Euler systems for higher K-theory of number fields

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1 Introduction

In this paper we construct elements in higher K-groups of a number field which behave similarly to the Euler system of Gauss sums investigated by K.Rubin in [Ru2]. Let p^m be a power of an odd prime number p. Choose a square free integer L whose prime factors split completely in the cyclotomic field $\mathbf{Q}(\mu_{p^m})$. Let $\bar{r}$ be a prime ideal of $\mathbf{Q}(\mu_{p^m L})$ above a rational prime r. For L and $\bar{r}$ as above we construct elements $\Lambda(L, \bar{r})$ which belong to K-groups with coefficients in $\mathbf{Z}/p^m$ of the field $\mathbf{Q}(\mu_L)$. These elements have the following properties which we prove in Section 3 of the paper.

Theorem A

(1) *If* Tr_l: $K_{2n+1}(\mathbf{Q}(\mu_L); \mathbf{Z}/p^m) \to K_{2n+1}(\mathbf{Q}(\mu_{L/l}); \mathbf{Z}/p^m)$ *is the transfer map in* K*-theory, then:*

$$Tr_l(\Lambda(L, \bar{r})) \;=\; \Lambda(L/l, \hat{r})^{1-l^n Fr_l^{-1}}.$$

Here Fr_l *is the Frobenius automorphism for a prime* l *dividing* L.

(2) *If* red: $K_{2n+1}(\mathbf{Q}(\mu_L); \mathbf{Z}/p^{m+1}) \;\to\; K_{2n+1}(\mathbf{Q}(\mu_L); \mathbf{Z}/p^m)$ *is the change of coefficients map in* K*-theory, then:*

$$red(\Lambda(L, \bar{r})) = \begin{cases} \Lambda(L, \tilde{r})^p & \text{if } m \geq m_0 \\ \Lambda(L, \tilde{r}) & \text{if } m < m_0, \end{cases}$$

where m_0 *is an integer depending on* r *and* p.

(3) *If* ∂: $K_{2n+1}(\mathbf{Q}(\mu_L); \mathbf{Z}/p^m) \to \bigoplus_{\nu | p} K_{2n}(k_\nu; \mathbf{Z}/p^m)$ *is the boundary map in the localization sequence, then:*

$$\partial(\Lambda(L, \bar{r})^{b_L^n \gamma_p}) \;=\; [N_{\bar{r}/\tilde{r}}(\pi_{\bar{r}}(B^n(\xi_{p^m})))]^{\Theta_n(b_L)}.$$

Here $\Theta_n(b_L)$ *denotes the* n*th Stickelberger element defined for an integer* b_L*, and* $N_{\bar{r}/\tilde{r}}(\pi_{\bar{r}}(B^n(\xi_{p^m})))$ *is an element of the group* $K_{2n}(k_{\tilde{r}}; \mathbf{Z}/p^m)$.

We would like to mention that our construction of elements $\Lambda(L, \bar{r})$ suggests how to obtain similar elements in K-groups of modular curves. We plan to address the matter elsewhere. Using $\Lambda(L, \bar{r})$ we were able to make a similar reasoning as K.Rubin in [Ru2], to get some results about the size of the group $D_{n+1}(\mathbf{Q})$ of divisible elements in the p-torsion part of $K_{2n}(\mathbf{Q})$ for n odd. A proof of the following statement which we give in Section 4 of this paper does not use "the Main Conjecture" in Iwasawa theory.

Theorem B
The number of elements of $D_{n+1}(\mathbf{Q})$ *divides the* p*-part of the number* $w_{n+1}(\mathbf{Q})\zeta_{\mathbf{Q}}(-n)$ *for primes* p *relatively prime to* $w_{n+1}(\mathbf{Q})$*, where* n *is odd and* $n \not\equiv -1 \, mod \, (p-1)$.

While proving Theorem B [cf. Corollary 3, below] we had to reformulate some of Rubin's results from [Ru2] for the field $\mathbf{Q}(\mu_{p^m})$ and $m > 1$. Our computations show a striking analogy between arithmetic of class groups of cyclotomic fields and arithmetic of the group $D_{n+1}(\mathbf{Q})$. In the last section of the paper we prove some results about the group $D_{n+1}(F)$ for a number field F and n even.

Theorem C
If F *is a totally real, abelian number field and* p *does not divide the conductor of* F*, then we have the following formula for the number of elements in* $D_{n+1}(F)$.

$$|D_{n+1}(F)| \;=\; [K_{2n+1}^{et}(\mathcal{O}_F[\frac{1}{p}]) \, ; \, C_{2n+1}^{et}]$$

Here C_{2n+1}^{et} *stands for the image in etale* K*-theory of Soulé's group of cyclotomic elements defined for the field* F.

In addition, let us point out that the classical conjecture of Kummer and Vandiver can be reformulated in terms of the group $D_{n+1}(\mathbf{Q})$ (n even), cf. Proposition 8. On the other hand, as we know from [Ba2], the Quillen-Lichtenbaum conjecture for $K_{2n}(\mathbf{Z})$ (n odd) has an equivalent statement using $D_{n+1}(\mathbf{Q})$. For these reasons we think that it is crucial to investigate the groups $D_{n+1}(F)$ and their arithmetic.

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2 Euler systems for higher K-theory

Euler systems introduced by A.Kolyvagin and K.Rubin may be used to construct interesting systems of elements in algebraic K-theory. Let $F/\mathbf{Q}$ be an abelian field extension. We fix an odd prime p and a natural number m. Let S be the set of square free natural numbers such that they are divisible only by primes l that split completely in F and such that $l \equiv 1 \, mod \, p^m$. We will use Euler systems as in [Ru3].

Definition 1 *Let $\alpha(L) \in F(\mu_{p^m L})$ for $L \in S$ be elements satisfying:*

(1) $\alpha(L) \in F(\mu_{p^m L})^{\times}$ *so* $\alpha(L) \neq 0$

(2) *if* $l|L$ *then* $N_l(\alpha(L)) \equiv (Fr_l - 1)\alpha(L/l) \;\; mod \;\; F(\mu_{p^m(L/l)})^{\times M}$ *or*

(2') *if* $l|L$ *then* $N_l(\alpha(L)) \equiv (1 - Fr_l^{-1})\alpha(L/l) \;\; mod \;\; F(\mu_{p^m(L/l)})^{\times M}$*, where M is a natural number divisible by p^m.*

We call a set $\{\alpha(L) \in F(\mu_{p^m L});\, L \in S\}$ with properties (1) and (2) or (1) and (2') an Euler system for the field F.

Remark 1. $\alpha(L)$ depends on p^m but p^m is fixed so we do not denote it. N_l and Fr_l are as in Rubin's papers [loc. cit].

Remark 2. The Euler system of cyclotomic units denoted by ξ_r in [Ru1] satisfies conditions (1) and (2). The Euler system of Gauss sums from [Ru2] satisfies (1) and (2').

2.1 Bott elements

Let A be a domain which contains a primitive root of unity of order p^m denoted by ξ_{p^m}. Let $K_*(-;\mathbf{Z}/p^m)$ denote the K-theory with $\mathbf{Z}/p^m$ coefficients defined by Browder and Karoubi. The Bockstein exact sequence in K-theory gives the following exact sequence.

$$K_2(A;\mathbf{Z}/p^m) \to K_1(A)[p^m] \to 0$$

Observe that: $K_1(A)[p^m] = A^*[p^m]$. In [So1] C.Soulé shows that there is a canonical Bott element $B(\xi_{p^m}) \in K_2(A; \mathbf{Z}/p^m)$ (denoted by α_n in [So1]) such that ξ_{p^m} is the image of $B(\xi_{p^m})$ by the surjective map of the above exact sequence. By definition $B(\xi_{p^m})$ is the image of ξ_{p^m} by the following composition of maps. Let μ_{p^m} denote the group of p^mth roots of unity.

$$\mu_{p^m} \xrightarrow{\cong} \pi_1(BGL_1(A))[p^m] \xrightarrow{\cong} \pi_2(BGL_1(A); \mathbf{Z}/p^m) \to K_2(A; \mathbf{Z}/p^m)$$

Observe that if $\varphi : A \to A$ is a ring homomorphism, then we have: $B(\varphi(\xi_{p^m})) = \varphi(B(\xi_{p^m}))$. But $\varphi(\xi_{p^m}) = \xi_{p^m}^c$ for some $c \in \mathbf{Z}$, hence $\varphi(B(\xi_{p^m})) = B(\xi_{p^m})^c$. If A is the ring of integers in a number field and $\wp$ is its prime ideal such that $\wp$ does not divide p, then $\pi_\wp(B(\xi_{p^m}))$ is the Bott element in the cyclic group $K_2(k_\wp; \mathbf{Z}/p^m)$. Here $\pi_\wp : K_2(A; \mathbf{Z}/p^m) \to K_2(k_\wp; \mathbf{Z}/p^m)$ denotes the reduction map on K-groups.

2.2 General construction

Note that by assumptions on p and L we have $F(\mu_{p^m}) \cap F(\mu_L) = F$ for all $L \in S$. Let $\mathcal{O}_{p^m L}$ denote the ring of integers in $F(\mu_{p^m}, \mu_L)$ and $\mathcal{O}_{p^m}$ denote the ring of integers in $F(\mu_{p^m})$.

Definition 2 *Let $\Lambda(L) \in K_{2n+1}(F(\mu_L), \mathbf{Z}/p^m)$ be given by the following formula.*

$$\Lambda(L) = Tr_{F(\mu_{p^m L})/F(\mu_L)}(\alpha(L) * B(\xi_{p^m})^{*n}),$$

where $\alpha(L)$ is as in Definition 1. We call the set of elements $\{\Lambda(L);\, L \in S\}$ an Euler system in K-theory of F. Here $B(\xi_{p^m})^{*n}$ is the Bott element multiplited by itself n times using product in K-theory. Further on we will denote it for simplicity by $B^n(\xi_{p^m})$. The symbol Tr denotes the transfer map in K-theory with finite coefficients. Note that in the above formula $B(\xi_{p^m})$ belongs to $K_2(\mathcal{O}_{p^m L}, \mathbf{Z}/p^m)$.

Remark 3. Sometimes it may be also interesting to consider elements $\alpha(L) \in F(\mu_{p^m L})$ which depend on p^m. It has been done by Kurihara in [Ku] where the author uses Soulé's cyclotomic elements to construct an Euler system in etale cohomology. Proposition 3 in Section 3.3 below describes behaviour of the Euler system of Gauss sums under norm maps $N_m : F(\mu_{p^{m+1} L}) \to F(\mu_{p^m L})$.

Remark 4. Observe that condition (1) of Definition 1 is needed for good definition of $\Lambda(L)$ because $K_1(F(\mu_{p^m L})) = F(\mu_{p^m L})^\times$. For $\Lambda(L)$ we have the following analogue of condition (2) of Definition 1.

Proposition 1 *If $Tr_l : K_*(F(\mu_L); \mathbf{Z}/p^m) \to K_*(F(\mu_{L/l}); \mathbf{Z}/p^m)$ denotes the transfer homomorphism, then: $Tr_l(\Lambda(L)) = (l^{-n} Fr_l - 1)\Lambda(L/l)$.*

Proof.

$$Tr_l(Tr_{F(\mu_{p^m L})/F(\mu_L)}(\ldots)) = Tr_{F(\mu_{p^m L})/F(\mu_{L/l})}(\ldots) =$$
$$= Tr_{F(\mu_{p^m L/l})/F(\mu_{L/l})} Tr_{F(\mu_{p^m L})/F(\mu_{p^m L/l})}(\ldots).$$

We used above various transfer maps between K-groups of the following tower of fields.

$$\begin{array}{ccccc} F(\mu_{p^m}) & \longrightarrow & F(\mu_{p^m L/l}) & \longrightarrow & F(\mu_{p^m L}) \\ \uparrow & & \uparrow & & \uparrow \\ F & \longrightarrow & F(\mu_{L/l}) & \longrightarrow & F(\mu_L) \end{array}$$

Diagram 1

Observe that the Bott element $B(\xi_{p^m}) \in K_2(\mathcal{O}_{p^m L}; \mathbf{Z}/p^m)$ is the image of the Bott element from $K_2(\mathcal{O}_{p^m L/l}; \mathbf{Z}/p^m)$. By the projection formula we get the following equalities.

$$Tr_{F(\mu_{p^m L})/F(\mu_{p^m L/l})}(\ldots) = N_l \alpha(L) * B^n(\xi_{p^m}) = [(Fr_l - 1)\alpha(L/l)] * B^n(\xi_{p^m}) =$$

$$= [Fr_l \alpha(L/l) * l^{-n} Fr_l B^n(\xi_{p^m})][\alpha(L/l)^{-1} * B^n(\xi_{p^m})] =$$

$$= (l^{-n} Fr_l - 1)[\alpha(L/l) * B^n(\xi_{p^m})].$$

In the above computations we used the equality: $Fr_l(B^n(\xi_{p^m})) = [B^n(\xi_{p^m})]^{l^n}$ which follows from Section 2.1. It completes the proof because Fr_l commutes with the transfer map $Tr_{F(\mu_{p^m L/l})/F(\mu_{L/l})}$. **Q.E.D.**

Remark 5. If we used condition (2') of Definition 1 in the last proposition, then we would get the following equality.

$$Tr_l(\Lambda(L)) = (1 - l^n Fr_l^{-1})\Lambda(L/l)$$

Example. Put

$$\alpha(L) = (1 - \xi_{p^m} \prod_{l|L} \xi_l)(1 - \xi_{p^m}^{-1} \prod_{l|L} \xi_l).$$

These elements were used by K.Rubin. They satisfy conditions **ES1.**$-$**ES4.** of [Ru1]. Hence they give well defined Euler system $\Lambda(L)$ for higher K-groups in the sense of Definition 2. It is the Euler system whose image by the Dwyer-Friedlander map is the Euler system in etale cohomology of $Spec\,\mathbf{Z}[1/L]$ considered by Kurihara in [Ku, Section 5].

3 The Euler system $\{\Lambda(L, \bar{r});\ L \in S\}$

3.1 Stickelberger elements

For convenience of the reader we recall from [Ba1, p. 328-329] definition of the higher Stickelberger elements. Let $F/\mathbf{Q}$ be an abelian extension with conductor f. Let σ_a denote an automorphism of $\mathbf{Q}(\mu_f)$ such that for a root of unity ξ of order f one has: $\sigma_a(\xi) = \xi^a$. Denote by $(a; F)$ the restriction of σ_a to F. Let b be an integer which is relatively prime to the number $w_{n+1}(\mathbf{Q}(\mu_f))$. Define

$$\Theta_n(b) = (b^{n+1} - (b; F)) \sum_{1 \leq a \leq f,\, (a,f)=1} \zeta_f(a, -n)(a; F)^{-1},$$

where ζ_f is the partial zeta function. Let us write the element $\Theta_n(b)$ as:

$$\Theta_n(b) = \sum_{1 \leq a \leq f,\, (a,f)=1} \Delta_{n+1}(a, b, f)(a; F)^{-1}.$$

By [CS, Theorem 1.2] we know that $\Theta_n(b) \in \mathbf{Z}[G(F/\mathbf{Q})]$. For $F = \mathbf{Q}$ ($f = 1$) we put $\Theta_n(b) = (b^{n+1} - 1)\zeta_{\mathbf{Q}}(-n)$, where $\zeta_{\mathbf{Q}}$ is the Riemann zeta. By [CS, Theorem 3] we have: $\Delta_{n+1}(a,b,f) \equiv (ab)^n \Delta_1(a,b,f) \, mod \, f \prod_{p|f} p^{v_p(n)}$.

