BOUNDS FOR CHARACTERISTIC FUNCTIONS OF POLYNOMIALS IN ASYMPTOTICALLY NORMAL VARIABLES

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Abstract. Let f be a polynomial in k variables and S_n be a normed sum of independent identically distributed random vectors $X_1, X_2, \dots, X_n$ taking values in $\mathbb{R}^k$. Upper bounds for $|\mathbf{E}\exp\{it\,f(S_n)\}|$ are derived provided that either the distribution of X_1 has a nondegenerate discrete component or the distribution of S_{n_0} has an absolutely continuous component for some n_0. The dependence on t and n is given in an explicit form. The proofs are based on some number-theoretic results.

1. Introduction and notation

Let $X = (X^{(1)}, \dots, X^{(k)})$ denote a random vector taking values in a k-dimensional Euclidian space $\mathbb{R}^k$. Consider a sequence $X_1, X_2, \dots$ of independent copies of X. Let

$$S_n = n^{-1/2}(X_1 + \cdots + X_n)$$

and $S_n = (S_n^{(1)}, \cdots, S_n^{(k)})$.
In this paper we study the characteristic function

$$g_n(t,a) = \mathbf{E}\exp\{it\,f(S_n + a)\}, \qquad a \in \mathbb{R}^k,$$

for large values of n and t, where

$$f(x) = \sum_{m_1=0}^{m} \cdots \sum_{m_k=0}^{m} \alpha(m_1, \dots, m_k)\,\big(x^{(1)}\big)^{m_1} \cdots \big(x^{(k)}\big)^{m_k}, \qquad m \geq 2.$$

Estimates of $g_n(t,a)$ can be based on the so-called symmetrization inequality (see Götze (1979) and Götze (1989), especially Corollary 3.26 and the remark after it referring to Weyl (1916); see also Yurinskii (1981)).

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In the case $k = 1$, for example, this approach leads to the following inequality (cf. Lemmas 3.1, 4.1, 4.2 in Yurinskii (1983)). Let

$$f(x) = \alpha x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_0.$$

For any nondegenerate random variable X, positive numbers A, ε and any integer $m \geq 2$ there exist positive constants c_1 and c_2 depending on A, ε, m and the distribution of X such that for all $a \in \mathbb{R}^1$

$$\big|g_n(t,a)\big| \leq c_1\big(n^{-A} + |\alpha t|^{-1/(m2^m)}\big) \tag{1}$$

provided that

$$c_2|\alpha t| \leq n^{(m-1)/2-\varepsilon}. \tag{2}$$

Remark. If X has a discrete distribution it is well known that $g_n(t,a)$ is an almost periodic function (see e.g. Besicovitch (1932), p.6, and Cramer (1970), p.26) and hence

$$\overline{\lim_{t\to\infty}}\,\big|g_n(t,a)\big| = 1.$$

Thus, $|t|$ has to be restricted to a finite interval depending on n and f.
The aim of this paper is to show that in (1) the exponent $1/(m2^m)$ of $|t|^{-1}$ can be replaced by a larger value in the following two cases:
a) the distribution of X has a nondegenerate discrete component (see Theorems 1 and 2 below),
b) for some n_0 the distribution of S_{n_0} has an absolutely continuous component (see Theorems 3 and 4).
Moreover, we get also bounds for $|\mathbf{E}\exp\{it\,f(Y+a)\}|$, where Y has a Gaussian distribution in $\mathbb{R}^k$ (see Theorems 5, 6 and 7).
Our proofs are based on inequalities for trigonometric sums and integrals.
In the case *a)* we use Vinogradov's type estimates for trigonometric sums (see Prachar (1978) and Lemma 6 below). In the case *b)* we apply the Vinogradov inequality for trigonometric integrals (see Lemma 4, ch.2, section II in Vinogradov (1985)) and its generalizations to the multidimensional case (see Lemma 4 below). Our refinements of (1) are similar in form to the improvements given by the Vinogradov method for trigonometric sums as compared to Weyl's results (see the Introduction of Section II in Vinogradov (1985)).

2. Main results

Let P, P_1, P_2 denote three probability measures in $\mathbb{R}^k$ and let γ be a positive number, $0 < \gamma \leq 1$. The measure P_1 is called a *component* of P of *weight* γ if

$$P = \gamma P_1 + (1-\gamma)P_2.$$

In Theorems 1 and 2 we consider the one-dimensional case and polynomials

$$f(x) = \alpha x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_0, \qquad m \geq 2.$$

Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with a distribution which admits a component P_1 such that for two points $a_1 < a_2$ we have*

$$P_1\{a_1\} = P_1\{a_2\} = \tfrac{1}{2}\,.$$

Let δ with $0 < \delta \le 1$, denote the weight of P_1 and define

$$\beta_m = \frac{1}{12(m-1)^2 \ln(m-1)}.$$

Then for any n: $n > 4/\delta$, and $m > \widehat{c}$ we have

$$\sup_{a \in \mathbb{R}^1} \Big| \mathbf{E} \exp\big\{ it\, f(S_n + a)\big\}\Big| \le 2\exp(-n\delta^2/8) + 15\,\widetilde{c}(m)\,(n\delta)^{-\beta_m}(\ln n)^{3/2}$$
$$+\, 17\,(a_2 - a_1)^{-m/(m-1)}\delta^{-1/2}\left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/(m-1)} \tag{3}$$

provided that

$$0 < |\alpha t| \le \left[\frac{\pi}{3m\sqrt{\delta}(a_2 - a_1)^m}(\ln n)^{-1/2}\right] n^{(m-1)/2}\,, \tag{4}$$

where the constants $\widehat{c}$ and $\widetilde{c}(m)$ are the same as in Lemma 6 below.

Remark. Let us denote by L the square bracket on the right-hand side of (4). We have

$$\left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/(m-1)} \ge |\alpha t|^{-m/(m-1)^2} L^{-1/(m-1)^2}.$$

Thus, the last summand on the right-hand side of (3) is of order $O\big(|t|^{-m/(m-1)^2}\big)$ for large $|t|$.

Theorem 2. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with a distribution F admitting a nondegenerate discrete component. Then for any $\varepsilon > 0$ and integer $m \ge 2$ there exist absolute constants c_4, c_5 and constants c_2, c_3 depending on F, ε and m such that for all $n \ge 1$ and any t in the range (2)*

$$\sup_{a \in \mathbb{R}^1} \Big| \mathbf{E} \exp\big\{ it\, f(S_n + a)\big\}\Big| \le c_3 \left(n^{-c_4/(m^2 \ln m)} + \left(\frac{\sqrt{n}}{|\alpha t|}\right)^{c_5/m}\right). \tag{5}$$

We shall now discuss the case of continuous components. Assume that a random vector Y in $\mathbb{R}^k$ has a distribution P_Y which admits an absolutely continuous component of weight λ. Then P_Y has an absolutely continuous component of weight γ for any γ with $0 < \gamma < \lambda$ and such that the Lebesgue density of the component is bounded. Let $\gamma = \lambda/2$. This implies (see Rudin (1962), p. 4 or Gelfand et al (1946), § 16) that $P_Y * P_Y$ has a component of weight $(\lambda/2)^2$ with uniformly continuous density function on $\mathbb{R}^k$, say $r(x)$. If $r(a_0) = r_0 > 0$ for some a_0 then there exists η such that for all x with $|x - a_0| \le \eta$ we have $r(x) \ge r_0/2$. Thus, $P_Y * P_Y$ admits a component which is uniformly distributed on a ball.

