Section 3 – 5 we are going to work with the following model:

We consider $s \geq 2$ particles on the non-negative integers $\{0, 1, \ldots, n\}$. Each unit of time we are allowed to knock exactly one of the particles which then describes a symmetric nearest-neighbor random walk. Let 0 and n be absorbing states of the random walk. It is well known (e.g. see [Chu **78**]) that if we only have one particle, say on state i, then the expected running time i.e. the number of knocks till absorption is equal to $i\,(n-i)$.

In Sections 3 – 5 we answer the following questions:

- *"How long does it take for $s = 2$ till either one particle will be absorbed at 0 or both will be absorbed at n ?"* (1)

  As a consequence of the law of large numbers one of these alternatives is reached in finite time almost surely. This means we are interested in the minimal and maximal running times.

- *"How long does it take for $s = 3$ till the majority of the particles will be absorbed in one of the states 0 or n ?"* (2)

  Again we are interested in extremal strategies.

- *"How long does it take for arbitrary $s \geq 3$ till either one particle will be absorbed at 0 or all of them will be absorbed at n ?"* (3)

If we label the positions of the particles with i and j for $s = 2$ and for convenience $0 \leq i \leq j \leq n$, the first question corresponds to the following equations:

$$F_n(i,j) = \begin{cases} 0\,, & \text{if} \quad i = 0\,, \\ i\,(n-i)\,, & \text{if} \quad j = n\,, \\ 1 + min\Big[\, \frac{1}{2}\big(F_n(i-1,j) + F_n(i+1,j)\big), & \\ \qquad\qquad \frac{1}{2}\big(F_n(i,j-1) + F_n(i,j+1)\big)\Big], & \text{if} \quad i < j\,, \\ 1 + \frac{1}{2}\big(F_n(i-1,i) + F_n(i,i+1)\big), & \text{if} \quad i = j\,. \end{cases} \tag{4}$$

For $s=3$ we denote the position of the third particle by k, so that $0 \le i \le j \le k \le n$. Then the second question corresponds to the equations below. We remark that we use again F_n for the expected running time, because the only difference is the occurrence of the third particle.

$$F_n(i,j,k) = \begin{cases} 0, & \text{if } \ i,j=0\,, \\ 0, & \text{if } \ j,k=n\,, \\ j\,(n-j), & \text{if } \ 1 \le j \le n-1, \ i=0 \ \text{ and } \ k=n\,, \\ F_n(j,k), & \text{if } \ 1 \le j \le k \le n-1 \ \text{ and } \ i=0\,, \\ F_n(n-j,n-i), & \text{if } \ 1 \le i \le j \le n-1 \ \text{ and } \ k=n\,, \\ 1+min\Big[\frac{1}{2}\big(F_n(i-1,j,k)+F_n(i+1,j,k)\big), & \\ \qquad \frac{1}{2}\big(F_n(i,j-1,k)+F_n(i,j+1,k)\big), & \\ \qquad \frac{1}{2}\big(F_n(i,j,k-1)+F_n(i,j,k+1)\big)\Big]\,, & \text{if } \ 1 \le i < j < k \le n-1\,, \\ 1+min\Big[\frac{1}{2}\big(F_n(i-1,i,k)+F_n(i,i+1,k)\big), & \\ \qquad \frac{1}{2}\big(F_n(i,i,k-1)+F_n(i,i,k+1)\big)\Big]\,, & \text{if } \ 1 \le i = j < k \le n-1\,, \\ 1+min\Big[\frac{1}{2}\big(F_n(i-1,j,j)+F_n(i+1,j,j)\big), & \\ \qquad \frac{1}{2}\big(F_n(i,j-1,j)+F_n(i,j,j+1)\big)\Big]\,, & \text{if } \ 1 \le i < j = k \le n-1\,, \\ 1+\frac{1}{2}\big(F_n(i-1,i,i)+F_n(i,i,i+1)\big), & \text{if } \ 1 \le i = j = k \le n-1\,. \end{cases} \tag{5}$$

Obviously, if we replace min by max in (4) and (5), respectively, we obtain the maximal expected running time which is denoted by $G_n(i,j)$ in the model for $s=2$.

Since our proofs are based on the fact that an extremal strategy fulfills equations (4) or (5) we have to show that the equations have unique solutions. In Lemma 2.1 below we will show this for equation (4). The proofs concerning equation (5) and the cases, where min is replaced by max follow the same arguments.

Lemma 2.1 *The solution of System (4) if it exists, is unique.*

Proof: Suppose there are two different solutions $\alpha_{(i,j)}$ and $\beta_{(i,j)}$ of equation (4). Then, without loss of generality, there exists a configuration (i,j) and $\varepsilon > 0$ such that $\alpha_{(i,j)} - \beta_{(i,j)} = \varepsilon$ is maximal. If there are several such configurations we choose one with maximal difference $j-i$. If there are still more than one such configuration we take anyone of them e.g. the one with minimal i. Obviously, $i \ne 0$ and $j \ne n$. If $i<j$ and

$$\beta_{(i,j)} = 1 + \tfrac{1}{2}\left(\beta_{(i,j-1)} + \beta_{(i,j+1)}\right)\,, \quad \text{then, since } \alpha_{(i,j)} \text{ fulfills (4),}$$

$$\varepsilon \;=\; \alpha_{(i,j)} - \beta_{(i,j)} \;\le\; \tfrac{1}{2}(\alpha_{(i,j-1)} - \beta_{(i,j-1)}) + \tfrac{1}{2}(\alpha_{(i,j+1)} - \beta_{(i,j+1)})\,. \tag{6}$$

If now one of the summands on the right hand side of (6) is strictly greater than $\varepsilon/2$, then (i,j) is not the configuration with maximal difference between α and β in contradiction to our assumptions. Hence both summands equal $\varepsilon/2$, but then $\alpha_{(i,j+1)} - \beta_{(i,j+1)} = \varepsilon$, which is again in contradiction to our assumption that j and i have maximal difference.