3.2 Gauss sums $\lambda_{b_L}(j(\overline{r}))$

The following section is based on the construction of map Λ from [Ba1]. We will use the Euler system of Gauss sums described by Rubin in [Ru2]. Let F be an abelian number field with conductor f. We pick a prime number r such that $(r, fp^m L) = 1$. Choose a natural number b_L such that $(b_L, w_{n+1}(\mathbf{Q}(\mu_{fp^m L}))) = 1$. Let $j : F(\mu_{p^m L}) \to \mathbf{Q}(\mu_{\overline{f}})$ be the natural imbedding, where $\bar{f}$ is the conductor of $F(\mu_{p^m L})$. We are going to use elements $\lambda_{b_L}(j(\overline{r}))$ from [C], see also Definition 3 below. By [C, Proposition 3.8] $(\lambda_{b_L}(j(\overline{r}))) = \overline{r}^{\Theta_0(b_L)}$. Put $\alpha(L) = \alpha(L, \overline{r}) = \lambda_{b_L}(j(\overline{r}))$ and $\Lambda(L) = \Lambda(L, \overline{r})$ as in Definition 2. We will use the following notation from [Ba1].

1. $\gamma_p = 1 + p^n \sigma_p^{-1} + p^{2n} \sigma_p^{-2} + \ldots$ if p does not divide f. Put $\gamma_p = 1$ if $p|f$ where σ_p was introduced in [Ba1, p.335].
2. $\overline{r}$ is a prime ideal in $F(\mu_{p^m L})$ over a rational prime r. $\tilde{r}$ is the prime in $F(\mu_L)$ below $\overline{r}$.
3. $N_{\overline{r}/\tilde{r}} : K_{2n}(k_{\overline{r}}; \mathbf{Z}/p^m) \to K_{2n}(k_{\tilde{r}}; \mathbf{Z}/p^m)$ is the norm homomorphism.
4. $\pi_{\overline{r}} : K_{2n}(\mathcal{O}_{F(\mu_{p^m L})}; \mathbf{Z}/p^m) \to K_{2n}(k_{\overline{r}}; \mathbf{Z}/p^m)$ is the reduction map.
5. $\partial : K_{2n+1}(F(\mu_L); \mathbf{Z}/p^m) \to \bigoplus_v K_{2n}(k_v; \mathbf{Z}/p^m)$ is the boundary map from the localization sequence.
6. $\Theta_n(b_L)$ is the higher Stickelberger element for $F(\mu_L)$.
7. $\Theta_n^*(b_L)$ is the higher Stickelberger element for $F(\mu_{p^m L})$.

Proposition 2

$$\partial(\Lambda(L, \bar{r})^{b_L^n \gamma_p}) = [N_{\overline{r}/\tilde{r}}(\pi_{\overline{r}}(B^n(\xi_{p^m})))]^{\Theta_n(b_L)}$$

Proof. Our proof follows main idea of the proof of [Ba1, Theorem 1]. We do a sketch of the proof. We want to point out that the proof of Theorem 1 of [Ba1] is not completely correct. The mistake will be removed in the present sketch. Consider the following commutative diagrams.

$$\begin{array}{ccc} K_{2n+1}(F(\mu_{p^m L}); \mathbf{Z}/p^m) & \xrightarrow{\partial'} & \bigoplus_w K_{2n}(k_w; \mathbf{Z}/p^m) \\ \downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle N} \\ K_{2n+1}(F(\mu_L); \mathbf{Z}/p^m) & \xrightarrow{\partial} & \bigoplus_v K_{2n}(k_v; \mathbf{Z}/p^m) \end{array}$$

Diagram 3.

Let $\mathcal{O}'$ denote $\mathcal{O}_{F(\mu_{p^m L})}$ and let $\sigma_a^{-1} = (a^{-1}; F(\mu_{p^m L})) = (a; F(\mu_{p^m L}))^{-1}$.

$$\begin{array}{ccc} K_{2n}(\mathcal{O}';\mathbf{Z}/p^m) & \xrightarrow{\pi_{\overline{r}}} & K_{2n}(k_{\overline{r}};\mathbf{Z}/p^m) \\ \downarrow{\scriptstyle \sigma_a^{-1}} & & \downarrow{\scriptstyle \sigma_a^{-1}} \\ K_{2n}(\mathcal{O}';\mathbf{Z}/p^m) & \xrightarrow{\pi_{\overline{r}^{\sigma_a^{-1}}}} & K_{2n}(k_{\overline{r}^{\sigma_a^{-1}}};\mathbf{Z}/p^m) \end{array}$$

Diagram 4.

It follows from commutativity of Diagram 4 that: $\pi_{\overline{r}^{\sigma_a^{-1}}}(B^n(\xi_{p^m})) = [\pi_{\overline{r}}(B^n(\xi_{p^m}))]^{a^n\sigma_a^{-1}}$. The following diagram also commutes cf. Diagram 4.8 in [Ba1].

$$\begin{array}{ccc} K_{2n}(k_{\overline{r}};\mathbf{Z}/p^m) & \xrightarrow{\sigma_a^{-1}} & K_{2n}(k_{\overline{r}^{\sigma_a^{-1}}};\mathbf{Z}/p^m) \\ \downarrow{\scriptstyle N_{\overline{r}/\tilde{r}}} & & \downarrow{\scriptstyle N_{\overline{r}^{\sigma_a^{-1}}/\tilde{r}^{\sigma_a^{-1}}}} \\ K_{2n}(k_{\tilde{r}};\mathbf{Z}/p^m) & \xrightarrow{\sigma_a^{-1}} & K_{2n}(k_{\tilde{r}^{\sigma_a^{-1}}};\mathbf{Z}/p^m) \end{array}$$

Diagram 5.

The equalities below follow from commutativity of Diagrams 3, 4 and 5.

$$\partial(\Lambda(L,\bar{r})^{b_L^n\gamma_p}) = [\partial(\Lambda(L,\bar{r}))]^{b_L^n\gamma_p} =$$

$$= [\partial \circ Tr_{F(\mu_{p^mL})/F(\mu_L)}(\alpha(L;\overline{r}) * B^n(\xi_{p^m}))]^{b_L^n\gamma_p} =$$

$$= [N \circ \partial'(\alpha(L;\overline{r}) * B^n(\xi_{p^m}))]^{b_L^n\gamma_p} = [N([\pi_{\overline{r}}B^n(\xi_{p^m})]^{\Theta_n^*(b_L)})]^{\gamma_p} =$$

$$= [N_{\overline{r}/\tilde{r}} \circ \pi_{\overline{r}}(B^n(\xi_{p^m}))]^{\Theta_n(b_L)}.$$

The mistake in [Ba1] was in computation of $\partial'(...)$ above. The correction goes as follows. Let $\overline{f}$ ($\tilde{f}$) be the conductor of $F(\mu_{p^mL})$ ($F(\mu_L)$ resp.). We will compute the images of ∂' for each ω (a prime ideal of $\mathcal{O}'$). First if $\omega = \overline{r}$ take all c mod $\overline{f}$ such that: $(c;F(\mu_{p^mL}))\overline{r} = \overline{r}$. Because $(c;F(\mu_{p^mL}))$ is the restriction of σ_c to $F(\mu_{p^mL})$ we will continue to use the σ_c notation. Observe that: $B^n(\xi_{p^m})^{(c;F(\mu_{p^mL}))} = B^n(\xi_{p^m})^{c^n}$ cf. Section 2.1.

$$\partial'(\alpha(L;\overline{r}) * B^n(\xi_{p^m})) = (\sum_{\substack{c mod \overline{f} \\ \overline{r}^{\sigma_c}=\overline{r}}} \Delta_1(c,b_L,\overline{f}))[k_{\overline{r}}] * \pi_{\overline{r}}(B^n(\xi_{p^m})) =$$

$$= \pi_{\overline{r}}(B^n(\xi_{p^m}))^{\sum_{\substack{c mod \overline{f} \\ \overline{r}^{\sigma_c}=\overline{r}}} \Delta_1(c,b_L,\overline{r})c^n\sigma_c^{-1}}.$$

Similarly if $\omega \neq \overline{r}$ and $\omega = \overline{r}^{\sigma_a^{-1}}$, then:

$$\partial'(\alpha(L;\overline{r}) * B^n(\xi_{p^m})^{a^n\sigma_a^{-1}}) = (\sum_{\substack{c mod \overline{f} \\ \overline{r}^{\sigma_c}=\overline{r}}} \Delta_1(ac,b_L,\overline{f}))[k_{\overline{r}^{\sigma_a^{-1}}}] * \pi_{\overline{r}^{\sigma_a^{-1}}}(B^n(\xi_{p^m})^{a^n\sigma_a^{-1}}) =$$

$$= \pi_{\overline{r}}(B^n(\xi_{p^m}))^{\sum_{\substack{c mod \overline{f} \\ \overline{r}^{\sigma_c}=\overline{r}}} \Delta_1(ac,b_L,\overline{f})c^na^n\sigma_{ac}^{-1}}.$$

If ω is not a $G(\mathbf{Q}(\mu_{\overline{f}})/\mathbf{Q})$ conjugate of $\overline{r}$, then $\partial'(...) = 0$. From now on the computation goes as in [Ba1]. **Q.E.D.**

Let b be a positive integer such that $(b, p^m w_{n+1}(\mathbf{Q}(\mu_f))) = 1$. Put $M = p^m$. Take L such that and $(L, bMw_{n+1}(\mathbf{Q}(\mu_f)) = 1$. For a moment we assume that $F = \mathbf{Q}(\mu_f)$ and do all computations for this case. At the end of the section we will describe how to deal with the general case. The following notation and comments will be useful in the proof of next lemma.

1. Let be given a character:
$$\epsilon_{L,\overline{r}} : k_{\overline{r}}^{\times} \longrightarrow \mu_{p^m Lf}$$
such that $\epsilon_{L,\overline{r}}(x) \equiv x^{\frac{N\overline{r}-1}{p^m Lf}} \mod \overline{r}$, and $\overline{r}$ denotes a prime ideal in $F(\mu_{p^m L})$ above r.

2. Let $g(L, \overline{r}, \xi_r) = \sum_{x \in k_{\overline{r}}^*} \epsilon_{L,\overline{r}}(x) \xi_r^{tr(x)}$.

3. Observe that b is odd. We can choose a natural number b_L such that
$$b_L \equiv b \, mod \, w_{n+1}(\mathbf{Q}(\mu_f)) p^m ML.$$
In this way σ_{b_L} is an element of $G(F(\mu_{p^m L})/\mathbf{Q})$ whose restriction to $G(F(\mu_{p^m})/\mathbf{Q})$ is σ_b. Let $\tilde{\sigma}_{b_L}$ be the unique element from $G(F(\mu_{p^m Lr})/\mathbf{Q}(\mu_r))$ which restricts in $G(F(\mu_{p^m L})/\mathbf{Q})$ to σ_{b_L}. Observe that b_L is relatively prime to $w_{n+1}(\mathbf{Q}(\mu_{p^m Lf}))$.

4. Let $N(\overline{r}) = r^k$. Then denote $N(\overline{r})^* = r^k$, $((-r)^k$ resp.) if $r \equiv 1 \mod 4$ ($r \equiv 3 \mod 4$, resp.). We know that $\mathbf{Q}(\sqrt{N(\overline{r})^*}) \subset \mathbf{Q}(\mu_r)$, [C, p. 308].

5. Assume that the prime ideal in $F(\mu_{p^m})$ below $\overline{r}$ splits completely in $F(\mu_{p^m L})$, e.g., when $r \equiv 1 \mod L$. Put $\hat{r}$ to be the prime ideal in $F(\mu_{p^m(L/l)})$ below $\overline{r}$. Observe that $N(\overline{r}) = N(\hat{r})$. Hence $k_{\overline{r}} = k_{\hat{r}}$ and $\epsilon_{L,\overline{r}}^l = \epsilon_{L/l,\hat{r}}$.

6. In the same way we define $b_{L/l}$. In this way $\sigma_{b_{L/l}}$ is the restriction of σ_{b_L} to $F(\mu_{p^m L/l})$. We see that restriction of $\tilde{\sigma}_{b_L}$ to $F(\mu_{p^m(L/l)r})$ equals to $\tilde{\sigma}_{b_{L/l}}$. In addition we have: $b_L \equiv b_{L/l} \, mod \, p^m M(L/l) w_{n+1}(\mathbf{Q}(\mu_f))$.

7. Fix a natural number l' such that:
$$ll' \equiv 1 \, mod \, fp^m(L/l)r.$$

Lemma 1
$$N_l \lambda_{b_L}(\overline{r}) \equiv \lambda_{b_{L/l}}(\hat{r})^{1-Fr_l^{-1}} mod F(\mu_{p^m(L/l)})^{\times M}$$

Proof. By [C, Lemma 3.10] we have:
$$N_l(\lambda_{b_L}(\overline{r})) = N_l(g(L, \overline{r}, \xi_r))^{b_L - \tilde{\sigma}_{b_L}} (N(\overline{r})^{*(l-1)(1-b_L)/2}).$$
But $N_l(g(L, \overline{r}, \xi_r))^{b_L - \tilde{\sigma}_{b_L}} = [\prod_{\sigma \in G_l} (\sum_{x \in k_{\overline{r}}^*} \epsilon_{L,\overline{r}}(x)^{\sigma} \xi_r^{tr(x)})]^{b_L - \tilde{\sigma}_{b_L}} =$
$$= [\prod_{\varphi : k_{\overline{r}}^* \rightarrow \mu_l} (\sum_{x \in k_{\overline{r}}^*} \varphi(x) \epsilon_{L,\overline{r}}(x) \xi_r^{tr(x)})]^{b_L - \tilde{\sigma}_{b_L}} [\sum_{x \in k_{\overline{r}}^*} \epsilon_{L,\overline{r}}(x)^{ll'} \xi_r^{tr(x)}]^{\tilde{\sigma}_{b_L} - b_L}.$$
By Davenport-Hasse distribution relation [La] the last expression equals:
$$[(\sum_{x \in k_{\hat{r}}^*} \epsilon_{L/l,\hat{r}}(x) \xi_r^{tr(x)})]^{b_L - \tilde{\sigma}_{b_L}} [\epsilon_{L,\overline{r}}(l)^{-l} (-N(\overline{r})^{(l-1)/2})]^{b_L - \tilde{\sigma}_{b_L}} \times$$
$$\times [(\sum_{x \in k_{\hat{r}}^*} \epsilon_{L/l,\hat{r}}(x) \xi_r^{tr(lx)})^{-Fr_l^{-1}}]^{b_L - \tilde{\sigma}_{b_L}}$$

which further equals:

$$[(\sum_{x\in k_{\hat{r}}^*}\epsilon_{L/l,\hat{r}}(x)\xi_r^{tr(x)})^{1-Fr_l^{-1}}]^{b_L-\tilde{\sigma}_{b_L}}[\epsilon_{L/l,\hat{r}}(l)^{l'-1}]^{b_L-\tilde{\sigma}_{b_L}}[N(\overline{r})^{(l-1)/2}]^{b_L-1}.$$

So we have: $N_l(\lambda_{b_L}(\overline{r}))=[g(L/l,\hat{r},\xi_r)^{1-Fr_l^{-1}}]^{b_L-\tilde{\sigma}_{b_L}}=$

$$=[g(L/l,\hat{r},\xi_r)^{1-Fr_l^{-1}}]^{b_{L/l}-\tilde{\sigma}_{b_{L/l}}}[(\sqrt{N(\hat{r})^*})^{b_{L/l}-\tilde{\sigma}_{b_{L/l}}}]^{Fr_l^{-1}-1}\equiv$$

$$\equiv(\lambda_{b_{L/l}}(\hat{r}))^{1-Fr_l^{-1}}\, mod\, F(\mu_{p^m(L/l)})^{\times M}$$

because $b_L-b_{L/l}\equiv 0 \text{ mod } fp^mM(L/l)$ **Q.E.D.**

Remark 6. Proposition 2 and Lemma 1 remind [Ru2, Prop. 1.1(ii) and (i), resp].

