REDUCTION OF REGULAR MULTIGRAPHS

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Abstract.

A multigraph is a graph with loops and multiple edges permitted. If all vertices of a multigraph have the same degree r, it is called r-regular.

The purpose of this paper is to give reduction procedures for the classes of r-regular multigraphs, where r is any natural number.

Moreover we present reduction procedures for the classes of r-regular multigraphs without loops.

A reduction procedure works as follows. Given an r-regular multigraph M, there is a sequence $M_1, \ldots, M_n$ of r-regular multigraphs s.t. $M = M_1$, and M_{i+1} can be obtained from M_i by one of finitely many wellspecified operations. M_n is the disjoint union of finitely many copies of multigraphs in a fixed finite set called the base.

We prove, that the list of operations we present is minimal, i.e. each operation is necessary to carry out the reduction procedure.

1. Reducing regular multigraphs

We shall present reduction procedures for r-regular multigraphs. Precisely, given any r-regular multigraph M, there is a sequence $M_1, \ldots, M_n$ of r-regular multigraphs s.t. $M_1 = M$ and M_{i+1} is obtained from M_i by deleting a vertex or a multiedge and by restoring regularity by defining new edges. M_n is the disjoint union of finitely many copies of multigraphs in a fixed finite set called the base, which we will specify depending on r.

A solution for simple r-regular graphs and a short history of the subject is given in [2].

This gives a natural characterisation of these classes of graphs by "prime elements" and generating rules. The bases we use have exactly one element, the "prime element" of the class, and all the other elements are generated from this element by the operations.

Furthermore, a reduction procedure for a class of graphs leads also to a device to give easy proofs by induction. Examples for such an approach are Johnson [3] and Statman [4].

Definition 1.1. *Let M be a multigraph with vertex set VM, having maximum degree less or equal to r. For $v \in VM$ the r-***deficiency of** *v is $d^r_M v := r - \deg_M v$, the* **maximum deficiency** *is $d^r M := \max\{d^r_M v | v \in VM\}$, and $s^r(M) := \sum_{v \in VM} d^r_M v$ is the r-***deficiency of** *M.*

First we consider r-regular graphs where r is even.

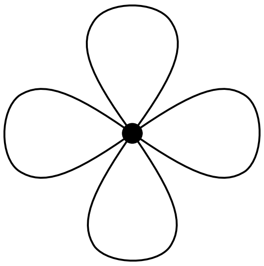


Figure 1.1 The base for 8-regular multigraphs $\{B_8\}$

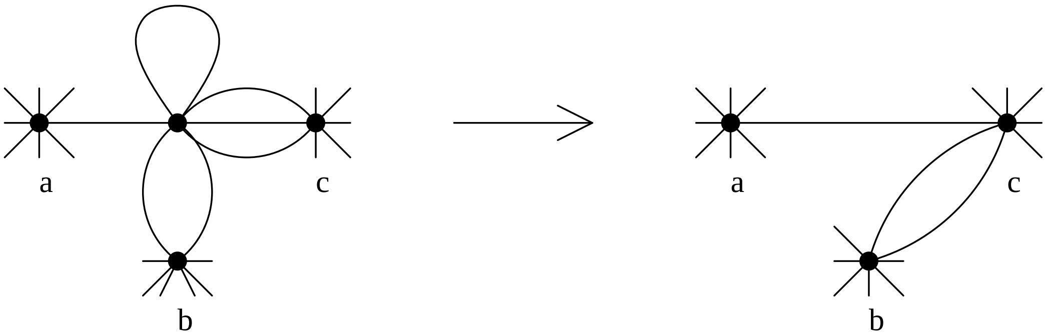


Figure 1.2 Reduction of 8-regular multigraphs

The base will contain only one multigraph, the multigraph B_r having one vertex and $\frac{r}{2}$ loops. There is only one rule for reduction, delete a vertex and restore regularity by adding new edges. We have to show, that this is always possible.

