# Stable $K$ - theory and topological Hochschild homology of $A_\infty$ rings

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# 1 Stable $K$-theory and topological Hochschild homology

The purpose of this note is to expand on material presented by the third-named author in a lecture at Northwestern University in 1988. The material that concerns us is the description of a program to prove that the analog of the trace map induces a natural homotopy equivalence of infinite loop spaces between stable $K$-theory and topological Hochschild homology of $A_\infty$ ring spectra. B. Dundas and R. McCarthy [2] have given a proof of the result (formulated for simplicial rings), using a strategy totally different from what we will describe here.

In this section we recall the definitions of these objects and prove a few of the basic properties. For us a spectrum will always be a connected strict $\Omega$-spectrum [8, page 11]. The definition of an $A_\infty$-ring spectrum is found in [9, page 248]. Expanding this definition in terms of diagrams satisfying certain properties suggests the definitions of $A_\infty$-module spectra and of $A_\infty$-bimodule spectra over a given $A_\infty$-ring. For details we must refer the reader to [3] and [4]. From now on, a ring is an $A_\infty$-ring spectrum and a module over a ring is an $A_\infty$-module spectrum over the given $A_\infty$-ring, unless we make an explicit qualification. The algebraic $K$-theory of an $A_\infty$ ring was first defined in [9, page 272]; other definitions of the algebraic $K$-theory of $A_\infty$-rings are found in [15].

Our definition of stable $K$-theory is suggested by remarks found in [19, page 388] concerning the stable $K$-theory of simplicial rings. First comes a piece of notation. If $M$ is a module over $R$, then $M\langle m\rangle$ is the module $M$ shifted up by $m$ dimensions. More precisely,

$$M\langle m\rangle = M \wedge S^m,$$

where $\wedge$ indicates the smash product of the spectrum with a pointed space [8, page 16].

**Definition 1.1** *We let $R$ be a ring and let $M$ be an $R$-bimodule. Define*

$$K^s(R;M) = \lim_m \Omega^{m+1}\mathrm{fiber}(K(R\oplus M\langle m\rangle) \longrightarrow K(R)).$$

In section 3 we will need to know how well $\Omega^{m+1}\mathrm{fiber}(K(R\oplus M\langle m\rangle) \longrightarrow K(R))$ approximates $K^s(R;M)$. Proposition 1.3 of [22, page 38] generalizes to the $A_\infty$ context and leads to the following result.

**Proposition 1.2** *Let*

$$F(R;M)(m) = \mathrm{fiber}(K(R \oplus M\langle m\rangle) \longrightarrow K(R)).$$

*Then the canonical inclusion*

$$\Omega^{m+1}F(R;M)(m) \longrightarrow K^s(R;M)$$

*is $m$-connected.*

$\square$

From [4] we take the following definition of topological Hochschild homology of a ring $R$ with coefficients in the bimodule $M$.

**Definition 1.3** *Let $R$ be a ring and $M$ an $R$-bimodule. Then $THH(R;M)$ is the realization of a simplicial spectrum $THH_\bullet(R;M)$ whose $k$-simplices satisfy*

$$THH_k(R;M) \simeq R^{\wedge k} \wedge M.$$

As is by now well-known, the technical point is to organize the smash products $R^{\wedge k} \wedge M$, $k \geq 0$, into a simplicial object. For instance, the first problem is that they are all *a priori* spectra defined over different universes. However, these problems are solved in [3] and in [4], and the construction behaves as one would expect. For instance, we have the following result.

**Proposition 1.4** *Take $M = R \wedge R^{op}$, the free bimodule of rank one. We have a natural equivalence of spectra*

$$THH(R; R \wedge R^{op}) \xrightarrow{\simeq} R$$

*for any ring $R$.*

**Proof:** The proof mimics in the category of spectra the usual manipulations in the category of rings. First we have for each $k \geq 0$ an isomorphism

$$R^{\wedge k} \wedge (R \wedge R^{op}) \cong R \wedge R^{\wedge k} \wedge R$$

which brings the $R^{op}$ factor around in front. Filtering these isomorphisms through the construction which yields topological Hochschild homology, we obtain a natural homotopy equivalence