Let us discuss now the general case of any $F/\mathbf{Q}$ abelian with conductor f. In this case let $\overline{f}$ ($\hat{f}$ resp.) denote the conductor of $F(\mu_{p^mL})/\mathbf{Q}$ ($F(\mu_{p^mL/l})/\mathbf{Q}$ resp.). Observe that $\overline{f}=l\hat{f}$. Let $\overline{r}$ be a prime ideal of $F(\mu_{p^mL})$ above $\hat{r}$ in $F(\mu_{p^mL/l})$ such that $\hat{r}$ splits completely in $F(\mu_{p^mL})$. The idea is to imbed the above fields into $\mathbf{Q}(\mu_{\overline{f}})$ ($\mathbf{Q}(\mu_{\hat{f}})$ resp.). Let us call these imbeddings j. There is one to one correspondence between ideals dividing $j(\overline{r})$ and these dividing $j(\hat{r})$. Again we see that the primes dividing $\hat{r}$ split completely in $\mathbf{Q}(\mu_{\overline{f}})$. Now $\lambda_{b_L}(j(\overline{r}))$ is a product of lambdas over corresponding ideals. The same for $j(\hat{r})$. We should observe that:

$$[\prod_{\wp|j(\hat{r})}g(L/l,\wp,\xi_r)]^{\hat{f}}\in F(\mu_{p^mL/l}).$$

The rest of computation is identical. We adjust appropriately the assumptions and notations (1) - (7) to this case, e.g., we take $(b;F(\mu_{p^m}))$ instead of σ_b etc. So Lemma 1 follows also in this case. The proof of the lemma shows that for $b_L=b$ we have:

$$N_l\lambda_b(\overline{r})=\lambda_b(\hat{r})^{1-Fr_l^{-1}}.$$

Corollary 1 *In the notation of (1) - (7) we have:* $Tr_l\Lambda(L,\overline{r})=\Lambda(L/l,\hat{r})^{1-l^nFr_l^{-1}}$.

Proof. It follows immediately by Lemma 1, Proposition 1 and Remark 5. **Q.E.D.**

3.3 A property of Gauss sums

In this section we define elements in K-theory of $F=\mathbf{Q}(\mu_f)$ using a system of Gauss sums which comes from characters with values in f_mth roots of unity with m changing and $f_m=l.c.m(f,p^m)$. Corollary 2 below describes how the elements in $K_{2n+1}(F;\mathbf{Z}/p^m)$ obtained in this way behave under the reduction of coefficients in K-theory. Let us fix notation which we use in this section. Let r and p denote different and fixed odd prime numbers such that $(r,fp)=1$. Let $F_m=\mathbf{Q}(\mu_{f_m})$ and let N_m be the norm map from F_{m+1} down to F_m, and let d_m denote the order of r mod f_m. Let $\bar{r}$ ($\tilde{r}$ resp.) be a prime in F_{m+1} (in F_m respectively) which lies over a rational prime r. We assume that $\bar{r}|\tilde{r}$. Notice that $\mathbf{F}_{r^{d_m}}$ is the residue field of the ideal $\tilde{r}$. Put m_0 to be the biggest m such that $r^{d_1}-1\equiv 0\, mod\, f_m$.

The trace map from $\mathbf{F}_{r^{d_{m+1}}}$ (resp. $\mathbf{F}_{r^{d_m}}$) down to $\mathbf{F}_r$ will be denoted by tr. Let b be a natural number such that $(b, pw_{n+1}(F)) = 1$. Put $\sigma_{b,m}$ to be the automorphism from $G(F_m/\mathbf{Q})$ which acts on μ_{f_m} by raising roots of unity to the bth power. Let $\tilde{\sigma}_{b,m}$ denote the unique automorphism from $G(F_m(\mu_r)/\mathbf{Q}(\mu_r))$ whose restriction to $G(F_m/\mathbf{Q})$ is equal to $\sigma_{b,m}$. Let $\epsilon_{m,\tilde{r}} : \mathbf{F}^{\times}_{r^{d_m}} \rightarrow \mu_{f_m}$ be a character such that: $\epsilon_{m,\tilde{r}}(x) \equiv x^{\frac{r^{d_m}-1}{f_m}} \, mod\, \tilde{r}$. In this section we denote a corresponding Gauss sum by:

$$g'(m,\tilde{r}) = \sum\nolimits_{x\in\mathbf{F}^{\times}_{r^{d_m}}} \epsilon_{m,\tilde{r}}(x)\xi_r^{tr(x)}.$$

Definition 3 *Let* $g(m,\tilde{r}) = (N(\tilde{r})^*)^{-\frac{1}{2}} g'(m,\tilde{r})$ *and* $\lambda_b(m,\tilde{r}) = \frac{g(m,\tilde{r})^b}{g(m,\tilde{r}^{\sigma_{b,m}})} = g(m,\tilde{r})^{b-\tilde{\sigma}_{b,m}}$.

Note that $\lambda_b(m,\tilde{r})$ is the number $\lambda_b(\tilde{r})$ (for the field F_m) in the notation of Section 3.2.

Remark 7. Numbers g and g' depend on the choice of a root of unity ξ_r. λ_b is independent of this choice. Note that $\lambda_b(m,\tilde{r}) \in F_m^{\times}$ by [C, p. 309]. The following proposition and corollary show how the numbers $\lambda_b(m,\tilde{r})$ behave under norm map N_m and how elements of K-groups of F constructed out of them by the method introduced in Section 2.3 transform under reduction of coefficients in K-theory.

Proposition 3

(i) *If* $m \geq m_0$, *then* $N_m \lambda_b(m+1,\bar{r}) = [\lambda_b(m,\tilde{r})]^p$.

(ii) *If* $m < m_0$, *then* $N_m \lambda_b(m+1,\bar{r}) = \lambda_b(m,\tilde{r})$.

Proof. (i) Let $R_m = \frac{N_m g(m+1,\bar{r})}{g(m,\tilde{r})^p} \in F_m(\mu_r)$. We will prove that $R_m \in \mu_{f_m}$. From this fact and Definition 3 we get (i). Our proof of this goes in two steps. Let $\tau \in G(F_m(\mu_r)/\mathbf{Q})$ be the complex conjugation. We will prove:

(1) $R_m^{\tau} = R_m^{-1}$

(2) $R_m \in \mathcal{O}^{\times}_{F_m(\mu_r)}$

Let us prove (1). It follows from definitions that:

$$R_m^{\tau} = \frac{N_m((g'(m+1,\bar{r})(N(\bar{r})^*)^{-\frac{1}{2}})^{\tau})}{(g'(m,\tilde{r})(N(\tilde{r})^*)^{-\frac{1}{2}})^{\tau p}} =$$

$$= \frac{N_m(r^{d_{m+1}} g'(m+1,\bar{r})^{-1})(\pm(N(\bar{r})^*)^{-\frac{1}{2}})}{(r^{d_m} g'(m,\tilde{r})^{-1}(\pm(N(\tilde{r})^*)^{-\frac{1}{2}}))^p} = [\frac{N_m(g'(m+1,\bar{r})(N(\bar{r})^*)^{-\frac{1}{2}})}{(g'(m,\tilde{r})(N(\tilde{r})^*)^{-\frac{1}{2}})^p}]^{-1} = R_m^{-1}.$$

Above we have used: $g'(m+1,\bar{r})^{\tau} = r^{d_{m+1}} g'(m+1,\bar{r})^{-1}$ and $g'(m,\tilde{r})^{\tau} = r^{d_m} g'(m,\tilde{r})^{-1}$, [La, p.4 **GS2**]. We also used: $((N(\bar{r})^*)^{-\frac{1}{2}})^{\tau} = \pm(N(\bar{r})^*)^{-\frac{1}{2}}$ and $((N(\tilde{r})^*)^{-\frac{1}{2}})^{\tau} = \pm(N(\tilde{r})^*)^{-\frac{1}{2}}$. The signs $(\pm)$ above always agree because $d_{m+1} = pd_m$ and p is odd. Also $N_m(-1) = (-1)^p = -1$. In addition $r^{d_{m+1}}((N(\bar{r})^*)^{-\frac{1}{2}}) = (N(\bar{r})^*)^{\frac{1}{2}}$ $(= (-1)^{d_{m+1}}(N(\bar{r})^*)^{-\frac{1}{2}}$ resp.) if $r \equiv 1 \mod 4$ ($r \equiv 3 \mod 4$, resp.). Similarly for $\tilde{r}$ with d_m instead of d_{m+1}. But $(-1)^{d_{m+1}} = (-1)^{d_m}$. It proves (1).

Next we show (2). By [C, Proposition 3.8] we have: $(g'(m,\tilde{r})^k) = \tilde{r}^{k\Theta_{0,m}}$, where k is a multiple of f_m and $\Theta'_{0,m} = \sum_{0<a<f_m,(a,f_m)=1}(1-\frac{a}{f_m})\sigma_a^{-1}$ is the Stickelberger element for the field F_m [Wa, p.93]. We will use the 0-th Stickelberger elements of Coates and Sinnott [Wa, p.95]:

$$\Theta_{0,m} = \sum_{0<a<f_m,(a,f_m)=1}(\frac{1}{2}-\frac{a}{f_m})\sigma_a^{-1} = \Theta'_{0,m} - \frac{1}{2}N$$

where $N = \sum_{\sigma\in G(F_m/\mathbf{Q})}\sigma$. By [C, p. 308] we get:

$$(g(m+1,\bar{r})^{w_1(F_{m+1})}) = \bar{r}^{w_1(F_{m+1})\Theta_{0,m+1}}.$$

Let Res be the restriction map $G(F_{m+1}/\mathbf{Q}) \longrightarrow G(F_m/\mathbf{Q})$. Since $Res(\Theta_{0,m+1}) = \Theta_{0,m}$ by [CS, p.159], one obtains:

$$(N_m g(m+1,\bar{r})^{w_1(F_{m+1})}) = (\mathcal{N}_m(\bar{r}^{w_1(F_{m+1})\Theta_{0,m+1}})) =$$

$$= (\mathcal{N}_m(\bar{r}))^{w_1(F_{m+1})Res\Theta_{0,m+1}} = \tilde{r}^{pw_1(F_{m+1})\Theta_{0,m}},$$

where $m \geq m_0$. We also used: $\mathcal{N}_m(\wp^\sigma) = \mathcal{N}_m(\wp)^{Res\sigma}$, where $\mathcal{N}_m(\wp)$ is the norm of a prime ideal $\wp$ from F_{m+1} calculated down to F_m. It follows that:

$$(R_m^{w_1(F_{m+1})}) = (\frac{N_m g(m+1,\bar{r})^{w_1(F_{m+1})}}{g(m,\tilde{r})^{pw_1(F_{m+1})}}) = \frac{\tilde{r}^{pw_1(F_{m+1})\Theta_{0,m}}}{\tilde{r}^{pw_1(F_{m+1})\Theta_{0,m}}} = \mathcal{O}_{F_m}$$

hence $R_m^{w_1(F_{m+1})} \in \mathcal{O}_{F_m}^\times$ and $R_m \in \mathcal{O}_{F_m(\mu_r)}^\times$. In order to show that R_m is an rf_m-th root of unity, by [Wa, Lemma 1.6], it is enough to show that for every $\sigma \in G(F_m(\mu_r)/\mathbf{Q})$ one has $|R_m^\sigma| = 1$. Namely for any σ one has $|R_m^\sigma|^2 = R_m^\sigma R_m^{\sigma\tau} = R_m^\sigma R_m^{-\sigma} = 1$. So $R_m \in \mu_{f_m r}$. Because $(r, f_m) = 1$ and $\mathcal{O}_{F_m} \cap \mu_{f_m r} = \mu_{f_m}$ we see that $R_m \in \mu_{f_m}$. Notice that $b - \tilde{\sigma}_{b,m}$ anihilates μ_{f_m}. It proves (i).

