INTERMEDIATE SUBGROUPS IN CHEVALLEY GROUPS

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Let G be a group and D be a subgroup of G. In this paper we are interested in the description of the lattice

$$\mathcal{L}(D,G) = \{H \mid D \leq H \leq G\}$$

of subgroups of G which contain D. We call subgroups $H \in \mathcal{L}(D,G)$ subgroups **intermediate** between D and G. We are primarily interested in the parametrization of intermediate subgroups and their behaviour with respect to set-theoretic and group-theoretic operations like intersection, conjugation, passage to normalizer of H or to normal closure of D in H, etc.

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We will speak about the structure of $\mathcal{L}(D,G)$ as opposed to the study of other classes of subgroups, like, for example, subgroups, generated by a given type of elements/subgroups (say, transvections, reflections, root elements, ...), defined by group theoretic properties (say, abelian, nilpotent, soluble, ...), ...

Little can be said about the structure of the lattice $\mathcal{L}(D,G)$ in general. In fact it is proven in [Tu] that *any* algebraic lattice is isomorphic to an interval in the subgroup lattice of an *infinite* group. This is why, to get sensible results, one has to specialize the class of groups considered. It turns out that in amazingly many examples arising in real life the lattice $\mathcal{L}(D,G)$ has a very transparent structure. There are some abstract conditions on a subgroup D, which are responsible for that (see § 10 and 14).

In our talk the group G will be a Chevalley group or its extension by diagonal automorphisms. Many results are only known for the classical groups, and some will be stated only for the general linear group to save space. We concentrate on results which are valid for *arbitrary* fields, not just finite or algebraically closed. This is why we speak only of *normal* types. While for a finite group theorist there is a good reason to consider also the twisted groups, for us they are no better than all other forms of semisimple algebraic groups.

Although we say something about generalizations to *commutative* rings, we do not discuss here results which *essentially* depend on the nature of the ground ring, or use some deeper ring theory. For the linear and unitary groups there are non-commutative generalizations of some of the results in the present paper, which depend on stability conditions, finiteness conditions, or non-commutative localizations. On the other side, for arithmetical rings there is a theory with many results which have no immediate counterparts in the theory we discuss. We leave all these things beyond the horizon. Some indications may be found in [HO], [PR], [Va16], [Va19], [Z2].

Several general remarks are in order here. First, one should note that many of the results we state below are easy – sometimes trivial – for finite fields. However for infinite fields they may be extremely difficult. People working in finite groups do not usually realize how much the finiteness condition – not even the Classification, but such trivial things as counting arguments or Sylow's theorem – make life easier. The most striking example of this is perhaps the theorem of O.King [Ki2] on the overgroups of diagonal subgroup in SL_2.

In some other cases the situation is strictly opposite: the results are easy to prove for infinite fields using Zariski topology and 'general position' arguments – which deliver no information whatsoever for the finite case. Sometimes it is easy to account for both the finite and the infinite case using *different* proofs for finite and infinite cases. However a 'natural' proof should cover *all* cases when a result holds by the same method. In many situations such natural proofs are still lacking.

Another point is that we describe *all* overgroups of a given group and not just the irreducible ones. Moreover we describe them up to inclusion – not just up to conjugacy. In many situations applying the classification of irreducible subgroups containing a long root unipotent element, a quadratic element or the like, from the very start we know the *irreducible* subgroups from $\mathcal{L}(D,G)$ *up to conjugacy*. However our work only starts rather than finishes at this point. One conjugacy class of such overgroups may lead to parametric families. Even the reduction of the general case to the irreducible one is not always trivial.

This survey complements [Kn], [V5], [V16], [V17], [V19], [Z1] – [Z3] rather than updates them. Many more details about the finite groups of Lie type as well as the corresponding algebraic groups, and especially on their maximal subgroups, may be found in [A1] – [A4], [KL1], [KL2], [Kn], [LSa], [LSe], [S6], [S10] – [S15], [T]. A wealth of information concerning linear case may be found in [Z1] – [Z3]. Parabolic subgroups are discussed in [V5]; overgroups of tori in [V16], [V17]; overgroups of subsystem subgroups in [V16]. Consult [K3], [L2], [L6] for different views of the subject. In [V19] one can find general background on Chevalley groups over rings and many additional references. The works [BKH], [BV3], [HV], [Sv], [V14], [V15], [V18] also contain extensive lists of related references, not fully reproduced here.

§ 1. Notation

In this section we introduce the notation used in the rest of this paper. See [Bo], [C1], [H2], [St], [V19] for more details and further references.

Let Φ be a reduced root system of rank l, P be a lattice lying between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. From this data one can construct an affine group scheme $G_P(\Phi,\ )$ over $\mathbb{Z}$, such that for any algebraically closed field K the value $G_P(\Phi,K)$ of this functor on K is the semisimple algebraic group over K corresponding to the pair Φ, P, which is called the *Chevalley-Demazure group scheme of type* (Φ,P). Let $T=T_P(\Phi,\ )$ be a split maximal torus of the Chevalley-Demazure group scheme $G_P(\Phi,\ )$. The values $G_P(\Phi,R)$ and $T_P(\Phi,R)$ of these functors on a commutative ring R with 1 ("the groups of rational points $G_P(\Phi,\ )$ and $T_P(\Phi,\ )$ with the coefficients in R") are called the *Chevalley group of type* (Φ,P) over R and its *split maximal torus* respectively. Usually we omit P in the notation and speak about a "*Chevalley group* $G=G(\Phi,R)$ *of type* Φ *over* R".

Let $L=L_{\mathbb{C}}$ be a complex semisimple Lie algebra of type Φ. Fix an order on Φ with Φ^+, Φ^- and $\Pi=\{\alpha_1,\dots,\alpha_l\}$ being the sets of *positive, negative* and *fundamental roots* respectively. Let $\{e_\alpha,\ \alpha\in\Phi;\ h_\alpha,\ \alpha\in\Pi\}$ be a

Chevalley base of L and $L_{\mathbb{Z}}$ be its the integral span. Now for an arbitrary commutative ring R we set $L_R = L_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$.

Let now $\pi\colon L \to \mathrm{gl}(V)$ be a representation of the complex semisimple Lie algebra $L = L_{\mathbb{C}}$ in a finite dimensional vector space V over $\mathbb{C}$. We often omit the symbol π in the action of G on V and for $x \in L$ and $u \in V$ write xu instead of $\pi(x)u$. Let $V_{\mathbb{Z}}$ be an *admissible* lattice in V (i.e. $V_{\mathbb{Z}}$ is invariant under the action of divided powers $e_\alpha^{(m)} = e_\alpha^m/m!$). Set $x_\alpha(\xi) = x_\alpha^\pi(\xi) = \exp(\xi\pi(e_\alpha))$, where $\alpha \in \Phi$, $\xi \in R$, and 'exp' is defined by the usual formula. For arbitrary commutative ring R we set $V_R = V_{\mathbb{Z}} \otimes R$. The group

$$\mathrm{E}_\pi(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\rangle \le \mathrm{GL}(V_R).$$

is the *elementary Chevalley group of type* Φ *over* R in the representation π. Thus if the lattice of weights $P(\pi)$ of the representation π equals P, then the groups $E_P(\Phi, R)$ and $G_P(\Phi, R)$ from the very start appear together with their representation in the free R-module $V_R = V_{\mathbb{Z}} \otimes R$, which we denote by the same letter π.

The group G_π is called *simply connected* if $P(\pi) = P(\Phi)$ and *adjoint* if $P(\pi) = Q(\Phi)$. We write G_{sc} and G_{ad} for simply connected and adjoint groups respectively. For the cases $\Phi = \mathrm{E}_8$, F_4 and G_2 one has $P(\Phi) = Q(\Phi)$ and thus $G_{\mathrm{sc}} = G_{\mathrm{ad}}$. However from our point of view these groups behave like simply connected groups of other types, rather than like the adjoint ones. It will be technically convenient to say that *there are no adjoint groups of types* $\Phi = \mathrm{E}_8$, F_4 *and* G_2.

When $R = K$ is a field the Chevalley group $G_\pi(\Phi, K)$ is generated by $E_\pi(\Phi, K)$ and certain semisimple elements. In many important cases, notably, when K is an algebraically closed field, or when $G_\pi = G_{\mathrm{sc}}$ is simply connected, one has $G_\pi(\Phi, K) = E_\pi(\Phi, K)$. However in general it is not true: adjoint groups are strictly larger than their elementary subgroups. For example, the adjoint group of type A_l coincides with $\mathrm{PGL}(l+1, K)$, while its elementary subgroup coincides with $\mathrm{PSL}(l+1, K)$. One can lift this distinction to the level of simply connected groups by considering 'diagonal extensions' of G_{sc}, the so called *extended Chevalley groups*, see [BMo], [V17]. Thus, for example, the extended (simply connected) group of type A_l coincides with $\mathrm{GL}(l+1, K)$.

To any root $\alpha \in \Phi$ there correspond *unipotent root elements* $x_\alpha(\xi)$, $\xi \in K$. If we want to stress that these root unipotents correspond to a given choice of a split maximal torus we call them *elementary root unipotents*. In general any conjugate of an elementary root unipotent is called a *root unipotent*. It is *long* or *short* depending on whether the root α is long or short.

Let now $\alpha \in \Phi$ and $\epsilon \in K^*$. As usual we set $h_\alpha(\epsilon) = w_\alpha(\epsilon)w_\alpha(1)^{-1}$, where $w_\alpha(\epsilon) = x_\alpha(\epsilon)x_{-\alpha}(-\epsilon^{-1})x_\alpha(\epsilon)$. The elements $h_\alpha(\epsilon)$ – and their conjugates – are called *semisimple root elements*. They are called *long* or *short* depending on whether the root α is long or short.

For a fixed $\alpha \in \Phi$ the map $x_\alpha : \ \xi \mapsto x_\alpha(\xi)$ is a homomorphism of the additive group R^+ of R to the one-parameter subgroup $X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\}$, which is called the *elementary unipotent root subgroup* corresponding to α. In fact x_α is an isomorphism of R^+ on X_α. When it does not lead to a confusion we omit the epithets "elementary" and "unipotent" and speak about *root elements* and *root subgroups*.

Let now $N = N(\Phi, K)$ be the subgroup of G, generated by $T = T(\Phi, K)$ and all the elements $w_\alpha(1)$, $\alpha \in \Phi$. It is very well known that if $|K| \geq 4$ the group N coincides with the normalizer of T in G (see [St]). The quotient group N/T is canonically isomorphic to the Weyl group W and for any $w \in W$ we fix a preimage n_w of w in N. Usually when we speak about subgroups containing T we tend to identify w and n_w.

Recall that we have fixed an order of Φ, with Φ^+ being the corresponding set of positive roots respectively. Set

$$U = U(\Phi, R) = \big\langle x_\alpha(\xi), \ \alpha \in \Phi^+, \ \xi \in R \big\rangle .$$

The product $B = \mathrm{B}(\Phi, R)$ of the groups T and U is called the *standard Borel subgroup* of G (corresponding to the given choice of T and Φ^+), U is the *unipotent radical of* B.

§ 2. Aschbacher classes

A convenient framework for systematization of known results on intermediate subgroups is provided by Aschbacher classes $\mathcal{C}_1 - \mathcal{C}_8$. These classes were introduced in the study of maximal subgroups of the finite classical groups in [A1]. The groups from these classes are 'obvious' maximal subgroups of a finite classical group. The famous subgroup structure theorem of Aschbacher says that a maximal subgroup is either in one of these classes or in the class $\mathcal{S}$ consisting of almost simple groups in certain absolutely irreducible representations (subject to some further restrictions, see [A1], [KL2]).

The maximality of groups from Aschbacher classes was completely studied *modulo the Classification* by P.Kleidman and M.W.Liebeck, see [KL1], [KL2], where one can find all the details and many further references (of course, for most of these groups their maximality was established long before independently of the Classification).

Many of these classes have natural analogues also in exceptional groups. They arose in the subgroup structure theorem for the finite exceptional groups, proven by G.M.Seitz, M.W.Liebeck and others, compare [LSe] and references there. See also [S12], [S13], [S15], [Te] for the general picture of maximal subgroups in exceptional groups.