$$THH_\bullet(R; R \wedge R^{op}) \longrightarrow B_\bullet(R, R, R),$$

where the latter object is a two-sided bar construction in which $R$ is acting on itself first by right multiplication and second by left multiplication. Then iterated multiplications induce a map of the bar construction to the constant simplicial object which is $R$ in every dimension, and this is a simplicial homotopy equivalence. Therefore, passing to realizations and composing, we obtain the desired equivalence

$$THH(R; R \wedge R^{op}) \xrightarrow{\simeq} R.$$

□

The trace map

$$K^s(R; M) \longrightarrow HH(R; M)$$

from stable $K$-theory to Hochschild homology, where $R$ and $M$ are a simplicial ring and bimodule, respectively, and where $HH(R; M)$ is the realization of the usual Hochschild homology, is defined in [19, page 392]. The definition may be formally carried over to the context of $A_\infty$-objects.

The result we want to discuss is the following theorem.

**Theorem 1.5** *Let $R$ be an $A_\infty$-ring and let $M$ be an $A_\infty$-bimodule over $R$. Then the trace map*

$$K^s(R; M) \longrightarrow THH(R; M)$$

*is a homotopy equivalence.*

The program to prove this result involves a number of reductions so that one is reduced to verifying the result in a special case. These reductions are described in the following propositions. Recall that for a discrete ring $R$ flat over the integers one may define bifree bimodules to be those bimodules isomorphic to $R \otimes L \otimes R$, where $L$ is a free abelian group of finite rank. Transferring the notion to $A_\infty$ rings and bimodules in the most natural way, it becomes the following definition.

**Definition 1.6** *For a ring $R$ the bifree bimodule of rank 1 is the bimodule $R \wedge R^{op}$. A bimodule $M$ is bi-free if it is weakly homotopy equivalent to a bimodule of the form $R \wedge L \wedge R^{op}$ where $L$ is a spectrum with the property that $L$ has a finite filtration by subspectra in which the quotients of successive layers are finite wedges of shifted sphere spectra $S\langle m \rangle$, $m \geq 0$.*

We have the following result.

**Proposition 1.7** *If the theorem is true for $M$ any bifree bimodule over $R$, then it is true for every bimodule over $R$.*

**Proof:** Every bimodule $M$ has a functorial resolution by bifree modules, given by combining the free bimodule – and the Bar – construction. That $THH(R;M)$ may be computed from resolution of $M$ by bifree $R$-bimodules for an arbitrary $R$-bimodule $M$, is a relatively simple argument based on the fact that the two ways of geometrically realizing a bisimplicial set yield the same space.

That $K^s(R;M)$ may also be retrieved from the stable $K$-theory of $R$ with coefficients in bi-free $R$-modules is also true, but the argument is more difficult.

First let $U_\bullet \longrightarrow M$ be a resolution of $M$ be bi-free $R$-modules. The realization of this map is a homotopy equivalence, and our definition of $K$-theory is homotopy invariant [15], so

$$K^s(R;|U_\bullet|) \longrightarrow K^s(R;M)$$

is also a homotopy equivalence.

What we have to do now is to commute the formation of the geometric realization and the formation of stable $K$-theory. Given the Volodin definition of $K$-theory for $A_\infty$-rings [5] we may generalize Lemma 1.2.2 of [6] to prove that the relative $K$-theory of a surjection of simplicial $A_\infty$-rings whose kernel has the property that its square is zero may be calculated degreewise. In our case, if we define simplicial spaces $F_\bullet(m)$ by

$$F_p(m) = \text{fibre}(K(R \oplus U_p\langle m\rangle) \longrightarrow K(R),$$

then the result implies that there is a natural homotopy equivalence

$$|F_\bullet(m)| \simeq \text{fibre}(K(R \oplus |U_\bullet|\langle m\rangle) \longrightarrow K(R)).$$

Taking limits,

$$\lim_m \Omega^{m+1} F_p(m) = K^s(R;U_p),$$

so we may assemble all these natural equivalences into one commuting diagram in which the vertical arrows are trace maps.

$$\begin{array}{ccc} |p \longmapsto K^s(R;U_p)| & \simeq & K^s(R;M) \\ \downarrow & & \downarrow \\ |p \longmapsto THH(R;U_p)| & \simeq & THH(R;M) \end{array}$$

So, by a wellknown principle, to deduce that the righthand vertical arrow is a homotopy equivalence, it suffices to show that for each $p$ the lefthand vertical arrow is a homotopy equivalence.

