χ^*-extremal graphs and the lexicographic product

Guogang Gao*   Xuding Zhu†

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Abstract

The star-chromatic number and the fractional-chromatic number are two generalizations of the ordinary chromatic number of a graph. We say a graph G is χ^*-*extremal* if its star-chromatic number is equal to its fractional-chromatic number. We prove that χ^*-extremal graphs G have the following nice property: For an arbitrary graph H the star-chromatic number $\chi^*(G[H])$ of the lexicographic product $G[H]$ is equal to the product of $\chi^*(G)$ and $\chi(H)$. Then we show that several classes of circulant graphs are χ^*-extremal. Thus for these circulant graphs G and arbitrary graphs H, if $\chi^*(G)$ and $\chi(H)$ are known then we can easily determine the star-chromatic number (hence the ordinary chromatic number) of the lexicographic product $G[H]$.

*Département d'IRO, Université de Montréal, C.P. 6128, Succursale A, Montréal, Québec H3C 3J7.

†Sonderforschungsbereich 343, Diskrete Strukturen in der Mathematik, Universität Bielefeld, 33501, Bielefeld, Germany.

1 Introduction

Let k and d be positive integers such that $k \geq 2d$. A (k,d)-coloring of a graph $G = (V, E)$ is a mapping $c : V \mapsto Z_k = \{0, 1, \ldots, k-1\}$ such that, for each edge $uv \in E$, $|c(u) - c(v)|_k \geq d$, where $|x|_k = \min\{|x|, k - |x|\}$. This generalizes the usual notion of a k-coloring: an ordinary k-coloring of G is just a $(k,1)$-coloring of G. Thus, the chromatic number $\chi(G)$ of G is the smallest k for which there is a $(k,1)$-coloring. The *star-chromatic number* of G, $\chi^*(G)$, is defined as

$$\chi^*(G) = \inf\{k/d : G \text{ has a } (k,d)\text{-coloring}\}.$$

The star-chromatic number of a graph was first introduced by Vince [10]. His approach uses some methods and results from continuous mathematics. A purely combinatorial treatment of the star-chromatic number of a graph was supplied by Bondy and Hell [1], and an alternate definition of the star-chromatic number can be found in [12]. It was proved in [1, 10], among other things, that

$$\chi(G) - 1 < \chi^*(G) \leq \chi(G), \quad i.e., \quad \chi(G) = \lceil \chi^*(G) \rceil.$$

Thus the chromatic number of a graph is determined by its star-chromatic number. On the other hand, two graphs with the same chromatic number could have different star-chromatic numbers. In this sense, the star-chromatic number of a graph captures its structure more precisely than the ordinary chromatic number.

Another generalization of the ordinary chromatic number is the fractional-chromatic number of a graph.

Definition 1 A mapping c from the collection $\mathcal{S}$ of independent sets of a graph G to the interval $[0,1]$ is a fractional-coloring if for every vertex x of G we have $\sum_{S \in \mathcal{S} \text{ s.t. } x \in S} c(S) = 1$. The value of a fractional-coloring c is $\sum_{S \in \mathcal{S}} c(S)$. The *fractional-chromatic number* $\chi_f(G)$ of G is the infimum of the values of fractional-colorings of G.

Obviously the condition $\sum_{S\in\mathcal{S} \text{ s.t. } x\in S} c(S) = 1$ can be replaced by the condition that $\sum_{S\in\mathcal{S} \text{ s.t. } x\in S} c(S) \geq 1$. If in the definition the mapping c is from $\mathcal{S}$ to the set $\{0,1\}$ instead of to the interval $[0,1]$, then c becomes an ordinary coloring of G.

The class of graphs G for which $\chi^*(G)$ equals $\chi_f(G)$ turns out to have some nice properties. We prove that if G is a graph with $\chi^*(G) = \chi_f(G)$, then for an arbitrary graph H we have $\chi^*(G[H]) = \chi^*(G)\chi(H)$, where $G[H]$ is the lexicographic product of G and H, (cf. definition in Section 2). Since $\chi(G[H]) = \lceil \chi^*(G[H]) \rceil$, we have $\chi(G[H]) = \lceil \chi^*(G)\chi(H) \rceil$ if G is χ^*-extremal. Then we show that $\chi^*(G) = \chi_f(G)$ for many circulant graphs G. Thus for these circulant graphs G and arbitrary graphs H, if $\chi^*(G)$ and $\chi(H)$ are known then we can easily determine the star-chromatic number $\chi^*(G[H])$, as well as the chromatic number $\chi(G[H])$, of the lexicographic product $G[H]$. This generalizes a result of Stahl [9] concerning the chromatic number of the lexicographic product of graphs.