Here we are interested in subgroups with non-trivial lattice of overgroups. Most of the large subgroups of Chevalley groups studied so far are obtained by dropping those additional conditions in the definition of Aschbacher classes, which guarantee maximality. We can also combine constructions of several Aschbacher classes and still get large subgroups. Here we informally describe how it works. Let V be a minimal representation of a group G of characteristic p, say the natural representation of a classical group. Then the classes $\mathcal{C}_1 - \mathcal{C}_8$ are *roughly* described as follows (see [A1] and [KL2] for the details):

$\mathcal{C}_1$: Stabilizers of subspaces $U < V$;

$\mathcal{C}_2$: Stabilizers of direct sum decompositions $V = U_1 \oplus \ldots \oplus U_t$ into <u>similar</u> summands;

$\mathcal{C}_3$: Stabilizers of field extensions of <u>prime</u> degree;

$\mathcal{C}_4$: Stabilizers of tensor decompositions $V = U_1 \otimes U_2$;

$\mathcal{C}_5$: Stabilizers of subfields of <u>prime</u> degree;

$\mathcal{C}_6$: Normalizers of some l-subgroups, $l \neq p$;[1]

$\mathcal{C}_7$: Stabilizers of tensor decompositions $V = U_1 \otimes \ldots \otimes U_t$ into <u>similar</u> factors;

$\mathcal{C}_8$: Classical subgroups.

It is clear what we should do to get large subgroups: we have to omit extra-conditions (the underlined ones) introduced to enforce maximality. The classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are fused and we get summand-wise stabilizers of direct sum decompositions $V = U_1 \oplus \ldots \oplus U_t$. In the classical cases these are basically subsystem subgroups. The classes $\mathcal{C}_4$ and $\mathcal{C}_7$ are fused and give us factor-wise stabilizers of tensor decompositions $V = U_1 \otimes \ldots \otimes U_t$. In the definitions of $\mathcal{C}_3$ and $\mathcal{C}_5$ we have only to drop the word 'prime'.

We can also combine procedures used in forming various Aschbacher classes to get other large subgroups. For example, we can fix a direct decomposition $V = U_1 \oplus \ldots \oplus U_t$ and then instead of taking the whole stabilizer of this decomposition consider field extension subgroups on each summand, etc.

In this paper we discuss at some length the classes $\mathcal{C}_1 - \mathcal{C}_3$ and their combinations (sections 3 – 18). This is a very interesting class of subgroups, which includes for example, maximal tori and subsystem subgroups. These classes are discussed for all Chevalley groups, or, at least, for all classical Chevalley groups.

[1] It is not immediately clear what should be a correct analogue of the class $\mathcal{C}_6$ for infinite fields (not to say rings) and we do not discuss this class any further.

Due to the lack of time (during the talk), space (in this survey) and fading of the transparencies we can not discuss the classes $\mathcal{C}_4 + \mathcal{C}_7$, $\mathcal{C}_5 + \mathcal{C}_8$ and $\mathcal{S}$ with the same amount of details. We restrict ourselves to stating a handful of results (sections 19 – 21) showing that the same patterns for distribution of intermediate subgroups apply also in these cases. We do not even attempt to properly describe the classes themselves for any group other than GL_n. There is of course a more serious reason for this: the definitive results for most of these classes in the natural generality are still not available. However the results are forthcoming and the author hopes to be able to address this topic in later publications.

§ 3. Standard subgroups

In this section we describe what can be considered an analogue of the Aschbacher classes $\mathcal{C}_1 + \mathcal{C}_2$ for all Chevalley groups. Sections 4 – 13 will be dedicated to the analysis of overgroups of certain large groups from this class.

The groups U and U^- considered above are special cases of groups $E(S) = E(S, R)$ which we introduce now for any closed subset S in Φ. Recall that a subset S in Φ is called *closed* if for any two roots $\alpha, \beta \in S$ such that $\alpha + \beta \in \Phi$, one has $\alpha + \beta \in S$. Define $E(S) = E(S, R)$ as the subgroup generated by all the elementary root subgroups X_α, $\alpha \in S$, with respect to T:

$$E(S) = E(S, R) = \langle x_\alpha(\xi),\ \alpha \in S,\ \xi \in R \rangle.$$

Then U and U^- coincide with $E(\Phi^+)$ and $E(\Phi^-)$ respectively. The groups $E(S)$ are particularly important when the set S is *special* (alias *unipotent*), i.e. $S \cap (-S) = \emptyset$. In this case $E(S)$ is just the product of all X_α, $\alpha \in S$, in any fixed order. Set $G(S) = T(\Phi, R)E(S)$.

Suppose that $S \subseteq \Phi$ is any closed set of roots. Then S is the disjoint union of its *reductive* (alias *symmetric*) part S^r which consists of $\alpha \in S$ such that $-\alpha \in S$ and its *unipotent* part S^u which consists of $\alpha \in S$ such that $-\alpha \notin S$. The set S^r is a closed subsystem of roots while the set S^u is special. Moreover S^u is an *ideal* in S, i.e. if $\alpha \in S$, $\beta \in S^u$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in S^u$. We avoid the common notations S^+ and S^- since we prefer to reserve them for $S^+ = S \cap \Phi^+$ and $S^- = S \cap \Phi^-$. It is easy to see that the group $G(S)$ is the semidirect product of the reductive subgroup $G(S^r)$ (a *Levi subgroup* of $G(S)$) and the unipotent subgroup $E(S^u)$ (the *unipotent radical* of $G(S)$) and analogously $E(S)$ is the semidirect product of $E(S^r)$ and $E(S^u)$.

Two sets of roots $S_1, S_2 \subseteq \Phi$ are called *conjugate* if there exists an element w of the Weyl group $W = \mathrm{W}(\Phi)$ such that $wS_1 = S_2$. If the sets S_1 and S_2

are conjugated then there exists an $n \in N$ such that $nG(S_1)n^{-1} = G(S_2)$. We say that an element $w \in W$ normalizes S if $wS = S$. The set $X(S)$ of all $w \in W$ which normalize S is called the *Weyl normalizer* of S. It is clear that $X(S)$ contains the Weyl subgroup $W(S) = W(S^r)$. Moreover in the case when $S = S^r$ is a root subsystem $X(S)$ coincides with the normalizer of $W(S)$ in W (see [C2]). Denote by $N(S)$ the subgroup of G generated by $G(S)$ and n_w, $w \in X(S)$. A theorem of Tits (see [T2]) implies that almost always $N(S)$ coincides with the normalizer of $G(S)$ in G. The only cases when this equality may fail are those of the fields $K = \mathbb{F}_2, \mathbb{F}_3$.

§ 4. Parabolic subgroups

The first example (for many the *only* one) where one sees intermediate subgroups is the description of parabolic subgroups due to J.Tits [T1], [Bu].

Recall that we have fixed an order on the root system which determines Π, Φ^+ and Φ^-. A *standard parabolic* subset P is a closed set of roots containing Φ^+. The standard parabolic subsets are pair-wise not conjugate and correspond bijectively to all of the subsets $J \subseteq \Pi$ of the fundamental system. Namely if $J \subseteq \Pi$ is such a subset, then we may define P_J to be the smallest closed set of roots containing Φ^+ and $-J$. The most important parabolic subsets are the maximal ones. A maximal parabolic subset corresponds to a set $J = J_r$, $1 \leq r \leq l$, which contains all the fundamental roots apart from α_r. The corresponding parabolic set P_{J_r} is maximal among the closed subsets and will be denoted P_r.

Standard parabolic subgroups of the Chevalley group $G = G(\Phi, K)$ are the subgroups which contain the standard Borel subgroup $B = B(\Phi, K)$ while *parabolic subgroups* are those conjugated to the standard parabolic ones. A classical theorem of Tits describes the lattice $\mathcal{L}(B, G)$.

Theorem. *The map $P_J \mapsto G(P_J)$ is a bijection between the sets of standard parabolic subsets and of standard parabolic subgroups.*

What is remarkable about this theorem is that there are no exceptions, i.e. it applies to *all* fields. This is quite unusual. Most of the analogous results for subgroups other than B break down for some small characteristics or cardinalities of the ground field.

Another theorem of Tits asserts that overgroups of B have remarkable properties with respect to conjugation.

Definition. *A subgroup B is called* **abnormal** *in G if for any $x \in G$ one has $x \in \langle B, xBx^{-1} \rangle$.*

It is easy to see that saying that B is abnormal in G is equivalent to saying that

i) Every subgroup containing B is self-normalizing,

ii) Any two distinct subgroups containing B are not conjugate.

Now another main result of [T1] may be stated as follows.

Theorem. *B is abnormal in G.*

The first examples of abnormal subgroups of which one would think, are, of course,

– Sylow normalizers, i.e. normalizers of the Sylow subgroups in G.
– Carter subgroups of finite soluble groups.

Actually in the case when characteristic of K is positive, B is precisely the normalizer of U, which is a Sylow p-subgroup of G.

§ 5. Subgroups of the Borel subgroup

The next example is that of overgroups of T in B. This description may be summarized in the following theorem. Throughout the section we assume that

$K \neq \mathbb{F}_2$ for $\Phi = A_l$, $l \geq 4$; $\Phi = \mathrm{E}_l$;
$K \neq \mathbb{F}_2, \mathbb{F}_3$ for $\Phi = \mathrm{A}_3$, B_l, C_l, $l \geq 2$, D_l, $l \geq 3$, F_4, G_2;
$K \neq \mathbb{F}_2, \mathbb{F}_4$ for $\Phi = A_2$;
K is perfect if $\operatorname{char} K = 2$ and $\Phi = \mathrm{C}_l$, $l \geq 1$.

Recall that $\mathrm{A}_1 = \mathrm{C}_1$, $\mathrm{B}_2 = \mathrm{C}_2$ and $\mathrm{D}_2 = \mathrm{C}_1 \times \mathrm{C}_1$. In this paper we *always* treat the group $\mathrm{SL}(2, K)$ as being *symplectic*, not linear. The reason is that like for other symplectic groups (but unlike $\mathrm{SL}_{n\geq 3}$) conjugating a long root element $x_\alpha(\xi)$ by an element from T we can only multiply ξ by ε^2 for some $\varepsilon \in K^*$, whereas for all other cases we can multiply ξ by any element from K^*. This is why $\square$'s play such a role in the theory of symplectic groups and non-perfect fields of characteristic 2 always pop up as exceptions. Actually to state the first theorem as it is, we have to assume moreover that

$\operatorname{char} K \neq 2$ if $\Phi = \mathrm{B}_l$, C_l, $l \geq 2$, F_4,
$\operatorname{char} K \neq 2, 3$ if $\Phi = \mathrm{G}_2$.

However if we replace closed sets of roots by quasi-closed ones [BT], the theorem below remains valid without these last assumptions. The same applies to the two other theorems. The example which follows describes the lattice $\mathcal{L}(T, B)$.

Theorem. *The map $S \mapsto G(S)$ is a bijection between the set of closed subsets of Φ^+ and the set of subgroups of B containing T.*

This result was stated under different disguises in [S2], [CPS], [B1], [B2], [Sz1], [V1], [Az], [V21]. Analyzing the proofs of this result, we see that they give also the following result.

Theorem. *T is abnormal in B.*

The results show that the behaviour of T as a subgroup of B is exactly the same as behaviour of B as a subgroup of G. The only difference is that while the results of Tits hold for *arbitrary* fields, here exceptions appear: it is easy to check that all excluded cases are actually exceptional.

Any overgroup of an abnormal subgroup is itself abnormal. Since every abnormal subgroup of G contained in B must contain T, the results above clearly imply

Theorem. *B is a minimal abnormal subgroup in G.*

Actually this result holds also for most of the excluded cases. For example, this is obviously true for $K = \mathbb{F}_2$, since in this case $B = U$ is nilpotent. The cases $K = \mathbb{F}_3, \mathbb{F}_4$ require somewhat more careful analysis.

The importance of Borel subgroups suggests that it might be interesting to study minimal abnormal subgroups. Every maximal subgroup is either normal or abnormal. Classification of minimal abnormal subgroups might give a new view of the maximal subgroups.

Problem. *Classify minimal abnormal subgroups of finite groups of Lie type up to conjugacy.*

§ 6. Overgroups of split maximal tori

In § 4 and § 5 we described the lattices $\mathcal{L}(B, G)$ and $\mathcal{L}(T, B)$. Now we combine these results and describe the lattice $\mathcal{L}(T, G)$. This result is already *terribly* much more difficult, than the previous ones. In this and following sections we give just a vague idea of how it works. One can find many further details and an almost exhaustive bibliography in the surveys of A.E.Zalesskii [Z2], A.S.Kondratiev [Kn] and the author [V16], [V17] (see also [V10], [V14], [V18], [HV]).

Throughout the section we assume that

$|K| \geq 7$ if $\Phi = \mathrm{A}_l$, $l \geq 2$, D_l, $l \geq 3$ or if G is adjoint;

$|K| \geq 13$ otherwise.

Recall that according to our agreement in § 1, there are no adjoint groups of types E_8, F_4, G_2. The numbers 7 and 13 in these conditions are explained as follows: in the proof we must pick up a nonzero element ε distinct from 5 prohibited values. But when we work with SL_2 the *square* of this element should also avoid these 5 values, this then what prohibits altogether eleven values. Thus the proof should work starting from the field of 12 elements. For safety we make it 13.