Theorem 3. *Let*

$$f_0(x) = \left(x^{(1)}\right)^m + \cdots + \left(x^{(k)}\right)^m, \qquad m \ge 2.$$

and let $X_1, X_2, \ldots$ denote a sequence of i.i.d. random vectors in $\mathbb{R}^k$. Assume that for some n_0 the distribution of $X_1 + \cdots + X_{n_0}$ has an absolutely continuous component. Let η denote a real number such that the distribution of $X_1 + \cdots + X_{2n_0}$ has a component of weight δ uniformly distributed on the ball $\{x\colon |x - a_0| \le \eta\}$. Then for all $n \ge 6n_0/\delta$ we have

$$\sup_{a \in \mathbb{R}^k} \left| \mathbf{E} \exp\left\{it\, f_0(S_n + a)\right\}\right| \le 2 \exp\left(\frac{-n\delta^2}{24 n_0}\right) + c^{**}(k)\, \eta^{-k} (n_0 \delta^{-1})^{k/2} |t|^{-k/m},$$

where

$$c^{**}(k) = (486/\pi)^{k/2} \left(\left(\frac{k}{2} + 1\right)^{k/2} + 2^{k+1} \Gamma^2\left(\frac{k}{2} + 1\right)\right).$$

and $\Gamma(\cdot)$ is the Gamma-function.

For components with differentiable densities we obtain the following result which is close to Theorem 3.

Theorem 4. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors in $\mathbb{R}^k$ with Lebesgue density p such that*

$$p = \alpha p_0 + (1 - \alpha)\, p_1, \qquad 0 < \alpha \le 1,$$

where p_0 is continuously differentiable. Let

$$V_0 = \int_{\mathbb{R}^k} \left|\operatorname{grad} p_0(x)\right| dx.$$

Then for $n \ge 2(k+1)/\alpha$, $m \ge 2$ and $t \ne 0$ we have

$$\sup_{a \in \mathbb{R}^k} \left| \mathbf{E} \exp\left\{it\, f_0(S_n + a)\right\}\right| \le c(k)\, V_0^k \left(\frac{2}{\alpha}\right)^{k/2} |t|^{-k/m} + 2 \exp\left(-\frac{\alpha^2 n}{8}\right), \qquad (6)$$

where $c(k) = 1.1 \cdot (17.5)^k$. If $\alpha = 1$ then the second summand on the right-hand side of (6) can be omitted.

Consider now the general polynomial

$$f(x) = \sum_{m_1 = 0}^{m} \cdots \sum_{m_k = 0}^{m} \alpha(m_1, \ldots, m_k)\, x_1^{m_1} \cdots x_k^{m_k} \qquad (7)$$

with $\alpha(0, \ldots, 0) = 0$. Let M stand for *the degree* of $f(x)$, i.e.

$$M = \max\{m_1 + \cdots + m_k \,:\, \alpha(m_1, \ldots, m_k) \ne 0\}.$$

Obviously, $km \geq M$. Define

$$\alpha_{**}(f) = \max \left|\alpha(m_1, \ldots, m_k)\right| > 0,$$

where *maximum* is taken over all $m_1, \cdots, m_k$ such that

$$0 \leq m_1, \cdots, m_k \leq m \quad ; \qquad m_1 + \cdots + m_k = M.$$

Theorem 5. *Let Z be a random vector in $\mathbb{R}^k$ with independent standard normal coordinates $Z^{(i)}$, $i = 1, 2, \ldots$. For $f(x)$ defined by (7) with $M \geq 2$ we have*

$$\sup_{a \in \mathbb{R}^k} \left|\mathbf{E} \exp\left\{it\, f(Z+a)\right\}\right| \leq c_6\, |\alpha_{**} t|^{-1/m} \ln^{c_7}(2 + |\alpha_{**} t|),$$

where

$$c_6 = k\,(2/\pi)^{1/2} + 2^{(13k - 3M/m)/2}(2\pi)^{1/m - k/2}(1.25M + 1)^{k-1}$$

and $c_7 = (3k - M/m)/2 - 1 \geq 0$.

Let now

$$f_1(x) = x^{(1)} \cdots x^{(k)}, \qquad k \geq 2.$$

Let Z as above be a random vector in $\mathbb{R}^k$ with independent standard normal coordinates. We denote

$$w_k(t) = \mathbf{E} e^{it f_1(Z)}.$$

It is easy to see that $w_k(t)$ is a mixture with respect to $\sigma > 0$ of the characteristic functions of normal distributions, i.e. $\exp\{-t^2/(2\sigma^2)\}$. Therefore, $w_k(t)$ is a real nonnegative even function tending monotonically to zero as $|t| \to \infty$.

Theorem 6. *For any $k \geq 2$ and all $|t| \geq 1$ we have*

$$l_k |t|^{-1} \ln^{k-2}(|t|) \leq w_k(t) \leq L_k |t|^{-1} \ln^{k-2}(2 + |t|),$$

where

$$l_k = 2^{-1/2}\left(\frac{2}{(k-2)^2 \pi e}\right)^{(k-2)/2}$$

and

$$L_{2m} = 3^{m-1},\ L_{2m+1} = 2.6 \cdot 3^{m-1}, \qquad m \geq 1.$$

Remark 1. Applying Theorem 5 we can get only

$$w_k(t) \leq c(k)\, |t|^{-1} \ln^{k-1}(2 + |t|).$$

Remark 2. The asymptotic formula

$$w_k(t) = L_k^* |t|^{-1} \ln^{k-2} |t| \cdot (1 + o(1)) \qquad \text{as} \qquad |t| \to \infty,$$

where L_k^* is a positive constant, holds for $k = 2, 3, 4$ (see Lemma 9 and Corollary of Lemma 11 below). It is probably true for $k > 4$ also.

We consider here the case when a polynomial in $\mathbb{R}^k$ has the form

$$f_2(x) = \left(x^{(1)}\right)^{m_1} \cdots \left(x^{(k)}\right)^{m_k} \tag{8}$$

with $m_j \geq 1,\ j = 1, 2, \ldots, k$.
Let Z be again a random vector in $\mathbb{R}^k$ with independent standard normal coordinates $Z^{(j)},\ j = 1, 2, \ldots, k$.

Theorem 7. *Assume that among $m_1, m_2, \ldots, m_k$ in (8) there exists the unique largest number, say $m = m_1$, and among all others $m_j,\ j = 2, 3, \ldots, k$ there exists at least one odd number, say m_2. For $0 < \theta < 1$ we denote*

$$z_\theta = \mathbf{E}\left|Z^{(1)}\right|^{-\theta}.$$

We have

$$|t|^{1/m}\left|\mathbf{E}\exp\{itf_2(Z)\}\right| \uparrow c_8, \qquad \text{as} \quad |t| \uparrow \infty,$$

where

$$c_8 = m^{-1} z_{1/m}^{-1} z_{(m-1)/m} z_{m_2/m} \cdots z_{m_k/m}.$$

3. Lemmas

a) *Binomial distribution.*

Lemma 1. *For any p: $0 \leq p \leq 1$, and $\varepsilon > 0$*

$$\sum_{|m-np|\geq n\varepsilon} \binom{n}{m} p^m (1-p)^{n-m} \leq 2\, e^{-n\varepsilon^2/2}.$$

For a proof see Uspensky (1937), p. 102.

Lemma 2. *For any integer $n \geq 1$*

$$\max_{0\leq m\leq n} \binom{n}{m} 2^{-n} \leq \sqrt{\frac{2}{\pi n}}.$$

Proof is trivial application of Stirling's formula.

b) *Uniform distribution on the unit ball in $\mathbb{R}^k$.*

Let U denote a random vector on $\mathbb{R}^k$ which has an uniform distribution on the unit ball

$$|x|^2 := \left(x^{(1)}\right)^2 + \cdots + \left(x^{(k)}\right)^2 \leq 1.$$

Its Lebesgue density, say $q(x)$, is given by $q(x) = \Gamma(k/2+1)\pi^{-k/2}$, if $|x| \leq 1$, and $q(x) = 0$ otherwise.

