THE HALL AND COMPOSITION ALGEBRAS OF CYCLIC QUIVER

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1. Introduction

Let Q_n be the quiver of type $\widetilde{A}_{n-1}$ of cyclic orientation with n vertices

$$\begin{array}{ccccccc}
 & & 1 & & 2 & & \\
 & & \circ & \rightarrow & \circ & & \\
 & \nearrow & & & & \searrow & \\
n\ \circ & & & & & & \circ \\
 & \nwarrow & & & & \swarrow & \\
 & & \circ & \cdots & \circ & &
\end{array}$$

conventionally,indices concerning with the vertices of Q_n are taken from $\mathbf{Z}/n\mathbf{Z}$, the residue calsses of integers modulo n.

Let k be a field and $\mathcal{T}$ be the category of locally nilpotent finite-dimensional representations of Q_n. As is remarked in [12], $\mathcal{T}$ may be considered as the category of finite length modules over some ring, and we will call the objects in $\mathcal{T}$ modules. Let S_i be the simple module corresponding to the vertex i of Q_n . Let M be any module and $\mathbf{r}M$ be the radical of M, let $\mathbf{r}^{t+1}M = \mathbf{r}(\mathbf{r}^t M)$, if M is indecomposable with top S_i, then it has an unique filtration of submodules

$$M \supset \mathbf{r}M \supset \cdots \supset \mathbf{r}^t M \supset \mathbf{r}^{t+1} M \cdots \supset \mathbf{r}^{j-1} M \supset \mathbf{r}^j M = 0$$

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such that $\mathbf{r}^t M/\mathbf{r}^{t+1}M \cong S_{i+t}$ for $t = 0, 1, \cdots, m-1$ and M is unique up to isomorphism with this property since any indecomposable module in $\mathcal{T}$ is serial. Let $M_{i,j}$ be the unique indecomposable R module with top S_i and composition length j, and $\{[M_{i,j}] | 1 \le i \le n, 1 \le j \le \infty\}$ is a complete set of the isomorphism classes of indecomposable R modules. By the Krull-Schmidt theorem,each M in $\mathcal{T}$ can be written uniquely as a direct sum of indecomposable representations

$$M \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} a_{i,j} M_{i,j}$$

where $a_{i,j}$ is a nonnegative integer and $a_{i,j}M_{i,j}$ denotes a direct sum of $a_{i,j}$ copies of $M_{i,j}$,and $a_{i,j}$ is unique determined by M. Let $\mathbf{Z}^+$ be the set of nonnegative integers,and let $((\mathbf{Z})^+)^{n\infty} = \{\mathbf{a} = (a_{i,j}) | 1 \le i \le n, 1 \le j \le \infty, a_{i,j} \in \mathbf{Z}^+, a_{i,j} \ne 0$ for all but finite many $j\}$. We have a bijection from the isomorphism classes of modules in $\mathcal{T}$ onto $((\mathbf{Z})^+)^{n\infty}$,

$$M = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} a_{i,j} M_{i,j} \longmapsto \mathbf{a}(M) = (a_{i,j})$$

with the inverse

$$\mathbf{a} = (a_{i,j}) \longmapsto M(\mathbf{a}) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} a_{i,j} M_{i,j}.$$

The following order is defined in [1]. Assume that $\mathbf{a} = (a_{i,j}), \mathbf{b} = (b_{i,j}) \in ((\mathbf{Z})^+)^{nm}$.

Define $\mathbf{a} \prec_2 \mathbf{b}$ if $\sum_{i=1}^{n}\sum_{j=1}^{m} j a_{i,j} < \sum_{i=1}^{n}\sum_{j=1}^{m} j b_{i,j}$, or $\sum_{i=1}^{n}\sum_{j=1}^{m} j a_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{m} j b_{i,j}$ and there is a pair (s,t) such that $a_{i,j} = b_{i,j}$ for all $j > t$ or $j = t$ and $i < s$ and $a_{s,t} > b_{s,t}$.

Since there are only finite many $\mathbf{a} = (a_{i,j}) \in ((\mathbf{Z})^+)^{n\infty}$ such that $\sum_{i=1}^{n}\sum_{j=1}^{m} j a_{i,j} = d$ for a given nonnegative integer d, $\prec_2$ is a total order of the semi-additive group $((\mathbf{Z})^+)^{nm}$ with the smallest element $(0, 0, \cdots, 0)$. And $((\mathbf{Z})^+)^{n\infty}$ is totally ordered semi-additive group with respect to these orders.

In the rest of this paper, we assume that k is a field of q elements.