Now we prove (ii). Observe that for $m < m_0$, $d_m = d_1$. So $N(\bar{r}) = N(\tilde{r})$. Hence the prime ideal $\tilde{r}$ of F_m splits completely in F_{m+1}. Let $k = max\{m \geq 1; F_m = F_1\}$. Then $k = max\{m \geq 1; f_m = f_1\}$. We see immediately that $m_0 \geq k$. For $m < k$ we have $\bar{r} = \tilde{r}$, $F_{m+1} = F_m$, $N_m = id$, $g(m+1,\bar{r}) = g(m,\tilde{r})$. So this case is obvious.
If $m \geq k$ we proceed as follows. Consider:

$$N_m(g'(m+1,\bar{r})) = \prod_{k=0}^{p-1}\sum_{x\in\mathbf{F}_{r^{d_{m+1}}}}\epsilon_{m+1,\bar{r}}(x)^{1+kf_m}\xi_r^{tr(x)} =$$

$$= \prod_{\varphi:\mathbf{F}^*_{r^{d_{m+1}}}\to\mu_p}\sum_{x\in\mathbf{F}_{r^{d_{m+1}}}}\epsilon_{m+1,\bar{r}}(x)\varphi(x)\xi_r^{tr(x)} =$$

By Davenport-Hasse distribution relation [La]:

$$= (\sum_{x\in\mathbf{F}_{r^{d_{m+1}}}}\epsilon_{m+1,\bar{r}}(x)^p\xi_r^{tr(x)})\epsilon_{m+1,\bar{r}}(p^{-p})\prod_{\varphi:\mathbf{F}^*_{r^{d_{m+1}}}\to\mu_p}(\sum_{x\in\mathbf{F}_{r^{d_{m+1}}}}\varphi(x)\xi_r^{tr(x)}) =$$

$$= g'(m,\tilde{r})\epsilon_{m,\tilde{r}}(p^{-1})(-1)N(\bar{r})^{\frac{p-1}{2}}$$

because $\epsilon_{m+1,\bar{r}}^p = \epsilon_{m,\tilde{r}}$. Observe that the operator N_m applied to g' at the begining of the computation is in $\mathbf{Z}[G(F_{m+1}(\mu_r)/F_m(\mu_r))]$. Hence

$$N_m(g(m+1,\bar{r})) = \pm g'(m,\tilde{r})\epsilon_{m,\tilde{r}}^{-1}(p)((N(\bar{r})^*)^{-1/2})^p = \pm g(m,\tilde{r})\epsilon_{m,\tilde{r}}^{-1}(p)$$

Now everything follows by application of $b - \tilde{\sigma}_{b,m}$ to the above equation. **Q.E.D.**

Remark 8. Let $m \geq m_0$. Observe that R_m does not depend on the choice of ξ_r. Indeed:

$$\frac{r^{d_{m+1}}-1}{f_{m+1}} - \frac{r^{d_m}-1}{f_m} = \frac{r^{d_m}-1}{pf_m}(1 + r^{d_m} + r^{2d_m} + ... + r^{(p-1)d_m} - p) =$$

$$= \frac{(r^{d_m}-1)^2}{pf_m}(1 + (1 + r^{d_m}) + (1 + r^{d_m} + r^{2d_m}) + ... + (1 + r^{d_m} + ... + r^{(p-2)d_m})) =$$

$$= \frac{(r^{d_m}-1)^2}{pf_m}(\frac{p(p-1)}{2} + (r^{d_m}-1)(....)).$$

Hence we get:

$$\frac{r^{d_{m+1}}-1}{f_{m+1}} \equiv \frac{r^{d_m}-1}{f_m}\ mod\ \frac{r^{d_m}-1}{f_m}(r^{d_m}-1).$$

It shows that for any $x \in \mathbf{F}_{r^{d_m}}$ (e.g., if $x \in \mathbf{F}_r$) we have: $\epsilon_{m,\tilde{r}}(x) = \epsilon_{m+1,\bar{r}}(x)$, where we understand the imbedding $\mathbf{F}_{r^{d_m}} \rightarrow \mathbf{F}_{r^{d_{m+1}}}$ as the natural map $\mathcal{O}_{F_m}/\tilde{r} \rightarrow \mathcal{O}_{F_{m+1}}/\bar{r}$. The claim of the remark follows.

Let $\Lambda(m) = Tr_{F_m/F}(\lambda_b(m,\tilde{r}) * B^n(\xi_{p^m})) \in K_{2n+1}(F;\mathbf{Z}/p^m)$ be as in Definition 2. Let r_{m+1} be the reduction of coefficients on K-groups induced by the projection $\mathbf{Z}/p^{m+1} \rightarrow \mathbf{Z}/p^m$. Then we have the following.

Corollary 2

(i) *If $m \geq m_0$, then $r_{m+1}(\Lambda(m+1)) = [\Lambda(m)]^p$.*

(ii) *If $m < m_0$, then $r_{m+1}(\Lambda(m+1)) = \Lambda(m)$.*

Proof. It goes by repetition of the argument given in the proof of Proposition 1. For simplicity we denote Bott element from $K_2(\mathcal{O}_{F_m};\mathbf{Z}/M)$ by B_m. Indicies after Tr indicate transfer maps in K-groups of fields with finite coefficients. Let j_m be the map induced in K-theory by inclusion of fields $F_m \longrightarrow F_{m+1}$. By Proposition 3(i) and basic properties of transfer maps given in [So3] we have the following equalities:

$$r_{m+1}(\Lambda(m+1)) = Tr_{F_{m+1}/F} \circ r_{m+1}(\lambda_b(m+1,\bar{r}) * B_{m+1}^n) =$$

$$= Tr_{F_m/F} \circ Tr_{F_{m+1}/F_m}(r_{m+1}(\lambda_b(m+1,\bar{r})) * r_{m+1}B_{m+1}^n) =$$

$$= Tr_{F_m/F} \circ Tr_{F_{m+1}/F_m}(r_{m+1}(\lambda_b(m+1,\bar{r})) * j_m B_m^n) =$$

$$= Tr_{F_m/F}(N_m\lambda_b(m+1,\bar{r}) * B_m^n) = Tr_{F_m/F}(\lambda_b(m,\tilde{r})^p * B_m^n) = \Lambda(m)^p,$$

in case if $m \geq m_0$. It proves (i). (ii) follows in the same way. **Q.E.D.**

4 The Euler system $\{\Lambda(L,\bar{r});\ L \in S\}$ and $K_*(Z)$

4.1 Divisible elements in $K_{2n}(\mathbf{Q})$ for n odd.

For an abelian p-torsion group A we denote by $div\, A = \bigcap_{k\geq 1} p^k A$ the subgroup of divisible elements. It follows from the localization sequence that $K_{2n+1}(F)$ does not have nontrivial divisible elements. On the other hand the group $K_{2n}(F)$ may have nontrivial divisible elements cf. [Ba2, Section II]. It follows from the exact sequence:

$$0 \to K_{2n}(\mathcal{O}) \to K_{2n}(F) \to \bigoplus_{\nu} K_{2n-1}(k_\nu) \to 0$$

that the group $D_{n+1}(F) = div\,[K_{2n}(F)_p]$ is contained in $K_{2n}(\mathcal{O})_p$ [loc. cit]. Hence $D_{n+1}(F)$ is finite. The first author together with M.Kolster proved in [Ba2, Theorem 3] that for n odd, p odd and F totally real one has the following formula for the number of elements of the group $D_{n+1}(F)$.

$$|D_{n+1}(F)| \;=\; |\frac{w_{n+1}(F)\zeta_F(-n)}{\prod_{\nu|p} w_n(F_\nu)}|_p^{-1}$$

Proof of the formula given in [Ba2] relies on "the Main Conjecture" in Iwasawa theory proven by A.Wiles and also on a result of P.Schneider. We would like to point out that for n odd, p an odd prime and $F = \mathbf{Q}$ one has: $|D_{n+1}(\mathbf{Q})| \;=\; |w_{n+1}(\mathbf{Q})\zeta_{\mathbf{Q}}(-n)|_p^{-1}$ which is the number of elements in $K_{2n}(\mathbf{Z})_p$ predicted by the Quillen-Lichtenbaum conjecture. Hence in this case the conjecture is equivalent to saying that $K_{2n}(\mathbf{Z})_p$ is divisible in $K_{2n}(\mathbf{Q})_p$. In this section we are going to investigate the number of divisible elements in $K_{2n}(\mathbf{Q})_p$ for n odd using the Euler system of elements $\Lambda(L,\bar{r})$. In the last section of the paper we prove some results concerning $D_{n+1}(F)$ for n even.

4.2 Kolyvagin operator and elements $\delta_n(L)$

In this section $F = \mathbf{Q}$, $E = \mathbf{Q}(\mu_{p^m})$ and $M = p^m$. Let $G_L = G(E(\mu_L)/E)$. We denote by $w : (\mathbf{Z}/p)^\times \to \mathbf{Z}_p^\times$ the Teichmüller character defined by a congruence: $w(c_0) \equiv c_0 \, mod\, p$. Let $\Delta = \{\sigma_c \in G(E/\mathbf{Q});\ c \equiv c_0 \, mod\, p,\ c \equiv w(c_0)\, mod\, M\}$. Let D_L denote the Kolyvagin operator [Ru2, p.310].

Lemma 2

(i) $\Lambda^*(L,\bar{r})^{D_L} \in K_{2n+1}(E(\mu_L);\mathbf{Z}/M)^{G_L}$ *where* $\Lambda^*(L,\bar{r})$ *is the Euler system from Section 3 constructed for the base field* E.

(ii) $\Lambda(L,\bar{r})^{D_L} \in K_{2n+1}(F(\mu_L);\mathbf{Z}/M)^{G_L}$

(iii) $\Theta_n^*(b_L)D_L \in (\mathbf{Z}/M)[G(E(\mu_L)/\mathbf{Q})]^{G_L}$

(iv) $\Theta_n(b_L)D_L \in (\mathbf{Z}/M)[G(F(\mu_L)/\mathbf{Q})]^{G_L}$.

Proof. (i) Proof by induction on number of divisors of L as in [Ru2]. $\Lambda^*(L,\bar{r})^{D_L(\sigma_l-1)} = \Lambda^*(L,\bar{r})^{D_{L/l}(l-1-N_l)} \;=\; i(\Lambda^*(L/l,\tilde{r})^{D_{L/l}(l^n Fr_l^{-1}-1)}) \;=\; i(\Lambda^*(L/l,\tilde{r})^{D_{L/l}(Fr_l^{-1}-1)}) \;=\; 0$ because

$l \equiv 1$ mod M and by induction hypothesis. First step of the induction is obvious. We denote by i the natural map on K-theory induced by the inclusion $F(\mu_{L/l}) \subset F(\mu_L)$. We see that $i \circ Tr_l = N_l$ on K-groups.

(iii) By (i) $\Lambda^*(L,\bar{r})^{D_L(\sigma_l - 1)} = 0$ in $K_{2n+1}(E(\mu_L); \mathbf{Z}/M)$. Pick $r \equiv 1$ mod M. Then by Proposition 2:

$$\partial'(\Lambda^*(L,\bar{r})^{b_L^n D_L(\sigma_l-1)}) = \pi_{\overline{r}}(B^n(\xi_{p^m}))^{\Theta_n^*(b_L) D_L(\sigma_l - 1)}.$$

By assumption on r the element $\pi_{\overline{r}}(B^n(\xi_{p^m}))$ has order M. So $\Theta_n^*(b_L)D_L(\sigma_l - 1) \equiv 0 \, mod \, M$ for each l.

(ii) and (iv) are proven in the same way as (i) and (iii). **Q.E.D.**

Now similarly to K. Rubin we can conclude by Lemma 2(iii) and (iv) that there are elements $\delta_n^*(L) \in (\mathbf{Z}/M)[G(E/\mathbf{Q})]$ and $\delta_n(L) \in \mathbf{Z}/M$ such that:

$$\Theta_n^*(b_L) D_L \equiv N_L \delta_n^*(L) \, mod \, M$$

$$\Theta_n(b_L) D_L \equiv N_L \delta_n(L) \, mod \, M.$$

We can also repeat [Ru2, Lemma 2.1] for E as the base field to get ane element $\delta_0^*(L)$ in $(\mathbf{Z}/M)[G(E/\mathbf{Q})]$ such that: $\Theta_0^*(b_L)D_L \equiv N_L \delta_0^*(L) \, mod \, M$.

Lemma 3 $Res_{E/\mathbf{Q}}(\delta_n^*(L)) = (1 - p^n)\delta_n(L)$

Proof. Applying $Res_{E(\mu_L)/\mathbf{Q}(\mu_L)}$ to the congruence which defines $\delta_n^*(L)$ we get:

$$(1 - p^n)\Theta_n(b_L) D_L \equiv N_L(Res_{E/\mathbf{Q}}(\delta_n^*(L))) \, mod \, M.$$

On the other hand we have:

$$(1 - p^n)\Theta_n(b_L) D_L \equiv (1 - p^n) N_L(\delta_n(L)) \, mod \, M.$$

Lemma 3 follows by comparison of these congruencies. **Q.E.D.**

Our next objective is to find a relationship between $\delta_n^*(L)$ and $\delta_0^*(L)$. Observe that:

$$\Theta_n^*(b_L) = \sum_{(a,p^m L)=1, 1 \le a < p^m L} \Delta_{n+1}(a, b_L, p^m L)\sigma_a^{-1}.$$

Let us write:

$$D_L = \sum_{(a_2,L)=1, 1 \le a_2 < L} d_{a_2} \sigma_{a_2}^{-1}$$

$$N_L = \sum_{(a_2,L)=1, 1 \le a_2 < L} \sigma_{a_2}^{-1}$$

$$\delta_n^*(L) = \sum_{(a_0,p^m)=1, 1 \le a_0 < p^m} N_{a_0} \sigma_{a_0}^{-1}$$

$$\delta_0^*(L) = \sum_{(a_0,p^m)=1, 1 \le a_0 < p^m} n_{a_0} \sigma_{a_0}^{-1}.$$

Lemma 4

$$N_{a_0} \equiv a_0^n b_L^n n_{a_0} \, mod \, M$$

Proof. $\Theta_n^*(b_L)D_L = \sum_{(a_2,L)=1, 1\leq a_2<L} \sum_{(a,p^mL)=1, 1\leq a<p^mL} \Delta_{n+1}(a, b_L, p^m L)\sigma_a^{-1} d_{a_2} \sigma_{a_2}^{-1} =$

$$= \sum_{(a_2,L)=1, 1\leq a_2<L} [\sum_{a \leftrightarrow (a_0,a_1)} \Delta_{n+1}(a, b_L, p^m L) d_{a_2} \sigma_{a_0}^{-1} \sigma_{a_1}^{-1}] \sigma_{a_2}^{-1} =$$

$$= \sum_{(a',L)=1, 1\leq a'<L} \sum_{(a_0,p^m)=1, 1\leq a_0<p^m} [\sum_{a_1 a_2 \equiv a' mod L} \Delta_{n+1}(a, b_L, p^m L) d_{a_2}] \sigma_{a_0}^{-1} \sigma_{a'}^{-1}.$$

In the summation above we replaced $a \in (\mathbf{Z}/p^m L)^\times$ by $(a_0, a_1) \in (\mathbf{Z}/p^m)^\times \times (\mathbf{Z}/L)^\times$ using the obvious isomorphism: $\mathbf{Z}/p^m L \rightarrow \mathbf{Z}/p^m \times \mathbf{Z}/L$. On the other hand:

$$N_L \delta_n^*(L) = \sum_{(a',L)=1, 1\leq a'<L} \sum_{(a_0,p^m)=1, 1\leq a_0<p^m} N_{a_0} \sigma_{a_0}^{-1} \sigma_{a'}^{-1}.$$

Hence comparison of the above computations gives:

$$N_{a_0} \equiv \sum_{a_1 a_2 \equiv a' mod L} \Delta_{n+1}(a, b_L, p^m L) d_{a_2} \, mod \, M$$

for all a_0 and all a'. In the same way we get:

$$n_{a_0} \equiv \sum_{a_1 a_2 \equiv a' mod L} \Delta_1(a, b_L, p^m L) d_{a_2} \, mod \, M.$$

In addition, by the Coates-Sinnott congruence [CS, Theorem 1.3. (ii)] we have:

$$N_{a_0} \equiv \sum_{a_1 a_2 \equiv a' mod L} a^n b_L^n \Delta_1(a, b_L, p^m L) d_{a_2} \equiv a_0^n b_L^n n_{a_0} \, mod \, M$$

because $a \equiv a_0 \, mod \, M$. **Q.E.D.**

4.3 Elements $K(L, \bar{r})$

Fix $r \equiv 1$ mod $p^m L$ and fix $\hat{r}$ an ideal of E over r. Choose a prime ideal $\bar{r}$ in $E(\mu_L)$ lying over $\hat{r}$. We know that $\lambda_{b_L}(\bar{r})^{D_L} \in (E(\mu_L)^\times / E(\mu_L)^{\times M})^{G(E(\mu_L)/E)}$ by Lemma 1 and [Ru2, Lemma 2.1] applied to these elements. By [Ru2, Lemma 2.2] the following map:

$$(E^\times / E^{\times M})^\chi \rightarrow ((E(\mu_L)^\times / E(\mu_L)^{\times M})^{G(E(\mu_L)/E)})^\chi$$

is an isomorphism for each odd character $\chi \neq w$ of Δ. Similarly to [Ru2, Prop. 2.3] define $k(L, \hat{r}) \in E^\times$ by a congruence: $k(L, \hat{r}) \equiv \lambda_{b_L}(\bar{r})^{D_L e_\chi} \, mod \, E(\mu_L)^{\times M}$. Observe that $k(L, \hat{r})$ does not depend on the choice of the prime $\bar{r}$ over $\hat{r}$. Define also $K(L, \bar{r}) \in K_{2n+1}(\mathbf{Q}; \mathbf{Z}/M)$ by the equality:

$$K(L, \bar{r}) = Tr_{E/\mathbf{Q}}(k(L, \hat{r}) * B^n(\xi_{p^m})).$$

Proposition 4

(i) *If $l \not| Lr$, then* $[k(L, \hat{r})]_l = 0$.