The case $\beta_{(i,j)} = 1 + \frac{1}{2}\,\left(\beta_{(i-1,j)} + \beta_{(i+1,j)}\right)$ and the case $i = j$ are treated similarly. ■

3 Random Walks on a Line with 2 Particles

We give an answer to Question (1) of Section 2.

3.1 Running Time using the Best-Case Strategy

We show that the optimal strategy is the one which always knocks the left of both particles. Optimal means that the running time is minimal till one of the two final states is reached.
This strategy will be denoted by F, and the expected running time to the constellation (i,j) will be denoted by $F_n(i,j)$, with $0 \le i \le j \le n$.

Theorem 3.1 *In the random walk model with 2 particles it is uniquely optimal always to knock the left particle to obtain the minimal expected running time in order to answer Question (1).*

The minimal expected running time is for all $0 \le i \le j \le n$ given by:

$$F_n(i,j) = i\left(\frac{(n-j)(n-1)}{n} + (n-i) - \sum_{t=1}^{n-j-1} \frac{2\,t}{t+j}\right).$$

Proof: Equations (4) are fulfilled, if:

$$\begin{array}{lll}
\text{(i)} & F_n(i,j) = 1 + \frac{1}{2}\big(F_n(i-1,j) + F_n(i+1,j)\big) & \text{for all } 1 \le i < j \le n\,, \\
\text{(ii)} & F_n(i,j) < 1 + \frac{1}{2}\big(F_n(i,j-1) + F_n(i,j+1)\big) & \text{for all } 1 \le i < j \le n-1\,, \\
\text{(iii)} & F_n(i,i) = 1 + \frac{1}{2}\big(F_n(i-1,i) + F_n(i,i+1)\big) & \text{for all } 1 \le i \le n-1\,.
\end{array}$$

ad (i) :

$$F_n(i,j) - 1 - \frac{1}{2}\big(F_n(i-1,j) + F_n(i+1,j)\big) \stackrel{!}{=} 0$$

Just replacing the formulas of Theorem 3.1 and elementary transformations yield after shortening the sums:

$$i\,\frac{n\,(n-i)}{n} - 1 - \frac{1}{2}\left(\frac{n\,(n-i+1)}{n}\,(i-1) + \frac{n\,(n-i-1)}{n}\,(i+1)\right) \quad = \quad 0.$$

After another elementary transformation we obtain our desired solution.

ad (ii) :

$$F_n(i,j) \overset{!}{<} 1 + \frac{1}{2}\big(F_n(i,j-1) + F_n(i,j+1)\big)$$

Expanding and factoring of both summands yields the following inequality:

$$i\left(\frac{(n-j)\,(n-1)}{n}\right) \quad < \quad 1 + \frac{i}{2}\left(\frac{(n-j+1)\,(n-i+1) + (n-j-1)\,(n-1)}{n} - \frac{2}{j}\right).$$

So we get:

$$0 \quad < \quad 1 - \frac{i}{j} \quad .$$

By assumption is $i < j$, and so this inequality is proved.

ad (iii) : Follow the same lines as in (i) and show that:

$$F_n(i,i) - 1 - \frac{1}{2}\big(F_n(i-1,i) + F_n(i,i+1)\big) \overset{!}{=} 0.$$

After shortening the left side is reduced to:

$$-i\left(\sum_{t=1}^{n-i-1} \frac{2\,t}{t+i}\right) - 1 - \frac{1}{2}\left((-2\,n + 2\,i) - (i-1)\left(\sum_{t=1}^{n-i-1} \frac{2\,t}{t+i}\right) - i\left(\sum_{t=1}^{n-i-2} \frac{2\,t}{t+i+1}\right)\right),$$

some more equivalent transformations yield:

$$= 2\,n - 2\,i - 2 - \left(\frac{2\,i+2}{i+1} + \frac{2\,i+4}{i+2} + \frac{2\,i+6}{i+3} + \cdots + \frac{2\,i + 2\,(n-i-1)}{n-1}\right)$$

$$= 2\,n - 2\,i - 2 - \underbrace{(2 + 2 + 2 + \cdots + 2)}_{n-i-1 \text{ summands}} \quad = \quad 0.$$

By definition $F_n(0,j) = 0$. ∎

Now we want to consider the asymptotical behavior of the best strategy:

By estimating the sum of the exact formula of Theorem 3.1 we get the asymptotical one. One possible kind of estimation is an application of the formula of Euler-MacLaurin (see [Kno **64**]) which replaces sums by integrals, and this way it is responsible for the occurrence of the natural logarithm in the following formula.

We denote the asymptotical formula in this paragraph by f_n. This notation is used throughout the rest of this paper, i.e. exact formulas are denoted by capitals and asymptotic ones by small letters.

$$f_n(i,j) = i^2 - ij + 2\,ij\,\log\left(\frac{n}{j}\right) \qquad \text{for all } 1 \le i \le j \le n\,.$$

For $i = 0$ we define: $f_n(0,j) := 0$.

3.2 Running Time using one of the Worst-Case Strategies

Now, we are looking for a strategy which maximizes the expected running time in order to answer Question (1). We denote this strategy by G as introduced in section 2. We are right, if we conjecture that strategy G requires always to knock the further right of both particles.

Theorem 3.2 *In our random walk model with 2 particles it is optimal always to knock the further right of the 2 particles to obtain a maximal expected running time.*

In contrast to the best-case strategy this strategy is not unique.

The maximal expected running time is easy to verify for all $1 \le i \le j \le n$:

$$G_n(i,j) = i(n-i) + j(n-j) - \frac{(n-i)(n-j)}{n}. \tag{7}$$

For $i = 0$ it holds $G_n(0,j) = 0$ by assumption.