□

Using the following proposition, we reduce to the case of the bifree bimodule of rank one.

**Proposition 1.8** *If $L$ is a spectrum satisfying the conditions of definition 1.6, then there are canonical equivalences*

$$K^s(R;M) \wedge L \longrightarrow K^s(R;M \wedge L)$$

*and*

$$THH(R;M) \wedge L \longrightarrow THH(R;M \wedge L).$$

**Proof:** For topological Hochschild homology this result is an immediate consequence of the definition. For stable $K$-theory, note that both sides are homology theories in the variable $L$ after passage to homotopy groups. For $K^s(R;M \wedge L)$ this is exactly the universal property enjoyed by the stablization, and it is true by definition for $K^s(R;M) \wedge L$. Now both sides agree for $S$, and so an easy induction using the filtration of $L$ ends the proof.

□

Combining these reduction steps one sees that it is enough to prove that the trace map induces an equivalence

$$K^s(R;R \wedge R^{op}) \longrightarrow THH(R;R \wedge R^{op}) \simeq R.$$

However, even to manage this, we have to change our ring $R$ somewhat.

**Proposition 1.9** *We may assume that the $A_\infty$-algebra $R$ over a given $E_\infty$-ring $k$ has $k$ as a retract. Precisely, the projection*

$$k \times R \longrightarrow R$$

*induces isomorphisms*

$$K^s(k \times R; 0 \times M) \longrightarrow K^s(R;M)$$

*and*

$$THH(k \times R; 0 \times M) \longrightarrow THH(R;M).$$

**Proof:** For this argument we use a plus construction definition of the algebraic $K$-theory of an $A_\infty$-ring as described in [9, page 272] and in [15, section 8]. However, our notation follows that of the definition given for the $K$-theory of simplicial rings found in [22].

We will always have rings with units

$$\eta : k \longrightarrow R$$

so that the natural way to view $k \times R$ is as a $k$-algebra with unit given by the diagonal. Then each of the projections from $k \times R$ to the factors is a $k$-algebra map, so that $k$ is a retract of $k \times R$ as a $k$-algebra. Note also that using the projection to $R$ to make $M$ into a $k \times R$-bimodule delivers the advertised structure. Using the projections, then, the diagram

$$\begin{array}{ccc} B\widehat{GL}(k \times R \oplus M\langle n \rangle)^+ & \longrightarrow & B\widehat{GL}(R \oplus M\langle n \rangle)^+ \\ \downarrow & & \downarrow \\ B\widehat{GL}(k \times R)^+ & \longrightarrow & B\widehat{GL}(R)^+ \end{array}$$

is seen to be equivalent to the diagram

$$\begin{array}{ccc} B\widehat{GL}(k)^+ \times B\widehat{GL}(R \oplus M\langle n \rangle)^+ & \longrightarrow & B\widehat{GL}(R \oplus M\langle n \rangle)^+ \\ \downarrow & & \downarrow \\ B\widehat{GL}(k)^+ \times B\widehat{GL}(R)^+ & \longrightarrow & B\widehat{GL}(R)^+ \end{array}$$

which is obviously homotopy cartesian. This proves the part of the proposition dealing with stable $K$-theory.

The result we need follows from Proposition 4.20 of [1]. Alternatively, one may argue directly that the projections to the factors of a product ring $R_1 \times R_2$ induce a homotopy equivalence

$$THH(R_1 \times R_2; M_1 \times M_2) \longrightarrow THH(R_1; M_1) \times THH(R_2; M_2),$$

as follows. It suffices to show that this arrow induces an isomorphism after passage to (spectral) homology with any field coefficients.