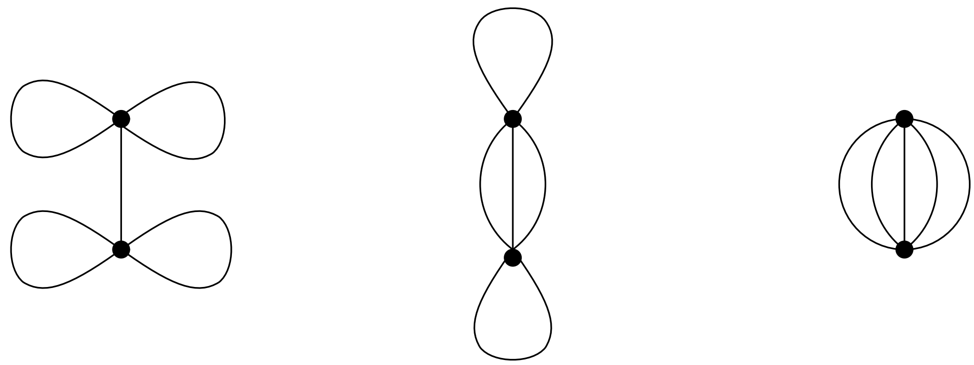
Proposition 1.1 *Let r be even and M be an r regular multigraph not the disjoint union of copies of B_r. Then there are a vertex of M and an r-regular multigraph N on the vertices of $M - v$ having $M - v$ as a subgraph.*

Proof. Let v be a vertex of M having k loops, $0 \leq k < \frac{r}{2}$. We delete v together with all edges incident with v.

$s^r(M - v) = r - 2k = 2\ell$. So we may restore regularity by adding ℓ new edges.

$\square$

Next we consider the case where r is odd.



$B_5^{1,2}$ $B_5^{3,1}$ $B_5^{5,0}$

Figure 1.3 The base for 5-regular multigraphs

The base will contain the multigraphs $B_r^{2k+1,\frac{r-(2k+1)}{2}}$ on two vertices having $2k+1$ edges joining them and $\frac{r-(2k+1)}{2}$ loops for each vertex, for $0 \leq k \leq \frac{r-1}{2}$.

The rule for reduction is as follows. Find two vertices v and w joined by at least one edge. Delete v and w together with all edges incedent with v or w and restore regularity by adding new edges. We have to show that this is always possible.

Proposition 1.2 *Let r be odd and M be an r-regular and connected multigraph not in the base for r regular multigraphs. Then there are vertices v and w of M, and an r-regular multigraph N s.t. $s^r(M-\{v,w\}) \neq 0$, N has the same vertices as $M-\{v,w\}$, and $M-\{v,w\}$ is a subgraph of N .*

Proof. M is not a copy of a multigraph in the base. Hence there are $v, w \in VM$ s.t. $s^r(M-\{v,w\}) \neq 0$. Let k_v be the number of loops of v , k_w the number of loops of w , j the number of edges joining v and w . So we have:

$0 \neq s^r(M-\{v,w\}) = r - 2k_v - j + r - 2k_w - j = 2(r - k_v - k_w - j) = 2\ell$. Thus we may restore regularity by adding ℓ new edges.

□

2. Reducing regular multigraphs without loops

Now we turn to multigraphs without loops. We would like to describe a reduction procedure for this class of graphs.

M_r will be the only element in the base, where M_r is the r-multiedge i.e. the r-regular multigraph on two vertices joined by r edges. The reductions are as follows.

Reduction 0 : Remove a vertex together with all its incident edges.

This rule applies when r is even only.

Reduction p : $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$.

Remove a p-multiedge together with all edges incident, where p is minimal for multiedges which are induced subgraphs.

It is understood for all reductions, that they are applicable only if we can solve the deficiency problem, by filling up with new edges or by splitting edges if necessery. That this is possible we will prove.