Proof: Applying the same formalism as in the proof of Theorem 3.1 to the function $G_n(i,j)$, we have to show:

$$\begin{array}{rll} \text{(i)} & G_n(i,j) \ge 1 + \frac{1}{2}\big(G_n(i-1,j) + G_n(i+1,j)\big) & \text{for all } 1 \le i < j \le n\,, \\ \text{(ii)} & G_n(i,j) = 1 + \frac{1}{2}\big(G_n(i,j-1) + G_n(i,j+1)\big) & \text{for all } 1 \le i < j \le n-1\,, \\ \text{(iii)} & G_n(i,i) = 1 + \frac{1}{2}\big(G_n(i-1,i) + G_n(i,i+1)\big) & \text{for all } 1 \le i \le n-1\,. \end{array}$$

ad (i) :

$i \ge 2$:

$$\begin{aligned} G_n(i,j) &= i\,(n-i) + j\,(n-j) - \frac{(n-i)\,(n-j)}{n} \\ &\overset{!}{\ge} 1 + \frac{1}{2}\Bigg((i-1)\,(n-(i-1)) + j\,(n-j) - \frac{(n-(i-1))\,(n-j)}{n} \\ &\qquad + \ (i+1)\,(n-(i+1)) + j\,(n-j) - \frac{(n-(i+1))\,(n-j)}{n}\Bigg) \\ &= i\,(n-i) + j\,(n-j) - \frac{(n-i)\,(n-j)}{n}\,. \end{aligned}$$

In each line even "=" holds.

$i = 1$:

$$G_n(1,j) \ge 1 + \tfrac{1}{2}\big(G_n(0,j) + G_n(2,j)\big) \quad \text{for all } 2 \le j \le n\,.$$

We consider the following inequality: $G_n(1,j) - 1 - \frac{1}{2}\big(G_n(2,j)\big) \ge 0\,.$

Calculation of the left side yields:

$$
\begin{aligned}
& n{-}1 + j\,(n-j) - \frac{(n-1)\,(n-j)}{n} - 1 - \frac{1}{2}\Big(2\,(n-2) + j\,(n-j) - \frac{(n-2)\,(n-j)}{n}\Big) \\
= \; & n{-}1 + j\,(n-j-1+\frac{j}{n}) - 1 - \frac{1}{2}\left(2\,(n-2) + j\,(n-j) - (n-2-j+\frac{2\,j}{n})\right) \\
= \; & j\,(n-j) + j\,(1-\frac{1}{n}) - \left(\frac{n}{2} + \frac{j}{2}\,(n-j) + \frac{j}{2}\,(1-\frac{2}{n})\right) \\
= \; & \frac{j}{2}\,(n-j) + \frac{j}{2} - \frac{n}{2} \quad = \quad j\,(n-j+1) - n \quad \geq \; 0.
\end{aligned}
$$

Only for $j = n$ equality holds.

ad (ii) : The transformations are analogous to part $\big((\text{i}), i \geq 2\big)$ of the proof, because of the symmetry in the variables.

ad (iii) : (iii) is easily verified.

We notice that in all cases, except for $\big((\text{i}), i = 1\big)$, it is possible to show identity. Thus we proved that in every situation of the worst-case, except for $\big((\text{i}), i = 1,\, 2 \leq j \leq n-1\big)$ it does not matter which of the 2 particles should be knocked. ■

We consider the asymptotical behavior once more:

Again we use small letters to distinguish between exact and asymptotical formulas. As easily checked, the following formula holds for g_n :

$$g_n(i,j) = i\,(n-i) + j\,(n-j) \qquad \text{for all } \; 1 \leq i \leq j \leq n\,.$$

We obtain this formula by neglecting the last summand of $G_n(i,j)$, because its contribution is decreasing in relation to the total amount by increasing n. This summand has a negative sign, hence the values of $g_n(i,j)$ are strictly greater than $G_n(i,j)$ for all $1 \leq i \leq j \leq n{-}1$.

In the following we will denote this model as the *[1:2]–model*. More generally, we will talk about a *[r:(s–r+1)]–model*, when we ask, whether r of s particles will be absorbed at 0, or $s-r+1$ particles will be absorbed at n.

4 Random Walks on a Line with 3 Particles

Now we consider the [2:2]–model according to the agreement above together with Question (2). The following strategy agrees with our intuition in the [2:2]–model:

The best strategy is the one which requires always to knock the central particle, because Question (2) has a symmetric character. To construct a formula for $f_n(i,j,k)$ we consider situations of type (i,i,k) and (i,k,k), which have the same expected value besides,

because of this symmetry. Type (i,i,k) means that the left 2 particles are standing on the same position and that the third particle is standing right of them, analogously for (i,k,k).
To determine the formula of type (i,i,k)-constellations, where $0 \le i \le k \le n$ we consider the following differences:

$$F_n(i,i,n) - F_n(i,i,k) \quad \text{for} \quad i \le k \le n-1,$$

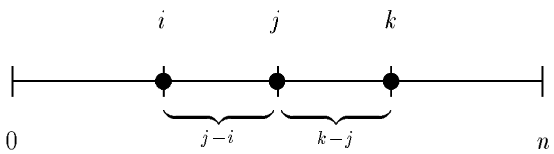
where $F_n(i,i,n) = F_n(n-i, n-i)$, because one particle is already absorbed at the right barrier. This agrees with the [2:1]–model for which we can derive a formula from Subsection 3.1 to determine the expected values of knocks.
A formal equation is for all $0 \le i \le j \le n$:

$$F_n(i,i,k) =$$
$$F_n(n-i,n-i) + (n-k)\left(\frac{(2\ i)(n-1)}{n} - \sum_{r=1}^{i-1} \frac{2\ (i-r)^2}{(n-r)^2}\right) - \sum_{s=1}^{n-k-1} \frac{2\ s\ i\ (2\ s + 2\ k - i)}{(k+s)^2}.$$

Hence, we get the following approach for the general cases, where $0 \le i \le j \le k \le n$:

$$F_n(i,j,k) \;\; = \;\; (k-j)\ (j-i) + \frac{k-j}{k-i}\ F_n(i,i,k) + \frac{j-i}{k-i}\ F_n(i,k,k),$$

where $(k-j)\ (j-i)$ denotes the expected running time of the central particle till it reaches i or k. The probability for reaching i is $\frac{k-j}{k-i}$ and the probability for reaching k is $\frac{j-i}{k-i}$.