To see this one needs to use the spectral sequence arising from the skeletal filtration of the realization of a simplicial spectrum. The spectral sequence starts with

$$E^1_{p,q} = h_p(X_q) \Rightarrow h_*|X|,$$

where $h$ is any homology theory. Since any homology theory $HF$ with field coefficients satisfies the strict Künneth theorem, one has, in fact, that for $THH_\bullet(R;M)$ the $E^1$-term of the spectral sequence is the Hochschild complex for the graded ring $HF_*R$ with coefficients in the bimodule $HF_*M$. In the case at hand, then, the spectral sequence may be pushed easily to the next term, giving us

$$E^2_{*,*} = HH_*(HF_*R; HF_*M) \Rightarrow HF_*THH(R;M).$$

Now we have

$$HF_*(R_1 \times R_2) = HF_*R_1 \times HF_*R_2$$

and

$$HF_*(M_1 \times M_2) = HF_*M_1 \times HF_*M_2,$$

so that the $E^2$-term of the spectral sequence associated with the domain is the Hochschild homology a product ring with coefficients in a product module. Now the spectral sequence associated to the diagonal of the target has

$$E^2_{*,*} = HH_*(HF_*R_1; HF_*M_1) \times HH_*(HF_*R_2; HF_*M_2).$$

But the graded version of Theorem 6.2 of [7, page 295] says that projection to the factors of any $F$-algebra $A_1 \times A_2$ induces an isomorphism

$$HH_*(A_1 \times A_2; N_1 \times N_2) \cong HH_*(A_1; N_1) \times HH_*(A_2; N_2)$$

for any $A_1$-bimodule $N_1$ and any $A_2$-bimodule $N_2$, so that the induced map of spectral sequences is an isomorphism at $E^2$. The desired homology equivalence follows. $\square$

We will make one more technical manouver before we prove the theorem. Let $k \oplus R$ denote the $k$ algebra obtained by forgetting that $R$ has a unit and then adding a new unit. When $R$ and $k$ are real rings with elements, there is an isomorphism

$$c : k \oplus R \longrightarrow k \times R$$

given by

$$(x, r) \longmapsto (x, x + r).$$

This map makes sense even for rings up to homotopy, and is an $A_\infty$ map, respectively, an $E_\infty$ map in the commutative context, which is a homotopy equivalence. Since our definitions are homotopy invariant,

$$K^s(k \oplus R; c^*(M)) \xrightarrow{\simeq} K^s(k \times R; M)$$

and similarly for topological Hochschild homology.

Now we can give a quick overview of the proof of the theorem, breaking it into three major steps. Every ring $R$ is an algebra over the sphere spectrum $S$, so we show that for a ring $A$ of the form $A = S \oplus R$, there is a natural equivalence

$$K^s(A; A \wedge A^{op}) \simeq A.$$

First we use methods of abstract $K$-theory to analyze the $K$-theory of certain generalized free products which approximate the rings appearing in the definition of stable $K$-theory. The approximations and a partial analysis are explained in the next section. Then we go to work on the results of the general analysis, using the fact that the result is already true for $A = S$. This case is obtained from the fact that stable homotopy theory splits off of the stable algebraic $K$-theory of the one point space, the main result of [19], with the fact that the other term in the splitting is in fact trivial, the main result of [20]. Finally, we show that the trace delivers compatibility with the equivalence

$$THH(A; A \wedge A^{op}) \simeq A$$

of proposition 1.4. We then have the result for all $A$-bimodules by propositions 1.7 and 1.8. In particular, we have the result for $A$-bimodules $c^*(M)$ pulled back from $R$, which, in view of proposition 1.9, is just the result for $R$ and $M$.

## 2 Generalized free products

In this section we give a definition of a generalized free product of rings-up-to-homotopy and present the example which is important for stable $K$-theory. We also state a decomposition theorem for the algebraic $K$-theory of generalized free products.