The characteristic function of U is given by

$$\mathbf{E}\, e^{i(v,U)} = \frac{\Gamma(k/2+1)\cdot 2^{k/2}}{|v|^{k/2}} \cdot J_{k/2}\big(|v|\big),$$

where $J_\nu(z)$ denotes the Bessel function of the order ν.
Let

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots$$

be the positive solutions of the equation $J_\nu(z) = 0$. We only need $\nu = k/2,\ k = 1, 2, 3, \dots$ It is known (see e.g. Watson (1966), § 15.3) that

$$j_{k/2,1} > \sqrt{\frac{k}{2}\left(\frac{k}{2}+2\right)} > \frac{k}{2}.$$

Lemma 3. *For* $k = 1, 2, 3, \dots$

$$\big|\mathbf{E}\, e^{i(v,U)}\big| \le \exp\left\{-\frac{|v|^2}{2(k+2)}\right\}, \qquad \text{if} \quad |v| \le j_{k/2,1}, \tag{9}$$

and for $|v| > j_{k/2,1}$

$$\big|\mathbf{E}\, e^{i(v,U)}\big| \le 0.75. \tag{10}$$

Moreover,

$$\int_{\mathbb{R}^k} \big|\mathbf{E}\, e^{i(v,U)}\big|^2 dv \le 2^k \pi^{k/2} \Gamma\left(\frac{k}{2}+1\right). \tag{11}$$

Proof. We have (see Watson (1966), § 15.41 and § 15.51) for all complex z

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{l=1}^{\infty}\left(1 - \frac{z^2}{j_{\nu,l}^2}\right).$$

Therefore, for any $v \in \mathbb{R}^k$

$$\mathbf{E}\, e^{i(v,U)} = \prod_{l=1}^{\infty}\big(1 - |v|^2 \cdot j_{k/2,l}^{-2}\big) \tag{12}$$

and

$$\sum_{l=1}^{\infty} j_{k/2,l}^{-2} = \frac{1}{2(k+2)}\,.$$

This and the inequality $1 - z \le e^{-z}$ together imply (9).
For $|v| \ge k/2 \ge 1/2$

$$\big|\mathbf{E}\, e^{i(v,U)}\big| \le \frac{\Gamma(k/2+1)\, 2^{k/2}}{|v|^{k/2}} \sqrt{\frac{2}{\pi|v|}}\left(1 - \frac{k^2}{4|v|^2}\right)^{-1/4} := G_k\big(|v|\big) \tag{13}$$

(see ibid. §13.74). For $|v| \geq j_{k/2,1}$ the right-hand side of (13) does not exceed $G_k(\sqrt{(k/2)(k/2+2)})$. Applying Stirling's formula to $\Gamma(k/2+1) = (k/2)\,\Gamma(k/2)$ we get

$$G_k\left(\sqrt{\left(\frac{k}{2}\right)\left(\frac{k}{2}+2\right)}\right) \leq 2\,e^{-1+3/(k+4)+1/(6k)} \cdot \left(\frac{2}{e}\right)^{k/2} \cdot \left(\frac{k}{4}+1\right)^{1/4}. \tag{14}$$

If $k \geq 1$ increases, the right-hand side of (14) decreases and for $k = 4$ it is equal to

$$e^{10/24} \cdot (2/e)^3 \cdot 2^{1/4} < 0.75.$$

For $k = 2$ or 3 we have $j_{1,1} = 3.83171\ldots$ and $j_{3/2,1} = 4.493409\ldots$ (see Abramowitz and Stegun (1964)).
Hence $G_2(j_{1,1}) = 0.21654\ldots$ and $G_3(j_{3/2,1}) = 0.15307\ldots.$ Moreover, for $k = 1$ we have $j_{1/2,1} = \pi$ and for $|v| \geq \pi$

$$\left|\mathbf{E}\,e^{i(v,U)}\right| = \left|\frac{\sin v}{v}\right| \leq \frac{1}{\pi} = 0.31830\ldots.$$

Thus, (10) is proved.
Consider now the integral in (11). We have

$$\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k}\left|\mathbf{E}\,e^{i(v,U)}\right|^2 dv = q * q(0) \leq \sup_x q(x) = \Gamma\left(\frac{k}{2}+1\right)\pi^{-k/2},$$

what is equivalent to (11). This concludes the proof of Lemma 3.

Corollary of Lemma 3. *For all integers $n \geq 2$ and r such that $2 \leq r \leq n$ we have*

$$\int_{\mathbb{R}^k}\left|\mathbf{E}\,e^{i(v,U/\sqrt{n})}\right|^r dv \leq c^*(k)\left(\frac{n}{r}\right)^{k/2}, \tag{15}$$

where

$$c^*(k) = 2^k\pi^{k/2}\left(\left(\frac{k}{2}+1\right)^{k/2} + 2^{k+1}\Gamma^2\left(\frac{k}{2}+1\right)\right).$$

Proof. The integral on the left-hand side of (15) equals

$$\begin{aligned} n^{k/2}\int_{\mathbb{R}^k}\left|\mathbf{E}\,e^{i(v,U)}\right|^r dv &= n^{k/2}\left(\int_{|v|<j_{k/2,1}} + \int_{|v|\geq j_{k/2,1}}\right)\left|\mathbf{E}\,e^{i(v,U)}\right|^r dv \\ &\leq \left(\frac{n}{r}\right)^{k/2}(2\pi)^{k/2}(k+2)^{k/2} + n^{k/2}(0.75)^{r-2}\cdot 2^k\pi^{k/2}\Gamma\left(\frac{k}{2}+1\right). \end{aligned} \tag{16}$$

Note, that

$$\max_{r>0} r^{k/2}(0.75)^r < 2^k\Gamma\left(\frac{k}{2}+1\right). \tag{17}$$

Then (16) and (17) imply (15).

c) *Inequalities for probabilities of large deviations.*

We need simple upper bounds for $P(|S'_n| > x)$ and $P(|Y| > x)$, where $S'_n = n^{-1/2}(U_1+\cdots+U_n)$ and $U_1,\ldots,U_n$ is a sequence of i.i.d. random vectors having the uniform distribution on the unit ball, i. e. $U_n \stackrel{d}{=} U$, and Y has a Gaussian distribution with zero mean and a nondegenerate covariance matrix $\Lambda = (\lambda_{j,l})$, $1 \le j,l \le k$. Here and everywhere below $X \stackrel{d}{=} Z$ means that random variables X and Z have the same distribution.
Note that similarly to (12) we get

$$\mathbf{E}\,\exp(v,U) = \prod_{l=1}^{\infty}\left(1+|v|^2\cdot j_{k/2,l}^{-2}\right) \le \exp\left(\frac{|v|^2}{2\,(k+2)}\right).$$

Now we have (see Prohorov (1968))

$$\begin{aligned} P\big(|S'_n|>x\big) &\le e^{-x^2/2}\mathbf{E}\,\exp\left(|S'_n|^2/2\right)\\ &= e^{-x^2/2}\mathbf{E}\left(\frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k} e^{(S'_n,u)}\cdot e^{-|u|^2/2}du\right)\\ &\le e^{-x^2/2}\frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k} e^{|u|^2/(2(k+2))-|u|^2/2}\,du\\ &= e^{-x^2/2}\left(1-\frac{1}{k+2}\right)^{-k/2}. \end{aligned}$$

The second factor is an increasing function of k. Hence we have for all $x \ge 0$

$$P\big(|S'_n|>x\big) < e^{1/2-x^2/2}.$$

This rough estimate will suffice for our purposes.
For $P(|Y| > x)$ we have

$$P\big(|Y|>r\sigma\big) \le 2\,\exp\left(-r^2/2\right),$$

where $\sigma^2 = \operatorname{tr}\Lambda = E\,|Y|^2$. The estimate can be derived from the inequalities in Yurinskii (1970).

d) *Trigonometric integrals and sums.*

We use the notation introduced before Theorem 5, in particular we consider $f(x)$ of the general form (7).
Note that for any $a \in \mathbb{R}^k$ and $f_a(x) := f(x+a)$ we have

$$\alpha_{**}(f_a) = \alpha_{**}(f).$$

Additionally we define

$$\alpha_*(f) = \max_{0\le m_1,\ldots,m_k\le m}\big|\alpha(m_1,\ldots,m_k)\big|.$$

Introduce for $m > 0$, $k \geq 1$ and $\alpha > 0$

$$\lambda(\alpha) = \min\big(1,\ 32^k \alpha^{-1/m} \ln^{k-1}(\alpha + 2)\big).$$

It is easy to see that $\lambda(\alpha)$ is a non-increasing function.