2 Lexicographic product of graphs

It is clear from the definitions that $\chi_f(G) \leq \chi^*(G) \leq \chi(G)$ for any graph G. There exist graphs G such that $\chi_f(G) = \chi^*(G) = \chi(G)$. There also exist graphs G such that $\chi_f(G) < \chi^*(G) < \chi(G)$, (cf. [1, 3, 10, 12]). A. Vince first posed the problem [10]: For which graphs G, we have $\chi^*(G) = \chi(G)$? Some sufficient conditions were presented in [2, 12] for a graph G to have the property that $\chi(G) = \chi^*(G)$. However it seems a very difficult problem to characterize all such graphs.

In this paper we are interested in graphs G for which $\chi^*(G) = \chi_f(G)$. We call such graphs χ^*-*extremal*, i.e., a graph G is χ^*-extremal if $\chi^*(G) = \chi_f(G)$.[1] We prove in this section that χ^*-extremal graphs G have a very nice property, namely, if G is χ^*-extremal and H is an arbitrary graph, then the star-chromatic number $\chi^*(G[H])$ of the lexicographic product $G[H]$ is equal to the product of $\chi^*(G)$ and $\chi(H)$.

Given graphs G and H, the *lexicographic product* $G[H]$ has vertex set

[1]The notion of χ^*-extremality of a graph is suggested by Gert Sabidussi.

$\{(g,h) : g \in V(G), h \in V(H)\}$ and two vertices $(g,h),(g',h')$ are adjacent if and only if either gg' is an edge of G or $g = g'$ and hh' is an edge of H. (The lexicographic product is also known as the wreath product [4, 12].)

Theorem 1 *For any graphs G and H, $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$.*

Proof. Suppose that $\chi_f(G) = r$ and $\chi_f(H) = s$. Let c be a fractional-coloring for G of value r and c' be a fractional-coloring for H of value s. Define an assignment c'' for the independent sets of $G[H]$ as follows: For an independent set S of $G[H]$ of the form $S = \{(g,h) : g \in X, h \in Y$ where X and Y are independent sets of G and H respectively$\}$ we let $c''(S) = c(X) \cdot c'(Y)$, and for any other independent set S of $G[H]$ we let $c''(S) = 0$. It is easy to verify that c'' is a fractional-coloring of $G[H]$ of value rs. Therefore $\chi_f(G[H]) \leq \chi_f(G)\chi_f(H)$.

On the other hand if c'' is a fractional-coloring of $G[H]$, we shall prove that the value of c'' is at least rs. Let g be an arbitrary vertex of G. For an independent set S of $G[H]$, we let $S_g = \{h \in H : (g,h) \in S\}$. Then S_g is an independent set of H. For any independent set X of H, we define $c'(X) = \sum_{S_g = X} c''(S)$. Then for each vertex $h \in H$, we have $\sum_{h \in X} c'(X) = \sum_{(g,h) \in S} c''(S) \geq 1$. Thus c' is a fractional-coloring of H, and hence the value of c', which is equal to $\sum_{h \in H, (g,h) \in S} c''(S)$, must be at least $\chi_f(H) = s$. For an independent set S of $G[H]$, we let $G(S) = \{g \in G : \exists h \in H, (g,h) \in S\}$. Obviously $G(S)$ is an independent set of G if S is an independent set of $G[H]$. For an independent set Y of G, we let $c(Y) = \sum_{G(S)=Y} c''(S) \cdot 1/s$. Then for any $g \in G$, $\sum_{g \in Y} c(Y) = \sum_{h \in H, (g,h) \in S} c''(S) \cdot 1/s \geq 1$. Therefore c is a fractional-coloring of G. Hence the value of c, which is equal to the product of $1/s$ and the value of c'', is at least $\chi_f(G) = r$. Therefore the value of c'' is at least rs. ∎

Theorem 2 *If G is a χ^*-extremal graph and H is an n-chromatic graph then*

$$\chi^*(G[H]) = n\chi^*(G).$$

Proof. It was proved in [12] that for any graph G and any n-chromatic graph H, we have

$$\chi^*(G[H]) = \chi^*(G[K_n]) \leq n\chi^*(G).$$

By Theorem 1, $\chi_f(G[K_n]) = \chi_f(G)\chi_f(K_n) = n\chi_f(G)$. Since $\chi_f(G[K_n]) \leq \chi^*(G[K_n])$, we have

$$n\chi_f(G) \leq \chi^*(G[H]) \leq n\chi^*(G).$$

Theorem 2 follows from the assumption that $\chi_f(G) = \chi^*(G)$. ∎

Since $\chi(G[H]) = \lceil \chi^*(G[H]) \rceil$, we have an immediate consequence.