**Definition 2.1** *Let $\alpha : C \longrightarrow A$ and $\beta : C \longrightarrow B$ be a pair of maps of rings. Let $D$ be a ring and let $A \longrightarrow D$ and $B \longrightarrow D$ be homomorphisms such that*

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

*commutes. The ring $D$ is a generalized free product of $A$ and $B$ over $C$ if the diagram*

$$\begin{array}{ccc} D \wedge_C D & \longrightarrow & D \wedge_A D \\ \downarrow & & \downarrow \\ D \wedge_B D & \longrightarrow & D \wedge_D D, \end{array}$$

*is homotopy cartesian in the category of spectra, where the objects in the corners are derived tensor products. For instance, at the upper left corner*

$$D \wedge_C D = |B_\bullet(D, C, D)|,$$

*the derived tensor product of $D$ with itself over $C$. Shorthand notation for such a ring $D$ will be $D = A *_C B$.*

The form of the definition is motivated by the problem of establishing the framework for analysing the $K$-theory of such a ring $D = A *_C B$. Recalling that $K$-theory is homotopy invariant, we write the definition in terms of derived tensor products, the homotopy invariant notion of tensor product introduced in [12] and discussed further in [3] and [4].

On the other hand, the free product can in some cases be written down following the recipe given for certain discrete rings in [17]. We will see this now. Let $C$ denote a commutative ring. That is, let $C$ be an $E_\infty$-ring spectrum, and let $A'$ be a $C$-algebra. Put $A = C \oplus A'$. Generalizing the construction of the preceding section, $A$ is then the $C$-algebra obtained by forgetting that $A'$ has a unit and then adjoining a unit. Let $G(m)$ denote the realization of the Moore loop group of a simplicial $m+1$-sphere. $G(m)$ is a topological group homotopy equivalent to the loop space of an ordinary $m+1$-sphere. Let $B(m) = C \wedge G(m)_+$ be the $A_\infty$-ring spectrum obtained by smashing together the spectrum $C$ and the space $G(m)_+$. We think of $B(m)$ as the group algebra of $G(m)$ over $C$. Notice that if we put $B'(m) = C \wedge G(m)$, then there is a $C$-bimodule splitting

$$B(m) \cong C \oplus B'(m).$$

Now we try out the formula

$$D(m) = C \oplus A' \oplus B'(m) \oplus A' \wedge_C B'(m) \oplus B'(m) \wedge_C A' \oplus A' \wedge_C B'(m) \wedge_C A' \oplus \ldots$$

for $D(m) = A *_C B(m)$ given in [17], where we substitute for normal tensor products the derived tensor products of $C$-bimodules

$$A' \wedge_C B'(m) = |B_\bullet(A', C, B'(m))|,$$

and so on, as introduced in the definition above.

**Proposition 2.2** *The ring $D(m)$ defined above is a generalized free product of $A$ and $B(m)$ over $C$.*

**Proof:** For the purposes of verifying that $D(m)$ has the correct homotopy type, we need to rewrite $D(m)$ some what. To simplify things, write $B$ for $B(m)$, $D$ for $D(m)$, and so on. Collecting together all terms which have $A'$ on the left we obtain a $C$-submodule $A''$, and doing likewise for $B'$, obtaining $B''$, we may write $D = C \oplus A'' \oplus B''$. Collecting differently and using certain algebraic properties of derived tensor products, we also obtain homotopy equivalences

$$D = A \oplus A \wedge_C B'' \text{ and } D = B \oplus B \wedge_C A''$$

of $A$- and $B$-modules, respectively. Now we examine the square

$$\begin{array}{ccc} D \wedge_C D & \longrightarrow & D \wedge_A D \\ \downarrow & & \downarrow \\ D \wedge_B D & \longrightarrow & D \wedge_D D, \end{array}$$

and verify that it is homotopy cartesian. Plugging into this square the alternate expressions for $D$, we obtain

$$\begin{array}{ccc} D \wedge_C (C \oplus A'' \oplus B'') & \longrightarrow & D \wedge_A (A \oplus A \wedge_C B'') \\ \downarrow & & \downarrow \\ D \wedge_B (B \oplus B \wedge_C A'') & \longrightarrow & D \wedge_D D. \end{array}$$

Appealing again to natural algebraic properties of the derived tensor products, one may verify that the canonical map between the vertical fibres is a homotopy equivalence. It follows that the original diagram is homotopy cartesian, as needed.