Lemma 4 (see Arkhipov et al. (1987) Theorem 5, § 3, ch. 1). *Let*

$$I_k(f) = \int_0^1 \cdots \int_0^1 \exp\big\{2\pi i\, f(x)\big\}\, dx.$$

Then

$$\big|I_k(f)\big| \leq \lambda(\alpha_*).$$

Corollary 1 of Lemma 4. *For any real $h > 0$ and $a \in \mathbb{R}^k$ with $k \geq 2$*

$$\left| \int_{-h}^{h} \cdots \int_{-h}^{h} \exp\big\{it\, f(x+a)\big\}\, dx \right| \leq (2h)^k \lambda\big((2\pi)^{-1}|t|(2h)^M \alpha_{**}(f)\big),$$

Remark 1. If $k > 1$ and the polynomial $f(x)$ can be represented as a sum of polynomials depending on different blocks of variables, e.g.

$$\widetilde{f}(x) = x^{(1)}\big(x^{(2)}\big)^6 + \big(x^{(3)}\big)^3\big(x^{(4)}\big)^2,$$

then in fact we can get larger exponent by $|t|^{-1}$ than one derived from Corollary 1 not taking into account the factor of the form $\ln^{k-1}(|t| + 2)$. For example for $\widetilde{f}(x)$ mentioned above we can obtain $|t|^{-1/2}$ instead of $|t|^{-1/6}$. See also Lemma 7 below.

Remark 2. The onedimensional case was studied by Vinogradov (1985).

Corollary 2 of Lemma 4. *For* $k = 1$, $m \geq 2$, $h > 0$ *and any* $a \in \mathbb{R}^1$

$$\left| \int_{-h}^{h} \exp\big\{it\, \widehat{f}(x+a)\big\}\, dx \right| \leq \frac{32 \cdot (2\pi)^{1/m}}{|\alpha t|^{1/m}} < \frac{81}{|\alpha t|^{1/m}}, \tag{18}$$

where

$$\widehat{f}(x) = \alpha\, x^m + \alpha_{m-1} x^{m-1} + \cdots + \alpha_1 x.$$

Remark. In some special cases the absolute constant in the right-hand side of (18) can be replaced by a smaller one.

Lemma 5 (cf.(18)). *Let $m \geq 2$ and*

$$f(x) = \alpha x^m + \alpha_1 x + \alpha_0.$$

Then for all real a, b and $t \neq 0$

$$\left| \int_a^b e^{itf(x)} dx \right| \leq 6 \cdot \left(\frac{8}{3}\right)^{1/3} |\alpha t|^{-1/m} < 8.3204 \cdot |\alpha t|^{-1/m}.$$

Proof. By the van der Corput Lemma we have in the case $m = 2$

$$\left| \int_a^b e^{itf(x)} dx \right| \le \frac{8}{\sqrt{|t| \min |f''(x)|}} = \frac{8}{\sqrt{2} |\alpha t|^{1/2}} ,$$

where minimum is taken over all $x : a \le x \le b$.
Consider now the case $m \ge 3$. Let $h > 0$ and put

$$N = m/2, \qquad \gamma = 8\big(m\,(m-1)\,|\alpha t|\big)^{-1/2}, \qquad \lambda(h) = h + \gamma h^{-(N-1)}.$$

It is easy to check that $\lambda(h)$ has the unique minimum at some point h_m and

$$\lambda(h_m) = d_m |\alpha t|^{-1/m},$$

where

$$d_m = \frac{m}{m-2} \left(\frac{16(m-2)^2}{m\,(m-1)} \right)^{1/m} .$$

For $m \ge 3$ we have

$$d_m \le d_3 = 3 \left(\frac{8}{3} \right)^{1/3} .$$

Write for $\theta > 0$

$$I(\theta) = \int_0^\theta \exp \big\{ it\, f(x) \big\}\, dx.$$

We have $|I(\theta)| \le h_m$ if θ: $0 < \theta \le h_m$. By the van der Corput Lemma for $\theta > h_m$

$$\big| I(\theta) \big| \le h_m + \frac{8}{\sqrt{|t|\, |f''(h_m)|}} = \lambda(h_m) \le d_3 |\alpha t|^{-1/m}.$$

This inequality holds also for $\theta < 0$. Since

$$\left| \int_a^b e^{itf(x)} dx \right| \le \big| I(b) \big| + \big| I(a) \big|,$$

Lemma 5 is proved.

For trigonometric sums we shall use the following

Lemma 6 (see Prachar (1978), Satz 4.2, § 4, Kap. VIII)**.** *Let $P \ge 1$ and Q be integers,*

$$S = \sum_{Q < x \le Q+P} \exp \big\{ 2\pi i \,(\alpha x^m + \alpha_{m-1} x^{m-1} + \cdots + \alpha_0) \big\},$$

where $\alpha, \alpha_{m-1}, \ldots, \alpha_0$ are real numbers.
Assume that

$$0 < 2m\, P\, |\alpha| \le 1.$$

Then there exists a constant $\widehat{c} \geq 3$ not depending on m, P, $Q\,\alpha$, $\alpha_{m-1}, \ldots, \alpha_0$ such that for $m \geq \widehat{c}$ we have (below $m_1 = m-1$)

$$|S| \leq \widetilde{c}(m)\, P^{1-1/(6m_1^2 \ln m_1)} \ln P + 2\,|\alpha|^{-1/m_1},$$

where

$$\widetilde{c}(m) = 2\, e^{15 m_1 \ln^2 m_1}.$$

Remark. In fact, the statement of Lemma 6 one can find also in the first edition of the Prachar book (1978) published in 1957.
In both editions it is one of the three theorems included in the section *Die Methode von Vinogradoff.*

e) *Functions of random vectors having absolute integrable characteristic functions.*

Let $Y = (Y^{(1)}, \ldots, Y^{(k)})$ be a random vector taking values in $\mathbb{R}^k$. We denote by $g_Y(v)$, $v = (v^{(1)}, \ldots, v^{(k)})$, the characteristic function of Y. Let $p_Y(x)$, $x = (x^{(1)}, \ldots, x^{(k)})$ denote the Lebesgue density of Y (if it exists) and let Λ_Y denote the covariance matrix of Y (if $\mathbf{E}|Y|^2 < \infty$) Assume now that

$$J_Y = \int_{\mathbb{R}^k} \big|g_Y(v)\big|\, dv < \infty. \tag{19}$$

We recall some simple inequalities for J_Y. First, if $g_Y(v) \geq 0$ then $J_Y = (2\pi)^k p_Y(0)$. For example, for a Gaussian random vector Y we have

$$J_Y = (2\pi)^{k/2} (\det \Lambda)^{-1/2}.$$

Furthermore, for all Y such that $\mathbf{E}|Y|^2 < \infty$ we get (see Cramer (1946), § 11.12)

$$J_Y \geq \int_{Q_Y \leq 2} \Big(1 - \frac{1}{2}\, Q_Y(v)\Big)\, dv = \left(\Gamma\Big(\frac{k}{2} + 1\Big)\right)^{-1} 2^{k/2} \pi^{k/2} (\det \Lambda)^{-1/2},$$

where

$$Q_Y(v) = \mathbf{E}(v, Y - \mathbf{E}\, Y)^2.$$

For the functions of random vectors we consider at first two special cases.

Case 1.