Corollary 1 *If G is a χ^*-extremal graph and H is an n-chromatic graph then*

$$\chi(G[H]) = \lceil n\chi^*(G) \rceil.$$

∎

The following result of Stahl, whose proof in [9] is complex, is now an easy consequence of Corollary 1:

Corollary 2 *If H is an n-chromatic graph then*

$$\chi(C_{2k+1}[H]) = 2n + \lceil n/k \rceil.$$

Proof. It is easy to verify that $\chi^*(C_{2k+1}) = \chi_f(C_{2k+1}) = 2 + 1/k$. Therefore $\chi(C_{2k+1}[H]) = 2n + \lceil n/k \rceil$ by Corollary 1. ∎

It is well known, [7, 9], that for any graph G, there is an integer n such that $\chi(G[K_n]) = n \cdot \chi_f(G)$. Thus if G is not χ^*-extremal, then we have $\chi^*(G[K_n]) \leq \chi(G[K_n]) = n \cdot \chi_f(G) < n \cdot \chi^*(G)$. Combining this observation with Theorem 2, we obtain:

Theorem 3 *A graph G is χ^*-extremal if and only if $\chi^*(G[H]) = \chi^*(G)\chi(H)$ for all graphs H.*

Note that the above theorem can be restated that a graph G is χ^*-extremal if and only if $\chi^*(G[K_n]) = n\chi^*(G)$ for all $n \geq 2$.

3 χ^*-extremal circulant graphs

For general graphs G, both $\chi_f(G)$ and $\chi^*(G)$ are difficult to calculate, and it seems also a very difficult problem to determine whether or not G is χ^*-extremal. In this section we concentrate on circulant graphs. We prove that some classes of circulant graphs are χ^*-extremal, and also construct some circulant graphs which are not χ^*-extremal.

Let p be a positive integer and let N be a subset of $\{1, 2, \ldots, p-1\}$ such that $i \in N$ implies $p-i \in N$. For brevity, we write $-i$ for $p-i$. The circulant graph $G(p, N)$ has vertices $0, 1, \ldots, p-1$ and $i \sim j$ if and only if $i - j \in N$, where subtraction is carried out modulo p.

Let $\alpha(G)$ be the *independence number* of the graph G, i.e., $\alpha(G)$ is the size of a maximum independent set of G. It is known, and easy to prove, that if a graph G is vertex transitive, then $\chi_f(G) = |V(G)|/\alpha(G)$. Therefore for any circulant graph $G = (p, N)$, we have $\chi_f(G) = p/\alpha(G)$. Thus to prove $\chi^*(G) = \chi_f(G)$ for a circulant graph $G = (p, N)$, it is sufficient to prove that $\chi^*(G) = p/\alpha(G)$, or equivalently $\chi^*(G) \leq p/\alpha(G)$, as we know that $\chi^*(G) \geq \chi_f(G)$.

Given a circulant graph $G = G(p, N)$ and an integer t, we let

$$\lambda_t(G) = \min\{|ti|_p : i \in N\}$$

and let

$$\lambda(G) = \max\{\lambda_t(G) : t = 1, 2, \ldots\},$$

where the multiplication ti is carried out modulo p. (Recall that $|x|_p = \min\{|x|, p - |x|\}$.)

Theorem 4 *Suppose G is a circulant graph. If $\lambda(G) \geq \alpha(G)$ then $\lambda(G) = \alpha(G)$ and G is χ^*-extremal.*

Proof. For any integer t, the mapping Δ on $\{0, 1, \ldots, p-1\}$ defined as $\Delta(i) = ti \pmod p$ is a $(p, \lambda_t(G))$-coloring of G. Therefore $\chi^*(G) \leq p/\lambda_t(G)$ for any integer t, and hence $\chi^*(G) \leq p/\lambda(G)$. Since $\chi_f(H) = |V(H)|/\alpha(H) \leq \chi^*(H)$ for any circulant graph H, thus if $\lambda(G) \geq \alpha(G)$ then we must have $\lambda(G) = \alpha(G)$ and $\chi^*(G) = \chi_f(G)$. ∎