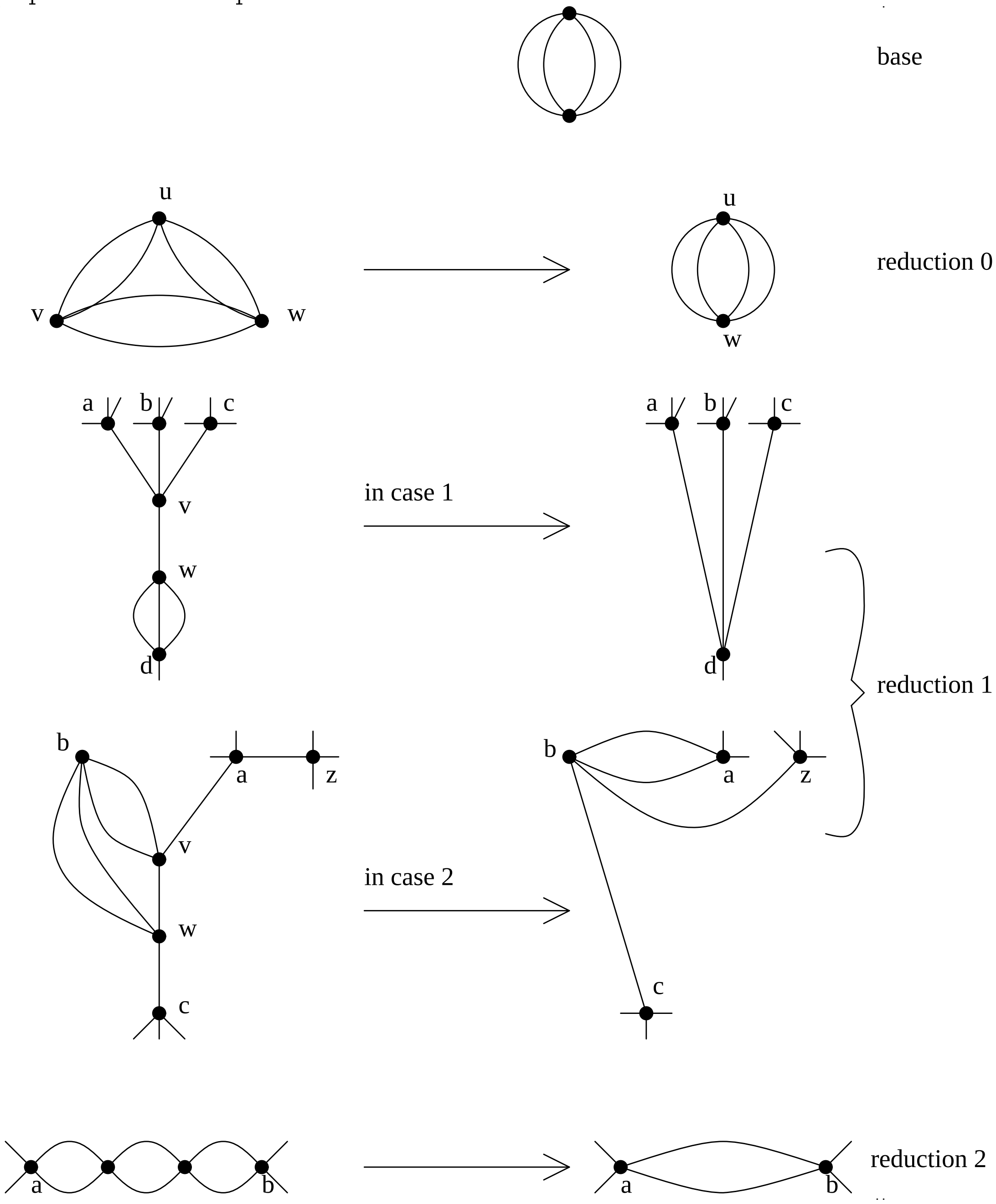


Figure 2.1 The reduction of 4-regular multigraphs without loops. For reduction 1 holds case 1 if $d^r_{M-M_1} \leq r-p$ for all $u \in \{a,b,c,d\}$ and case 2 holds if $d^r_{M-M_1} > r-p$ for some $u \in \{a,b,c\}$.

Theorem 2.2. *Every r -regular multigraph without loops is reducible to a disjoint union of copies of M_r using reduction 0, reduction $1,\ldots,$ and reduction $\lfloor \frac{r}{2} \rfloor$.*

Proof. Let M be a component of an r-regular multigraph without loops. We assume that M is not isomorphic to M_r. We fix M_p s.t. $1 \leq p \leq r$, M_p is an induced subgraph of M, and p is minimal. Let $\{v, w\}$ be the set of vertices of M_p and let $\{w, v_1, \ldots, v_{n-1}\}$ be the set of neighbors of v in M. From our assumptions we deduce $n \geq 2$ and $1 \leq p \leq \lfloor \frac{r}{2} \rfloor$.

Case 1. $d^r_{M-M_p} v_i \leq r - p$ for all $i \in \{1, \ldots, n-1\}$.

Let N be the set of vertices of M adjacent to w but not adjacent to v. $s := \sum_{u \in N} d^r_{M-M_p} u$. It follows that s is less or equal to $r - p$. Thus every element of $\{d^r_{m-M_p} v_1, \ldots, d^r_{M-M_p} v_{n-1}, s\}$ is less or equal to $r-p$. We order this set in a way s.t. we get a decreasing sequence $\{d_i\}_{i=1}^n$. For this sequence of numbers $\leq r - p$ the following holds:

$2(r-p) = \sum_{i=1}^n d^r_{M-M_p} v_i + s = \sum_{i=1}^n d_i$ and

for $1 \leq k \leq n$: $\sum_{i=1}^n d_i \leq k(r-p) \leq k(k-1)(r-p) + \sum_{i=1}^n \min\{(r-p)k, d_i\}$.