Moreover we make the same assumption on the characteristic of the ground field as in § 5. Again these assumptions are not necessary if we are ready to accept quasi-closed sets of roots (see [BT]) instead of the closed ones in the statement (of course, we must keep the condition from § 5 for the symplectic case).

Theorem. *For any intermediate subgroup F, $T \leq F \leq G$, there exists a unique closed set of roots $S \subseteq \Phi$ such that*

$$G(S) \leq F \leq N(S).$$

It is easy to check that the fields K, $|K| \leq 5$, are exceptions for all the cases and that the fields K, $|K| \leq 11$, are exceptions for the symplectic case (see [Ko1], [VD]). It is quite plausible though, that the theorem holds for all non-symplectic cases whenever $|K| \geq 7$ (this conjecture has been formulated in [V6], [V17]; in [S4] G.M.Seitz posed the problem of the description of overgroups of a split maximal torus for all fields K, $|K| \geq 4$).

In a classical paper [BT] by A.Borel and J.Tits this theorem was proven for the case of an algebraically closed field K. Of course in [BT] only *closed connected* subgroups of G were considered (without any restrictions on the characteristics whatsoever) and it was proven that they are exhausted by $G(S)$ for quasi-closed sets of roots $S \subseteq \Phi$. But in fact a classical theorem of Chevalley (see, for example [H1] or [Sp]) implies that for an algebraically closed field any subgroup containing T is closed in the Zariski topology.

For the group $\mathrm{GL}(n, K)$ (i.e. an extended Chevalley group of type A_{n-1}) this theorem was proven in 1976 in a paper of Z.I.Borewicz [B3], see also [BV1], [V2]. In [V4], [V8] other proofs of this (and somewhat stronger) results were proposed, based on the calculations with Bruhat and Bruhat-like decompositions.

The next important step was a paper by G.M.Seitz [S3] published in 1979, where he proved that the standard description holds for a finite field K such that $\operatorname{char} K \neq 2$ and $|K| \geq 13$. The arguments in [S3] used the finiteness of the ground field in a very essential way and they could not be generalized to an infinite field. In fact such deep results as, say, theorems on 2-fusion have been used (this explains the restriction imposed on the characteristic).

In 1979–80 analogous results were obtained by the author and E.V.Dybkova for extended Chevalley groups of types C_l and D_l, i.e. the general symplectic group $\mathrm{GSp}(2l, K)$ and the general orthogonal groups $\mathrm{GO}(2l, K)$ (see [VD], [V3]).

In the same paper [VD] the author and E.V.Dybkova proved the following reduction result for the symplectic group. Let $\operatorname{char} K \neq 2$ and $|K| \geq 7$. Then the description of overgroups of a split maximal torus in $\mathrm{Sp}(2l, K)$ is standard if and only if the description of overgroups of a split maximal torus in $\mathrm{SL}(2, K)$ is standard.

Analogous results for the extended Chevalley group of type B_l i.e. for the odd orthogonal groups $\mathrm{GO}(2l+1, R)$ were obtained at the same time but were not published until [V14]. Actually it was noted in [V14] that the results of [V3] remain valid for the ordinary orthogonal groups $\mathrm{SO}(n, R)$. Of course this is due to the fact that the groups $\mathrm{SO}(n, R)$ are already extended by some (though not all in general) diagonal automorphisms.

The theorem for the adjoint groups of types E_6 and E_7 has been proven by the author. A sketch of its proof is presented in [V11], [V12], [V20] (see also [Va17] and references there).

A method which allowed us to prove the theorem for simply connected groups of non-symplectic types over an infinite field was developed by the author in the period 1983–1989. A crucial point was that for all non-symplectic groups one could reduce the problem to the *extended* Chevalley groups of smaller rank and for the extended groups the analogous problems had already been solved.

For the special linear group $\mathrm{SL}(n, K)$, $n \geq 3$ over an infinite field K this proof has been published in the first part of [V6], while the subsequent parts contained a proof of standardness under assumption $|K| \geq 7$. For $\Phi = \mathrm{D}_l$ standardness under this assumption is proven in [V18].

The case SL_2 was the last one to be solved and in § 9 we discuss the proof for this case. This proof shows that analysis of *abstract* subgroups over an *infinite* field may be quite a challenge even for such small groups as $\mathrm{SL}(2, K)$.

§ 7. The group GL_n

Here we prove the theorem from § 6 for *adjoint* groups of type A_{n-1}. Instead of working in $\mathrm{PGL}(n, K)$ itself we prefer to handle the corresponding *extended* simply connected group $\mathrm{GL}(n, K)$. The *natural* proof for the general linear group, obtained in [Bo3], [BV1], [V2], works for all fields K, $|K| \geq 7$, (see also [V4], [V8], where this proof is transcribed in terms of Bruhat decomposition). Here we reproduce this proof in a slightly modified form.

Let $G = \mathrm{GL}(n, K)$ be the general linear group and $D = \mathrm{D}(n, K)$ be its subgroup of diagonal matrices. It is easy to prove that if there is a counter-example to the standard description of subgroups in G containing D, then there is a *primitive irreducible* counter-example. Thus it remains only to prove that if H is a primitive irreducible subgroup in G which contains D but is not contained in N, then it coincides with G. By McLaughlin's theorem it suffices to prove that H contains a non-trivial elementary transvection (then H contains a root subgroup and the normal subgroup in H, generated by it, either coincides with $\mathrm{SL}(n, K)$, or is conjugate to $\mathrm{Sp}(n, K)$, but no conjugate of $\mathrm{GSp}(n, K)$ is normalized by D for $n \geq 4$).

Thus it remains to prove that if x is a non-monomial matrix, then the subgroup $X = \langle D, x\rangle$, generated by D and x contains an elementary transvection. The proof is based on the analysis of pseudo-reflections contained in X. Recall that e is the identity matrix of degree n and e_{ij}, $1 \le i, j \le n$, is a standard matrix unit, i.e. the matrix which has 1 in the position (i, j) and zeros elsewhere. Let now $d_r(\varepsilon) = e + (\varepsilon - 1)e_{rr}$, $1 \le r \le n$, $\varepsilon \in K^*$, be an 'elementary' pseudo-reflection. Fix some r and consider the conjugate $y = y(\varepsilon) = xd_r(\varepsilon)x^{-1}$ of $d_r(\varepsilon)$ by x:

$$y(\varepsilon) = \begin{pmatrix} 1+\alpha_1(\varepsilon-1)\beta_1 & \alpha_1(\varepsilon-1)\beta_2 & \dots & \alpha_1(\varepsilon-1)\beta_n \\ \alpha_2(\varepsilon-1)\beta_1 & 1+\alpha_2(\varepsilon-1)\beta_2 & \dots & \alpha_2(\varepsilon-1)\beta_n \\ \vdots & \vdots & & \vdots \\ \alpha_n(\varepsilon-1)\beta_1 & \alpha_n(\varepsilon-1)\beta_2 & \dots & 1+\alpha_n(\varepsilon-1)\beta_n \end{pmatrix}$$

where $\alpha_i = x_{ir}$ are the entries of the r-th column of x, while $\beta_j = x'_{rj}$ are the entries of the r-th row of the inverse matrix $x^{-1} = (x'_{ij})$.

Now fix some s, $1 \le s \le n$. The s-th rows of matrices $xd_r(\varepsilon)x^{-1}$ and $xd_r(\eta)x^{-1}$ are proportional, apart from the diagonal entries. But the diagonal entries may be corrected by multiplication by some $d_s(\theta)$. Indeed, choose $\varepsilon, \eta \in K^*$ in such a way, that $\varepsilon - 1$, $\eta - 1$, $1 + \alpha_s(\varepsilon - 1)\beta_s$, $1 + \alpha_s(\eta - 1)\beta_s$, $\varepsilon - \eta$, $\varepsilon - \eta^{-1}$ are all distinct from zero (we can do this, since our field contains at least 7 elements). Now set

$$\theta = (\varepsilon - 1)(\eta - 1)^{-1}(1 + \alpha_s(\varepsilon - 1)\beta_s)^{-1}(1 + \alpha_s(\eta - 1)\beta_s).$$

A direct calculation shows that the product

$$z = \big(xd_r(\varepsilon)x^{-1}\big)d_s(\theta)\big(xd_r(\eta)x^{-1}\big)^{-1}$$

has the same s-th row as the identity matrix and thus lies in a proper parabolic subgroup. Clearly, $t_{is}(z_{is}) \in X$ for all $i \ne s$. On the other side z_{is}, $i \ne s$, differs from $\alpha_i\beta_s$ only by an invertible factor. Thus either we get a nontrivial transvection in X, or (since we may vary both r and s) we have $x_{ir}x'_{rj} = 0$ for all $i \ne j$, in which case x is monomial, a contradiction.

§ 8. The group $\mathrm{SL}_{n\ge 3}$

In this section we prove the theorem from § 6 for the case of *simply connected* Chevalley group $G = \mathrm{SL}(n, K)$ of type A_{n-1}, $n \ge 3$, over an *infinite* field K, see [V6], Part I. Actually the proof works also for *large* finite fields of cardinality, say $> 6n + 1$. However considerable additional efforts are needed to prove the result in *natural* generality, i.e. for all fields K, $|K| \ge 7$ (see [V6], Parts II – IV).

Let $D = \mathrm{SD}(n, K)$. As in the case of the general linear group it is easy[2] to reduce the proof to the analysis of a primitive irreducible counterexample. In other words, as before we have only to prove that if x is a non-monomial matrix, then the group $X = \langle D, x\rangle$ contains an elementary transvection.

Let $d_{ij}(\varepsilon)$, $1 \leq i \neq j \leq n$, $\varepsilon \in K^*$ be a 'two-dimensional semisimple element'

$$d_{ij}(\varepsilon) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{jj} = d_i(\varepsilon)d_j(\varepsilon^{-1}).$$

From the point of view of Chevalley groups $d_{ij}(\varepsilon)$ is a semisimple root element in $\mathrm{SL}(n, K)$. The 'natural' proof is based on the analysis of the conjugates of such element in a counterexample.

However the analysis of an infinite (or a large finite) field is easier since in this case G contains non-trivial homologies, i.e. matrices with $n-1$ equal eigenvalues. Namely, take any $\varepsilon \in K^*$ such that $\varepsilon^n \neq 1$. Then the matrix

$$d^r(\varepsilon) = \mathrm{diag}(\varepsilon, \ldots, \varepsilon, \varepsilon^{1-n}, \varepsilon, \ldots, \varepsilon) = \varepsilon d_r(\varepsilon^{-n}) \in G$$

is noncentral and has $n - q$ equal eigenvalues.

The conjugates of such matrices look – up to a scalar factor – exactly like the conjugates of $d_r(\varepsilon) \in \mathrm{GL}(n, K)$. We may again fix an r, $1 \leq r \leq n$, and an $\varepsilon \in K^*$, $\varepsilon^n \neq 1$ and consider the conjugate of $d^r(\varepsilon)$ by x:

$$y(\varepsilon) = xd^r(\varepsilon)x^{-1} = \varepsilon\big(\delta_{ij} + \alpha_i(\varepsilon^{-n} - 1)\beta_j\big), \qquad 1 \leq i, j, \leq n,$$

where $\alpha_i = x_{ir}$ and $\beta_j = x'_{rj}$, as before.

Fix some $1 \leq i \neq j \leq n$ and take an index $h \neq i, j$. We may assume that $\alpha_h, \beta_h, \alpha_i, \beta_i \neq 0$ (otherwise the problem is reduced to a proper parabolic subgroup). Set

$$\eta = \alpha_i(\varepsilon^{-n} - 1)\beta_i\big(1 + \alpha_h(\varepsilon^{-n} - 1)\beta_h\big)^{-1},$$

where invertibility of $1 + \alpha_h(\varepsilon^{-n} - 1)\beta_h$ prohibits at most n further values of ε. Prohibiting not more than $2n$ further values of ε we may even assume, that $\eta \neq \pm 1$. Now we may consider the matrix

$$z = y(\varepsilon)d_{hi}(\eta)y(\varepsilon^{-1}) \in X.$$

---

[2]This 'easy' refers to large fields. For $|K| \geq 7$ the reduction uses description of overgroups of $D(n, K)$ in $\mathrm{GL}(n, K)$, see [V21]. This is a general pattern: in the proofs for simply connected groups quite often we have to use the corresponding results for adjoint groups.