(ii) $[k(L;\hat{r})]_r = \delta_0^*(L)e_\chi \hat{r}$

(iii) *If* $l \not| Lr$, *then* $\partial_l(K(L,\bar{r})) = 0$.

(iv)

$$\partial_r(K(L,\bar{r})^{b_L^n\gamma_p}) = \begin{cases} (N_{\bar{r}/r}(\pi_{\bar{r}}B^n(\xi_{p^m})))^{\delta_n(L)}; & if\ \chi = w^{-n} \\ 0; & if\ \chi \neq w^{-n}. \end{cases}$$

Proof. (i) and (ii) are proven similarly to [Ru2, Proposition 2.3]. (iii) follows from (i). To prove (iv) we denote by i the map between Quillen localization sequences induced by imbeddings: $\mathbf{Q} \to \mathbf{Q}(\mu_L)$ and $E \to E(\mu_L)$. We also put $\partial_r, \partial_r'', \partial_r'$ to be the boundary operators in the Quillen localization sequences for $\mathbf{Q}, \mathbf{Q}(\mu_L), E(\mu_L)$, respectively. One has:

$$i\partial_r(K(L,\bar{r})^{b_L^n\gamma_p}) = \partial_r'' i(K(L,\bar{r})^{b_L^n\gamma_p}) =$$

$$= \partial_r''(Tr_{E(\mu_L)/\mathbf{Q}(\mu_L)} \circ i((k(L,\hat{r}) * B^n(\xi_{p^m}))^{b_L^n\gamma_p})) = N'[\partial_r'(\lambda_{b_L}(\bar{r})^{D_L e_\chi} * B^n(\xi_{p^m}))^{b_L^n\gamma_p}] =$$

$$= (N'(\pi_{\bar{r}}B^n(\xi_{p^m})))^{\Theta_n(b_L)D_L e_{\chi w^n}} = (N'(\pi_{\bar{r}}B^n(\xi_{p^m})))^{N_L\delta_n(L)e_{\chi w^n}} =$$

$$= i(N_{\bar{r}/r}(\pi_{\bar{r}}B^n(\xi_{p^m}))^{\delta_n(L)e_{\chi w^n}}).$$

The second equality above holds because $i \circ Tr_{E/\mathbf{Q}} = Tr_{E(\mu_L)/\mathbf{Q}(\mu_L)} \circ i$.
But $i : K_{2n}(\mathbf{F}_r; \mathbf{Z}/M) \to \oplus_{\wp|r} K_{2n}(k_\wp; \mathbf{Z}/M)$ (induced by $\mathbf{Q} \to \mathbf{Q}(\mu_L)$) is an imbedding and $Res_{E(\mu_L)/\mathbf{Q}(\mu_L)}(e_{\chi w^n}) = 0$ if $\chi w^n \neq 1$ and $Res_{E(\mu_L)/\mathbf{Q}(\mu_L)}(e_{\chi w^n}) = 1$ if $\chi w^n = 1$. So (iv) follows. **Q.E.D.**

Next proposition follows easily from Proposition 3 and Corollary 2. Recall the number m_0 from Proposition 3.

Proposition 5 *Let* $K(m)$ $(K(m-1))$ *denote the element* $K(L,\bar{r})$ $(K(L,\tilde{r}))$ *defined for the field* $\mathbf{Q}(\mu_{p^m L})$ $(\mathbf{Q}(\mu_{p^{m-1}L})$, *resp). Here* $\tilde{r}$ *denotes the prime ideal in* $\mathbf{Q}(\mu_{p^{m-1}L})$ *below* $\bar{r}$.

(i) *If* $m-1 \geq m_0$, *then* $r_m(K(m)) = [K(m-1)]^p$.

(ii) *If* $m-1 < m_0$, *then* $r_m(K(m)) = K(m-1)$.

Definition 4 *Let* $\wp|l$ *be a prime ideal of* E. *Define*

$$\varphi_\wp : \{x \in E^\times/E^{\times M}; [x]_l = 0\} \to (\mathbf{Z}/M)[G(E/\mathbf{Q})]$$

$$\varphi_\wp(x) \equiv \sum_{\sigma \in G(E/\mathbf{Q})} v_{\wp^\sigma}(\bar{x})\sigma \bmod M,$$

where $\bar{x}$ *is a lifting of* x *to* $E^\times$ *prime to* l.

We have the following analog for $E/\mathbf{Q}$ of [Ru2, Theorem 2.4].

Theorem 1 *Let* $L \in S$ *and let* $\hat{r}$ *and* $\hat{r}'$ *be primes of* E *over distinct rational primes* $r \equiv r' \equiv 1 \bmod M^2L$. *Let* $A(L)$ *be the* p *part of the ideal class group of* $E(\mu_L)$ *and let* $\bar{r}$ *and* $\bar{r}'$ *be ideals of* $E(\mu_L)$ *above* $\hat{r}$ *and* $\hat{r}'$ *respectively whose projections into* $A(L)^\chi$ *are the same. Then:* $\varphi_{\hat{r}}(k(L,\hat{r}')) \equiv \delta_0^*(Lr)e_\chi \bmod \delta_0^*(L)e_\chi(\mathbf{Z}/M)[G(E/\mathbf{Q})]$.

Proof. Observe that we have:

$$k(L,\hat{r}) = \lambda_{b_L}(\bar{r})^{D_L e_\chi} = G(\epsilon_{L,\bar{r}};\psi)^{(b_L-\tilde{\sigma}_{b_L})D_L e_\chi} = G(\epsilon_{L,\bar{r}}^{-1};\psi)^{(\tilde{\sigma}_{b_L}-b_L)D_L e_\chi},$$

where we have used a more standard notation for Gauss sums: $G(\epsilon_{L,\bar{r}};\psi) = g(L,\bar{r},\xi_r)$ and $\psi(x) = \xi_r^{Trx}$ in order to indicate that g depends on $\epsilon_{L,\bar{r}}$. It follows that we are able to use proof of Theorem 2.4. of [Ru2] if we adjust it to our situation. Below is a list of necessary adjustments.

1) Lemma 2.5 holds without any change in our case.

2) Corollary 2.6 holds if one puts: $s'(L,r) = \sum_{a\in Z/ML^\times} v_r(-\frac{1}{(at)!})\tau_a^{-1}\, mod\, M$, where $t = \frac{r-1}{ML}$ for $r \equiv 1\, mod\, M^2L$, $\Theta'(L,r,\chi) = e_\chi(\sigma_{b_L} - b_L)s'(L,r)$, and put also:

$$\Theta(L,\chi) \;=\; e_\chi\Theta(L) \;=\; \frac{1}{ML}(\sigma_{b_L} - b_L)s(L).$$

3) In Proposition 2.7 we need to take E instead of F, r as above, $\hat{r}$ a prime of E above r, and $\pi \in E^\times$. The conclusion of the proposition says in our case that:

$$\varphi_{\hat{r}}(\frac{k(L,\hat{r})}{\pi}\delta_0^*(L)e_\chi) \;=\; \delta_0^*(Lr)e_\chi,$$

where $\varphi_{\hat{r}}$ is as in Definition 4.

4) Proposition 2.8 reformulates as follows.

Let $L \in S$, $\hat{r}$, $\hat{r}'$ be primes of E lying above distinct rational primes r and r' such that $r \equiv r' \equiv 1\, mod\, M$. Let $A(L)$ be the p-part of the ideal class group of $E(\mu_L)$ and let us suppose that $\bar{r}$ and $\bar{r}'$ are prime ideals of $E(\mu_L)$ above $\hat{r}$ and $\hat{r}'$, respectively, such that they map to the same class in $A(L)^\chi$. Then we have:

$$k(L,\hat{r}')/k(L,\hat{r}) \;\in\; \frac{E^{\times\delta_0^*(L)e_\chi}E^{\times M}}{E^{\times M}}.$$

Proof of Theorem 1 now goes in the same way as the proof of Theorem 2.4 in [Ru2]. **Q.E.D.**

Let $\mathcal{O}_E$ be the ring of integers in E, and let $\wp$ be a prime ideal in $\mathcal{O}_E$ with a residue field $k_\wp$. Let $\pi_\wp : \mathcal{O}_E \to k_\wp$ be the reduction map. As usual, $\pi_\wp$ denotes also the map induced on K-theory. Observe that the following diagram commutes.

$$\begin{array}{ccc} K_{2n+1}(\mathcal{O}_E[1/Lr];\mathbf{Z}/M) & \stackrel{\oplus_{\wp|l}\pi_\wp}{\longrightarrow} & \bigoplus_{\wp|l} K_{2n+1}(k_\wp;\mathbf{Z}/M) \\ \downarrow Tr & & \downarrow \oplus N_\wp \\ K_{2n+1}(\mathbf{Z}[1/Lr];\mathbf{Z}/M) & \stackrel{\pi_l}{\to} & K_{2n+1}(\mathbf{F}_l;\mathbf{Z}/M) \end{array}$$

Diagram 6

Notice that $K(L,\bar{r}) \in K_{2n+1}(\mathbf{Z}[1/Lr];\mathbf{Z}/M)$. Take $l \not| Lr$. Let g_l be a generator of $\mathbf{F}_l^\times$. For any prime ideal $\wp$ of E such that $\wp|l$ define $v_\wp$ as in [Ru2] by the congruence: $k(L,\hat{r}) \equiv g_l^{v_\wp(k(L,\hat{r}))}$ mod $\wp$.

Theorem 2 *Let $(l, Lr) = 1$ and let $l \equiv 1 \, mod \, M^2 L$. Then there is an integer number k such that:* $\pi_l(K(L,\bar{r})^{b_L^n \gamma_p}) = N_\wp(g_l * \pi_\wp(B^n(\xi_{p^m})))^{\delta_n(Ll)+\delta_n(L)k}$.

Proof. $\pi_l(K(L,\bar{r})^{b_L^n\gamma_p}) = (\pi_l(K(L,\bar{r})))^{b_L^n\gamma_p} = (\prod_{\wp | l} N_\wp(\pi_\wp(k(L,\hat{r}) * B^n(\xi_{p^m}))))^{b_L^n\gamma_p} =$

$$= (\prod_{1\le a<p^m,(a,p)=1} N_{\wp^{\sigma_a^{-1}}}(g_l^{v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r}))} * \pi_{\wp^{\sigma_a^{-1}}}(B^n(\xi_{p^m}))))^{b_L^n\gamma_p} =$$

$$= (\prod_{1\le a<p^m,(a,p)=1} N_{\wp^{\sigma_a^{-1}}}((g_l * \pi_{\wp^{\sigma_a^{-1}}}(B^n(\xi_{p^m})^{a^n\sigma_a^{-1}}))^{v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r}))}))^{b_L^n\gamma_p} =$$

$$= (\prod_{1\le a<p^m,(a,p)=1} (N_\wp(g_l * \pi_\wp(B^n(\xi_{p^m})))^{a^n v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r}))\sigma_a^{-1}})^{b_L^n\gamma_p} =$$

$$= ((N_\wp(g_l * \pi_\wp(B^n(\xi_{p^m}))))^{\sum_{1\le a<p^m,(a,p)=1} a^n v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r}))\sigma_a^{-1}})^{b_L^n\gamma_p}$$

$$= ((N_\wp(g_l * \pi_\wp(B^n(\xi_{p^m}))))^{\sum_{1\le a<p^m,(a,p)=1} a^n v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r}))})^{b_L^n\gamma_p}.$$

Theorem 2 follows from Lemma 3 and Lemma 5 which is stated below. Observe that: $(1-p^n)Res_{E/\mathbf{Q}}(\gamma_p) \equiv 1 \, modM$. From now on we assume that $\chi = w^{-n}$ and that $\chi \ne w$, i.e., $n \not\equiv -1 \, mod \, (p-1)$.

Lemma 5

$$\sum_{1\le a<p^m,(a,p)=1} b_L^n a^n v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r})) \equiv Res_{E/\mathbf{Q}}(\delta_n^*(Ll)) \, mod \, Res_{E/\mathbf{Q}}(\delta_n^*(L))(\mathbf{Z}/M)$$

Proof. To simplify this proof we will write $\equiv$ instead of $\equiv$ mod M. We work with the character $\chi = w^{-n}$. Denote

$$\delta_0^*(Ll) = \sum_{1\le a<p^m,(a,p)=1} n_a\sigma_a^{-1},$$

$$\delta_0^*(L) = \sum_{1\le a<p^m,(a,p)=1} m_a\sigma_a^{-1}.$$

By Theorem 1 from previous section we have: $\varphi_\wp(k(L,\hat{r})) \equiv \delta_0^*(Ll)e_\chi + \delta_0^*(L)e_\chi\eta$, where $\eta \in (\mathbf{Z}/M)[G(E/\mathbf{Q})]$. Let us write $\eta = \sum_{1\le a<p^m,(a,p)=1} l_a\sigma_a^{-1}$.

$$\delta_0^*(Ll)e_\chi = (\sum_{1\le a'<p^m,(a',p)=1} n_{a'}\sigma_{a'}^{-1})(\frac{1}{p-1}\sum_{\sigma_c\in\Delta} w^{-n}(\sigma_c)\sigma_c^{-1}) =$$

$$= \frac{1}{p-1}\sum_{1\le a<p^m,(a,p)=1}(\sum_{a'c\equiv a} n_{a'}c^{-n})\sigma_a^{-1}.$$

In the same way:

$$\delta_0^*(L)\eta e_\chi = (\sum_{1\le a_0<p^m,(a_0,p)=1} m_{a_0}\sigma_{a_0}^{-1})(\sum_{1\le a_1<p^m,(a_1,p)=1} l_{a_1}\sigma_{a_1}^{-1})(\frac{1}{p-1}\sum_{\sigma_c\in\Delta} w^{-n}(\sigma_c)\sigma_c^{-1}) =$$

$$= \frac{1}{p-1} \sum_{1\le a<p^m,(a,p)=1} [\sum_{a'c\equiv a} (\sum_{a_0a_1\equiv a'} m_{a_0} l_{a_1}) c^{-n}] \sigma_a^{-1}.$$

So by Theorem 1:

$$\sum_{1\le a<p^m,(a,p)=1} a^n v_{\wp^{\sigma_a^{-1}}}(k(L,\hat{r})) \equiv$$

$$\equiv \frac{1}{p-1} \sum_{1\le a<p^m,(a,p)=1} (\sum_{a'c\equiv a} a'^n n_{a'}) + \frac{1}{p-1} \sum_{1\le a<p^m,(a,p)=1} \sum_{a'c\equiv a} (\sum_{a_0a_1\equiv a'} a'^n m_{a_0} l_{a_1}) \equiv$$

$$\equiv \sum_{1\le a<p^m,(a,p)=1} a^n n_a + \sum_{1\le a<p^m,(a,p)=1} \sum_{a_0a_1\equiv a} a_0^n m_{a_0} a_1^n l_{a_1} \equiv$$

$$\equiv \sum_{1\le a<p^m,(a,p)=1} a^n n_a + (\sum_{1\le a_0<p^m,(a_0,p)=1} a_0^n m_{a_0})(\sum_{1\le a_1<p^m,(a_1,p)=1} a_1^n l_{a_1}).$$

Hence multiplying by b_L^n and using Lemma 4 we get Lemma 5. **Q.E.D**

4.4 Etale cohomology and Chebotarev density theorem

In this section we collect some results on the structure of etale cohomology groups and elements $K(L,\bar{r})$ which are crucial for applications of the Euler system of elements Λ to K-theory of the rationals, presented in the next section. From now on we work in etale K-theory and etale cohomology. Let K_M denote the element of

$$K_{2n+1}^{et}(\mathbf{Z}[\frac{1}{Lpr}];\mathbf{Z}/M) \cong H_{et}^1(\mathbf{Z}[\frac{1}{Lpr}];\mathbf{Z}/M(n+1))$$

obtained from $K(L,\bar{r}) \in K_{2n+1}(\mathbf{Z}[\frac{1}{Lpr}];\mathbf{Z}/M)$ via the map of Dwyer and Friedlander cf. [DF]. Observe that all properties of $K(L,\bar{r})$ which we have proved until now imply similar statements for elements K_M. It follows immediately from functorial properties of the map of Dwyer and Friedlander.