By a theorem of V. Chungphaisan which may be found e.g. in Berge [1,p. 121] there exists a multigraph M' having no loops s.t. $\{d_i\}_{i=1}^n$ is the degree sequence of M'. So we remove M_p and all edges incident with M_p from M and construct an isomorphic copy of M' on $\{v_n, \ldots, v_{n-1}\} \cup N$ s.t. the multigraph without loops M^* we obtain is r-regular. M^* has less vertices than M.

Case 2. There is an $i \in \{1, \ldots, n-1\}$ s.t. $d^r_{M-M_p} v_i > r - p$ holds.

We fix such an v_i. Let q be the number of edges joining v and v_i, and let ℓ be the number of edges joining w and v_i. By the assumption of this case we have $q + \ell > r - p$. If we suppose that there is an u, adjacent to v_i, and different to v and w, then the number of edges joining u and v_i would be less or equal to $r - (q+\ell) < r - (r-p) = p$, which contradicts the fact, that we have choosen p minimal. Hence v_i has no other neighbors than v and w. It follows that $r = \ell + q$ and $d^r_{M-M_p} v_i = r$.

We compute $s^r(M - \{v, w, v_i\}) = r - (p+q) + r - (p+\ell) = 2r - (2p+q+\ell) = r - 2p$. So, if we delete M_p, v_i becomes an isolated point having deficiency r, but $s^r(M - (M_p \cup \{v_i\})) = s^r(M - (v, w, v_i)) = r - 2p$, which does not fit. We have to increase the deficiency in $M - \{v, w, v_i\}$.

We have two subcases. $s^r(M - \{v, w, v_i\})$ is zero only in case r is even and $p = \frac{r}{2}$. If r is odd or $p \neq \frac{r}{2}$ there has to be a vertex u in $M - \{v, w, v_i\}$. For such an u the following holds:

$d^r_{M-M_p} u \leq s^r(M - \{v, w, v_i\}) = r - 2p < r$. Hence u is joined to a vertex z different from v and w. By the minimality of p there are at least p edges joining u and z.

We remove M_p together with all edges incident with M_p from M and furthermore we delete p edges joining u and z. The resulting multigraph has deficiency $2r$ and v_i has deficiency r. So we may restore regularity by joining v_i by r new edges to deficient vertices different from v_i.

If r is even and $p = \frac{r}{2}$ then $\{v, w, v_i\}$ is the vertex set of M. In this case $p = q = \ell = \frac{r}{2}$. We delete v_i and all edges incident with v_i and join v and w by $\frac{r}{2}$ new edges.

In both subcases the multigraph we obtain is r -regular, has no loops and is smaller than M .

$\square$

Remark. If r is even we can avoid deleting edges of $M-\{v,w,v_i\}$ in the second case. Let $r=2r'$, since $d^r_{M-M_p}v_i=r$ and $s^r(M-M_p)=2(r-p)$ it follows $\sum_{\substack{j=1\\j\neq i}}^{n-1} d^r_{M-vw}v_j+s=r-2p$. We remove M_p , put in a new vertex u and $r'-p$ edges between u and $\{v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{n-1}\}\cup N$ and $r'+p$ edges between u and v_i . Then u is incident with $r'-p+r'+p=r$ edges and

$$d^r_{M-M_p}v_i-(r'+p)=r'-p=s^r(\{v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_{n-1}\}\cup N(r'-k)\ .$$

So we join v_i and N by $r'-p$ new edges. The resulting graph is an r -regular multigraph M^* without loops and $|VM^*|=|VM|-1$ and hence this reduction is impossible if r is odd.

3. Minimality of the reductions

Now we show no multigraph in the bases and no reduction rule can be avoided.

Multigraphs having even degree $2r$:

The base we used contains B_{2r} , the multigraph having one vertex and r loops, and nothing else. There is only one reduction rule. Clearly the base and the set of rules are minimal.

Multigraphs having odd degree $2r+1$:

The base contains all multigraphs having degree $2r+1$ and two vertices. So the base is minimal as long as we demand reductions to reduce the number of vertices.

That all reductions $1\leq p\leq r$ are needed is shown by figure 3.1.

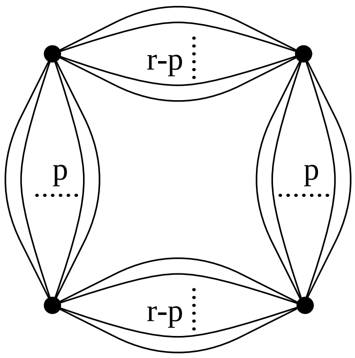


Figure 3.1

Multigraphs without loops. The arguments are as for multigraphs.

References

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