A straightforward calculation shows that $z_{hi} = 0$ and

$$z_{ij} = (\eta^{-1} - \eta)\alpha_i(\varepsilon^{-n} - 1)\beta_j.$$

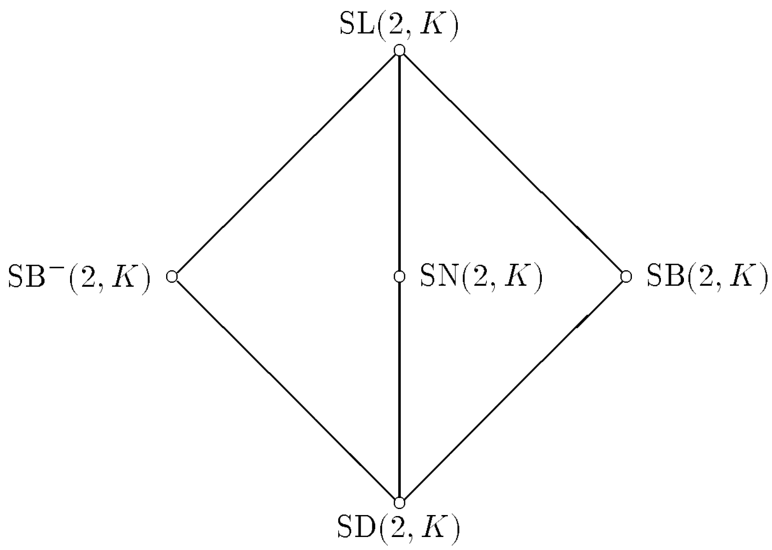
It is easy to check, that $z_{hi} = 0$ implies that $t_{ij}(z_{ij}z'_{jj}) \in X$. If ε is not a root of another polynomial of degree $2n$, we can assume that $z'_{jj} \neq 0$, and thus $t_{ij}(\alpha_i\beta_j) \in X$. Now the proof is completed exactly as in the case of GL_n.

The proof for the case $|K| \geq 7$ follows roughly the same lines, but there are some further complications. For example, the group G does not in general contain non-trivial homologies. As a result, it is impossible to conclude directly from the fact that $x_{ij} = 0$ for some $x \in X$ that x is monomial. This is why one is forced to consider more complicated dependencies among the entries of matrices from a counterexample (see [V6], III, IV).

The proofs of the theorem from § 6 for groups of other types are based on roughly the same ideas as the ones reproduced in this section and the preceding one. Of course, they are technically much more complicated. For classical groups one can use their minimal modules. For the exceptional ones we have to play with the Bruhat decomposition of semi-simple root elements and the like, see [V17] and references there. A sketch of the proof for the adjoint case may be found in [V11], [V12], [V20].

§ 9. The group SL_2

For the group $\mathrm{SL}(2, K)$ the theorem from § 6 asserts that there are only three proper intermediate subgroups between $\mathrm{SD}(2, K)$ and $\mathrm{SL}(2, K)$:



where $\mathrm{SB}(2,K)$ and $\mathrm{SB}^-(2,K)$ are the upper and the lower standard Borel subgroups in $\mathrm{SL}(2,K)$, while $\mathrm{SN}(2,K)$ is the monomial subgroup.

Of course for a finite field K we know *all* subgroups of $\mathrm{SL}(2,K)$, but for an infinite field this result turned out to be incredibly tough. In [VD] the author and E.V.Dybkova proved the theorem for $\mathrm{SL}(2,K)$ (and thus also for $\mathrm{Sp}(2l,K)$) under the additional assumption that $|K^*| > |K^*/K^{*2}|$ and observed that $\mathrm{SN}(2,K)$ was maximal in $\mathrm{SL}(2,K)$ whenever $-1 \in K^{*2}$. In [K1] O.King proved the maximality of $\mathrm{SN}(2,K)$ in full generality (compare also [L5] where the maximality of $\mathrm{SN}(2,K)$ was proved under essentially the same additional assumptions $|K^*/K^{*2}| < \infty$ or $-1 \in K^{*2}$ as in [VD]). In 1986 the author noticed that a slight modification of this proof allows to prove the theorem for $\mathrm{SL}(2,K)$ (and thus also for $\mathrm{Sp}(2l,K)$) under the additional assumption that $-1 \in K^{*2}$ (see [V10], [V16]). Finally in [K2] O.King was able to prove the theorem for $\mathrm{SL}(2,K)$ in full generality. Actually in this paper O.King describes overgroups of $\mathrm{SD}(2,K)$ also for fields of characteristic 2 in terms of the K^2-submodules in K.

Here we reproduce a proof of O.King [K2]. Let F be a subgroup of $\mathrm{SL}(2,K)$ which contains $\mathrm{SD}(2,K)$ but is not contained in the three subgroups above. Passing from F to a subgroup which is conjugate to F by a diagonal matrix from $\mathrm{GL}(2,K)$, we may assume that F contains a matrix of the form

$$f = \begin{pmatrix} 1 & 1 \\ \alpha & 1+\alpha \end{pmatrix}, \qquad \alpha \in K^*.$$

Passing, if necessary, from F to $\langle T, fTf^{-1}\rangle$, we may assume further that α does not belong to a fixed *finite* subset of K^*. We denote the root element $h_\alpha(\varepsilon)$ simply by $h(\varepsilon)$:

$$h(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \qquad \varepsilon \in K^*.$$

Now if $\operatorname{char} K \neq 2,3,5$, O.King chooses an $\alpha \neq 0, 1, \pm 2, \pm 3, -4, 5, -6$ and forms the following product:

$$\begin{aligned} g =& f^{-1} h\left(\frac{2}{\alpha}\right) f^{-1} h\left(\frac{\alpha+4}{\alpha+2}\right) fh\left(\frac{3}{4}\right) f^{-1} h\left(\frac{3}{\alpha-3}\right) f^{-1} \\ & h\left(\frac{\alpha+6}{\alpha+3}\right) fh\left(\frac{5}{6}\right) f^{-1} h\left(\frac{\alpha-2}{\alpha-5}\right) fh(3) f^{-1} h\left(\frac{\alpha+3}{\alpha+2}\right) fh\left(\frac{2}{3}\right) \\ & f^{-1} h\left(\frac{\alpha-1}{\alpha-2}\right) fh(\alpha-3) fh\left(\frac{4}{3}\right) f^{-1} h\left(\frac{\alpha+2}{\alpha+4}\right) fh\left(\frac{\alpha}{2}\right) f. \end{aligned}$$

Then a direct calculation shows that $g_{21} = 0$ while g_{12} is a nonzero polynomial in α and thus equals zero only for finitely many values of α. This shows that some subgroup conjugated with F by a diagonal matrix contains a nontrivial elementary transvection. Thus $F = \mathrm{SL}(2, K)$. For $\operatorname{char} K = 3, 5$ O.King constructs analogous (but shorter) formulae expressing transvections in terms of f and $h(\varepsilon)$. For $\operatorname{char} K = 2$ both the proof and the statement must be modified considerably, since for a nonperfect field of characteristic 2 there may be infinitely many intermediate subgroups.

§ 10. Pronormal subgroups

Now we know what the analogue of the classification theorem for overgroups of T is. But what is the analogue of the conjugacy theorem? Before T was abnormal in B and B was abnormal in G. But T does not coincide with its normalizer N in G, so it cannot be abnormal.

Definition. *A subgroup T of a group G is called* **pronormal** *in G if for any $x \in G$ there exists such $y \in \langle T, xTx^{-1} \rangle$ that $yTy^{-1} = xTx^{-1}$.*

In other words, this means that any two subgroups conjugate to T in G are conjugated already in the subgroup they generate.

Let $N = N_G(T)$ be the normalizer of T in G. It is easy to see that saying that T is pronormal in G is equivalent to saying that

i) $N_G(H) = HN_N(H)$ for any subgroup H containing T;

ii) any two conjugate subgroups H and F containing T are conjugated by an element of N.

Actually both conditions may be conveniently fused in the following condition: for every two conjugate subgroups $T \leq F, H \leq G$ and any $x \in G$ such that $xFx^{-1} = H$ may be expressed in the form $x = wy$ for appropriate $w \in N$ and $y \in F$.

Now we can state the analogue of conjugacy theorem. We keep the same assumptions on K as in § 6.

Theorem. *T is pronormal in G.*

The first examples of pronormal subgroups which one would think, are, of course,

– Sylow subgroups of (normal subgroups in) finite groups,
– Hall subgroups of soluble (normal subgroups in) finite groups,
– U in G (see [S1]),
– maximal tori of algebraic groups (see [H1], [Sp]).

More generally a *weakly closed* subgroup of a Sylow subgroup T of G is pronormal in G, etc.

Pronormality is a common generalization of normality and abnormality. The normalizer of a pronormal subgroup is abnormal. In particular this theorem implies that N is abnormal in G. However, it is not, generally speaking, a minimal abnormal subgroup.

It turns out that pronormality plays a very important role in the description of the lattice of subgroups of an abstract group [B6]. In fact for any pronormal subgroup T in G the lattice of intermediate subgroups looks as follows. It is a disjoint union

$$\mathcal{L}(T,G)=\bigcup^{\emptyset}\mathcal{L}(F,N_G(F)),$$

taken over all subgroups F generated by some conjugates of T (either in G or in F itself, in this case this does not matter, see § 14 for the details). Intuitively the groups F are 'connected' with respect to T, in [B6] they are called T-complete. The normal closure T^H of T in any intermediate subgroup $H\in\mathcal{L}(T,G)$ coincides with one of the groups F above, while the normalizer $N_G(H)$ is contained in $N_G(T^H)$.

Thus to describe overgroups of a pronormal subgroup, one has to classify the T-complete subgroups F and to calculate the factor-groups $N_G(T)/T$.

However, as we shall see later, there are many further subgroups, for which $\mathcal{L}(T,G)$ may be decomposed in this way. Among these subgroups pronormal ones are characterized by the fact that they have large normalizers, see properties i) and ii) above. In fact, for a pronormal subgroup F the following two characteristic properties hold:

– every sandwich $\mathcal{L}(F,N_G(F))$ is isomorphic to an interval inside $\mathcal{L}(T,N)$ (in other words $N_G(F)/F$ is a section of N/T);

– two sandwiches $\mathcal{L}(F_1,N_G(F_1))$ and $\mathcal{L}(F_2,N_G(F_2))$ are conjugate if and only if they are conjugate by an element of N: $F_1\sim_G F_2$ if and only if $F_1\sim_N F_2$.

In particular, for a pronormal subgroup the calculation of all factor-groups $N_G(F)/F$ is reduced to the calculation of the 'Weyl group' N/T.

Thus, there are good chances for a nice description of overgroups of Sylow subgroups in a finite group. This is essentially done in [A5] for sporadic groups.

Problem. *Describe the overgroups of Sylow subgroups in finite groups of Lie type.*

The following question is purely speculative, since one cannot really expect a reasonable answer. However even new examples of pronormal subgroups could be of great value for understanding the subgroup structure of finite groups.

Problem. *Classify pronormal subgroups of finite groups of Lie type.*

§ 11. Generalizations to rings

In this section we mention how the results of the previous sections may be generalized to rings. Of course here the answers should take into account the structure of the ideals of the ground ring.

Definition. *A* **net** *of ideals in R of type Φ is a family $\sigma = (\sigma_\alpha)$, $\alpha \in \Phi$, of ideals σ_α in R such that $\sigma_\alpha \sigma_\beta \subseteq \sigma_{\alpha+\beta}$ whenever $\alpha, \beta \in \Phi$ are such that $\alpha + \beta \in \Phi$.*

Now we can assign to a net σ of type Φ in R subgroups $E(\sigma)$, $G(\sigma)$ and $N(\sigma)$ in $G(\Phi, R)$ by the same formulae, as in § 3, for a closed set of roots S. Namely, we set

$$E(\sigma) = \langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in \sigma_\alpha \rangle.$$

Now we can define $G(\sigma)$ as $T \cdot E(\sigma)$ and $N(\sigma)$ as the normalizer of $G(\sigma)$ in G. Since K_1 of a ring is non-trivial, these are *not* the correct definitions of $G(\sigma)$ and $N(\sigma)$, but they suffice for the instances in which we use them. In fact $G(\sigma)$ should be defined by certain linear congruences on entries of matrices describing the action of elements from G on a minimal module, see § 13 for the case of the general linear group.

In the next two sections we can avoid this problem by *defining* $N(\sigma)$ as $N_G(E(\sigma))$. In fact a considerable part of real work consists in proving that the correctly defined $N(\sigma)$ actually *does* coincide with the normalizer of $E(\sigma)$ for the cases we are interested in. This result is, for instance, a very wide generalization of the normality theorem of Suslin stating that the elementary subgroup $\mathrm{E}(n, R)$ is normal in $\mathrm{GL}(n, R)$, when R is commutative and $n \geq 3$. We do not discuss these aspects of the theory here, see [BV3], [V16] for the details and further references.