Altogether:

Theorem 4.1 *In the [2:2]–model with 3 particles there is an unique optimal strategy for the best-case; it states that we always have to knock that particle standing in the middle.*

The minimal expected running time is calculated as follows for all $0 \le i \le j \le k \le n$*:*

$$F_n(i,j,k) \;\; = \;\; (k-j)\ (j-i) +$$
$$\frac{k-j}{k-i}\left[F_n(n-i,n-i) + (n-k)\left(\frac{(2\ i)(n-1)}{n} - \sum_{r=1}^{i-1}\frac{2\ (i-r)^2}{(n-r)^2}\right) - \sum_{s=1}^{n-k-1}\frac{2\ s\ i\ (2\ s+2\ k-i)}{(k+s)^2}\right] +$$
$$\frac{j-i}{k-i}\left[F_n(k,k) + i\left(\frac{(2\ (n-k))(n-1)}{n} - \sum_{r=1}^{n-k-1}\frac{2\ (n-k-r)^2}{(n-r)^2}\right) - \sum_{s=1}^{i-1}\frac{2\ s\ (n-k)\ (2\ s+n+k-2\ i)}{(n-i+s)^2}\right].$$

We do not prove this theorem, but one possible technique to prove it, is to check the (in-)-equalities which follow from System (5). Analogously we have done this in Subsection 3.1 in the proof of the [1:2]–model and its (in-)equalities coming from System (4).

Of course, the asymptotical behavior for $n \to \infty$ is interesting in this section:

To estimate the exact equalities we use again the formula of Euler-MacLaurin. We only want to show one estimation, because the cases of type (i,i,k) and (i,k,k) follow the same arguments. So we estimate just in that case, for which all 3 particles are standing on the same position. For the three existing summands in the exact formulas we estimate:

$$\begin{array}{lrcl} \text{(i)} & f_n(n-i,n-i) & \approx & 2(n-i)^2\log\left(\frac{n}{n-i}\right)\,, \\ \text{(ii)} & (n-i)\left(\frac{(2i)(n-1)}{n} - \sum_{r=1}^{i-1}\frac{2(i-r)^2}{(n-r)^2}\right) & \approx & 4i^2-2in+4(n-i)^2\log\left(\frac{n}{n-i}\right)\,, \\ \text{(iii)} & -\sum_{s=1}^{n-i-1}\frac{2si(2s+i)}{(i+s)^2} & \approx & 2i^2-4in+6i^2\,\log\left(\frac{n}{i}\right)\,, \end{array}$$

where estimation (i) is known from Subsection 3.2. Summation of the three right hand sides yields for all $1 \le i \le n-1$:

$$f_n(i,i,i) \;=\; 3\,i^2\log\left(\left(\frac{n}{i}\right)^2\right) + 3\,(n-i)^2\,\log\left(\left(\frac{n}{n-i}\right)^2\right) - 6\,i\,(n-i)\,.$$

5 Random Walks on a Line with s Particles

In this section we consider the $[1{:}s]$–model for which we study Question (3).

We are going to investigate the asymptotical behavior of the best-case strategy:

The reader will not be astonished if we remark that the best-case strategy is unique and it is optimal always to knock that particle standing to the left of all other particles. We only consider constellations for which all s particles are standing on the same position. This in a way is the most interesting constellation, since if we always knock the very left particle it either will be absorbed at 0 or it meets the second particle from the left and so on. Hence we only present formulas for $f_n\underbrace{(i,i,\ldots,i)}_{s\text{ times}} := f_n(i^s)$.

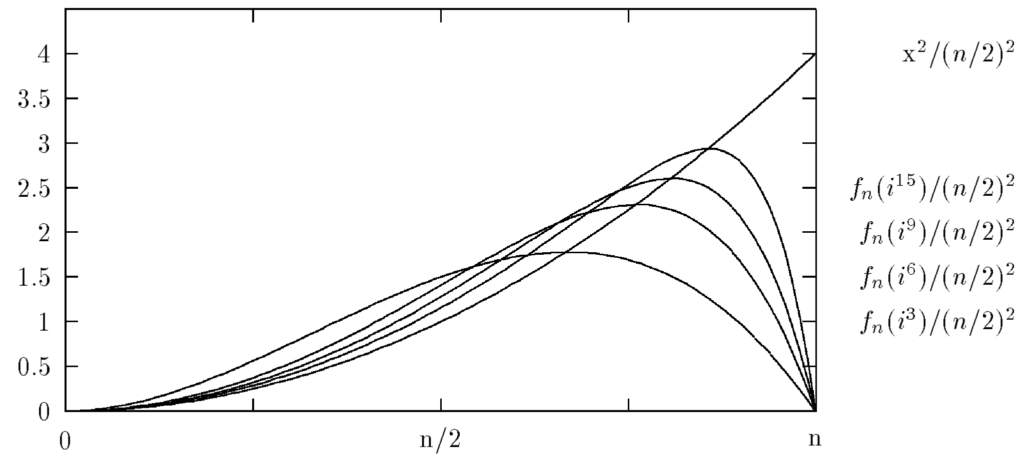
Observation 5.1:

On the basis of computer calculations for $3 \le s \le 25$ it is plausible to conjecture:

$$f_n(i^s) \;=\; \left(\frac{s}{s-2}\right)\, i^2\, \left(1-\left(\frac{i}{n}\right)^{s-2}\right) \qquad \text{for} \quad 0 \le i \le n.$$