For an integer $m \geq 2$ set

$$f_0(x) = \big(x^{(1)}\big)^m + \cdots + \big(x^{(k)}\big)^m.$$

Lemma 7. *Under the condition* (19) *we have for any $a \in \mathbb{R}^k$ and all real numbers $t \neq 0$,*

$$\big|\mathbf{E}\, e^{itf_0(Y+a)}\big| \leq 9^k (2\pi)^{-k} \left(\int_{\mathbb{R}^k} \big|g_Y(v)\big|\, dv\right) |t|^{-k/m}. \tag{20}$$

Proof (cf. Sadikova (1966)). Since $|g_Y(v)| = |g_{Y+a}(v)|$, we can assume that $a = 0$. Write

$$K(u) = \Big\{x\colon \max_{1\le j\le k} \big|x^{(j)}\big| \le u\Big\}.$$

Then

$$\begin{aligned}
&\int_{K(u)} e^{itf_0(x)} p_Y(x)\,dx \\
&= \int_{K(u)} e^{itf_0(x)} \Big\{\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-i(v,x)} g_Y(v)\,dv\Big\}\,dx \\
&\quad = \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} g_Y(v)\Big\{\prod_{j=1}^{k}\int_{-u}^{u} \exp\big(-iv^{(j)}x^{(j)} + it\,\big(x^{(j)}\big)^m\big)\,dx^{(j)}\Big\}\,dv. \qquad (21)
\end{aligned}$$

Applying Lemma 5 we find an upper bound for the modulo of the integral on the left-hand side of (21) which does not depend on u. We obtain (20) as u tends to infinity.

Example. If Y has a nondegenerate normal distribution then for any $a \in \mathbb{R}^k$

$$\Big|\mathbf{E}\exp\big\{it\,f_0(Y+a)\big\}\Big| \le c(k,\Lambda)\,|t|^{-k/m}, \qquad (22)$$

where

$$c(k,\Lambda) = (2\pi)^{-k/2} 9^k (\det\Lambda_Y)^{-1/2}.$$

Note that (22) has been obtained in Bentkus et al.(1993), Corollary 8.1, by another method and without specifying the constant factor $c(k,\Lambda)$.
In particular, for $k = 2$ we have

$$\begin{aligned}
&\Big|\mathbf{E}\exp\big\{it\,\big(Y^{(1)}+a^{(1)}\big)^m + \big(Y^{(2)}+a^{(2)}\big)^m\big\}\Big| \\
&\qquad \le \frac{81}{2\pi}\,\sigma_1^{-1}\sigma_2^{-1}(1-\rho^2)^{-1/2}|t|^{-2/m},
\end{aligned}$$

where σ_1 and σ_2 are the standard deviations of $Y^{(1)}$ and $Y^{(2)}$, respectively, and ρ is their correlation coefficient. We denote for $m \ge 2$

$$b_m = \begin{cases} \sqrt{2/\pi}\,\Gamma(1+1/m) & \text{if } m \text{ is even},\\ \sqrt{2/\pi}\,\Gamma(1+1/m)\,\cos(\pi/(2m)) & \text{if } m \text{ is odd}. \end{cases}$$

Clearly, $b_m < \sqrt{2/\pi}$ and $b_m \uparrow \sqrt{2/\pi}$ as $m \to \infty$.
The inequalities for characteristic functions of powers of normal random variables (see Bagirov (1990)) imply for independent $Y^{(1)}, \ldots, Y^{(k)}$ with zero means (cf.(22))

$$\sup_t |t|^{k/m}\Big|\mathbf{E}\exp\big\{it\,f_0(Y)\big\}\Big| = b_m^k\sigma_1^{-1}\cdots\sigma_k^{-1},$$

where $\sigma_j^2 = \operatorname{Var} Y^{(j)}$, $j = \overline{1,k}$.

Case 2.

If $k=1$, $m\ge 2$ and

$$f(x)=\alpha x^m+\alpha_{m-1}x^{m-1}+\cdots+\alpha_0$$

then using the same arguments as above for

$$\int_{-h}^{h}\exp\{it\,f(x+a)\}\,p_Y(x)\,dx=\int_{-h}^{h}\exp\{it\,f(x+a)\}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ivx}g(v)\,dv\right)dx$$

and (18) we obtain

$$\left|\mathbf{E}\,e^{itf(Y+a)}\right|\le\frac{32(2\pi)^{-1+1/m}}{|\alpha t|^{1/m}}\int_{-\infty}^{\infty}\left|g(v)\right|dv.$$

In the general case for the polynomials of the type (7) one can estimate the characteristic function using the inequality

$$\begin{aligned}\left|\mathbf{E}\,e^{itf(Y+a)}\right|&\le P\big(\max_{1\le j\le k}\left|Y^{(j)}\right|\ge h\big)\\&+(2h)^k\lambda\big((2\pi)^{-1}|t|(2h)^M\alpha_{**}\big)(2\pi)^{-k}\int_{\mathbb{R}^k}\left|g(v)\right|dv\end{aligned}\tag{23}$$

and taking h as a positive function of t tending to infinity as $|t|\to\infty$. If the first summand on the right hand side of (23) decreases exponentially as h increases, we get an estimate of the characteristic function of the order $|t|^{-1/m}\log^a|t|$ with some $a\ge 0$ as $|t|\to\infty$. The proof of Theorem 5 follows exactly along this line.

f) Special polynomials in independent normal variables

Lemma 8. *Let a random variable X equals a product of three independent random variables*

$$X=Z_1^m\cdot Z_2\cdot S,$$

where $m\ge 1$ is an integer, Z_1, Z_2 have standard normal distribution and $S>0$ is such that

$$\mathbf{E}S^{-1/m}<\infty.\tag{24}$$

Then

$$|t|^{1/m}\left|\mathbf{E}e^{itX}\right|\uparrow d\qquad\text{as}\quad|t|\uparrow\infty,$$

where

$$d=\mathbf{E}\left|Z_1\right|^{-(m-1)/m}\mathbf{E}S^{-1/m}.$$

Proof Obviously it is sufficient to prove the Lemma only for $t>0$. We have

$$Z_1^mZ_2S\stackrel{d}{=}\left|Z_1\right|^mZ_2S$$

and

$$
\begin{aligned}
t^{1/m}\big|\mathbf{E}e^{itX}\big| &= \int_0^\infty d\,Pr(S<s)\\
&\times\Big(t^{1/m}\int_0^\infty \exp\big(-t^2v^2s^2/2 - v^{2/m}/2\big)(2/\pi)^{1/2}m^{-1}v^{(1-m)/m}\,dv\Big)\\
&= \int_0^\infty s^{-1/m}d\,Pr(S<s)\\
&\times\Big(\int_0^\infty \exp\big(-\tau^2/2-\tau^{2/m}/[2(st)^{2/m}]\big)m^{-1}(2/\pi)^{1/2}\tau^{(1-m)/m}\,d\tau\Big),
\end{aligned}
$$

For any fixed $s>0$ the inner integral in the last equality increases and tends to

$$m^{-1}\mathbf{E}|Z_1|^{-(m-1)/m} = m^{-1}z_{(m-1)/m}$$

as $t\uparrow\infty$.
This together with (24) imply Lemma 8.

In Lemmas 9–11 below we use the notation introduced before Theorem 6. In the proofs of the Lemmas the Bessel functions $J_0,\, I_0,\, K_0,\, Y_0,$ appear. All necessary formulas for these functions and numerical data are taken from Abramowitz and Stegun (1964) and Prudnikov et al.(1988).

Lemma 9. *The characteristic function of* $Z^{(1)}Z^{(2)}$ *equals*

$$w_2(t) = (1+t^2)^{-1/2} \tag{25}$$

and the Lebesgue density $p(v)$ *of* $Z^{(1)}Z^{(2)}$ *for any* $v\neq 0$ *is defined by*

$$p(v) = \pi^{-1}K_0(|v|). \tag{26}$$

Proof. Since

$$Z^{(1)}Z^{(2)} \overset{d}{=} \frac{1}{2}\Big(\big(Z^{(1)}\big)^2 - \big(Z^{(2)}\big)^2\Big),$$

(25) follows from the fact that

$$\mathbf{E}\exp\big(it(Z^{(1)})^2\big) = (1-2it)^{-1/2}.$$

We get (26) from the inversion formula (see 9.6.25 in Abramowitz and Stegun (1964)).