Nets of type A_l were introduced by N.S.Romanovskii [Ro] in a special case and by Z.I.Borewicz in general [Bo1] – [Bo5], [BV1] – [BV3]. Corresponding subgroups in GL_n, SL_n, Sp_{2n} were studied in detail by Z.I.Borewicz and his students (we cannot reproduce a complete bibliography here, consult [Ne] for the early history of these ideas and [Z2], [VD], [V14], [V15], [V18] for further references). For arbitrary Φ, nets were defined by K.Suzuki [Sz1], [Sz2] in a special case and by the author in general [V1], [V5]. These papers dealt with rings close to fields (like semilocal ones, see below), where the above definitions of $G(\sigma)$ and $N(\sigma)$ are satisfactory. For arbitrary rings the "correct" groups were defined in [VP].

With these definitions we can generalize the contents of the previous sections to some classes of rings. In fact $\mathcal{L}(T, B)$ can be described for an almost arbitrary ring R – the only requirement is that R should have enough units: R should be additively generated by R^* (by R^{*2} in the symplectic case) and the ideal generated by $\varepsilon - 1$ (or $\varepsilon^2 - 1$, $\varepsilon^3 - 1$), where $\varepsilon \in R^*$, should coincide with R. Under these conditions every subgroup in B containing T coincides with $G(\sigma)$ for an appropriate net σ and T is pronormal in B (see [B1], [B2], [V1]).

It is somewhat more complicated with $\mathcal{L}(B, G)$. In fact there is no analogue of the Bruhat decomposition for rings other then direct sums of (skew-

)fields[3]. For rings the lattice $\mathcal{L}(B,G)$ was studied in [Ro], [B1], [B2], [Sz1], [Sz2], [V1], and several further papers (see [V5] for a complete bibliography). In these papers $\mathcal{L}(D,G)$ was completely described when R is a "zero-dimensional" ring satisfying the restrictions above. Not to go into details, we note that a safe example of zero-dimensional rings are semilocal ones.

Recall that a ring R is called *semilocal* if its factor-ring modulo the Jacobson radical J is artinian. For a commutative ring it is equivalent to saying that R has finitely many maximal ideals, or that $R/J \cong K_1 \oplus \ldots \oplus K_t$ is a direct sum of finitely many fields (in particular R is local if $t = 1$, i.e. if $R/J \cong K$ is a field). For semilocal rings the conditions imposed on their units amount simply to saying that $K_i \neq \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4$, etc. For such rings again all intermediate subgroups H, $B \leq H \leq G$, are of the form $G(\sigma)$ and B is abnormal in G. (This is not the case in general however, even for such *very* nice rings as Hasse domains the answer is much more complicated).

These results applied to *all* types. As for the lattice $\mathcal{L}(T,G)$ so far only the *adjoint* (or extended) *classical* groups were considered. Let R again be semilocal. Then only the small direct summands of R/J may cause trouble. If all K_i are distinct from $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$ and $\mathbb{F}_5$ (and $\operatorname{char} K \neq 2$ if $\Phi = \mathrm{B}_l$ or C_l), then for any intermediate subgroup H, $T \leq H \leq G$, there exists a unique net σ of ideals in R of type Φ such that $G(\sigma) \leq H \leq N(\sigma)$, see [B3], [BV1], [V2], [V3], [VD], [V14]. Moreover if R is *local* then T is pronormal in G (for a general semilocal ring T need not be pronormal – it is only *paranormal*, compare § 14), see [V9], [VD], [V14]. For the general linear group over *non-commutative* semilocal rings one has to prohibit one more direct summand of R/J, namely the ring $\mathrm{M}(2,\mathbb{F}_2)$ of matrices of degree 2 over the field of two elements.

There are presently no approaches to the description of overgroups of T for local rings, not only for the exceptional groups, but even for such innocent looking creatures as SL_n. The only case for which it was possible to go through for $\mathrm{SL}_{n\geq 3}$ are the so called *uniserial* rings, for which ideals are linearly ordered by inclusion [Ha]. Here we state the part of the result which works for the adjoint classical groups and does not work in other situations.

Let R be any commutative ring, J be its Jacobson radical. Consider the group B_J generated by B and all $x_\alpha(\xi)$, where $\alpha \in \Phi^-$, $\xi \in J$. In other words B_J is precisely the group $G(\tau)$ for the net τ defined by $\tau_\alpha = R$ if $\alpha \in \Phi^+$ and $\tau_\alpha = J$ otherwise. Then for the adjoint classical groups one can describe the lattice $\mathcal{L}(T, B_J)$ for almost arbitrary commutative rings R.

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[3] This applies to groups of finite degree. For the stable elementary group $\mathrm{E}(\Lambda) = \varinjlim \mathrm{E}(n,\Lambda)$ there is an analogue of Bruhat decomposition which holds for *any* ring Λ – Sharpe's decomposition.

Namely R should have enough units in the same sense as in the description of $\mathcal{L}(T,B)$ above. Assume that R is additively generated by R^* and that there exist such $\varepsilon, \eta \in R^*$ that $\varepsilon - 1$, $\eta - 1$, $\varepsilon - \eta$, $\varepsilon\eta - 1 \in R^*$. Then every intermediate subgroup H, $T \leq H \leq B_J$ is again of the form $G(\sigma)$ for an appropriate σ and T is abnormal in B_J. This is precisely what is still lacking for other groups, including $\mathrm{SL}_{n\geq 3}$.

The reason why the description of overgroups of T in G works only for *zero-dimensional* rings is that T consists entirely of semisimple elements. Of course, B has lots of unipotent elements, but their geometry is very poor. Semisimple elements may be very upsetting even for fields. Unipotents are much nicer and behave in a more predictable way. In the next sections we shall see in particular that when D has enough unipotents it is almost always possible to describe $\mathcal{L}(D,G)$ for *arbitrary* commutative rings.

Problem. *Describe $\mathcal{L}(T,G)$ for all Chevalley groups when R is a semilocal ring.*

§ 12. Overgroups of subsystem subgroups

In this section we introduce another pattern in the distribution of intermediate subgroups. Namely, we study the overgroups of *regularly embedded* semisimple subgroups in the sense of E.B.Dynkin. Such subgroups were christened *subsystem subgroups* by M.W.Liebeck and G.M.Seitz.

Namely let $\Delta \subseteq \Phi$ be an embedding of root systems. This embedding defines a regular embedding of Chevalley groups $G(\Delta,R) \leq G(\Phi,R)$ over any commutative ring R with 1. We are interested in the description of overgroups of $E(\Delta,R)$ in $G(\Phi,R)$. Of course to get such a description one has to assume something about Δ (and possibly also about R): the case $\Delta = \emptyset$ is upsetting.

The assumption which seems reasonable in all cases is that $\Delta^{\perp} = \emptyset$, i.e. that there are no roots in Φ orthogonal to all roots in Δ. However for commutative rings one usually needs stronger assumptions.

Now we state known results for the *classical* groups. First of all recall that the root subsystems of classical systems look as follows:

$$\Phi = \mathrm{A}_l: \qquad \Delta = \mathrm{A}_{k_1} + \ldots + \mathrm{A}_{k_r},$$

$$\Phi = \mathrm{B}_l: \qquad \Delta = \mathrm{A}_{k_1} + \ldots + \mathrm{A}_{k_r} + \mathrm{D}_{l_1} + \ldots + \mathrm{D}_{l_{s-1}} + B_{l_s},$$

$$\Phi = \mathrm{C}_l: \qquad \Delta = \mathrm{A}_{k_1} + \ldots + \mathrm{A}_{k_r} + \mathrm{C}_{l_1} + \ldots + \mathrm{C}_{l_s},$$

$$\Phi = \mathrm{D}_l: \qquad \Delta = \mathrm{A}_{k_1} + \ldots + \mathrm{A}_{k_r} + \mathrm{D}_{l_1} + \ldots + \mathrm{D}_{l_s},$$

Here $p+q = l$, $p = (k_1+1)+\ldots+(k_r+1)$, $q = l_1+\ldots+l_s$, and $k_1, \ldots, k_r \geq 0$, $l_1, \ldots, l_s \geq 0$ (see [Dn], table 9). It is assumed here that $A_0 = B_0 = D_1 = \emptyset$

is the empty root system. Set $n_i = 2l_i$ if the corresponding summand is C_{l_i} or D_{l_i} and $n_i = 2l_i + 1$ if it is B_{l_i}.

Then the corresponding groups $E(\Delta, R)$ are isomorphic to one of the following groups: to the group

$$\mathrm{E}(k_1 + 1, R) \oplus \ldots \oplus \mathrm{E}(k_r + 1, R)$$

in the case $\Phi = \mathrm{A}_l$; to the group

$$\mathrm{E}(k_1 + 1, R) \oplus \ldots \oplus \mathrm{E}(k_r + 1, R) \oplus \mathrm{Epin}(n_1, R) \oplus \ldots \oplus \mathrm{Epin}(n_s, R)$$

in the cases $\Phi = \mathrm{B}_l$ and $\Phi = \mathrm{D}_l$; and, finally, to the group

$$\mathrm{E}(k_1 + 1, R) \oplus \ldots \oplus \mathrm{E}(k_r + 1, R) \oplus \mathrm{Ep}(2l_1, R) \oplus \ldots \oplus \mathrm{Ep}(2l_s, R)$$

in the case $\Phi = \mathrm{C}_l$. Here $\mathrm{E}(n, R) = \mathrm{E}(\mathrm{A}_{n-1}, R)$ is the elementary subgroup of $\mathrm{SL}(n, R)$ generated by all elementary transvections (see [HO]), while $\mathrm{Epin}(2l + 1, R) = \mathrm{E}(B_l, R)$, $\mathrm{Epin}(2l, R) = \mathrm{E}(\mathrm{D}_l, R)$, and $\mathrm{Ep}(2l, R) = \mathrm{E}(\mathrm{C}_l, R)$ are analogous subgroups in $\mathrm{Spin}(n, R)$ and $\mathrm{Sp}(2l, R)$ respectively. For the orthogonal group $\mathrm{SO}(n, R)$ the elementary subgroups $\mathrm{Epin}(n, R)$ should be replaced by $\mathrm{EO}(n, R)$.

Now we are ready to state the results about overgroups of subsystem subgroups in the classical groups. Assume that $\Phi = \mathrm{A}_l, \mathrm{B}_l, \mathrm{C}_l$ or D_l and that R is a commutative ring, such that $2 \in R^*$ if Φ has roots of different length. Assume further that Δ is a subsystem in Φ such that $k_1, \ldots, k_r, l_1, \ldots, l_s$ of its irreducible summands are at least 2. In the orthogonal case assume moreover that all $k_1, \ldots, k_r$ are at least 4.

Theorem. *For any subgroup H in $G(\Phi, R)$ containing $E(\Delta, R)$ there exists a unique net σ of ideals in R such that*

$$E(\sigma) \leq H \leq N(\sigma).$$

In particular, for fields this theorem says that there exists a unique closed set of roots $S \subseteq \Phi$ such that $E(S) \leq H \leq N(S)$. Of course $E(\Delta, K)$ contains root subgroups and thus for fields this theorem may be deduced from the results of J.Mclaughlin, B.S.Stark, W.Kantor, Li Shang Zhi and F.Timmesfeld describing irreducible subgroups of classical groups generated by root subgroups (see [Ka1], [Ka2], [Ti1] – [Ti4] and references there). However a much easier proof than that works for all commutative rings. We sketch this proof for the case $\Phi = \mathrm{A}_l$ in the next section.

Now we discuss the bound on k_i, l_j in the theorem above, limiting ourselves to the case $\Phi = \mathrm{A}_l$ (analogous remarks apply to other series as well).

Observe that the theorem is not true any more if there are at least two summands of rank ≤ 1. Clearly, the case when we have at least two summands of rank 0 is intractable, so we assume that $k_i \geq 1$ for all i. Consider, for example the subsystem $\Delta = 2\mathrm{A}_1$ in $\Phi = \mathrm{A}_3$. Then clearly $E(2,K) \oplus E(2,K)$ is contained in $\mathrm{Sp}(4,K)$ which does not have the form predicted by the theorem.

For fields $\mathrm{Sp}(2l,K)$ is the only primitive irreducible counterexample *up to conjugacy*, but it is quite an exercise to explicitly list all the possibilities for *reducible* subgroups containing $E(\Delta,K)$ (see [Ko5]). Even worse, one immediately discovers that description of intermediate subgroups in our sense and their description up to conjugacy is not quite the same thing. For example, there exist infinite parametric families of intermediate subgroups conjugate to $\mathrm{Ep}(2l,K)$ (unlike the case of $k_i \geq 2$, when there are only finitely many $E(\Delta,K)$-complete intermediate subgroups).