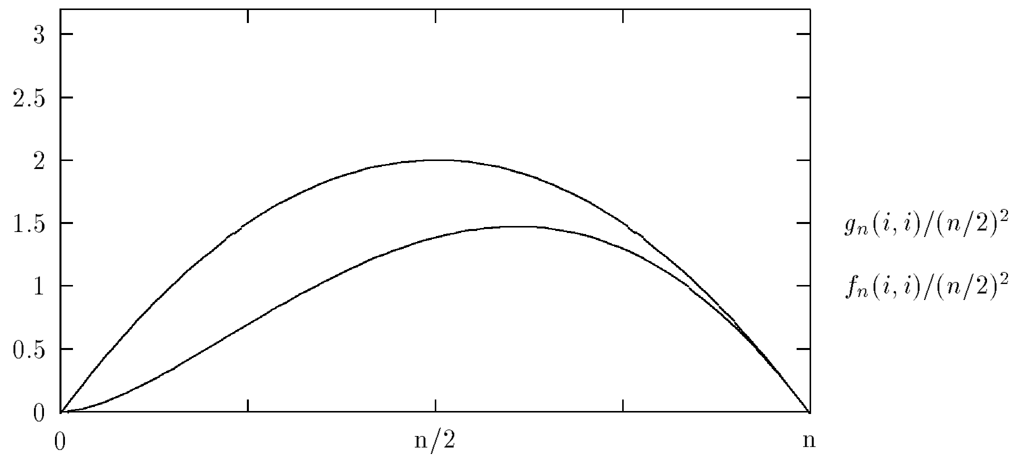
Note, that other than in the case of 2 particles the log-term vanishes for $s \ge 3$.

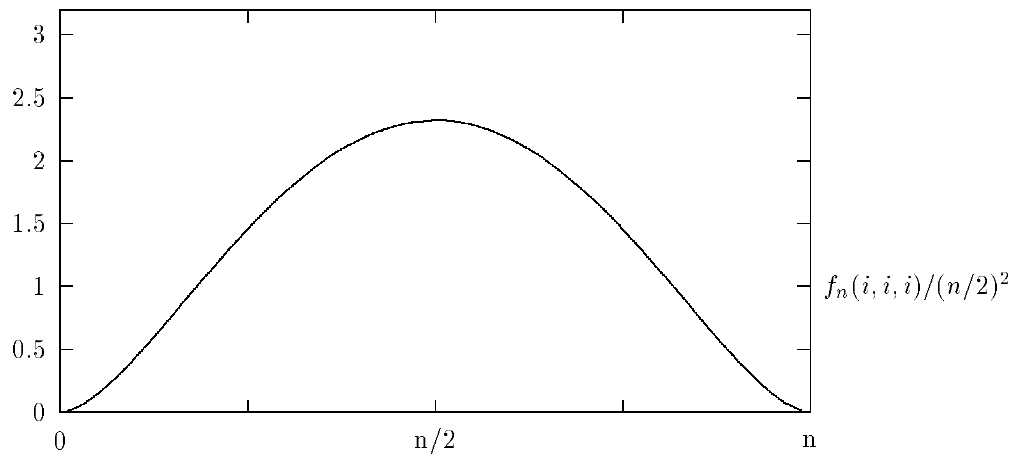
For this model we only present the plots for the best-cases with $s = 3, 6, 9, 15$ and for the function f(x)=x^2, because the initial parts of the asymptotical formulas of $f_n(i^s)$ tend to the right branch of the parable f(x)=x^2 for increasing s.
As mentioned in the introduction, the expected running time for one particle with initial position i equals $i\ (n-i)$. This product is maximal for $i = n/2$ and its maximal value is $(n/2)^2$ which justifies the choice of the normalizing constant above. Independent of the chosen strategy, the order of the running time is n^2.
On the abscissa i runs from 0 to n, and on the ordinate all values are normalized by $(n/2)^2$.



6 Comparison of different strategies

In this section we want to compare directly the presented best-case and worst-case strategies. Normalizing the expected number of knocks by $(n/2)^2$ enables us to compare different stationary strategies concerning their efficiencies. Letting n tend to ∞ and fixing $\frac{i}{n}$ and $\frac{j}{n}$ (and $\frac{k}{n}$ respectively), we see that for any stationary strategy S the sequences of the fractions $\left(\frac{S_n(i,j)}{\left(\frac{n}{2}\right)^2}\right)_{n\to\infty}$ and $\left(\frac{S_n(i,j,k)}{\left(\frac{n}{2}\right)^2}\right)_{n\to\infty}$ converge, where $i = i(n), j = j(n)$ and $k = k(n)$ are chosen such that $\frac{i}{n}, \frac{j}{n}$ and $\frac{k}{n}$ are arbitrary constants.
We present the plots of the best-case and worst-case formulas for the [1:2]–model and the plot of the best-case formula for the [2:2]–model in the following illustrations, i.e. the values for $f_n(i,i)$ and $g_n(i,i)$ and for $f_n(i,i,i)$ are shown there.





In the following subsections we only consider that case for which all these constants we write about above are $\frac{1}{2}$. This means that the initial positions of the random walks are $i = j\,(= k) = n/2$.

6.1 Comparison in the [1:2]–Model

We find an exact upper bound for this model directly from the following naive strategy:

- The worst strategy G_n from Subsection 3.2 needs $2\ (n/2)^2$ knocks by permanently knocking the right particle, as can be easily verified.

To improve this strategy we give an example of a better strategy, but it is still not optimal.

- First we knock any of the 3 particles till it is absorbed. If the absorption takes place at 0, it is possible to answer Question (2). But if the absorption takes place at the right barrier, the second particle has to be knocked till absorption, too. Both cases are of same probability, in the average the running time is $1.5\ (n/2)^2$.

- From the formula of Theorem 4.1 we can calculate the exact lower bound for the expected running time for $i = n/2$ using the best strategy:
$2\ \log(2)\ (n/2)^2\ \approx\ 1.3863\ (n/2)^2$. This value is optimal.