□

To analyze the $K$-theory of a such a free product $D$ in terms of its constituent rings, we introduce the category $\mathcal{MV}$ of Mayer-Vietoris presentations. Neglecting certain technical finiteness and cofibration conditions, the definition is as follows.

**Definition 2.3** *A Mayer-Vietoris presentation is a diagram*

$$\begin{array}{ccc} M_C & \longrightarrow & M_A \\ \downarrow & & \downarrow \\ M_B & \longrightarrow & M_D \end{array}$$

*of modules $M_?$ over the rings ? where the maps are maps of modules over the ring of the domain and satisfy the following homotopy condition:*

*The diagram of modules*

$$\begin{array}{ccc} M_C \wedge_C D & \longrightarrow & M_A \wedge_A D \\ \downarrow & & \downarrow \\ M_B \wedge_B D & \longrightarrow & M_D \wedge_D D, \end{array}$$

*where $\wedge_?$ indicates the derived tensor product over the ring ?, should be a homotopy-cartesian diagram of spectra.*

*A map of Mayer-Vietoris presentations*

$$MV \quad = \quad \begin{array}{ccc} M_C & \longrightarrow & M_A \\ \downarrow & & \downarrow \\ M_B & \longrightarrow & M_D \end{array} \quad \longrightarrow \quad MV' \quad = \quad \begin{array}{ccc} M'_C & \longrightarrow & M'_A \\ \downarrow & & \downarrow \\ M'_B & \longrightarrow & M'_D \end{array}$$

*is a quadruple of maps $M_? \longrightarrow M'_?$ of modules over the ring ? such that all squares commute.*

The first example which comes to mind is of course

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

Now the category $\mathcal{MV}$ is a category with cofibrations, notions of weak equivalence, and supports a cylinder functor. In other words, the Fibration Theorem (Theorem 1.6.4) and the Approximation Theorem (Theorem 1.6.7) of [18, pages 350, 354], respectively, apply, yielding the following theorem, one of the main technical results of [14].

**Theorem 2.4** *Given a generalized free product of $A_\infty$-rings*

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

*in which the $C$-bimodules $A$ and $B$ split as $A \cong C \oplus A'$ and $B \cong C \oplus B'$, respectively, there is a category with cofibrations and weak equivalences $\mathrm{Nil}(C; A', B')$ and functors*

$$\begin{array}{ccc} \mathrm{Nil}(C; A', B') & \longrightarrow & \mathcal{M}(C) \\ \mathrm{Nil}(C; A', B') & \longrightarrow & \mathcal{M}(A) \times \mathcal{M}(B) \end{array}$$

*to products of appropriate module categories such that the diagram*

$$\begin{array}{ccc} K(\mathrm{Nil}(C; A', B')) & \longrightarrow & K(C) \\ \downarrow & & \downarrow \\ K(A) \times K(B) & \longrightarrow & K(D) \end{array}$$

*is homotopy cartesian.*

$\square$

The paper [13] develops the cartesian square for simplicial rings extending ideas of Pierre Vogel, [16]. Most of the $K$-theoretical technicalities necessary for the result we need are addressed in this paper, and modifications necessary to treat the case of $A_\infty$-rings will be indicated in [14].

# 3 Approximations to stable $K$-theory

In this section we outline the steps in the key calculation that for a ring $A = S \oplus A'$ we have

$$K^s(A; A \wedge A^{op}) \simeq A.$$

To identify $K^s(A; A \wedge A^{op})$ in a stable range, it suffices to calculate

$$F(A; A \wedge A^{op})(m) = \mathrm{fiber}(K(A \oplus A \wedge A^{op}\langle m \rangle) \longrightarrow K(A)$$

since the canonical map

$$\Omega^{m+1} F(A; A \wedge A^{op})(m) \longrightarrow K^s(A; A \wedge A^{op})$$

is $m$-connected by proposition 1.2.

Now, working with $A = S \oplus A'$ enables us to reformulate the calculation of the relative $K$-theory in terms of the $K$-theory of a generalized free product. To do this, we specialize the example of the free product

$$D(m) = A *_C B(m) = A *_C C \wedge G(m)_+$$

given in the preceding section by taking $C = S$, the sphere spectrum.