On the other hand for *rings* the case when the irreducible summands of Δ have rank 1 is essentially untractable. In fact the theorem above would imply a description of subgroups in $G(\Delta,R)$ normalized by $E(\Delta,R)$ (compare [Go], [Sv]). As is well known, there is no hope to describe the normal structure of the groups $\mathrm{GL}(2,R)$ or $\mathrm{SL}(2,R)$ over general commutative rings (compare [W]). Even for such *very* nice rings as semilocal ones, or Hasse domains with infinite multiplicative groups, a complete description of normal subgroups was obtained only quite recently and involves much more complicated structures, than ideals [CK].

The 7-dimensional representation of the Chevalley group of type G_2 and the 8-dimensional representation of the Chevalley group of type B_3 are responsible for the stronger restrictions on the ranks of the irreducible summands of Δ in the orthogonal case. In fact, the embeddings $\mathrm{A}_2 \subset \mathrm{G}_2$ and $\mathrm{A}_3 \subset \mathrm{B}_3$ furnish counter-examples to the conclusion of the theorem above with the ranks of irreducible summands equal to two or to three.

There is no doubt that results analogous to the theorem above hold also for large subsystem subgroups of exceptional groups and that all the necessary tools are contained in [V19].

§ 13. The group GL_n over commutative rings.

Here, in the example of the general linear group $\mathrm{GL}(n,R)$, we explain the correct definition of $G(\sigma)$ and $N(\sigma)$ (see [B1] – [B5], [BV1] – [BV3]) and show how the proof of the preceding theorem works. The proof we reproduce below is a simplified version of the proof from [BV3].

Let Λ be an arbitrary associative ring with 1. A square array $\sigma = (\sigma_{ij})$, $1 \leq i,j \leq n$, of two-sided ideals σ_{ij} in Λ is called a net of ideals in Λ of degree n if $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ for all values of the indices i,j,r. A net σ is called a D-net if $\sigma_{ii} = \Lambda$ for all i. Clearly D-nets of degree n are precisely the nets of type A_{n-1} defined in § 11.

As in § 11 we can associate with σ the elementary net subgroup $E(\sigma)$ generated by all elementary transvections $t_{ij}(\xi)=e+\xi e_{ij}$, where $1\le i\neq j\le n$, $\xi\in\sigma_{ij}$.

For a given net σ we denote by $M(\sigma)$ the corresponding subring in the full matrix ring $\mathrm{M}(n,\Lambda)$, consisting of all matrices $x=(x_{ij})$ congruent to 0 modulo σ:

$$M(\sigma)=\big\{x=(x_{ij})\in\mathrm{M}(n,\Lambda)\mid\ x_{ij}\in\sigma_{ij},\ 1\le i,j\le n\big\}.$$

Then $e+M(\sigma)$ is a multiplicative system. The largest subgroup $G(\sigma)$ of the general linear group $G=\mathrm{GL}(n,\Lambda)$, contained in $e+M(\sigma)$ is called the net subgroup, corresponding to σ and is denoted by $G(\sigma)$. For a commutative ring R one has $G(\sigma)=G\cap\big(e+M(\sigma)\big)$.

As before $N(\sigma)$ is the normalizer of $G(\sigma)$ in G. In many important situations, in particular for the nets appearing in the theorem above, $N(\sigma)$ consists of the matrices $x\in G$ such that $x_{ir}\sigma_{rs}x'_{sj}\subseteq\sigma_{ij}$ for all i,j,r,s.

Let ν be an equivalence relation on the index set $\{1,\dots,n\}$. We can associate with ν a D-net $[\nu]$ by setting $[\nu]_{ij}=\Lambda$ if $i\overset{\nu}{\sim}j$ and $[\nu]_{ij}=0$ otherwise. Then the corresponding group $E(\nu)=E([\nu])$ is exactly one of the elementary subsystem subgroups $E(\Delta)$ considered in the preceding section. The condition on ranks of irreducible summands of Δ amounts to saying that all equivalence classes of ν contain at least *three* elements.

Now assume that R is commutative and $H\ge E(\nu)$. For $i\neq j$ denote by σ_{ij} the set of all $\xi\in R$ such that $t_{ij}(\xi)\in H$. Set, moreover, $\sigma_{ii}=R$. Clearly σ is the largest D-net of ideals such that $E(\sigma)\le H$. We have to prove that $H\le N(\sigma)$. This amounts to verifying the conditions $x_{ir}\sigma_{rs}x'_{sj}\subseteq\sigma_{ij}$ for all $x\in H$ and all i,j,r,s, $r\neq s$, $i\not\sim j$.

Take any pair of indices $r\neq s$ and any $\xi\in\sigma_{rs}$ and consider the following transvection $y=xt_{rs}(\xi)x^{-1}\in H$. We have to prove that all these transvections belong to $E(\sigma)$. Fix an index j and take two indices p,q equivalent to j and such that j,p,q are pairwise distinct. Consider the matrix

$$z=yt_{hp}(a_{qr})t_{hq}(-a_{pr})y^{-1}\in H.$$

Explicit calculations using the commutativity of R show that $z_{ih}=' \delta_{ij}$ for all $h\neq p,q$, while $z_{ip}=\delta_{ip}+b_{ij}a_{qr}$ and $z_{iq}=\delta_{iq}+b_{ij}a_{pr}$. Another commutation with matrices from $E(\nu)$ shows that $y_{ij}x_{pr}\in\sigma_{ij}$ for all $p\sim j$, $p\neq j$.

These inclusions hold for *all* transvections of the form $y=xt_{rs}(\xi)x^{-1}$, $x\in H$. But this then readily implies that all these transvections belong to $E(\sigma)$. Indeed consider another transvection $u=yt_{jp}(1)y^{-1}\in H$. The inclusions above applied to u instead of y show that $u_{ip}y_{jj}\in\sigma_{ip}=\sigma_{ij}$. But

$u_{ip} = y_{ip}y'_{pp}$, where $y'_{pp} = 1 - x_{pr}\xi x'_{sp}$, so that $u_{ip} \equiv y_{ij} (\mod \sigma_{ij})$. Thus $y_{ij}y_{jj} \in \sigma_{ij}$. Applying this last inclusion to u we get $u_{ip}u_{pp} \in \sigma_{ip} = \sigma_{ij}$. But $u_{ip}y_{pp} \equiv y_{ij} (\mod \sigma_{ij})$. Thus, finally, $y_{ij} \in \sigma_{ij}$ for all i, j.

Now to prove polynormality of $E(\nu)$ one has to prove that $E(\sigma)$ is normal in $N(\sigma)$ and that all conjugates of $E(\nu)$ contained in $N(\sigma)$ in fact lie already in $E(\sigma)$. These are relatively deep facts. For example, they contain Suslin's normality theorem as a very special case, see [BV3] for details.

Essentially the same proofs work for other classical cases as well (see [V10], [V15], [V16] for details and further references). In fact this result is a very wide generalization of the description of normal subgroups in classical groups over commutative rings (see references in [V19]).

§ 14. Paranormal and polynormal subgroups

We have seen in § 12 that for the subsystem subgroups the lattice of intermediate subgroups admits the same nice description as for the split maximal tori. However the subsystem subgroups are not pronormal in general, simply because there normalizer might be too small.

Take for example the subgroup

$$D = \mathrm{E}(m, K) \oplus \mathrm{E}(m, K) \oplus \mathrm{E}(2m, K) \leq \mathrm{GL}(4m, K).$$

Then clearly one cannot present all the matrices from the normalizer of $H = \mathrm{E}(2m, K) \oplus \mathrm{E}(2m, K) \geq D$ in the form hw, where $h \in H$ and $w \in N_G(D)$.

It turns out though, that the subsystem subgroups satisfy a weaker property, which guarantees the standard description of the intermediate subgroups. We introduce some notation first. Let $D \leq G$ and denote by

$$\Omega_G(D) = \{xDx^{-1},\ x \in G\}$$

the set of all conjugates of D in G.

Definition. *A subgroup F, $D \leq F \leq G$, is called D-***complete** *if it is generated by subgroups which are conjugate to D in F*

$$F = D^F = \langle X \in \Omega_F(D) \rangle.$$

The following very important notion was introduced by Z.I.Borewicz and O.N.Macedońska in [BM].

Definition. *A subgroup D of G is called* **paranormal** *if for any $x \in G$ one has*

$$D^{\langle D, xDx^{-1}\rangle} = \langle D, xDx^{-1} \rangle.$$

In other words, a subgroup D is paranormal if for any $x \in G$ the subgroup $H = \langle D, xDx^{-1} \rangle$ is D-complete. Clearly it follows that *all* subgroups generated by some conjugates of D in G are D-complete. It is obvious that a pronormal subgroup is paranormal, but as we have seen the converse is not true.

Theorem. *Under the assumptions of the above theorems the elementary subsystem subgroup $E(\Delta, R)$ is paranormal in $G(\Phi, K)$.*

Other remarkable examples of paranormal subgroups, which are not, generally speaking, pronormal, are

– maximal tori in *finite* Chevalley groups, [S6];

– diagonal subgroups (*split* maximal tori) in *classical* groups over semilocal rings, [BV1], [V3], [VD], [V14].

Paranormal subgroups exhibit the same behaviour of intermediate subgroups as pronormal ones, apart from normalizers and conjugacy. Indeed, for any paranormal subgroup D in G the lattice of intermediate subgroups is a disjoint union

$$\mathcal{L}(D, G) = \bigcup^{\emptyset} \mathcal{L}(F, N_G(F)), \qquad F = \langle X \in \Omega_G(D),\ X \leq F \rangle.$$

taken over all subgroups F generated by some conjugates of D in G. As in the case of pronormal subgroups the normal closure D^H of D in any intermediate subgroup $H \in \mathcal{L}(D, G)$ coincides with one of the groups F above, while the normalizer $N_G(H)$ is contained in $N_G(D^H)$.

In the paper [BM] another, a still wider, generalization of pronormal subgroups is introduced.

Definition. *A subgroup D of G is called* **polynormal** *if for any subgroup $H \leq G$ one has*

$$D^{D^H} = D^H.$$

Immediately from the definition it follows that for any polynormal subgroup D in G the lattice of intermediate subgroups is a disjoint union

$$\mathcal{L}(D, G) = \bigcup^{\emptyset} \mathcal{L}(F, N_G(F)), \qquad F = D^F,$$

taken over all D-complete subgroups F. The sandwiches for the polynormal subgroups are a priori *larger*, than for the paranormal ones. Moreover it is possible that one subgroup, generated by some conjugates of D lies between another such subgroup and the normalizer of the latter in G. This is counterintuitive and cannot arise for nice subgroups, like, say, groups of points of *connected* algebraic groups over *large* fields. This is why the author feels that this further generalization is less useful than the notion of a paranormal subgroup.

To the contrary paranormal subgroups arise quite naturally from the pronormal ones. Namely, if T is a pronormal subgroup in G, then a T-complete subgroup D may have small normalizer and is not necessarily pronormal in G, while on the other hand it is *always* paranormal.

Once more to describe overgroups of a paranormal or a polynormal subgroup D, one has to classify the D-complete subgroups F and to calculate the factor-groups $N_G(F)/F$. However, unlike for the case of pronormal subgroups, the calculation of these factors is in no way reduced to the calculation of N/D. This is precisely why G.Seitz has to change his torus when calculating the normalizer of an intermediate subgroup [S6].

§ 15. Field extension subgroups

In this section we start to discuss the groups related to the Aschbacher class $\mathcal{C}_3$. Let L be an extension of the ground field K of degree r. Then a vector space V of dimension m over L may be considered as a vector space of dimension $n = mr$ over K. Every L-linear endomorphism of the space is K-linear. Picking up a K-base in V we get an embedding $\mathrm{GL}(m, L) < \mathrm{GL}(n, K)$ (compare the next section for a more detailed description of the case $m = 1$).

One can define analogous embeddings for other groups of Lie type, sometimes there are several different patterns. For example, $G = \mathrm{Sp}(2n, K)$ has not only subgroups $\mathrm{Sp}(2m, L)$, where as above L/K is an extension of degree $r = n/m$, but also subgroups $\mathrm{U}(n, L)$, where L is a quadratic extension of K. For the *finite* classical groups one can find complete lists of all such embeddings in [A1], [KL2]. However the main goal of these works is the search for maximal subgroups. This is why they concentrate on the case when L/K is an extension of *prime* degree.

In this section we will be interested in the case when $m \geq 2$, i.e. when $\mathrm{GL}(m, L)$ contains unipotent elements. The image of a transvection from $\mathrm{GL}(m, L)$ is a *quadratic* unipotent element. The case $m = 1$ is *very* different and will be discussed in the next three sections.

We are interested in subgroups of $\mathrm{GL}(n, K)$ containing $\mathrm{SL}(m, L)$. For $m \geq 3$ the answer is quite uniform and may be stated as follows.