Lemma 6 *Let S be a finite set of prime numbers which contains p, r and all prime divisors of L. Assume that $K_M \in H_{et}^1(\mathbf{Z}_S,\mathbf{Z}/M(n+1))$ is nonzero and that p does not divide $w_{n+1}(\mathbf{Q})$. Then in $H_{et}^1(\mathbf{Z}_S,\mathbf{Z}/M(n+1))$ there is a free $\mathbf{Z}/M$-module which contains K_M.*

Proof. Let M_1 be a power of p such that $M \mid M_1$. Consider a commutative diagram.

$$\begin{array}{ccc} H_{et}^1(\mathbf{Z}_S;\mathbf{Z}/M_1(n+1)) & \rightarrow & H_{cont}^2(\mathbf{Z}_S;\mathbf{Z}_p(n+1))[M_1] \\ \downarrow red & & \downarrow \\ H_{et}^1(\mathbf{Z}_S;\mathbf{Z}/M(n+1)) & \rightarrow & H_{cont}^2(\mathbf{Z}_S;\mathbf{Z}_p(n+1))[M] \end{array}$$

Diagram 7

In the diagram the right vertical map is multiplication by $\frac{M_1}{M}$ and the left vertical map comes from reduction of coefficients. Horizontal maps in the diagram come from Bockstein sequence in continous cohomology on $Spec\,\mathbf{Z}_S$. Let t be the order of K_M. We choose $M_1 = \frac{M^2}{t}$. Since $M \le M_1 \le M^2$, Proposition 5(ii) implies that $red(K_{M_1}) = K_M$. Recall that the group $H_{et}^1(\mathbf{Z}_S;\mathbf{Z}_p(n+1))$ has $|w_{n+1}(\mathbf{Q})|_p^{-1}$ elements, cf. [So3]. By assumption p

does not divide the number $w_{n+1}(\mathbf{Q})$. Hence the lower horizontal map in Diagram 7 is an isomorphism. Let x be the image of K_{M_1} under the upper horizontal map and let y denote the image of K_M under the lower map of the diagram. We have $y = x^{\frac{M_1}{M}} = x^{\frac{M}{t}}$. It implies equality: $1 = y^t = x^M$. Hence $x \in H^2_{cont}(\mathbf{Z}_S; \mathbf{Z}_p(n+1))[M]$. By surjectivity of the lower horizontal map we can pick T from $H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$ which maps onto x. Then $T^{\frac{M}{t}}$ maps onto y and injectivity of the lower horizontal map implies that $K_M = T^{\frac{M}{t}}$. K_M is nonzero by assumption, hence T is nonzero, too. It follows that K_M belongs to a free $\mathbf{Z}/M$-module generated by T which is contained in $H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$. **Q.E.D.**

In the sequel we keep assumptions of Lemma 6, i.e., we assume that $p \not| w_{n+1}(\mathbf{Q})$ and that $K_M \neq 0$. Let p^s be the largest power of p such that $K_M = p^s y$, for some y from the group $H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$. By Lemma 6 the order of K_M equals $\frac{M}{p^s}$. Let $E = \mathbf{Q}(\mu_M)$ and let $\mathcal{O}_E$, ($\mathcal{O}_{E,S}$, respectively) be the ring of integers (S-integers) in E. From now on we choose S to be a finite set of prime numbers which contains: p, r, all prime divisors of L, and in addition has a property that $Pic\,\mathcal{O}_{E,S}$ vanishes. For a prime number $l \notin S$, and a prime ideal λ from $\mathcal{O}_{E,S}$ which lies above l, we have the following commutative diagram whose horizontal arrows are induced by reduction maps.

$$\begin{array}{ccc} H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1)) & \rightarrow & H^1_{et}(k_l; \mathbf{Z}/M(n+1)) \\ \downarrow & & \downarrow \\ H^1_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(n+1)) & \rightarrow & H^1_{et}(k_\lambda; \mathbf{Z}/M(n+1)) \end{array}$$

Diagram 8

We want to prove that the left, vertical map in the diagram is injective under our assumptions. The Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r(G(E/\mathbf{Q}); H^s_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(n+1))) \Longrightarrow H^{r+s}_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$$

gives the following exact sequence of low terms.

$$0 \rightarrow H^1(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) \rightarrow H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1)) \rightarrow H^1_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(n+1))^{G(E/\mathbf{Q})}$$

On the other hand, by Herbrand quotient we have:

$$\mid H^1(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) \mid = \mid H^0_{Tate}(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) \mid .$$

In addition: $\mid H^0_{Tate}(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) \mid \leq \mid H^0(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) \mid =$

$$= \mid H^0(G(\mathbf{Q}_\infty/\mathbf{Q}); \mathbf{Z}/M(n+1)) \mid \leq \mid H^0(G(\mathbf{Q}_\infty/\mathbf{Q}); \mathbf{Q}_p/\mathbf{Z}_p(n+1)) \mid = \mid w_{n+1}(\mathbf{Q}) \mid_p^{-1} .$$

The last equality follows from [So3]. Since by assumption p does not divide $w_{n+1}(\mathbf{Q})$, it follows that $H^1(G(E/\mathbf{Q}); \mathbf{Z}/M(n+1)) = 0$ and the left map in Diagram 8 is injective. The exact sequence of sheaves for etale cohomology on $Spec\,\mathcal{O}_{E,S}$:

$$0 \rightarrow \mathbf{Z}/M(1) \rightarrow \mathbf{G_m} \stackrel{M}{\rightarrow} \mathbf{G_m} \rightarrow 1$$

and assumption $Pic\,\mathcal{O}_{E,S} = 0$ show that one has an isomorphism:

$$\mathcal{O}^\times_{E,S}/\mathcal{O}^{\times M}_{E,S} \xrightarrow{\sim} H^1_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(1)).$$

Tensoring with $\mathbf{Z}/M(n)$ gives an isomorphism:

$$\mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M} \otimes \mathbf{Z}/M(n) \xrightarrow{\sim} H^1_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(n+1)).$$

In a similar way one gets an isomorphism:

$$k_\lambda^{\times}/k_\lambda^{\times M} \otimes \mathbf{Z}/M(n) \xrightarrow{\sim} H^1_{et}(k_\lambda; \mathbf{Z}/M(n+1)).$$

Under the last two isomorphisms the bottom horizontal map in Diagram 8 identifies easily with the map:

$$\mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M} \otimes \mathbf{Z}/M(n) \longrightarrow k_\lambda^{\times}/k_\lambda^{\times M} \otimes \mathbf{Z}/M(n)$$

which is induced by reduction at λ. Let ξ be a generator of $\mathbf{Z}/M(1)$. Then $\xi^{\otimes n}$ is a generator of $\mathbf{Z}/M(n)$. Let $\beta \otimes \xi^{\otimes n} \in \mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M} \otimes Z/M(n)$ be the image of K_M under the left vertical map from Diagram 8. We consider β to be the image of an element (also denoted by β) from $\mathcal{O}_{E,S}^{\times}$. It follows by Lemma 6 that $\beta \otimes \xi^{\otimes n}$ is of order $\frac{M}{p^s}$. We want to prove that the image of $\beta \otimes \xi^{\otimes n}$ under the lower horizontal map in Diagram 8 has also order $\frac{M}{p^s}$ for some choices of prime ideals λ. To choose $\lambda's$ with this property we will use a theorem of K.Rubin [Ru2, Theorem 3.1] which is based on Chebotarev density theorem. Observe that for our reason it is enough to consider the map $\mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M} \longrightarrow k_\lambda^{\times}/k_\lambda^{\times M}$ induced by the reduction. First we need two lemmas.

Lemma 7 *Let $\alpha \in \mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M}$ and let t be the maximal power of p such that α is in $\mathcal{O}_{E,S}^{\times t}/\mathcal{O}_{E,S}^{\times M}$. Then t is also the largest power of p such that $\alpha \in E^{\times t}/E^{\times M}$. In addition, the order of α is $\frac{M}{t}$.*

Proof. Let t_1 be the maximal power of p such that $\alpha \in E^{\times t_1}/E^{\times M}$. Then $\alpha = y^{t_1} z^M$ for some y and z from E. Hence $\alpha = (yz^{\frac{M}{t_1}})^{t_1}$ and obviously $yz^{\frac{M}{t_1}} \in \mathcal{O}_{E,S}^{\times}$. So $t_1 \mid t$ by the definition of t. On the other hand $\mathcal{O}_{E,S}^{\times} \subseteq E^{\times}$, hence $t_1 \mid t$ by definition of t_1.

Let now t_1 be the order of α in $\mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M}$. Then $\alpha^{t_1} = z^M$, for some $z \in \mathcal{O}_{E,S}^{\times}$. Hence $\alpha = z^{\frac{M}{t_1}}\eta$, where $\eta^{t_1} = 1$ and $\eta \in E$. Since all Mth roots of unity are in E, there is $\xi \in E$ such that $\eta = \xi^{\frac{M}{t_1}}$ and $\alpha = (z\xi)^{\frac{M}{t_1}}$. Obviously $z\xi \in \mathcal{O}_{E,S}^{\times}$. It follows by the definition of t that $\frac{M}{t_1} \mid t$ hence $\frac{M}{t} \mid t_1$. On the other hand, $\alpha^{\frac{M}{t_1}} = 1$ because $\alpha \in \mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M}$, hence t_1 divides $\frac{M}{t}$ by the definition of t_1. **Q.E.D.**

We apply Lemma 7 to $\alpha = \beta$ where β is the element of $\mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M}$ associated to K_M in our previous discussion. It follows that $t = p^s$, i.e., t - the largest power of p such that $\beta \in \mathcal{O}_{E,S}^{\times t}/\mathcal{O}_{E,S}^{\times M}$ equals p^s - the largest power of p which divides K_M in $H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$. We omit a proof of the next lemma which follows by an argument similar to that after Diagram 8.

Lemma 8 *If K_M is nonzero, then the highest powers of p which divide K_M in the groups $H^1_{et}(\mathbf{Z}_S; \mathbf{Z}/M(n+1))$ and $H^1_{et}(\mathbf{Q}; \mathbf{Z}/M(n+1))$ are the same.*

Theorem 3 *Let $\alpha \in \mathcal{O}_{E,S}^{\times}/\mathcal{O}_{E,S}^{\times M}$, $\wp \in Pic(\mathcal{O}_{E(\mu_L)})$, and let t be the maximal power of p such that α is in $\mathcal{O}_{E,S}^{\times t}/\mathcal{O}_{E,S}^{\times M}$. There exist infinitely many prime ideals λ in $\mathcal{O}_E$ such that there is a prime ideal in $\mathcal{O}_{E(\mu_L)}$ lying above λ, which is in the ideal class $\wp$. If l is the prime number below λ, then $l \equiv 1 \, mod \, M^2 L$. The image of α in $k_\lambda^{\times}/k_\lambda^{\times M}$ has order $\frac{M}{t}$.*

Proof. [Ru2, Theorem 3.1.]

Let π_l be the map induced on etale K-theory by the reduction $\mathbf{Z}_S \to k_l$. Pick λ and l as in Theorem 3. Let $g_l * \pi_\lambda(B^n(\xi_{p^m}))$ be the generator of $K^{et}_{2n+1}(k_\lambda; \mathbf{Z}/M) \cong H^1_{et}(k_\lambda; \mathbf{Z}/M(n+1))$. The following proposition will be crucial in the next section. It is an immediate corollary of what we have proved in this section.

Proposition 6 *If K_M is nonzero and p does not divide $w_{n+1}(\mathbf{Q})$, then for prime numbers l from Theorem 3 we have: $\pi_l(K_M) = [g_l * \pi_\lambda(B^n(\xi_{p^m}))]^{\nu t}$ for some $\nu \in (\mathbf{Z}/M)^\times$, where t is the largest power of p dividing K_M in the group $H^1_{et}(\mathbf{Q}; \mathbf{Z}/M(n+1))$.*

We finish this section with a result which may be of some independend interest. Let Y be a submodule of $\mathcal{O}^\times_{E,S}/\mathcal{O}^{\times M}_{E,S}$ which consists of all y such that any $\sigma_a \in G(E/\mathbf{Q})$ acts at y raising it to the a^{-n}th power. It follows easily from the definition that Y equals $H^1_{et}(\mathcal{O}_{E,S}; \mathbf{Z}/M(n+1))^{G(E/\mathbf{Q})}$. Note that by construction and Lemma 6 $K_M \in Y$.

Proposition 7 *If the element K_M is nonzero, then the module Y contains a free $\mathbf{Z}/M$-submodule.*

Proof. It follows by Lemma 6 and the HS spectral sequence argument after Diagram 8. **Q.E.D.**

4.5 An application of the Euler system

In this section we show how the elements $K(L, \bar{r})$ may be used in K-theory of $\mathbf{Q}$. We prove in Corollary 3 that the number of divisible elements in p-torsion part of the group $K_{2n}(\mathbf{Q})$ does not exceed the maximal power of p which divides the number $|w_{n+1}(\mathbf{Q})\zeta_{\mathbf{Q}}(-n)|_p^{-1}$ provided that p does not divide $w_{n+1}(\mathbf{Q})$. Note that Theorem 3 of [Ba2] implies equality of the two above mentioned powers of p. Though weaker, Corollary 3 has an advantage over [Ba2, Theorem3], since it does not use "the Main Conjecture" in Iwasawa theory. Its proof relies instead on the properties of elements $K(L, \bar{r})$ which we have obtained until now. Our argument parallels Rubin's proof of the Main Conjecture for the odd part of the ideal class group of $\mathbf{Q}(\mu_p)$ [Ru2, Section 4].