A summary of the results from the last two subsections is Theorem 6.1. **S** denotes the set of all stationary strategies as introduced in Section 1:

Theorem 6.1 *Let n be fixed. Let $S_n \in \mathbf{S}$ be chosen arbitrary. Then we have for all $S_n(i,j)$ with $1 \le i \le j \le n-1$:*

$$f_n(i,j)\ <\ F_n(i,j)\ \le\ S_n(i,j)\ \le\ G_n(i,j)\ <\ g_n(i,j)\,.$$

By definition we have

$$\begin{aligned} f_n(0,j)\ &=\ F_n(0,j)\ =\ S_n(0,j)\ =\ G_n(0,j)\ =\ g_n(0,j)\ =\ 0, \\ f_n(i,n)\ &=\ F_n(i,n)\ =\ S_n(i,n)\ =\ G_n(i,n)\ =\ g_n(i,n)\ =\ i\,(n-i). \end{aligned}$$

Proof: In Subsections (3.1) and (3.2) we have proved the weak inequalities $F_n(i,j) \le S_n(i,j)$ and $S_n(i,j) \le G_n(i,j)$ of the above chain of inequalities, because F and G are two contradictional optimal strategies.

We prove the strong inequality $f_n(i,j) < F_n(i,j)$ in two steps:

To show that f_n fulfills (4), we first verify three relations. Then we prove for all $1 \le i \le j \le n-1$ that $f_n < F_n$.

For $f_n(i,j)$ these three relations are necessary to show:

$$\begin{array}{lll} \text{(i)} & f_n(i,j) = 1 + \frac{1}{2}\big(f_n(i-1,j) + f_n(i+1,j)\big) & \text{for all } 1 \le i < j \le n\,, \\ \text{(ii)} & f_n(i,j) < 1 + \frac{1}{2}\big(f_n(i,j-1) + f_n(i,j+1)\big) & \text{for all } 1 \le i < j \le n-1\,, \\ \text{(iii)} & f_n(i,i) < 1 + \frac{1}{2}\big(f_n(i-1,i) + f_n(i,i+1)\big) & \text{for all } 1 \le i \le n-1\,. \end{array}$$

Note, that the operators "=" and "<" in (i) and (ii) are the same as in the proof of Theorem 3.1, but in (iii) we show "<" and not "=".

ad (i) : This is easy to verify by expanding.

ad (ii) :

$$1 + \frac{1}{2}\big(f_n(i, j-1) + f_n(i, j+1)\big)$$

$$= 1 + \frac{1}{2}\left(i\,(j-1) - i^2 - 2\,i\,(j-1)\,\log\left(\frac{j-1}{n}\right) + i\,(j+1) - i^2 - 2\,i\,(j+1)\,\log\left(\frac{j+1}{n}\right)\right),$$

because of the concavity of the logarithm we continue:

$$\geq 1 + \frac{1}{2}\left(2\,ij - 2\,i^2 - 4\,ij\,\log\left(\frac{j}{n}\right)\right)$$

$$= ij - i^2 - 2\,ij\,\log\left(\frac{j}{n}\right) + 1 \qquad > f_n(i, j).$$

ad (iii) : (iii) is similar to (ii).

At last we prove:
$f_n(i, j) < F_n(i, j)$ for all $1 \leq i \leq j \leq n-1$.

The assumption that $f_n(i, j) > F_n(i, j)$ holds for at least one pair (i, j) with $1 \leq i \leq j \leq n-1$ leads to a contradiction. The idea of the proof is the same as in the proof of Lemma 2.1 in Section 2.

By looking at the formulas $G_n(i, j)$, (7) and $g_n(i, j)$, (8) we easily check that $G_n(i, j) < g_n(i, j)$ for all $1 \leq i \leq j \leq n-1$. ■

6.2 Comparison in the [2:2]–Model

We want to present the exact lower bound for the expected running time for $n \to \infty$ in that special case for which all 3 particles are standing at $n/2$ at the beginning of the random walk. We have done the same for the [1:2]–model in the last subsection. Again, the given bound will be improved step by step.

- This upper bound is easy to find by activating anyone of the 3 particles till absorption and then using the worst-case strategy of the [1:2]– and [2:1]–model, respectively, depending on the location of absorption of the chosen particle at first. Hence, in the worst-case we need $(1+2)\,(n/2)^2 = 3\,(n/2)^2$ knocks for answering the Question (3).

- An obvious upper bound is $2.5\,(n/2)^2$, because if at first 2 particles will be absorbed at the same barrier we are done, otherwise (with probability $1/2$) we have to activate the last particle till absorption, too.

- A further strategy is to knock anyone of the 3 particles till absorption and then to use the best-case strategy of the [1:2]– and [2:1]–model, respectively, depending on the location of absorption of the chosen particle at first. Using this strategy the expected running time is $(1 + 2\ \log(2))\ (n/2)^2 \approx 2.3863\ (n/2)^2$.

- Knocking the central particle permanently yields the exact lower bound for the expected running time of each stationary strategy. We get it by inserting $i = j = k = n/2$ in Theorem (4.1): $6\ (2\ \log(2) - 1)\ (n/2)^2 \approx 2.317767\ (n/2)^2$.

- As a consequence of Lemma 8.2.3 in [Alt **91**] we obtain a lower bound of the expected running times of $2.25\ (n/2)^2$.

7 Intersecting Random Walks on a Circle

In this section we consider a second decision model with random walks. For this purpose consider the n-circle (with $n \in \mathbb{N}$) C_n and label its vertices with 0 to $n-1$ in the natural way.
Suppose we are given $s < n$ particles on C_n and each unit of time we are allowed to activate exactly one particle, which describes one step of a nearest neighbor random walk on C_n with transition probability 1/2. This process is stopped, if the positions of 2 particles coincide.