**Lemma 3.1** *In a stable range (up through $\approx 2m$) we have a natural chain of equivalences*

$$F(A; A \wedge A^{op})(m) \simeq \mathrm{fiber}(K(A *_S S \wedge G(m)_+) \longrightarrow K(A)).$$

**Proof:** There is a canonical $2m+1$-connected map

$$S^m \longrightarrow G(m)$$

analogous to the familiar map $S^m \longrightarrow \Omega S^{m+1}$. In the definition

$$A \oplus A \wedge A^{op}\langle m \rangle \cong A \oplus A \wedge S^m \wedge A,$$

where multiplication is zero on $A \wedge S^m \wedge A$, one may substitute $G(m)$ for $S^m$ obtaining a similar ring

$$D'(m) = A \oplus A \wedge G(m) \wedge A^{op}$$

and a $2m+1$-connected ring map

$$A \oplus A \wedge A^{op}\langle m \rangle \longrightarrow D'(m).$$

Writing $D(m) = A *_S S \wedge G(m)_+$ and using the recipe given in proposition 2.2, we see there is an obvious $2m+1$-connected retraction

$$D(m) \longrightarrow D'(m).$$

Furthermore, using the plus-construction definition of $K$-theory, as in the proof of proposition 1.9, one sees that passage to $K$-theory preserves connectivity of ring maps. (Cf. proposition 1.1, [22, page 36].) Then concatenating highly connected maps and relativizing finishes the proof of the lemma. $\square$

Continue with the specializations $C = S$, $B(m) = S \wedge G(m)_+$, and $D(m) = A *_S B(m)$ introduced above, and write

$$K(D(m), A) = \text{fiber}(K(D(m)) \longrightarrow K(A),$$

just renaming our approximation to $F(A; A \wedge A^{op})$.

Next observe that the retraction $B(m) \longrightarrow S$ induces a map of generalized free product diagrams

$$\begin{array}{ccccccc} S & \longrightarrow & A & & S & \longrightarrow & A \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ B(m) & \longrightarrow & D(m) & & S & \longrightarrow & A. \end{array}$$

Theorem 2.4 applies to both situations and the map of generalized free products induces a map of cartesian squares. The Nil-term in the second square is simply $K(S) \times K(S)$, so if we define

$$\widetilde{K}(\text{Nil}(S; A', B'(m))) = \text{fiber}(K(\text{Nil}(S; A', B'(m))) \longrightarrow K(S) \times K(S))$$

and

$$K(B(m), S) = \text{fiber}(K(B(m)) \longrightarrow K(S))$$

and take homotopy fibers of the map of cartesian squares, we get the following result.

**Proposition 3.2** *There is a fibration-up-to-homotopy*

$$\widetilde{K}(\mathrm{Nil}(S; A', B'(m))) \longrightarrow K(B(m), S) \longrightarrow K(D(m), A).$$

$\square$

From this fibration sequence one derives the sequence

$$\Omega^{m+1} K(B(m), S) \longrightarrow \Omega^{m+1} K(D(m), A) \longrightarrow \Omega^m \widetilde{K}(\mathrm{Nil}(S; A', B'(m)))$$

in which the middle term is our approximation to $K^s(A; A \wedge A^{op})$. The following result identifies the other two terms in the fibration sequence.

**Theorem 3.3** *In a stable range of dimensions $\phi(m)$ tending to infinity with $m$ we have $\phi(m)$-equivalences*

1. $\Omega^{m+1} K(B(m), S) \simeq S$.
2. $\Omega^m \widetilde{K}(\mathrm{Nil}(S; A', B'(m))) \simeq A'$.

**Remarks on the proof:** The stable range equivalence $\Omega^{m+1} K(B(m), S) \simeq S$ is derived by combining the fact that stable homotopy theory splits off of the stable algebraic $K$-theory of the one point space, the main result of [19], with the fact that the other term in the splitting is in fact trivial, the main result of [20]. The other identification follows after careful analysis of the reduced Nil-term $\widetilde{K}(\mathrm{Nil}(S; A', B'(m)))$ [14]. This part of the argument uses alternative descriptions of the Nil-term and manipulations patterned on those in [19], as well as two more applications of [20].

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