Lemma 10. *For* $K_0(v)$ *the following inequalities hold:*
a) for $0<v\le 1$

$$K_0(v) < 1.267\,\ln(2/v), \tag{27}$$

b) for $v\ge 1$

$$K_0(v)\le e^{1-v}K_0(1). \tag{28}$$

Proof. The bound (27) follows from the inequality

$$K_0(v) < \ln(2/v)\, I_0(v), \qquad 0 < v \le 1,$$

which in its turn is the corollary of 9.8.5 in Abramowitz and Stegun (1964). Furthermore, $e^v K_0(v)$ is a decreasing function for $v > 0$. It follows from the formula (see 9.6.23 in Abramowitz and Stegun (1964))

$$K_0(v) = \int_1^\infty e^{-vu}(u^2-1)^{-1/2}\, du.$$

Thus, we obtain (28).

Corollary of Lemma 10. *We have*

$$\frac{2}{\pi}\int_1^\infty K_0(v)\, dv < 0.269 \tag{29}$$

and for $t \ge 1$

$$\frac{2}{\pi}\int_0^{1/t} K_0(v)\, dv < 1.614\, t^{-1}\ln(2+t). \tag{30}$$

Proof. We get (29) immediately from Lemma 10, b)
For $t \ge 1$ we have

$$\begin{aligned}\frac{2}{\pi}\int_0^{1/t} K_0(v)\, dv &= \frac{2}{\pi t}\int_0^1 K_0(z/t)\, dz\\ &\le \frac{2.534}{\pi t}\int_0^1 \ln(2t/z)\, dz < 1.614\, t^{-1}\ln(2+t).\end{aligned}$$

This concludes the proof of the Corollary.

Lemma 11. *For* $t > 0$ *we have*

$$w_3(t) = (2\pi)^{-1/2} t^{-1} \exp\big((4t^2)^{-1}\big) K_0\big((4t^2)^{-1}\big) \tag{31}$$

and

$$w_4(t) = \pi(4t)^{-1}\Big(J_0^2\big((2t)^{-1}\big) + Y_0^2\big((2t)^{-1}\big)\Big). \tag{32}$$

Proof. It follows easily from 2.3.6.10 in vol.1 and 2.16.3.12 in vol.2 in Prudnikov et al. (1988).

Corollary of Lemma 11. *We have*

$$w_3(t) = (2/\pi)^{1/2} t^{-1}\ln t \cdot (1+o(1)), \qquad as \qquad t \to \infty$$

and

$$w_4(t) = \pi^{-1} t^{-1}\ln^2 t \cdot (1+o(1)), \qquad as \qquad t \to \infty.$$

Proof. The Corollary follows from (31), (32) and limit relations (see 9.6.13 and 9.1.13 in Abramowitz and Stegun (1964))

$$K_0(v) = -\ln v \cdot (1+o(1)) \quad \text{and} \quad Y_0(v) = 2/\pi \ln v \cdot (1+o(1)) \quad \text{as} \quad v \downarrow 0.$$

g) The reduction of the general problem to the special case.

We shall use also two simple Lemmas.

Lemma 12. *Let X and Y be independent random vectors taking values in $\mathbb{R}^k$. Then for any real Borel function f and for all real t*

$$\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\big\{it\,f(X+Y+a)\big\}\Big|\le\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\big\{it\,f(X+a)\big\}\Big|.$$

The proof is clear.

Let now $X_1, X_2, \ldots$ be a sequence of i.i.d. random vectors in $\mathbb{R}^k$. We take two integers n_1, n: $1\le n_1<n$, and put $n_2=[n/n_1]$,

$$\begin{aligned}
&Y_1=X_1+\cdots+X_{n_1},\\
&Y_2=X_{n_1+1}+\cdots+X_{2n_1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&Y_{n_2}=X_{(n_2-1)n_1+1}+\cdots+X_{n_2n_1},\\
&R=X_{n_2n_1+1}+\cdots+X_n,
\end{aligned}$$

(if $n=n_2n_1$ then $R=0$).
Let the distribution of Y_1 be a mixture of distributions Q_1 and Q_2 weighted with δ, $0<\delta\le1$, and $1-\delta$, respectively.
We can write

$$\begin{aligned}
\mathbf{E}\exp\big\{it\,f(X_1+\cdots+X_n+a)\big\}&=\sum_{j=0}^{n_2}\binom{n_2}{j}\delta^j(1-\delta)^{n_2-j}\\
&\times\mathbf{E}\exp\big\{it\,f(Y_{1,1}+\cdots+Y_{1,j}+Y_{2,j+1}+\cdots+Y_{2,n_2}+\widehat{R}+a)\big\},
\end{aligned}$$

where $Y_{1,1},\ldots,\widehat{R}$ are independent random vectors; $Y_{1,1},\ldots,Y_{1,j}$ have the same distribution Q_1 and $Y_{2,j+1},\ldots,Y_{2,n_2}$ have the same distribution Q_2 and $\widehat{R}\overset{\mathrm{d}}{=}R$. We divide the above sum into two parts according to the value of j: $|j-n_2\delta|\ge n_2\delta/2$ or $|j-n_2\delta|<n_2\delta/2$.
Applying Lemmas 1 and 12 we get

$$\begin{aligned}
\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\big\{it\,f(X_1+\cdots+X_n+a)\big\}\Big|&\le2\exp\Big(-\frac{n_2\delta^2}{8}\Big)\\
&+\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\big\{it\,f(Y_{1,1}+\cdots+Y_{1,n_3}+a)\big\}\Big|,
\end{aligned}$$

where $n_3=[n_2\delta/2]+1$. Replacing $f(x)$ by $f(x/\sqrt{n})$ we obtain

Lemma 13. *With the notation introduced above we have*

$$\begin{aligned}
\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\big\{it\,f(S_n+a)\big\}\Big|&\le2\exp\Big(-\frac{n_2\delta^2}{8}\Big)\\
&+\sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\Big\{it\,f\Big(\frac{Y_{1,1}+\cdots+Y_{1,n_3}}{\sqrt{n}}+a\Big)\Big\}\Big|
\end{aligned}\tag{33}$$