Theorem. *Let L/K be any field extension of degree r, $m \geq 3$, $n = mr$. Then for any subgroup $\mathrm{SL}(m, L) \leq H \leq G = \mathrm{GL}(n, K)$ there exists a unique intermediate subfield $K \leq E \leq L$, $[L : E] = d$ such that*

$$\mathrm{SL}(md, E) \leq H \leq N_G(\mathrm{SL}(md, E)).$$

This is a corollary from a result of Li Shang Zhi [L4]. In the case $m = 2$, $|K| \neq \mathbb{F}_2$, the description is similar, but there is another series of complete intermediate subgroups, namely the groups $\mathrm{Sp}(2d, E)$ for all intermediate subfields $K \leq E \leq L$, $[L : E] = d$. In the exceptional case of the $\mathrm{Sp}(4, \mathbb{F}_2)$ two further examples arise.

Li Shang Zhi has proven analogous results for overgroups of $\mathrm{Sp}(m, L)$ in $\mathrm{GL}(n, K)$. Nothing changes as compared with the case of overgroups of

$\mathrm{SL}(m, L)$, except, of course, that now the groups $\mathrm{Sp}(md, E)$ always appear as complete intermediate subgroups, and not just for $m = 2$. He also announced analogous results for overgroups of $\Omega(m, L)$.

For the *finite* case analogous results were obtained independently by R.W.Dye, see [D1] – [D3] and references there. Basically he was interested only in the extensions of prime degree. On the other hand R.W.Dye considered also the case of tori, which Li Shang Zhi had to exclude for infinite fields (by a very good reason, as we shall see in the following sections). In the language of these works the field extension subgroups are stabilizers of spreads. The method of proof is also geometric.

It seems plausible that using ideas analogous to those described in § 11 and § 13 one can obtain analogous results for commutative rings. For the fields of char $\neq 2$ the answer follows also from the theory of quadratic pairs and its infinite analogue developed by F.Timmesfeld.

§ 16. Maximal tori

In this section we describe how maximal tori look like for the case of the general linear group and briefly outline the current situation concerning description of their overgroups. In principle what we call a maximal torus is the group of K-points of a maximal torus in the corresponding algebraic group.

Maximal tori are extremely important subgroups, especially in the finite case. They play a crucial role both in the structure theory (since they control conjugacy classes of semisimple elements) and in the representation theory. One might find detailed information concerning tori in the finite case in [SS], [C2], [C3], [S5], [S6], [Kn].

The situation over infinite fields is more complicated. Our description below is utterly naive, for example we do not worry about separability. One can find a complete cohomological classification of maximal tori in the classical groups over *infinite* fields in the paper [Km] by K.Kariyama.

First consider the case $n = 1$ in the construction of a field extension subgroup from the preceding section. In other words we consider inclusion

$$L^* \leq \mathrm{GL}(n, K), \qquad [L : K] = n.$$

In fact, let $\omega_1, \dots, \omega_n$ be a base of L over K. Then for any $\alpha \in L$. One has

$$\alpha\omega_j = \sum_{i=1}^{n} \alpha_{ij}\omega_i, \qquad \alpha_{ij} \in K, \quad 1 \leq j \leq n.$$

Now the above embedding is described by the correspondence $\alpha \mapsto (\alpha_{ij})$.

Problem. *Describe overgroups of* L^* *in* $\mathrm{GL}(n, K)$.

As we shall soon see this problem is *extremely* difficult in general, and provides examples for description of subgroups which is much more complicated than anything we encountered so far. However to make it still more difficult we may combine it with the Aschbacher class $\mathcal{C}_1$.

Namely, consider a partition $n = n_1 + \ldots + n_t$. Let $L_1, \ldots, L_t$ be extensions of K of degrees $n_1, \ldots, n_t$ respectively. Then one can consider

$$T = L_1^* \oplus \ldots \oplus L_t^* \leq \mathrm{GL}(n, K).$$

Such a subgroup T is a maximal torus in $\mathrm{GL}(n, K)$.

Problem. *Describe overgroups of* $L_1^* \oplus \ldots \oplus L_t^*$ *in* $\mathrm{GL}(n, K)$.

When $t = n$ all the fields L_i coincide with K and we get precisely the problem considered in § 6.

Roughly speaking the situation with classification of overgroups of maximal tori is as follows. Here G is a simple K-defined algebraic group over a field K and T is an arbitrary K-defined maximal torus. We consider the lattice $\mathcal{L}(T_K, G_K)$ of subgroups lying between the groups of K points of T and G. For algebraically closed fields all tori are split and the results described in § 6 apply. The situation for the finite case will be outlined in the next section.

For the field $\mathbb{R}$ of real numbers all subgroups containing a maximal torus are closed in the real topology [Dj], [P]. In particular, there are only finitely many intermediate subgroups and they are easily classified using the corresponding results for Lie algebras.

For some special tori over local fields the problem was considered by S.L.Krupetskii, see [Kr1] – [Kr3] and references in [V17]. The qualitative answer for the general case was obtained by V.P.Platonov [P]. Let K be a local field with valuation v. Then any intermediate subgroup H is an open (in v-adic topology) subgroup in the group $(\overline{H})_K$ of K-points of its closure (in Zariski topology). This reduces the explicit classification to Bruhat-Tits theory. In particular there are not more than countably many intermediate subgroups.

For a global field K only the case of a *cyclic* maximal torus in one of the groups GL_n or SL_n has been considered so far by V.A.Koibaev and V.P.Platonov. In particular, Chebotarev's density theorem implies that there are *uncountably* many intermediate subgroups.

For arbitrary fields only the case of non-split tori in GL_2 has been considered by V.A.Koibaev, Z.I.Borewicz and others, see § 18 below for a description of this result.

§ 17. Overgroups of maximal tori: finite case

For a finite ground field there is an almost exhaustive description of subgroups in Chevalley groups which contain a maximal torus. This remarkable result is due to G.M.Seitz [S6]. The standing assumption in this section is that $K = \mathbb{F}_q$ be a finite field, $\operatorname{char} K \neq 2, 3$, $q \geq 13$. Seitz' results apply to all finite groups of Lie type, but for simplicity we state them only for the groups of normal types.

Let $\overline{G} = G(\Phi, \overline{K})$ be a Chevalley group of type Φ over the algebraic closure $\overline{K}$ of a finite field $K = \mathbb{F}_q$. Recall that the standard Frobenius endomorphism $\sigma = \sigma_q$ of $\mathrm{GL}(n, \overline{K})$ is obtained by raising the matrix entries to the q-th power, $\sigma : (x_{ij}) \mapsto (x_{ij}^q)$. A standard Frobenius endomorphism of $\overline{G}$ is the mapping $\sigma : \overline{G} \to \overline{G}$ induced by σ_q under some embedding $i : \overline{G} \to \mathrm{GL}(n, \overline{K})$, see [SS], [C3].

A maximal torus T in a finite Chevalley group $G = G(\Phi, K)$ has the form $T = \overline{T} \cap G$, where $\overline{T}$ is a σ-*invariant* maximal torus. Conjugacy classes of maximal tori correspond to the conjugacy classes of the Weyl group $W(\Phi)$, see [C2], [C3], [SS]. Let further $\overline{X}_\alpha$, $\alpha \in \Phi$, be the corresponding root subgroups. Since $\overline{T}$ is σ-invariant, these subgroups are permuted by σ.

Let $\Delta = \{\overline{X}_1, \dots, \overline{X}_t\}$ be a σ-orbit of root subgroups. We consider the span $\overline{X} = \overline{X}(\Delta) = \langle \overline{X}_1, \dots, \overline{X}_t \rangle$. The groups $X = O^{p'}(\overline{X}^\sigma)$ are called T-root subgroups by G.M.Seitz. They play the same role in the description of overgroups of T as the usual root subgroup do in the description of overgroups of a split torus.

Every T-root subgroup is either unipotent, or a group of Lie type over a certain extension of the ground field. If T is split, then all the T-root subgroups are unipotent, when T is minisotropic, all of them are semisimple. In general there are T-root subgroups of both types.

Now let S be a set of T-root subgroups which is closed in the following sense: if X, Y, Z are three T-root subgroups such that $X, Y \in S$ and $Z \leq \langle X, Y \rangle$, then $Z \in S$. Let $G(T, S)$ be the group generated by T and all $X \in S$. Let further $N(T, S)$ be the normalizer of $G(T, S)$ in G. Then the result of G.M.Seitz may be stated as follows.

Theorem. *Let T be an arbitrary maximal torus in $G = G(\Phi, K)$. Then for any subgroup H of G that contains T there exists a unique closed set S of T-root subgroups such that*

$$G(T, S) \leq H \leq N(T, S).$$

Actually Seitz has shown that the group $G(T, S)$ appearing in the theorem is is exactly the subgroup T^H generated by conjugates of T contained in H. This, plus standard arguments about the normalizers $N(T, S)$, amounts to the following result.

Theorem. *T is a paranormal subgroup of G.*

Another important result of [S6] asserts that the notion of a T-complete subgroup does not depend on the choice of a maximal torus T.

Theorem. *If H contains two maximal tori T_1 and T_2, then ${T_1}^H = {T_2}^H$.*

The proof in [S6] uses the Classification. In [S5], [Kn] one can find an explicit description of the maximal tori and T-root subgroups for the finite cases. Independently of the work of G.M.Seitz only the overgroups of Singer cycles in the classical groups (and their large subgroups) have been studied, see [Ka2], [He], [D1] – [D3], [De], [Wa].

§ 18. Overgroups of maximal tori: GL_2

In this section we describe the work of V.A. Koibaev, Z.I.Borewicz and others on the subgroups of the group $G = \mathrm{GL}(2, K)$ over an arbitrary field K which contain a maximal torus, see [Ko7], [BKH], [BK2], [Ko8]. The contents of this section show that for infinite fields the structure of the lattice of intermediate subgroups may be very different from the finite patterns.

Assume that K is an *infinite* field such that $\operatorname{char} K \neq 2$ and L/K is a quadratic extension of K. We may assume that $L = K(\sqrt{d})$, $d \in K \setminus K^2$. Then the corresponding maximal torus in G has the form

$$T = T(d) = \left\{ \begin{pmatrix} x & yd \\ y & x \end{pmatrix}, \quad x, y \in K,\ (x, y) \neq (0, 0) \right\}.$$

We are interested in the lattice $\mathcal{L}(T, G)$. For a split maximal torus we were practically done once we could prove that our intermediate subgroup H contains an elementary transvection. Here the situation is strictly opposite: at this point the real work just starts.

Theorem. *Let $T \leq H \leq G$. Then either $H \leq N_G(T)$, or H contains an elementary transvection.*

We can associate with any intermediate subgroup H an additive subgroup $A \varepsilon K^+$ setting:

$$A = A(H) = \left\{ \alpha \in K \mid \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \in H \right\}.$$

This subgroup is called the *module of transvections* of the subgroup H. Further we consider the corresponding *ring of multipliers*

$$R = R(H) = \{ \lambda \in K,\ \lambda A \subseteq A \}.$$

It is clear that R is a ring and A is an R-module.

The first natural question is which R and A do arise? It turns out that there is some absolute ring of multipliers R_0 and a subring $R \leq K$ has the form $R(H)$ if and only if it contains R_0. This R_0 may be defined as follows: it is the subring in K generated by all elements of the form

$$\frac{d}{x^2 - d}, \qquad x \in K.$$

For example, for $K = \mathbb{Q}$ this R_0 has the form $R_0 = {P_0}^{-1}\mathbb{Z}$, where $P_0 = P$,

$$P = \left\{ p \text{ odd prime },\ \left(\frac{d}{p}\right) = 1 \right\},$$

if $d \not\equiv 1(\mod 8)$ and $P_0 = P \cup \{2\}$ if $d \equiv 1(\mod 8)$.

At this point the following technical condition is imposed on the torus T: we assume that $2 \in {R_0}^*$. Under this condition the lattice of intermediate subgroups is completely described in [Ko8] (see also [BK2] for some preliminary results and [Ko7], [BKH] for a complete analysis of the case $K = \mathbb{Q}$). We reproduce this description below. Without this condition the answer is much more complicated.

Theorem. *For a subring $R \subseteq K$ and an R-submodule $A \subseteq K$ to have the form $R = R(H)$, $A = A(H)$ for some intermediate subgroup H, $T \leq H \leq G$, it is necessary and sufficient that $R \supseteq R_0$ and $dA^2 \subseteq R$.*

For the field $K = \mathbb{Q}$ this result holds also without assumption $2 \in R_0$. Thus, in particular, for *any* d there are $\aleph_1$ intermediate subgroups (as compared to only *five* of them for the case of a split maximal torus). Clearly, only *three* of these subgroups (the torus T itself, its normalizer and the whole group G) are algebraic, all others are Zariski dense in G.

The next question is to describe all intermediate subgroups with given R and A. Denote this sublattice of $\mathcal{L}(T,G)$ by $\mathcal{L}_{R,A} = \mathcal{L}_{R,A}(T,G)$ It turns out that there exists a unique smallest subgroup and a unique largest subgroup in $\mathcal{L}_{R,A}$.