We again ask the following questions:

Which is the best strategy to maximize the expected running time of the process ? (8)

Which is the best strategy to minimize the expected running time of the process ? (9)

Due to the symmetry of the random walks and the underlying graph Questions (8) and (9) are easily answered for the case of $s = 2$ particles. Obviously, every possible strategy yields the same expected meeting time, especially best-case and worst-case meeting times coincide and equal $(j - i)(n - j + i)$, if we start in the initial positions $0 \leq i \leq j \leq n$.

In this section we are going to answer Question (8) for the case of $s = 3$ and $s = 4$ particles, respectively. Question (9) will be answered for the case of $s = 3$ particles.
For the case of $s = 3$ and $s = 4$ particles, respectively, the worst-case meeting time G_n of an initial position fulfills

$$\begin{aligned} G_n(i,j,k) = 1 + \frac{1}{2}\max[G_n(i-1,j,k) &+ G_n(i+1,j,k), \\ G_n(i,j-1,k) &+ G_n(i,j+1,k), \\ G_n(i,j,k-1) &+ G_n(i,j,k+1)], \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
G_n(i,j,k,l) = 1 + \frac{1}{2}\max[G_n(i-1,j,k,l) &+ G_n(i+1,j,k,l),\\
G_n(i,j-1,k,l) &+ G_n(i,j+1,k,l),\\
G_n(i,j,k-1,l) &+ G_n(i,j,k+1,l)\\
G_n(i,j,k,l-1) &+ G_n(i,j,k,l+1)],
\end{aligned}
\tag{11}
$$

respectively (here and in the rest of the chapter sums and differences are computed "modulo n"). Similarly for 3 particles the best-case meeting time F_n satisfies

$$
\begin{aligned}
F_n(i,j,k) = 1 + \frac{1}{2}\min[F_n(i-1,j,k) &+ F_n(i+1,j,k),\\
F_n(i,j-1,k) &+ F_n(i,j+1,k),\\
F_n(i,j,k-1) &+ F_n(i,j,k+1)].
\end{aligned}
\tag{12}
$$

For the sake of simplicity we may (without loss of generality) restrict our attention the following models:
For analyzing Question (8) in the case of $s = 3$ and $s = 4$, respectively, we assume that $0 = i \le j \le k \le n-1$ and $0 = i \le j \le k \le l \le n-1$, respectively.

In order to solve Question (9) in the presence of 3 particles we investigate the situations $0 = i \le j \le k \le \left[\frac{n}{2}\right]$ (where $\left[\frac{n}{2}\right]$ denotes the smallest integer greater than or equal to $\frac{n}{2}$) and $0 = i < k - \left[\frac{n}{2}\right] < j < \left[\frac{n}{2}\right] < k$, i.e. we distinguish between the cases that all particles are "on the same side" of the circle and the case that they are not. Theorem 7.4 will show why these cases have to be treated separately.

While using these models, equations (10)–(12) have to be understood "modulo these models", that means we have to take into account that every new situation has to be described in the appropriate model.

First of all we state that under appropriate conditions the solutions of equations (10)–(12) are uniquely determined.

Lemma 7.1 *Under the initial conditions*

$$
G_n(0,0,k) = G_n(0,k,k) = F_n(0,0,k) = F_n(0,k,k) = 0 \tag{13}
$$

for $s = 3$ and

$$
G_n(0,0,k,l) = G_n(0,j,j,l) = G_n(0,j,l,l) = 0
$$

for $s = 4$, respectively, the solutions of the equations (10)–(12), if existent are unique.

The proof follows the same argumentation as the proof of Lemma 2.1.

Next we are going to answer Question (8) for the case of 3 and 4 particles and Question (9) for the case of 3 particles:

Theorem 7.2 *a) In the case of $s = 3$ particles the maximal expected running time is given by*

$$G_n(0,j,k) = [(j-1)(k-j-1) + (k-1)(n-k)]\delta_{\{0 \neq j \neq k\}}$$

(where δ denotes the indicator function). It is achieved by all strategies moving any particle not belonging to a cluster (i.e. any particle which does not have a neighboring particle), if possible, and not moving the central particle in a cluster of 3 particles.

b) In the case of $s = 4$ particles the maximal expected running time equals

$$G_n(0,j,k,l) = (l-k+j-1)(n-l+k-j-1)\delta_{\{0 \neq j \neq k \neq l\}}.$$

It is achieved by all strategies activating any particle not belonging to a cluster, if possible, activating any particle in the presence of two separated clusters of two particles and activating one of the outer particles in a cluster of 4 particles.

Proof: a) First of all observe that the initial conditions (13) are automatically satisfied because of the indicator function. For $0 = i < j-1 < j < k-1 < k < n-1$ it is easily checked that

$$\begin{aligned} G_n(0,j,k) &= 1 + \frac{1}{2}[G_n(0,j-1,k-1) + G_n(0,j+1,k+1)] \\ &= 1 + \frac{1}{2}[G_n(0,j-1,k) + G_n(0,j+1,k)] \\ &= 1 + \frac{1}{2}[G_n(0,j,k-1) + G_n(0,j,k+1)]. \end{aligned}$$

For $k = n-1$ we have for all $j \neq k, 0$

$$\begin{aligned} G_n(0,j,n-1) &= [(j-1)(n-j-2) + (n-2)]\delta_{\{j \neq 0, n-1\}} \\ &= j(n-j-1) \\ &= 1 + \frac{1}{2}[G_n(0,j+1,n-1) + G_n(0,j-1,n-1)]. \end{aligned}$$

and

$$\begin{aligned} j(n-j-1) &\geq 1 + (n-3) + (j-1)\frac{n-j-3}{2} \\ &= 1 + \frac{G_n(0,0,j) + G_n(0,j,n-2)}{2} \end{aligned}$$

since $j \geq 1$ and

$$\begin{aligned} j(n-j-1) &\geq 1 + (n-3) + (j-2)\frac{n-j-2}{2} \\ &= 1 + \frac{G_n(0,0,j+1) + G_n(0,j-1,n-2)}{2} \end{aligned}$$

with equality if and only if $j = 1$ or $j = n-2$.
The cases $j = 1$ and $j = k-1$ are treated similarly.

b) The proof for $s = 4$ follows the same lines. ■

Remark 7.3 *Notice that in both cases there is no unique strategy to obtain a maximal expected running time. However, the "natural" strategy, i.e. the strategy which at any time knocks that particle which has the longest distance to its nearest neighbor, is in the set of strategies leading to a maximal expected meeting time.*

Now we treat the question of how to obtain fastest contact.