4. Proof of Theorem 1

Let
$$f(x) = \alpha x^m + \alpha_{m-1} x^{m-1} + \dots + \alpha_0,$$
where $\alpha \neq 0$ and m is an integer, $m \geq 2$. Let $\varepsilon_1, \varepsilon_2, \dots$ be a sequence of i.i.d. random variables taking values 0 and 1 with the same probability 1/2.
We take here $n_1 = 1$, $n_2 = n$, $n_3 = [n\delta/2] + 1$. Put
$$\Delta = \frac{a_2 - a_1}{\sqrt{n}}, \qquad \tau = (2\pi)^{-1} t \Delta^m,$$
$$f_1(x) = \Delta^{-m} f(\Delta x) = \alpha x^m + \Delta^{-1}\,\alpha_{m-1} x^{m-1} + \dots + \Delta^{-m}\,\alpha_0.$$
Then
$$\frac{Y_{1,j}}{\sqrt{n}} \stackrel{d}{=} \frac{a_1}{\sqrt{n}} + \frac{a_2 - a_1}{\sqrt{n}} \cdot \varepsilon_j = \frac{a_1}{\sqrt{n}} + \Delta\,\varepsilon_j,$$
$$\begin{aligned} t\,f\left(\frac{Y_{1,1} + \dots + Y_{1,n_3}}{\sqrt{n}} + a\right) &\stackrel{d}{=} t\,f\left(\Delta\,(\varepsilon_1 + \dots + \varepsilon_{n_3}) + \frac{n_3 a_1}{\sqrt{n}} + a\right) \\ &= 2\pi\tau\, f_1(\varepsilon_1 + \dots + \varepsilon_{n_3} + a'). \end{aligned}$$
Thus,
$$\begin{aligned} &\sup_{a \in \mathbb{R}^k} \left| \mathbf{E} \exp\left\{ it\,f\left(\frac{Y_{1,1} + \dots + Y_{1,n_3}}{\sqrt{n}} + a\right)\right\}\right| \\ &\qquad \leq \sup_{a' \in \mathbb{R}^k} \left| \mathbf{E} \exp\left\{ 2\pi i \tau\, f_1(\varepsilon_1 + \dots + \varepsilon_{n_3} + a')\right\}\right|. \end{aligned} \tag{34}$$
To simplify the notation we write $r = n_3$. We have
$$\begin{aligned} &\mathbf{E} \exp\left\{2\pi i \tau f_1(\varepsilon_1 + \dots + \varepsilon_r + a')\right\} \\ &\quad = \sum_{j=0}^{r} \exp\left\{2\pi i \tau f_1(j + a')\right\} p_r(j) = \widetilde{S}, \end{aligned} \tag{35}$$
where
$$p_r(j) = \binom{r}{j} 2^{-r}.$$
The probabilities $p_r(j)$ do not decrease (do not increase resp.) when j changes from 0 to $[r/2] + 1$ (from $[r/2] + 1$ to n, resp.). Moreover, by Lemma 2
$$\max_{0 \leq j \leq r} p_r(j) \leq \sqrt{\frac{2}{\pi r}}\,.$$
We recall that according to Abel's partial summation we have for any real numbers a_l and b_l
$$\sum_1^N a_l b_l = a_N B_N - \sum_1^{N-1} B_l (a_{l+1} - a_l), \tag{36}$$

where

$$B_l = b_1 + \cdots + b_l, \qquad l = 1, 2, \dots, N.$$

If a sequence of non-negative real numbers $a_1, \dots, a_N$ is such that there exists l_0: $1 \le l_0 \le N$, for which

$$a_1 \le a_2 \le \cdots \le a_{l_0}, \qquad a_N \le a_{N-1} \le \cdots \le a_{l_0},$$

then (36) yields

$$\left| \sum_1^N a_l b_l \right| \le 3 a_{l_0} \max_j |B_j|.$$

We now estimate the sum $\widetilde{S}$ in (35).
Let

$$A = \left[\sqrt{r \ln r} \right] + 1.$$

Then by Lemmas 1, 2 and 6 for $m \ge \widehat{c}$

$$\begin{aligned}
\left|\widetilde{S}\right| &\le \sum_{|j-r/2|\ge A} p_r(j) \qquad + \left| \sum_{r/2+A>j>r/2-A} \exp\left\{2\pi i \tau\, f_1(j+a')\right\} p_r(j) \right| \\
&\le 2 \exp\left(-\frac{1}{2} r \left(\frac{A}{r}\right)^2 \right) \\
&\qquad + 3\sqrt{\frac{2}{\pi r}} \max_{r/2-A<k<r/2+A} \left| \sum_{r/2-A<j\le k} \exp\left\{2\pi i \tau\, f_1(j+a')\right\} \right| \\
&\le \frac{2}{\sqrt{r}} + 3\sqrt{\frac{2}{\pi r}} \left[\widetilde{c}(m)\, (2A)^{1-1/(6m_1^2 \ln m_1)} \ln(2A) + 2|\alpha\tau|^{-1/m_1} \right]
\end{aligned} \tag{37}$$

provided that

$$0 < 4mA\, |\alpha\, \tau| \le 1. \tag{38}$$

Finally, we estimate the summands on the right-hand side of (37). Under the assumptions of Theorem 1

$$r = \left[\frac{n\delta}{2}\right] + 1 > \frac{n\delta}{2} \ge 3.$$

It is easy to see that

$$\begin{aligned}
3\sqrt{\frac{2}{\pi r}} \cdot 2 \cdot |\alpha\tau|^{-1/m_1} &= 6\sqrt{\frac{2}{\pi}} \sqrt{\frac{n}{r}}\, (2\pi)^{1/m_1} (a_2 - a_1)^{-m/(m-1)} \left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/m_1} \\
&< 12\sqrt{2} (a_2 - a_1)^{-m/(m-1)} \delta^{-1/2} \left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/m_1}.
\end{aligned}$$

and

$$2A < 2(\sqrt{r \ln r} + 1) < 3\sqrt{r \ln n} < n^2.$$

We introduce

$$\theta_m = \frac{1}{6m_1^2 \ln m_1}$$

and obtain

$$(2A)^{(1-\theta_m)}\ln(2A) \le 3\,r^{(1-\theta_m)/2}(\ln n)^{1/2}\cdot 2\log n.$$

Hence the right-hand side of (37) is not larger than

$$2\sqrt{2}(n\delta)^{-1/2} + \frac{18\sqrt{2}}{\sqrt{\pi}}\,\widetilde{c}(m)\,(n\delta)^{-\theta_m/2}(\ln n)^{3/2} + 12\sqrt{2}\,(a_2-a_1)^{-m/(m-1)}\delta^{-1/2}\left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/m_1}. \tag{39}$$

Recall that

$$\widetilde{c}(m) > 1000, \qquad \theta_m < 1/12, \quad n\delta > 4.$$

Hence (39) is smaller than

$$15\widetilde{c}(m)\,(n\delta)^{-\theta_m/2}(\ln n)^{3/2} + 17\,(a_2-a_1)^{-m/(m-1)}\delta^{-1/2}\left(\frac{\sqrt{n}}{|\alpha t|}\right)^{1/m_1}.$$

The condition (38) is satisfied if

$$0 < |\alpha t| \le \frac{\pi}{3m}\cdot\frac{1}{\sqrt{\delta}}\cdot\frac{1}{(a_2-a_1)^m}\cdot n^{(m-1)/2}\cdot(\ln n)^{-1/2},$$

which together with (33) and (34) complete the proof of Theorem 1.

5. Proof of Theorem 2

By assumption of Theorem 1 there exist $a_1 \ne a_2$ such that

$$P(X_1 = a_1) = \varepsilon_1 > 0 \qquad \text{and} \qquad P(X_1 = a_2) = \varepsilon_2 > 0.$$

Put $\varepsilon_3 = \min(\varepsilon_1, \varepsilon_2)$. We define a probability measure P_1 by the formula

$$P_1(\{a_1\}) = P_1(\{a_2\}) = \tfrac{1}{2}.$$

Then the distribution F of X_1 can be represented as

$$F = 2\varepsilon_3 P_1 + (1-2\varepsilon_3)\,P_2,$$

where P_2 is a probability measure.
By Theorem 1 we get (5) for all n and m larger than some initial values. For all other possible values of n and m (5) follows from (1) provided that t satisfies (2).