Theorem 5 *If $N = \{\pm1, \pm2, \ldots, \pm(k-1)\}$ and $G = G(p, N)$, then G is χ^*-extremal.*

Proof. Let $t = \lfloor p/k \rfloor$. Then $\lambda_t(G) = t$. Hence $\lambda(G) \geq t$. Now we show that $\alpha(G) \leq t$. Let S be a subset of $\{0, 1, \ldots, p-1\}$ of size $t+1$. Then (by the pigeonhole principle) there exist $i, j \in S$ such that either $|i-j| < k$ or $p - |i-j| < k$. This implies $i - j \in N$ and hence S is not independent. Thus $\lambda(G) \geq t \geq \alpha(G)$. By Theorem 4, we have $\chi^*(G) = \chi_f(G)$. ∎

Theorem 6 *If $|N| \leq 3$ and $G = G(p, N)$, then G is χ^*-extremal.*

Proof. If $|N| < 3$, then G is either a matching or a disjoint union of cycles. It is obvious that $\chi^*(G) = \chi_f(G)$ in these cases. Suppose $|N| = 3$. Then p must be even and $N = \{i, p-i, p/2\}$ for some $i < p/2$. Let $d = \gcd(i, p)$. Now we consider two cases.

Case 1. p/d is even. It is easy to see that in this case G is the disjoint union of d copies of the circulant graph $H = G(p/d, \{1, p/d - 1, p/2d\})$. Therefore we only need to show that $\chi^*(H) = \chi_f(H)$. If $p/2d$ is odd, then H is a bipartite graph, and therefore $\chi^*(H) = \chi_f(H) = 2$. Suppose $p/2d = 2n$ is even. Then $\lambda(H) \geq \lambda_{2n-1}(H) = 2n-1$. We shall show that $\alpha(H) \leq 2n-1$. Otherwise suppose S is an independent set of H with $|S| = 2n$. Without loss of generality, we assume that $0 \in S$. Then $2n \notin S$ and therefore either $|S \cap \{0, 1, 2, \ldots, 2n-1, 2n\}| \geq n+1$ or $|S \cap \{0, 4n-1, 4n-2, \ldots, 2n+1, 2n\}| \geq n+1$. But this is impossible because each of the two sets $\{0, 1, \ldots, 2n\}$ and $\{0, 4n-1, \ldots, 2n\}$ induces an odd cycle of size $2n+1$. Thus we have proved that $\alpha(H) \leq 2n-1 \leq \lambda(H)$ and this implies that $\chi^*(H) = \chi_f(H)$ by Theorem 4.

Case 2. $p/d = 2m+1$ is odd. Obviously G contains an odd cycle of length $2m+1$. Therefore by Theorem 1 of [5], $\alpha(G) \leq \frac{m}{2m+1}p = md$. Let $t = m$ or $m+1$ whichever is odd, and let k be an integer such that $ki = d \pmod{p}$. An easy calculation shows that $\lambda_{tk}(G) = md$. Therefore $\lambda(G) \geq \lambda_{tk}(G) \geq \alpha(G)$. Thus $\chi^*(G) = \chi_f(G)$ by Theorem 4.

In both cases, G is χ^*-extremal. Hence the theorem is proved. ∎

Theorem 6 asserts that all circulant graphs of vertex degree at most three are χ^*-extremal. For circulant graphs of vertex degree four, we only obtain some partial results.

Theorem 7 *Suppose $G = G(p, N)$ is a circulant graph and $|N| = 4$.*

(1): *If p is odd, $N = \{\pm 1, \pm\lfloor p/2 \rfloor\}$, then G is χ^*-extremal.*

(2): *If $N = \{\pm 1, \pm k\}$, k is odd and $p > (k(k-3)+2)r/2$, where $r = p \pmod{k}$, then G is χ^*-extremal.*

(3): *If $N = \{\pm 1, \pm k\}$, k is even and $p > k(k-1)$, then G is χ^*-extremal.*

Proof. (1): Since G contains a triangle, we have $\alpha(G) \le p/3$ by Theorem 1 of [5]. Thus $\alpha(G) \le \lfloor p/3 \rfloor$. Let $t = \lfloor p/3 \rfloor$. It is easy to see that $\lambda_t(G) = t$. Therefore $\lambda(G) \ge \lambda_t(G) = t \ge \alpha(G)$. Hence by Theorem 4, $\chi^*(G) = \chi_f(G)$.