Let us introduce notation which will be used in the sequel. As in the previous section, p is an odd prime not dividing $w_{n+1}(\mathbf{Q})$. M is a power of p which will be specified by Lemma 9 and Theorem 4 below. $L = \prod l$ is a square-free integer whose prime divisors l are congruent to 1 modulo M. We keep L fixed until Corollary 3. For simplicity, we denote $D_{n+1}(\mathbf{Q})$ by A. Let ψ: $\bigoplus_{l \neq p} K^{et}_{2n}(\mathbf{F}_l; \mathbf{Z}/M) \to A$ be a map from the localization sequence [Ba2, Diagram 2.4] and let v_l denote $\psi(\xi^{\otimes n}_{p^m})$, where $\xi^{\otimes n}_{p^m}$ is a generator of $K^{et}_{2n}(\mathbf{F}_l; \mathbf{Z}/M) \cong H^0_{et}(\mathbf{F}_l; \mathbf{Z}/M(n))$. Let B denote a subgroup of A generated by the v_l's such that $l \mid L$. Finally, we fix (again until Corollary 3) a generator u of a cyclic summand of the quotient group A/B. Next we want to choose primes r and $\bar{r}$ in order to construct elements $K(L, \bar{r})$.

Lemma 9 *For M big enough, there is an exact sequence:*

$$K^{et}_{2n+1}(\mathbf{Q}; \mathbf{Z}/M) \xrightarrow{\partial} \bigoplus_{l \neq p} K^{et}_{2n}(\mathbf{F}_l; \mathbf{Z}/M) \xrightarrow{\psi} A \to 0.$$

The map ψ restricted to $\bigoplus_{l \equiv 1 \, mod \, M^2 L} K^{et}_{2n}(\mathbf{F}_l; \mathbf{Z}/M)$ has still image A.

Proof. Lemma follows by [Ba2, p.291] and [Ku, Proof of Prop. 1.1]. **Q.E.D.**

Let r be a prime number such that $r \equiv 1 \, mod \, M^2 L$ and that the image of v_r in A/B under projection map generates the same cyclic summand as u does. It can be chosen by Lemma 9. Let $\bar{r}$ be a fixed prime ideal in $E(\mu_L)$ lying above r. The following theorem is an analogue of Theorem 4.1 of [Ru2]. Let $d_n(L)$ be the largest power of p which divides number $\delta_n(L)$ defined in Section 4.2.

Theorem 4 *Let M be so big that it is divisible by $|A|d_n(1)p$. In addition, assume that $d_n(L) \mid d_n(1)$. Then there is a prime number l such that: $l \equiv 1 \, mod \, M^2 L$ and $\frac{d_n(L)}{d_n(Ll)} u = 0$ in the group A/B.*

Proof. It follows by assumptions that $d_n(L)p$ divides M. Hence by Proposition 4, $\partial_r(K_M^{b_L^n \gamma_p})$ is nonzero and consequently $K_M \neq 0$. Proposition 4 implies also that:

$$\partial(K_M^{b_L^n \gamma_p}) = \partial_r(K_M^{b_L^n \gamma_p}) + \sum_{l|L} \partial_l(K_M^{b_L^n \gamma_p}) = (N_{\bar{r}/r}(\pi_{\bar{r}}(B^n(\xi_{p^m}))))^{\delta_n(L)} + \sum_{l|L} \partial_l(K_M^{b_L^n \gamma_p}).$$

Let t be, as in the previous section, the highest power of p which divides K_M in the group $H^1_{et}(\mathbf{Q}; \mathbf{Z}/M(n+1))$. It follows from the last equalities that t divides $d_n(L)$. But $d_n(L) \mid M$ by assumption, hence t divides M. To proceed with the proof, apply the map ψ to both sides of the equalities, divide the result by t, and finally project down to A/B. We obtain in this way the following equality.

$$0 = \psi(\partial(K_M^{b_L^n \gamma_p})) = \frac{d_n(L)}{t} u$$

Since $u \neq 0$, the number $d_n(L)$ is divisible by pt. To conclude the proof it remains to find a prime l which is congruent to 1 modulo $M^2 L$ and such that $t = d_n(Ll)$. By Theorem 3 and Proposition 6 there is an l such that $l \equiv 1 \, mod \, M^2 L$ and for which

$$\pi_l(K_M^{b_L^n \gamma_p}) = [g_l * \pi_\lambda(B^n(\xi_{p^m}))]^{\nu_1 t},$$

for some $\nu_1 \in (\mathbf{Z}/M)^\times$. On the other hand Theorem 2 implies:

$$\pi_l(K_M^{b_L^n \gamma_p}) = [g_l * \pi_\lambda(B^n(\xi_{p^m}))]^{\delta_n(Ll) + k\delta_n(L)}$$

for some integer k. (Recall that, by assumption, p does not divide b_L.) Comparing the right hand sides of the last two equalities, we obtain that $t \equiv |\delta_n(Ll) + k\delta_n(L)|_p^{-1} \, mod \, M$. Since $pt \mid d_n(L)$, it follows easily that $t = d_n(Ll)$ for the prime l. **Q.E.D.**

Corollary 3 *The number of elements of the group $A = D_{n+1}(\mathbf{Q})$ divides the number $|w_{n+1}(\mathbf{Q})\zeta_{\mathbf{Q}}(-n)|_p^{-1}$ for primes p relatively prime to $w_{n+1}(\mathbf{Q})$, where n is an odd integer and $n \not\equiv -1 \, mod \, (p-1)$.*

Proof. Let $u_1, u_2, \ldots, u_a$ be generators of cyclic summands, one generator for a summand, in the decomposition of A/B into cyclic groups. Let c_i denote the order of the image of u_i in the quotient group $A/(B, u_1, u_2, ..., u_{i-1})$. We have obviously: $|A/B| = \prod c_i$. Assume that M is as in Theorem 4 and that $d_n(L)$ divides $d_n(1)$. Applying Theorem 4 inductively, we obtain prime numbers $l_1, l_2, ..., l_a$ which have the following properties.

1. $l_i \equiv 1 \, mod \, M^2 L$

2. the number $d_n(L_i)$ divides $d_n(L_{i-1})$, where we put $L_0 = L$, and $L_i = L \prod_{j \leq i} l_j$. In addition we have equality $\frac{d_n(L_{i-1})}{d_n(L_i)} u_i = 0$, in the group $A/(B, u_1, u_2, ..., u_{i-1})$ for $i = 1, 2, ..., a$.

It follows that $|A/B| = \prod c_i$ divides $\prod \frac{d_n(L_{i-1})}{d_n(L_i)}$ hence also $d_n(L)$. Putting $L = 1$ in the above implies that the number of elements in $A = D_{n+1}(\mathbf{Q})$ divides the number $d_n(1)$. Recall that $d_n(1) = |\delta_n(1)|_p^{-1} = |(b_1^{n+1} - 1)\zeta_{\mathbf{Q}}(-n)|_p^{-1}$ and $(b_1, pw_{n+1}(\mathbf{Q})) = 1$, by assumption on numbers b_L. Taking minimum over numbers b_1 gives the final claim by [C, Lemma 2.3]. **Q.E.D.**

5 A class number formula for higher K-groups

5.1 Divisible elements in $K_{2n}(F)$ for n even

In this section we prove an analogue of a class number formula of Kummer for higher K-groups of number fields. Let n be a positive integer and let F be a totally real, abelian number field with ring of integers $\mathcal{O}_F$. We assume for technical reasons that p does not divide the conductor of F. In [So1, 3.1.2] Soulé defined a subgroup of cyclotomic elements in $K_{2n+1}(\mathcal{O}_F) \otimes \mathbf{Z}_p$. The subgroup is generated by elements which are obtained from cyclotomic units of $\mathcal{O}_F$ in a similar way as the elements $\Lambda(L)$ were obtained in Section 2 from $\alpha(L)$. We denote Soulé's group of cyclotomic elements by $C_{2n+1} = C_{2n+1}(F)$. It follows by [So1, Theorem 1] and [So2, Remark 6.4] that the index of C_{2n+1} in $K_{2n+1}(\mathcal{O}_F) \otimes \mathbf{Z}_p$ is finite. Let $C_{2n+1}^{et} = C_{2n+1}^{et}(F)$ be the image of C_{2n+1} by the Dwyer-Friedlander map:

$$\rho : K_{2n+1}(\mathcal{O}_F) \otimes \mathbf{Z}_p \longrightarrow K_{2n+1}^{et}(\mathcal{O}_F[\frac{1}{p}]).$$

Since ρ is surjective [loc.cit., Theorem 8.7], it follows that the index of C_{2n+1}^{et} in the group $K_{2n+1}^{et}(\mathcal{O}_F[\frac{1}{p}])$ is finite. Let $D_{n+1}(F)$ be the group of divisible elements in $K_{2n}(F)_p$ from Section 4.1. In the case when n is even we have the following formula for the number of elements of $D_{n+1}(F)$. It is the main result of the section. A proof given below relies on "the Main Conjecture".

Theorem 5 *Let p and F be as above and let n be even. Then*

$$|D_{n+1}(F)| = [K_{2n+1}^{et}(\mathcal{O}_F[\tfrac{1}{p}]); C_{2n+1}^{et}]$$

Before presenting a proof of the theorem two remarks are in order.

Remark 9. We think that the formula is an anologue for K-groups of the number field F of the classical Kummer's formula $h^+ = [E; \mathcal{E}]$, [La, Theorem 5.1, p.85] for the second factor of class number of the cyclotomic field $\mathbf{Q}(\xi_{p^k})$. Here E and $\mathcal{E}$ are units and cyclotomic units in the ring $\mathbf{Z}[\xi_{p^k}]$. In this analogy $D_{n+1}(F)$ replaces the ideal class group. More on the analogy may be found in [Ba2, p.292].

Remark 10. If the map ρ is an isomorphism (as was conjectured by Dwyer and Friedlander), then a similar formula holds for $K_{2n+1}(\mathcal{O}_F) \otimes \mathbf{Z}_p$ and C_{2n+1}.

Our proof of Theorem 5 generalizes an argument of Bloch and Kato who showed that if F is $\mathbf{Q}$, then $|K_{2n}^{et}(\mathbf{Z}[\frac{1}{p}])| = [K_{2n+1}^{et}(\mathbf{Z}[\frac{1}{p}]; C_{2n+1}^{et}]$. Consider the following diagram cf. [BK, p. 387].

$$\begin{tikzcd}
0 \arrow[d] & \\
R_{-n} \cong D_{n+1}(F)^* \arrow[d] & 0 \arrow[d] \\
H^1(\mathcal{O}_F[\frac{1}{p}]; W^{-n}) \arrow[r, "a"] \arrow[d, "\rho_\Sigma(-n)"] & H^0(G; (\mathcal{P}/\mathcal{C})^*(-n)) \arrow[d] \\
\bigoplus_{\nu|p} H^1(F_\nu; W^{-n}) \arrow[r, "b"] \arrow[d] & H^0(G; \mathcal{P}^*(-n)) \arrow[d] \\
H^1(\mathcal{O}_F[\frac{1}{p}]; \mathbf{Z}_p(n+1))^* \arrow[r, "c"] \arrow[d] & H^0(G; \mathcal{C}^*(-n)) \arrow[d] \\
H^2(\mathcal{O}_F[\frac{1}{p}]; W^{-n}) \arrow[r, "d"] \arrow[d] & H^1(G; (\mathcal{P}/\mathcal{C})^*(-n)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Diagram 9

LEFT COLUMN. Left column of Diagram 9 is a fragment of the exact sequence from [Sch, Satz 5(ii), p.186]. We replaced in it Galois cohomology groups by etale cohomology groups of $Spec\,\mathcal{O}_F[\frac{1}{p}]$, since as is well-known one has isomorphisms [Sch]:

$$H^i(G_\Sigma; \mathbf{Z}/p^k(j)) \cong H^i(\mathcal{O}_F[\frac{1}{p}]; \mathbf{Z}/p^k(j))$$

and similarly for cohomology with coefficients in j-fold twists $\mathbf{Z}_p(j)$ or in $W^j = \mathbf{Q}_p/\mathbf{Z}_p(j)$. Here G_Σ is the Galois group of the maximal, p-ramified, algebraic extension of F [Sch, p.182]. F_ν is the completion of F at a finite prime ν dividing p. For an abelian group A, by $A^* = Hom_{cts}(A; \mathbf{Q}_p/\mathbf{Z}_p)$ we denote its Pontriagin dual. R_{-n} is the kernel of the natural map $\rho_\Sigma(-n)$, which by Satz 5 (ii) [Sch, p.197] is isomorphic to the dual of the group $D_{n+1}(F)$.

RIGHT COLUMN. $G = G(F_\infty/F)$, where $F_\infty = F(\mu_{p^\infty})$. Let $\mathcal{O}_\infty$ be the ring of integers in F_∞. We put

$$\mathcal{P} = \prod_{\nu|p} Gal(F_{\nu,\infty}^{ab}/F_{\nu,\infty})$$

where $F_{\nu,\infty}^{ab}$ denotes the maximal, abelian extension of the field $F_{\nu,\infty}$. $\mathcal{P}$ is given its obvious G-module structure. For $\mathcal{C}$ we take a G-module of cyclotomic units in $\mathcal{O}_\infty$ constructed by Gillard for a field F with p not dividing the degree $[F : \mathbf{Q}]$ in [Gi]. It was denoted by C in

[So1, p.394]. In order to describe an embedding of $\mathcal{C}$ into $\mathcal{P}$ we need more notation. Let $F_k = F(\xi_{p^k})$ and let $\mathcal{O}_k$ be its ring of integers. For a finite prime ν, by $U_{k,\nu}$ we denote the group of principal units in the local ring $\mathcal{O}_{k,\nu}$ [Wa, p.163]. Define

$$\mathcal{U} = \prod_{\nu|p} \lim_{\leftarrow} U_{k,\nu}$$

where inverse limits are taken over norm maps in the cyclotomic tower of F. $\mathcal{C}$ maps into $\mathcal{U}$ by the diagonal embedding [Gi, p.2]. It follows from the local class field theory that the G-module $\mathcal{U}$ imbeds into $\mathcal{P}$. It gives $\mathcal{C}$ a structure of a G-submodule of $\mathcal{P}$. The right column of Diagram 9 is the exact sequence obtained from

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}/\mathcal{C} \longrightarrow 0$$

by taking Pontriagin duals, Tate twists and finally applying cohomology.