6. Proof of Theorem 3

Let $U, U_1, U_2, \dots$ be a sequence of i.i.d. random vectors in $\mathbb{R}^k$ having the uniform distribution on the unit ball $\{x\colon\ |x| \le 1\}$, $x \in \mathbb{R}^k$. Using the notation from part 3, g) we write

$$n_1 = 2n_0, \qquad Y_{n,1} \stackrel{d}{=} \eta\, U_1 + a_0\,.$$

Then by (33)

$$
\begin{aligned}
\sup_{a\in\mathbb{R}^k} \Big|\mathbf{E}\exp\big\{it\, f_0(S_n+a)\big\}\Big| &\le 2\exp\Big(-\frac{n_2\delta^2}{8}\Big) \\
&\quad + \sup_{a\in\mathbb{R}^k}\Big|\mathbf{E}\exp\Big\{it\, f_0\Big(\frac{\eta(U_1+\dots+U_{n_3})}{\sqrt{n}}+\frac{n_3 a_0}{\sqrt{n}}+a\Big)\Big\}\Big| \\
&= 2\exp\Big(-\frac{n_2\delta^2}{8}\Big) + \sup_{a'\in\mathbb{R}^k}\Big|\mathbf{E}\exp\Big\{it\, f\Big(\frac{\eta(U_1+\dots+U_{n_3})}{\sqrt{n}}+a'\Big)\Big\}\Big|. \qquad (40)
\end{aligned}
$$

By (15) and (20)

$$
\begin{aligned}
&\sup_{a'\in\mathbb{R}^k}\Big|\mathbf{E}\exp\Big\{it\, f_0\Big(\frac{\eta(U_1+\dots+U_{n_3})}{\sqrt{n}}+a'\Big)\Big\}\Big| \\
&\qquad \le 9^k(2\pi)^{-k}|t|^{-k/m}\int_{\mathbb{R}^k}\Big|g_U\Big(\frac{\eta v}{\sqrt{n}}\Big)\Big|^{n_3}dv \\
&\qquad \le 9^k(2\pi)^{-k}c^*(k)\,|t|^{-k/m}\eta^{-k}\Big(\frac{n}{n_3}\Big)^{k/2}. \qquad (41)
\end{aligned}
$$

If $n\ge 6n_0/\delta$ then $n_2\ge n/(3n_0)$ and $n_3\ge \min[2, n\delta/(6n_0)$. Therefore, (40) and (41) prove Theorem 3.

Remark. We omit the proof of Theorem 4 as it is based on similar arguments. One should only replace $|g_U(v)|$ by

$$
\Big|\int_{\mathbb{R}^k} e^{i(v,x)}p_0(x)\,dx\Big|,
$$

which is easily seen to be less than $\big(\pi/(3|v|)\big)\,V_0$.

7. Proof of Theorem 5

It is clear that without loss of generality we may assume that $\alpha_{**}=1$. We take in (23)

$$
h=(2\ln(2+|t|))^{1/2}>1.
$$

Then the first summand on the right-hand side of (23) is smaller than

$$
k\pi^{-1/2}(2+|t|)^{-1}\ln^{-1/2}(2+|t|).
$$

Furthermore, since

$$
\ln(2+|t|)\le 0.125(2+|t|)^{5/2},
$$

we get $2h<(2+|t|)^{5/4}$ and

$$
\ln\big(2+(2\pi)^{-1}|t|(2h)^M\big)<(5M/4+1)\ln(2+|t|).
$$

Therefore, the second summand in the right-hand side of () is smaller than

$$
A|t|^{-1/m}\ln^B(2+|t|),
$$

where
$$A = 2^{(13k-3M/m)/2}(2\pi)^{1/m-k/2}(1.25M+1)^{k-1}$$
and $B = (3k - M/m)/2 - 1$.
The above arguments prove Theorem 5.

Remark 1. Let T be an orthogonal $(k \times k)$-matrix and $f_T(x) = f(Tx)$. The constant m in (7) may be different for f_T and f. For example, $m = 1$ for $x_1 x_2$, but $m = 2$ for
$$\frac{x_1 + x_2}{\sqrt{2}} \cdot \frac{x_1 - x_2}{\sqrt{2}} = \frac{x_1^2 - x_2^2}{2}.$$
Thus, the proper choice of T can improve estimate given by Theorem 5.

Remark 2. We can get the results similar to those obtained in Theorem 5 replacing Z by $n^{-1/2}(U_1 + \ldots + U_n)$. In this way we can prove for general polynomial (7) inequality similar to one given in Theorem 3 for f_0. In this case the first term of the estimate remains unchanged and the second has the order $O\big(|t|^{-1/m}\ln^b|t|\big)$ with some $b > 0$ for large $|t|$.

8. Proof of Theorem 6

We denote
$$A = v_3^2 \cdot \ldots \cdot v_k^2, \qquad B = v_3^2 + \ldots + v_k^2,$$
$$D = (2/\pi)^{(k-2)/2}, \qquad \delta = t^{-1/(k-2)} \qquad \text{for} \qquad t \geq 1.$$
Since $w_k(t)$ is even, it is sufficient to consider only $t \geq 1$.
The lower bound for $w_k(t)$ follows from the inequalities
$$\begin{aligned}
\mathbf{E}e^{itf(Z)} &= \int_0^\infty \ldots \int_0^\infty (1 + At^2)^{-1/2} D\, e^{-B/2}\, dv_3 \ldots dv_k \\
&\geq \int_\delta^1 \ldots \int_\delta^1 (2At^2)^{-1/2} D e^{-B/2}\, dv_3 \ldots dv_k \\
&\geq 2^{-1/2} D e^{-(k-2)/2} t^{-1} \left(\int_\delta^1 \frac{dv}{v} \right)^{k-2} = l_k t^{-1} \ln^{(k-2)} t.
\end{aligned}$$
We prove now the upper bound for $w_k(t)$ using mathematical induction in k separetaly for odd and even values of k.
It follows from (25) that $w_2(t) \leq t^{-1}$ for all $t > 0$.
We have also for $t \geq 1$
$$\begin{aligned}
w_3(t) &= (2\pi)^{-1/2} e^{1/4} t^{-1} K_0\big((4t^2)^{-1}\big) \\
&\leq 1.267 (2\pi)^{-1/2} e^{1/4} t^{-1} \ln(8t^2) < 2.6\, t^{-1} \ln(2+t).
\end{aligned}$$
Assume now that for some $k \geq 2$ the inequality
$$w_k(t) \leq L_k t^{-1} \ln^{k-2}(2+t),$$

holds for $t \geq 1$. Then

$$w_{k+2}(t) = \frac{2}{\pi} \int_0^\infty w_k(tv) K_0(v)\, dv$$
$$= \left(\int_0^{1/t} + \int_{1/t}^1 + \int_1^\infty \right) \ldots = I_1 + I_2 + I_3.$$

From (30) we get

$$I_1 \leq 1.614\, t^{-1} \ln(2+t).$$

Furthermore, by induction hypothesis and (29) we obtain

$$I_3 \leq \frac{2}{\pi}\, w_k(t) \int_1^\infty K_0(v)\, dv \leq 0.269\, L_k t^{-1} \ln^{k-2}(2+t).$$

We have for I_2

$$I_2 \leq \frac{2}{\pi}\, K_0(1/t) \int_{1/t}^1 w_k(tv)\, dv$$
$$\leq \frac{2}{\pi}\, K_0(1/t) L_k t^{-1} \int_{1/t}^1 v^{-1} \ln^{k-2}(2+tv)\, dv$$
$$< \frac{2}{\pi}\, K_0(1/t) L_k t^{-1} \ln^{k-2}(2+t) \ln t\,.$$

Since for $t \geq 1$ we have

$$2/\pi K_0(1/t) \leq 3.534/\pi \ln(2t) < 1.075 \ln(2+t),$$

we get

$$I_2 < 1.076 L_k t^{-1} \ln^k(2+t).$$

Thus, for $t \geq 1$

$$w_{k+2}(t) < 2.959 L_k t^{-1} \ln^k(2+t) < L_{k+2} t^{-1} \ln^k(2+t).$$

9. Proof of Theorem 7

Since m_2 is odd, we can write (see Bagirov (1991), Corollary 3)

$$Z_2^{m_2} \stackrel{d}{=} Z_2 S_1,$$

where the random variables on the right-hand side are independent and $S_1 \geq 0$. We have

$$Z_2^{m_2} Z_3^{m_3} \cdots Z_k^{m_k} \stackrel{d}{=} Z_2 S, \tag{42}$$

where

$$S \stackrel{d}{=} S_1 |Z_3|^{m_3} \cdots |Z_k|^{m_k}$$

and
$$Z_1^{m_1} Z_2^{m_2} \cdots Z_k^{m_k} \stackrel{d}{=} |Z_1|^m Z_2 S.$$
Combining the relations
$$S^{-1/m} |Z_2|^{-1/m} \stackrel{d}{=} |Z_2|^{-m_2/m} \cdots |Z_k|^{-m_k/m},$$
(42) and Lemma 8 we finish the proof of Theorem 7.

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