Theorem 7.4 *For $s = 3$ particles and the situations described above the shortest expected running time is*

$$F_n(0, j, k) = j(k-j) + (k(n-k) - \left[\frac{n^2-1}{4}\right])\delta_{\{k>[\frac{n}{2}]\}}.$$

(Again brackets of a number denote the smallest integer greater than or equal to this number.) It is obtained by all strategies moving j, in case that $0 < j < k \leq [n/2]$ and moving any particle, if $0 = i < k - \left[\frac{n}{2}\right] < j < \left[\frac{n}{2}\right] < k$.

Proof: We only consider the case of even n. The other case is treated identically. First assume that $0 < j < k < n/2$. Then clearly

$$\begin{aligned}
F_n(0, j, k) &= j(k-j) \\
&= 1 + \frac{1}{2}((j-1)(k-j+1) + (j+1)(k-j-1)) \\
&= 1 + \frac{1}{2}(F_n(0, j-1, k) + F_n(0, j+1, k)) \\
&< 1 + \frac{1}{2}(j(k-1-j) + j(k+1-j)) \\
&= 1 + \frac{1}{2}(F_n(0, j, k-1) + F_n(0, j, k+1)) \\
&= 1 + \frac{1}{2}((j+1)(k-j) + (j-1)(k-j) \\
&= 1 + \frac{1}{2}(F_n(0, j+1, k+1) + F_n(0, j-1, k-1)).
\end{aligned}$$

For $k = n/2$ we have to distinguish between $j > 1$ and $j = 1$. If $j > 1$ we again check that

$$F_n(0, j, n/2) = j(n/2 - j) = 1 + \frac{1}{2}(F_n(0, j-1, n/2) + F_n(0, j+1, n/2))$$

and

$$\begin{aligned}
j(n/2 - j) &< 1 + \frac{j(n/2 - 1 - j) + j(n/2 + 1 - j) - 1}{2} \\
&= 1 + \frac{1}{2}(F_n(0, j, n/2 - 1) + F_n(0, j, n/2 + 1))
\end{aligned}$$

as well as

$$j(n/2-j) < 1 + \frac{1}{2}(F_n(0, j+1, n/2+1) + F_n(0, j-1, n/2-1)).$$

If $j = 1$ again

$$F_n(0, 1, n/2) = n/2 - 1 = 1 + \frac{1}{2}(F_n(0, 0, n/2) + F_n(0, 2, n/2))$$

and

$$n/2 - 1 < n/2 - 1/2 = 1 + \frac{1}{2}(F_n(0, 1, n/2-1) + F_n(0, 1, n/2+1))$$

as well as

$$n/2 - 1 < n/2 - 1/2 = 1 + \frac{1}{2}(F_n(0, 0, n/2-1) + F_n(0, 2, n/2+1)).$$

The assertion for the second model, where $0 = i < k - \left[\frac{n}{2}\right] < j < \left[\frac{n}{2}\right] < k$ is proved analogously. ∎

Remark 7.5 *Observe that the best strategy to obtain the shortest expected running time is unique only in the case where all particles are "on the same side" of the circle. The natural strategy always to move that particle which has shortest distance to its two neighboring particles is in the set of the best strategies.*
Also observe that in the case $0 = i < k - \left[\frac{n}{2}\right] < j < \left[\frac{n}{2}\right] < k$ the strategies to obtain best-case and worst-case meeting times coincide.

8 Open Problems

In the present paper we only could solve some of the possible questions in the context of decision models with random walks. There is a variety of open problems left, e.g.:

- *"Is it possible to present an optimal running time for the [r:(s–r+1)]–model, i.e. we ask, whether r of s particles reach the left barrier or $s-r+1$ reach the right one ?"*
- *"How do the running times and optimal strategies change, if we have a drift, i.e. the transition probabilities for the random walk are not equal to $\frac{1}{2}$?"*
- *"What are the optimal strategies, if we are allowed to knock 2 particles of 3 or more particles in each unit of time ?"*
- *"What happens, if we substitute the circle C_n in Section 7 by a torus or even more general by a k-regular graph ?"*

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9 References

[Ald **91**] Aldous, D.J.: "Meeting times for independent Markov chains". Stoch. Proc. Appl. 38, pp. 185-193, 1991

[Ald **89**] Aldous, D.J.: "Random walks on graphs, Bibliography". Preprint, 1989

[Alt **91**] Althöfer, I.: "On pathology in game tree and other recursion tree models". Habilitationsschrift, Universität Bielefeld, 1991

[Chu **78**] Chung, K.L.: "Elementare Wahrscheinlichkeitstheorie und stochastische Prozesse". Übers. aus d. Engl. von H. Vogt, Berlin, Springer, 1978

[CTW **92**] Coppersmith, D., Tetali, P., Winkler, P.: "Collisions among random walks on a graph", SIAM J. on Discrete Math. 6/No.3, pp. 363-375, 1993

[Gro **92**] Grotendiek, T.: "Random Walk–Entscheidungsmodelle". Diplomarbeit, Universität Bielefeld, 1992

[Kno **64**] Knopp, K.: "Theorie und Anwendung der unendlichen Reihen". Die Grundlehren der Mathematischen Wissenschaften, Band 2, 5. bericht. Aufl., Springer, 1964

[TW **93**] Tetali, P., Winkler, P.: "Simultaneous reversible Markov chains". Bolyai Soc. Math. Stud., Combinatorics, Paul Erdös is Eighty (Vol. 1), pp. 433-451, 1993