(2): If p is even G is bipartite since G has a 2-coloring. Clearly G is χ^*-extremal in this case. We now assume that p is odd. Then G contains an odd cycle C_t, where $t = \lfloor p/k \rfloor + r$. It follows that $\alpha(G) \le \frac{(t-1)p}{2t}$ by Theorem 1 of [5]. An easy (though tedious) computation shows that $\lfloor \frac{(t-1)p}{2t} \rfloor = \frac{p-k}{2}$. So $\alpha(G) \le \frac{p-k}{2}$. It is routine to check that $\lambda_{(p+1)/2}(G) = (p-k)/2$. Therefore $\lambda(G) \ge \lambda_{(p+1)/2}(G) = (p-k)/2 \ge \alpha(G)$. Thus $\chi^*(G) = \chi_f(G)$ by Theorem 4.

(3): The graph G contains an odd cycle C_{k+1}. This implies that $\alpha(G) \le kp/(2k+2)$. Or simply $\alpha(G) \le \lfloor kp/(2k+2) \rfloor$. Let $t = \lfloor kp/(2k+2) \rfloor$. It can be proved that $\lambda_t(G) = t$. Therefore $\lambda(G) \ge \lambda_t(G) = t \ge \alpha(G)$. Thus $\chi^*(G) = \chi_f(G)$ by Theorem 4. ∎

Theorem 8 *Suppose that $N = \{\pm k, \pm(k+1), \ldots, \pm(2k-1)\}$ and $G = G(p, N)$. If $6k - 2 \le p \le 8k - 3$ then G is χ^*-extremal.*

Proof. This graph was studied by Sidorenko in [8], where it was served as a triangle-free regular graph whose independence number equals its degree, i.e., it was proved there that $\alpha(G) = 2k$. Now an easy calculation shows that $\lambda_2(G) = 2k$. Therefore $\lambda(G) \ge \alpha(G)$ and hence $\chi^*(G) = \chi_f(G)$ by Theorem 4. ∎

Theorem 9 *Suppose that* $k' = k + l \le p/2$, $N = \{\pm k, \pm(k+1), \ldots, \pm k'\}$ *and* $G = G(p, N)$. *If* $p - 2k' < \min\{k, l\}$ *then* G *is* χ^**-extremal.*

Proof. We first show that $\alpha(G) \le k$. Otherwise suppose $\alpha(G) \ge k+1$. Consider the vertices of G are placed in a circle in the order $0, 1, \cdots, p-1$. For a subset S of G, we define a *segment* of S to be a maximal subset of S of the form $\{i, i+1, \cdots, i+r\}$ (addition modulo p). We choose a maximum independent set S of G with the least number of segments. Since i is adjacent to $i+k$ and $|S| \ge k+1$, we see that S has at least two segments.

Consider one segment $I_1 = \{i, i+1, \cdots, i'\}$ of S, where $i' = i + r$. Let M be the set of vertices of G which are not in I_1 and also not adjacent to any vertex of I_1. Obviously $S \setminus I_1 \subset M$. It is easy to see that M consists of three segments,

$$\begin{aligned} X &= \{i'+1, i'+2, \cdots, i'+k-1\}, \\ Y &= \{i'+k'+1, i'+k'+2, \cdots, i+p-k'-1)\}, \\ Z &= \{i-k+1, i-k+2, \cdots, i-1\} \end{aligned}$$

(addition are carried out modulo p).

An easy calculation shows that $|I_1| + |Y| \le k < |S|$. Therefore $S \cap X \ne \emptyset$ or $S \cap Z \ne \emptyset$. Without loss of generality we assume that $S \cap X \ne \emptyset$. Let $I_2 = \{j, j+1, \cdots, j+s\}$ be a segment of S contained in X such that all the vertices of G between I_1 and I_2 are not in S. We do the following operation to S: Add the set $I = \{i'+1, i'+2, \cdots, j-1\}$ of all the vertices of G which are between I_1 and I_2 into S, and delete all the vertices of S which is adjacent to some vertex of I. The resulting set S' is certainly an independent set of G. We now show that $|S'| > |S|$, which then contradicts our assumption that S is a maximum independent set.

If $j - i' \le l$, then any vertex x of G which is adjacent to some vertices of $i'+1, i'+2, \cdots, j-1$ is either adjacent to i' or adjacent to j. Thus x is not in S. Therefore no vertex is deleted from S and hence $|S'| > |S|$.

Suppose then that $j - i' > l$, and we consider two cases.

Case 1. $j - i' - l \le p - 2k'$.