HORIZONTAL MAPS. Let M be the maximal, abelian, p-ramified p-extension of F_∞ and let $\mathcal{X} = G(M/F_\infty)$ be the Galois group of M with an ordinary G-action by conjugation [Wa, p.291]. Mapping $Gal(F_{\nu,\infty}^{ab}/F_{\nu,\infty})$ into $\mathcal{X}$ for each prime ν and multiplication in $\mathcal{X}$ give a map $\mathcal{P} \longrightarrow \mathcal{X}$ which sends $\mathcal{C}$ to zero. Let $\alpha : \mathcal{P}/\mathcal{C} \longrightarrow \mathcal{X}$ be the induced map. Let us define maps a and d in Diagram 9. Since $H^1(G; W^{-n}) = 0$ for $n \neq 0$, [Sch, Lemma 2.8.] and $cd_p(G) = 1$, it follows from exact sequences of low terms of the Hochschild-Serre spectral sequence of the covering $Spec\mathcal{O}_\infty[\frac{1}{p}] \longrightarrow Spec\mathcal{O}_F[\frac{1}{p}]$ that $H^i(\mathcal{O}_F[\frac{1}{p}]; W^{-n}) \cong H^{i-1}(G; H^1(\mathcal{O}_\infty[\frac{1}{p}]; W^{-n}))$ for $i = 1, 2$. Kummer pairing gives an isomorphism of G-modules $\mathcal{X}^* \cong H^1(\mathcal{O}_\infty[\frac{1}{p}]; \mathbf{Q}_p/\mathbf{Z}_p)$ which after twisting induces an isomorphism on cohomology:

$$H^{i-1}(G; H^1(\mathcal{O}_\infty[\frac{1}{p}]; W^{-n})) \cong H^{i-1}(G; \mathcal{X}^*(-n)).$$

Hence one obtains isomorphisms:

$$\beta_i : H^i(\mathcal{O}_F[\frac{1}{p}]; W^{-n}) \xrightarrow{\cong} H^{i-1}(G; \mathcal{X}^*(-n))$$

for $i = 1, 2$. We define map a (d, respectively) to be the composite

$$H^i(\mathcal{O}_F[\tfrac{1}{p}]; W^{-n}) \xrightarrow{\beta_i} H^{i-1}(G; \mathcal{X}^*(-n)) \xrightarrow{\alpha_i} H^{i-1}(G; (\mathcal{P}/\mathcal{C})^*(-n))$$

for $i = 1$ ($i = 2$, respectively), where maps α_i are induced by α. Next we define map b. We use the Hochschild-Serre spectral sequence attached to a covering $Spec(F_{\nu,\infty}) \longrightarrow Spec(F_\nu)$ for $\nu|p$. Observe that since by assumption p does not divide the conductor of F, we have $G \cong G(F_{\nu,\infty}/F_\nu)$, for $\nu|p$. From now on we identify these two groups. Since by [Sch, Lemma 2.8.] $H^1(G; H^0(F_{\nu,\infty}; W^{-n})) = H^1(G; W^{-n}) = 0$ for $n \neq 0$, the low-terms exact sequences give that $\bigoplus_{\nu|p} H^1(F_\nu; W^{-n}) \cong \bigoplus_{\nu|p} H^0(G; H^1(F_{\nu,\infty}; W^{-n}))$. We define map b to be the composite of the last isomorphism with a map:

$$\bigoplus_{\nu|p} H^0(G; H^1(F_{\nu,\infty}; W^{-n})) \xrightarrow{\cong} H^0(G; \mathcal{P}^*(-n))$$

which is induced on cohomology by an obvious isomorphism of G-modules:

$$\mathcal{P}^* \stackrel{\cong}{\to} \bigoplus\nolimits_{\nu|p} H^1(F_{\nu,\infty}; \mathbf{Q}_p/\mathbf{Z}_p).$$

Finally, let us define map c. We follow [So1, p.384ff]. Let $E = \lim_{\leftarrow} \mathcal{O}_k[\frac{1}{p}]^\times$ be the limit over norm maps of the groups of p-units in $\mathcal{O}_k = \mathcal{O}_{F(\xi_{p^k})}$. Using products in etale cohomology Soulé defined a map:

$$\tilde{\phi} : E(n)_G \to H^1(\mathcal{O}_F[\frac{1}{p}]; \mathbf{Z}_p(n+1))$$

[So1, loc.cit.]. We precompose Soulé's map $\tilde{\phi}$ with a map induced by the inclusion of $\mathcal{C}$ into E. It gives a map:

$$\tilde{c} : \mathcal{C}(n)_G \to H^1(\mathcal{O}_F[\frac{1}{p}]; \mathbf{Z}_p(n+1)).$$

Put $c = Hom_{cts}(\tilde{c}; \mathbf{Q}_p/\mathbf{Z}_p)$. We leave to the reader to check that Diagram 9 commutes. To finish the proof of Theorem 5 we need a definition and three lemmas.

Definition 5 *Let $f : A \longrightarrow B$ be a homomorphism of abelian groups with finite kernel and cokernel. Define $z(f) = \frac{|ker f|}{|coker f|}$.*

Lemma 10 *Let be given a commutative diagram of abelian groups and homomorphisms which has exact rows.*

$$\begin{array}{ccccccccccc}
0 & \to & B_1 & \to & B_2 & \to & \dots & \to & B_5 & \to & 0 \\
 & & \uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle f_2} & & \dots & & \uparrow{\scriptstyle f_5} & & \\
0 & \to & A_1 & \to & A_2 & \to & \dots & \to & A_5 & \to & 0
\end{array}$$

Diagram 10

Assume that $ker f_i$ and $coker f_i$, for i = 1, 2, ..., 5 are finite. Then

$$\frac{z(f_1)z(f_3)z(f_5)}{z(f_2)z(f_4)} = 1.$$

Lemma 11 *(Herbrand's Lemma) Let*

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow B \longrightarrow 0$$

be an exact sequence of G-modules such that groups of invariants and groups of coinvariants of A and B are finite. Then we have the following equality.

$$\frac{|A^G|}{|A_G|}\frac{|B_G|}{|B^G|} = \frac{|ker(M^G \longrightarrow N^G)|}{|ker(M_G \longrightarrow N_G)|}\frac{|coker(M_G \longrightarrow N_G)|}{|coker(M^G \longrightarrow N^G)|}$$

Proofs of Lemma 10 and Lemma 11 are elementary exercises which we leave to the reader. In the sequel we will need the following properties of horizontal maps in the Diagram 9.

Lemma 12

(i) *cokerc* $= 0$*, kerc is finite.*

(ii) *kera* $= D_{n+1}(F)^*$*, cokera is finite.*

(iii) *cokerd* $= 0$*, kerd is finite.*

(iv) $z(a) = z(d)$.

Proof. (i) $ker\tilde{c}$ and $coker\tilde{c}$ are finite groups because $\tilde{\phi}$ is a rational isomorphism of finitely generated abelian groups, [So1,p.384] and $\mathcal{C}(n)_G$ is a subgroup of finite index in $E(n)_G$, [So2, Theoreme 3 and Remarque 6.4.]. Hence, $kerc = (coker\tilde{c})^*$ and $cokerc = (ker\tilde{c})^*$ are also finite groups. It follows easily from description of $\mathcal{C}$ given in [Gi, p.9. (12)] that, for $n > 0$, the group of coinvariants $\mathcal{C}(n)_G$ is torsionfree. Hence its finite subgroup $ker\tilde{c}$ is zero.

(ii) and (iii) By a simple chase in Diagram 9 one gets a short exact sequence

$$0 \longrightarrow cokera \longrightarrow kerc \longrightarrow kerd \longrightarrow 0.$$

Hence, by (i) both groups $cokera$ and $kerd$ are finite. Moreover, commutativity of Diagram 9 gives that $kera = D_{n+1}(F)^*$ and $cokerd = 0$.

(iv) Take $M = \mathcal{X}^*(-n)$, $N = (\mathcal{P}/\mathcal{C})^*(-n)$ and apply Lemma 11 to the tautological exact sequence

$$0 \longrightarrow A \longrightarrow \mathcal{X}^*(-n) \xrightarrow{\alpha} (\mathcal{P}/\mathcal{C})^*(-n) \longrightarrow B \longrightarrow 0$$

of map $\mathcal{X}^*(-n) \longrightarrow (\mathcal{P}/\mathcal{C})^*(-n)$ which is induced by the map α. It follows easily from (i)-(iii) that assumptions of Lemma 11 are fulfilled in this case. By Lemma 11 we have:

$$\frac{|A^G|}{|A_G|}\frac{|B_G|}{|B^G|} = \frac{z(a)}{z(d)}.$$

Consider characteristic polynomials of Iwasawa modules from the above exact sequence. Denote by h_M the characteristic polynomial of a finitely generated module M, [Wa, Chapter 13]. By the multiplicativity of characteristic polynomials on exact sequences of Iwasawa modules, one gets that $h_A h_{(\mathcal{P}/\mathcal{C})^*(-n)} = h_{\mathcal{X}^*(-n)} h_B$. By "the Main Conjecture" the polynomials $h_{(\mathcal{P}/\mathcal{C})^*(-n)}$ and $h_{\mathcal{X}^*(-n)}$ differ by a unit from the ring $Z_p[[T]]$. In particular, we have $|h_{(\mathcal{P}/\mathcal{C})^*(-n)}(0)|_p^{-1} = |h_{\mathcal{X}^*(-n)}(0)|_p^{-1}$, where $| - |_p$ denotes the p-adic valuation. Hence $|h_A(0)|_p^{-1} = |h_B(0)|_p^{-1}$. Next, we apply [C, Lemma 9, p.349] to A and B. Together with the last equality it implies that:

$$\frac{|A^G|}{|A_G|}\frac{|B_G|}{|B^G|} = 1.$$

Hence we obtain $\frac{z(a)}{z(d)} = 1$. **Q.E.D.**

To finish the proof of Theorem 5 we apply Lemma 10 to Diagram 9. Lemma 12 and Lemma 10 imply that: $|kerc| = |D_{n+1}(F)^*|$. Obviously, $|D_{n+1}(F)^*| = |D_{n+1}(F)|$ because the group is finite. On the other hand, $Im\tilde{c}$ is isomorphic to the group C^{et}_{2n+1} of Soulé's cyclotomic elements because etale cohomology and etale K-theory of $Spec\, \mathcal{O}_F[\frac{1}{p}]$ are isomorphic by a product preserving map cf. [DF, Remark 8.8 and Proposition 5.4]. It implies that:

$$|ker c| = |coker \tilde{c}| = [H^1(\mathcal{O}_F[\frac{1}{p}]; \mathbf{Z}_p(n+1)); Im \tilde{c}] = [K^{et}_{2n+1}(\mathcal{O}_F[\frac{1}{p}]); C^{et}_{2n+1}].$$

Q.E.D.

5.2 The group $D_{n+1}(\mathbf{Q})$ and the Kummer-Vandiver conjecture

We finish with two propositions which make the analogy mentioned in Remark 9 more plausible.

Proposition 8 *Let $p > 2$ be a prime number and $F = \mathbf{Q}$. The Kummer-Vandiver conjecture is true for p iff $D_{n+1}(\mathbf{Q}) = 0$, for every positive and even n.*

Proof. Crucial step in this connection was made by M. Kurihara in [Ku]. We know by [Ku, Corollary 1.5] that the Kummer-Vandiver conjecture is true for p if and only if $H^2(\mathbf{Z}[1/p]; \mathbf{Z}_p(n+1)) = 0$ for every $n > 0$ and n even. Put $r = n+1$ in the paper to get this.

Now look at the exact sequence in Proposition 1.1 of [Ku] with $F = \mathbf{Q}$. Compare the exact sequence to the exact sequence in Proposition 2.1 of [Ku] (they are the same up to the last group on the right but the image of the right map ϕ is described below). On the other hand let us look at page 15 of [BZ]. The etale version of the sequence (*)

$$0 \to Im\, K_{2n+1}(F; \mathbf{Z}/p^k) \to \bigoplus_{\nu} K_{2n}(k_\nu; \mathbf{Z}/p^k) \to D(n)_p \to 0$$

at the top of the page is the same as the above mentioned sequence from [Ku]. It shows that $D_{n+1}(\mathbf{Q}) = D(n)_p = H^2(\mathbf{Z}[1/p]; \mathbf{Z}/p^k(n+1))$ for k big enough. The lines 9-12 at the page 15 of [BZ] explain why this equality is consistent with change of coefficients (on $D(n)_p$ we have identity maps to commute with the coefficients change maps on the right). Hence, taking the inverse limit on coefficients in the last equation we get:

$$D_{n+1}(\mathbf{Q}) = H^2(\mathbf{Z}[1/p]; \mathbf{Z}_p(n+1))$$

We implicitly used [Ba2, Theorem 3]. **Q.E.D.**

Since the Kummer-Vandiver conjecture is known to be true for all p such that $2 < p < 125000$, [Wa, p.159], we have the following.

Corollary 4 $D_{n+1}(\mathbf{Q}) = 0$ *for every* $2 < p < 125000$ *and every even and positive* n.

Remark 11. Let us look at the case of $F = \mathbf{F}_q(t)$. Put as usual $A = \mathbf{F}_q[t]$. Observe that by [So4, Théoréme 1] and by the homotopy invariance: $K_i(\mathbf{F}_p[t]) \cong K_i(\mathbf{F}_p)$, we have: $div\, K_i(\mathbf{F}_q(t)) = 0$ for every i. It would be interesting to connect this observation to the Vandiver conjecture in the function field case using the theory of Drinfeld modules.

We have another reformulation of the Kummer-Vandiver conjecture in terms of vanishing of a certain K-group.

Proposition 9 *The Kummer-Vandiver conjecture is true for an odd prime number p if and only if* $K_4^{et}(\mathbf{Z}[\frac{1}{p}, \xi_p + \xi_p^{-1}]) = 0$.

Proof. Let $B = \mathbf{Z}[\frac{1}{p}, \mu_p]$. Consider the sequence of sheaves on etale site of $Spec\, B$:

$$0 \to \mathbf{Z}/p(1) \to \mathbf{G_m} \xrightarrow{p} \mathbf{G_m} \to 1.$$

Let $A \cong H^1(B; \mathbf{G_m})_p \cong (Pic\, B)_p$. Applying cohomology to the exact sequence above one gets: $A/p \cong H^2(B; \mathbf{Z}/p(1))$. Consider the complex conjugation τ and let A^+ and B^+ denote the fixed point sets in the action of τ on A and B. Then $B^+ = \mathbf{Z}[\frac{1}{p}, \xi_p + \xi_p^{-1}]$ and by a simple transfer argument (p is odd) we obtain:

$$(A/p)^+ \cong H^2(B, \mathbf{Z}/p(1))^+ = H^2(B^+; \mathbf{Z}/p(1)).$$

On the other hand since $(A/p)^+ = A^+/p$, it follows that: $A^+/p \cong H^2(B^+; \mathbf{Z}/p(1))$. By [DF] we have:

$$K_4^{et}(B^+)/p \cong H^2(B^+; \mathbf{Z}_p(3))/p \cong H^2(B^+; \mathbf{Z}/p(3)) = H^2(B^+; \mathbf{Z}/p(1)) \otimes \mathbf{Z}/p(2).$$

It follows that $A^+ = 0$ if and only if $K_4^{et}(B^+) = 0$. **Q.E.D.**

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