Theorem. *The smallest subgroup in $\mathcal{L}_{R,A}$ is the group*

$$F = \left\langle T,\ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},\ \alpha \in A \right\rangle = T \begin{pmatrix} 1 & 0 \\ A & \Omega_0(A) \end{pmatrix},$$

where the subgroup $\Omega_0(A) \leq R^*$ is defined as follows:

$$\Omega_0(A) = \left\langle \frac{(d\alpha + x)^2 - d}{x^2 - d},\ x \in K,\ \alpha \in A \right\rangle.$$

Now consider the ideal Q in R generated by the elements

$$\frac{dx}{x^2-d}, \qquad x \in K,$$

and the set

$$S = \{s \in R,\ Qs \subseteq dA\}.$$

Clearly S is an ideal in R such that $QA \subseteq S \subseteq A$.

Theorem. *The largest subgroup in $\mathcal{L}_{R,A}$ is the group*

$$F^0 = T\begin{pmatrix} 1 & 0 \\ A & \Omega^0(A) \end{pmatrix},$$

where the subgroup $\Omega^0(A) \leq R^*$ is defined as follows:

$$\Omega^0(A) = \left\{\theta \in R^*,\ \theta^2 - 1 \in S\right\}.$$

Now we can explicitly describe the lattice $\mathcal{L}_{R,A}$.

Theorem. *Every subgroup H from $\mathcal{L}_{R,A}$ has the form*

$$H = T\begin{pmatrix} 1 & 0 \\ A & \Delta \end{pmatrix},$$

where Δ is a subgroup of R^* such that $\Omega_0(A) \leq \Delta \leq \Omega^0(A)$ and for all $x \in K$ and $\delta \in \Delta$ one has

$$1 + \frac{1}{\theta^2}\frac{d}{x^2-d}(\theta^2 - 1) \in \Delta.$$

The final theorem of this section shows that the usual operations of passing to the normalizer and to the normal closure of T inside an intermediate subgroup leave $\mathcal{L}_{R,A}$ invariant.

Theorem. *For any $H \in \mathcal{L}_{R,A}$ one has $N_G(H), T^H \in \mathcal{L}_{R,A}$.*

Of course this pattern in the description of intermediate subgroups is more complicated, than anything we have encountered so far, since passage from H to T^H may generate *infinite* descending chains. The paper [BKH] and unpublished works by V.A.Koibaev show that still more complicated patterns arise when $2 \notin R_0^*$ and for cyclic tori in the general linear groups $\mathrm{GL}(n, K)$.

§ 19. Overgroups of tensored subgroups

Let K be a field and $U_1, \dots, U_t$ be vector-spaces over K of dimensions $n_1, \dots, n_t$ respectively. Then their tensor product $V = U_1 \otimes \dots \otimes U_t$ has dimension $n = n_1 \dots n_t$. Thus one gets a natural embedding

$$D = \mathrm{GL}(n_1, K) \otimes \dots \otimes \mathrm{GL}(n_t, K) \leq G = \mathrm{GL}(n, K).$$

These groups are obtained by fusing the Aschbacher classes $\mathcal{C}_4$ and $\mathcal{C}_7$. We are interested in the lattice $\mathcal{L}(D, G)$.

Of course once more we are not concerned about their maximality, we just want them to be large enough. If one wants to get maximal subgroups one must either assume that $t = 2$ and $n_1 \neq n_2$ (this is the class $\mathcal{C}_4$), or else that $n_1 = \dots = n_t = m$ and take the normalizer of D (this is the class $\mathcal{C}_7$). This normalizer is denoted by $\mathrm{GL}(m, K) \uparrow S_t$. As an abstract group it is isomorphic to the wreath product $\mathrm{GL}(n, K) \wr S_t$, but in a primitive representation.

One can construct analogous subgroups also for other groups of Lie type. Sometimes they fall into different families. For example, both $\mathrm{Sp}(l, K) \otimes \mathrm{Sp}(m, K)$ and $\mathrm{SO}(l, K) \otimes \mathrm{SO}(m, K)$ lie in the orthogonal group $\mathrm{O}(n, K)$, $n = lm$. For the finite case all the details about such embeddings may be found in [KL2].

Again the group D contains quadratic unipotent elements, so that for a finite field of characteristic $\neq 2$ description of *irreducible* subgroups containing D *up to conjugacy* may be deduced from the theory of quadratic pairs.

We restrict ourselves to a special case of a result by Li Shang Zhi [L9].

Theorem. *Let $D = \mathrm{SL}(l, K) \otimes \mathrm{SL}(m, K)$, where $l, m \geq 3$. Let further, $G = \mathrm{GL}(n, K)$, $n = lm$. Then any subgroup H, $D \leq H \leq G$ either normalizes D or* $\mathrm{SL}(n, K)$.

When $l = 2$ or $m = 2$ there are additional examples of complete intermediate subgroups. Li Shang Zhi has obtained also analogous results for overgroups of $\mathrm{Sp}(l, K) \otimes \mathrm{Sp}(m, K)$, $l, m \geq 4$, in $\mathrm{GL}(n, K)$, $n = lm$, and announced similar results for other classical groups.

There is no doubt that using ideas similar to those described in § 11 and § 13 one can obtain analogous results for an arbitrary commutative ring R and arbitrary number of factors t under assumption $n_1, \dots, n_t \geq 3$. The answer involves ideals of R and is similar to, but easier than, that for the case of subsystem subgroups. As already mentioned, for the fields of char $\neq 2$ this answer should follow also from the theory of quadratic pairs and its infinite analogue developed by F.Timmesfeld.

§ 20. Overgroups of classical subfield subgroups

In this section we state results about the overgroups of the classical subgroups over subfields. These results touch Aschbacher classes $\mathcal{C}_5$ and $\mathcal{C}_8$. In the form we reproduce them below they are due to E.L.Bashkirov [Ba1] – [Ba7], but of course many special cases were known before.

Theorem. *Let K be an algebraic extension of a subfield L. Assume that either $n \geq 3$ or $n = 2$ and $\operatorname{char} K \neq 2$. Then for any subgroup H, such that $\mathrm{SL}(n, L) \leq H \leq G = \mathrm{GL}(n, K)$, there exists a unique intermediate subfield E, $L \leq E \leq K$, such that*

$$\mathrm{SL}(n, E) \leq H \leq N_G(\mathrm{SL}(n, E)).$$

The proof of this result in based on the following remarkable fact which is a generalization of Dickson's lemma. Recall that any subgroup in $\mathrm{GL}(n, K)$ conjugate to $\{t_{12}(\alpha),\ \alpha \in L\}$ is called an L-root subgroup. The following theorem essentially answers the question what can be generated by two L-root subgroups.

Theorem. *Let L be an infinite field and λ be an element algebraic over L. If $\operatorname{char} L = 2$ suppose that λ is separable over L. Then*

$$\left\langle \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda L & 1 \end{pmatrix} \right\rangle = \mathrm{SL}(2, L(\lambda)).$$

When λ is *transcendental* over L a theorem of J.Tits asserts that the group on the left hand side is isomorphic to the free product of two copies of L^+ (see [T3]). This lemma was first proven by E.L.Bashkirov by very elaborate explicit calculations [Ba1], [Ba4]. Later V.I.Chernousov found a much shorter algebro-geometric proof. Unfortunately this proof is not published, see [Z5] for an exposition.

Analogous results hold also for overgroups of other classical groups over subfields. As a pattern we state the following results from [Ba3] and [Ba7].

Theorem. *Let K be an algebraic extension of a subfield L. Assume that either $n = 2l \geq 4$ and $\operatorname{char} K \neq 2$. Then for any subgroup H, $\mathrm{Sp}(n, L) \leq H \leq G = \mathrm{GL}(n, K)$, there exists a unique intermediate subfield E, $L \leq E \leq K$, such that one of the following holds*

(1) $\mathrm{SL}(n, E) \leq H \leq N_G(\mathrm{SL}(n, E))$;

(2) $\mathrm{Sp}(n, E) \leq H \leq N_G(\mathrm{Sp}(n, E))$;

(3) $\mathrm{SU}(n, E) \leq H \leq N_G(\mathrm{SU}(n, E))$.

This theorem essentially contains description of irreducible subgroups in $\mathrm{GL}(n,K)$ generated by L-root subgroups. For $L \ncong \mathbb{F}_3, \mathbb{F}_9$ any such subgroup is conjugate to one of the following groups: $\mathrm{SL}(n,E)$, $\mathrm{SU}(n,E)$, $\mathrm{Sp}(n,E)$. Analogous result holds also for the case when $\operatorname{char} K = 2$, but it is much more delicate, since in the non-separable case it involves subgroups defined by *pairs* of intermediate subfields, see [Ba6].

Theorem. *Let K be an algebraic extension of a subfield L. Assume that either $n = 2l \geq 4$ and $\operatorname{char} K \neq 2$. Then for any subgroup H, $\Omega(n,L) \leq H \leq G = \mathrm{GL}(n,K)$, one of the following holds:*

(1) H contains an L-root subgroup;

(2) there exists a unique intermediate subfield E, $L \leq E \leq K$, such that

$$\Omega(n,E) \leq H \leq N_G(\Omega(n,E)).$$

These results generalize a number of preceding results, including in particular results of R.W.Dye, O.H.King and Li Shang Zhi (see references in [KL], [L8], [L10], [L11]). Compare also [B5], [Nu].

In fact one can consider a more general problem. Let Φ be an arbitrary root system and $R \leq S$ be an arbitrary embedding of rings with 1. Describe subgroups intermediate between $G(\Phi,R)$ and $G(\Phi,S)$. This problem is solved in several special cases. One of the typical situations considered by many authors is when S is a good ring, like, say a Dedekind domain, and R is its field of fractions. This situation was studied by N.S.Romanovskii, R.A.Schmidt, J.E.Cremona, A.E.Zalesskii and others, see [Ro], [Sch], [Cr], [Z4]. A.V.Stepanov established much more general results stated in terms of stability conditions (see references in [Sv]).

§ 21. Concluding remarks

In the preceding sections we treated only several typical situations. In fact for specific fields (like finite, algebraically closed, local, global, etc.) and for various classes of rings (commutative, semilocal, subject to stability conditions, Dedekind domains, Hasse domains, etc.) there are *many* further results in the same spirit.

In this survey we presented almost exclusively results of *geometric* nature, where the answer does not essentially depend on the ground field. (Apart possibly from very few small exceptions). For example, the description of overgroups of a *split* maximal torus is a geometric result, saying that everything looks exactly as in the absolute case. The theorem of Seitz from § 17 is another example of a geometric result: the answer is different from that for the absolute case, but again it does not depend on the *finite* ground field and may be modeled at the absolute level. On the other hand § 18 gives an example of an *arithmetic* result: the absolute case provides no insight whatsoever.

One explanation of the geometric answers is that in reality everything happens in a linear object, like an associative algebra, a Lie algebra or a Jordan algebra. Let us illustrate this in the example of $\mathrm{GL}(n, R)$. In the examples considered in §§ 4 – 17 the basic subgroups were always multiplicative groups of certain R-subalgebras in $\mathrm{M}(n, R)$ (in fact subalgebras defined by *linear* congruences). The situation in § 18 was quite different.

Observe though that for rings this distinction becomes somewhat shaky even in such utterly geometric results as the description of overgroups of a subsystem subgroup. Of course the *qualitative* answer does not essentially depend on the ground ring: only on its lattice of ideals. But as soon as one tries to actually *calculate* the factor-groups like $N(\sigma)/E(\sigma)$ one inevitably encounters K-functors. But explicit calculations of K-functors depend in a critical way on the *arithmetic* nature of a ring.

So far we have not touched the class $\mathcal{S}$. In this context one may observe that *simple* irreducible groups defined over the ground *field* rarely have many overgroups. For an algebraically closed field this was proven by E.B.Dynkin in characteristic 0 and by G.M.Seitz [S11] in the general case. Exceptional groups tend to have few simple subgroups which act irreducibly on their minimal modules, see [Te], [S15]. For the finite case see [LSS], [STe], [S14]. Some results of this sort have been known before, especially for finite fields, see, for example [Co] or [Su1], [Su2]. It is probable that results of such type remain valid for arbitrary fields provided the subgroup is isotropic enough.

On the other hand description of overgroups of a reducible subgroup H may be quite complicated even when H itself has a very transparent structure. One of the arising difficulties is that it is not always immediately clear how to connect the *irreducible* overgroups of H with the structure of the irreducible components of H. For example, I.D.Suprunenko classified subgroups of $\mathrm{GL}(n, p^m)$ containing $\mathrm{SL}(2, p)$ in a representation with two irreducible components. As an recent example we could mention [DP].

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