The set of those vertices which are adjacent to some vertices of I is contained in the set $A = \{i'+k+1, i'+k+2, \cdots, j-1+p-k\}$. We shall show that $|A \cap S| < |I|$, therefore the number of vertices added to S is greater than the number of vertices deleted from S and hence $|S'| > |S|$.

The vertex i' is adjacent to vertices $i' + k, i' + k + 1, \cdots, i' + k'$ and vertices $i' - k', i' - k' + 1, \cdots, i' - k$. The vertex j is adjacent to vertices $j + k, j + k + 1, \cdots, j + k'$ and vertices $j - k', j - k' + 1, \cdots, j - k$. Since $p - 2k' < l$ and $j - i' > l$, we have $j + k' > i' + p - k'$. Thus the set of those elements of S which is adjacent to some vertices of I is contained in the union of the following two sets:

$$\{i' + k' + 1, i' + k' + 2, \cdots, j + k - 1\}$$

and

$$\{i' - k + 1, i' - k + 2, \cdots, j - k' - 1\}.$$

Each of the two sets has $j - i' - l - 1$ elements. Hence the total number of elements deleted from S is at most $2(j - i' - l - 1) \leq p - 2k' + j - i' - l - 2 < j - i' - 2$. Thus $|A \cap S| < |I|$, and hence $|S'| > |S|$.

Case 2. $j - i' - l > p - 2k'$.

A similar calculation shows that the set of those elements of S which is adjacent to some vertices of I is contained in the union of the following two sets: $\{i'+k'+1, i'+k'+2, \cdots, i'-k'-1\}$ and $\{j+k'+1, j+k'+2, \cdots, j-k'-1\}$. Each of the two sets has $p - 2k' - 1$ elements. Hence the total number of elements deleted from S is at most $2(p-2k'-1) < j-i'-2$, which is less than $|I|$, i.e. the number of elements added to S. Hence $|S'| > |S|$, contradicting that S is a maximum independent set.

Therefore $\alpha(G) \leq k$. Since $\lambda(G) \geq \lambda_1(G) = k$, we have $\chi^*(G) = \chi_f(G)$ by Theorem 4. ∎

Given integers p and k, the graph G_p^k, which plays a special role in the discussion of the star-chromatic number, is actually the circulant graph $G(p, N)$ with symbol set $N = \{k, k+1, \ldots, p-k\}$. As a corollary of the above theorem, (where $k + l = \lfloor p/2 \rfloor$), we have $\chi^*(G_p^k) = \chi_f(G_p^k) = p/k$.

Remark For all circulant graphs G, the parameter $\lambda(G)$ can be determined in polynomial time. For the classes of circulant graphs G discussed above, we have $\chi^*(G) = |V(G)|/\alpha(G) = |V(G)|/\lambda(G)$. Therefore the independence number, the star-chromatic number and hence the chromatic number of these graphs can be determined in polynomial time. For general circulant graphs, the complexity of determining these parameters remains an open problem.

So far we have seen many χ^*-extremal circulant graphs. However there are also many circulant graphs which are not χ^*-extremal. Take any circulant graph H with $\chi_f(H) \neq \chi(H)$. (Such circulant graphs are abundant. For example, the odd cycles and most of the other χ^*-extremal circulant graphs discussed in this paper satisfy this inequality.) Then it is well known, and also easy to verify, that the complement $\overline{H}$ of H is a circulant graph, and for any integer $k \geq 2$, the disjoint union $\overline{G}$ of k copies of $\overline{H}$ is also a circulant graph. Now we show that the circulant graph G, which is the complement of $\overline{G}$, is not χ^*-extremal. In fact $G = K_k[H]$. Thus by Theorem 1, $\chi_f(G) = k \cdot \chi_f(H)$. However by Theorem 6 of [12], $\chi^*(G) = k \cdot \chi(H)$. Thus $\chi^*(G) > \chi_f(G)$ and G is not χ^*-extremal. If we let $H = C_5$ and $k = 2$ in the above discussion, we obtain the smallest such circulant graph. It is in fact the circulant graph $G = G(10, N)$ with $N = \{1, 3, 4, 5, 6, 7, 9\}$. This graph has star-chromatic number 6 and fractional-chromatic number 5.

Note that each of the non-χ^*-extremal circulant graphs as constructed above is of compound order. A non-χ^*-extremal circulant graph of prime order is found by a computer search. Let $G = G(13, N)$ with $N = \{1, 3, 4, 9, 10, 12\}$. Then $\chi_f(G) = 13/3$ and $\chi^*(G) > 13/3$.

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