Hall algebra is introduced by C M. Ringel. Let $F^L_{M,N}$ be the number of submodules L' of a finite module L in $\mathcal{T}$ with the property that $L' \cong N$ and $L/L' \cong M$. The Hall algebra $\mathcal{H}(\mathcal{T})$ is a free abelian group with a basis $\{u_{[M]}\}_{[M]}$ indexed by the isomorphism classes of finite moduless M in $\mathcal{T}$ with the multiplication

$$u_{[M]}u_{[N]} = \sum_L F^L_{M,N} u_{[L]}.$$

$\mathcal{H}(\mathcal{T})$ is an associative ring with the unit element,and if $F^L_{M,N} \neq 0$, then $\mathbf{dim} L = \mathbf{dim} M + \mathbf{dim} N$. The subalgebras $\mathcal{C}(\mathcal{T})$ and $\mathcal{L}(\mathcal{T})$ of $\mathcal{H}(\mathcal{T})$ generated by $\{u_{[S]} | S \ simple\}$ and $\{u_{[M]} | M \ semisimple\}$,respectively, are called the composition and respectively the Loewy algebra of $\mathcal{T}$. Denoted by $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ and $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ the rational extension of $\mathcal{H}(\mathcal{T})$ and $\mathcal{C}(\mathcal{T})$, respectively, that is $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) = \mathcal{H}(\mathcal{T}) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathcal{C}_{\mathbf{Q}}(\mathcal{T}) = \mathcal{C}(\mathcal{T}) \otimes_{\mathbf{Z}} \mathbf{Q}$.

The aim of this paper is to prove the following Theorem.

Main Theorem *If the rank of $\mathcal{T}$ is large or equal to 3, then* $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) = \mathcal{C}_{\mathbf{Q}}(\mathcal{T})[X]$.

In section 2, we prove that the basic element of the form $u_{[M_{i,ln}]}$ are contained in $< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), u_{[M_{i,n}]} >$, by give the explicit expression of them. Using the natural grading on $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ induced by the dimensional vector of the modules in $\mathcal{T}$, we proved that $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is generated by any element of grade $(1, 1, \cdots, 1)$ in $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) \backslash \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ in section 3. We find a centre element c of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ and proves that it is not in $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$, by using results in [11], thus we get that $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) = \mathcal{C}_{\mathbf{Q}}(\mathcal{T})[c]$ in section 4. In section 5, we prove that c is an indeterminant by comparing the Gelfand-Krikilov dimension of factor algebras of $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ and $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ with finite Gelfand-Kirilov dimension. Some tedious calculation for the proof of the results in section are left as an appendix.

C. M. Ringel also introduces the twisted Hall and composition algebras and proves that some twisted form $\mathcal{C}_*(\mathcal{T})$ of $\mathcal{C}(\mathcal{T})$ is identified with U^+, the +-part of the quamtum group $U_q = U_q(\tilde{A}_{n-1})$ of an affine Lie algebra of class $\widetilde{A}_{n-1}$([10], [11]). Our result remains true in the twisted cases, this gives another possiblity to study the quantum group $U_q^+(\widetilde{A}_{n-1})$ using Hall algebras.

2. The elements $u_{[M_{i,ln}]}$

Assume that $l \geq 2$ in this section, we prove that the element $u_{[M_{i,ln}]}$ of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$, lies in the subalgebra of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ generated by $\{u_{[S_1]}, \cdots, u_{[S_n]}, u_{[M_{i,n}]}\}$. Let $\tau(M)$ be the number of the indecomposable summands of a module M.

Lemma 2.1. Let $\mathbf{a} = (a_{s,t}), \mathbf{b} = (b_{s,t})$ with $a_{s,t} = 0 = b_{s,t}$ for all $s \not\equiv i \mod (n)$ or $t \not\equiv 0 \mod (n)$. Assume that there is a short exact sequence

$$0 \longrightarrow M(\mathbf{a}) \longrightarrow M(\mathbf{c}) \longrightarrow M(\mathbf{b}) \longrightarrow 0$$

and $\mathbf{c} = (c_{s,t})$, then $c_{s,t} = 0$ for all $s \not\equiv i \mod (n)$ or $t \not\equiv 0 \mod (n)$.

Proof. Note that *top* $M(\mathbf{a}) \cong \tau(M(\mathbf{a}))S_i$ and $\operatorname{soc} M(\mathbf{a}) \cong \tau(M(\mathbf{a}))S_{i-1}$, *top* $N(\mathbf{a}) \cong \tau(N(\mathbf{a}))S_i$ and $\operatorname{soc} N(\mathbf{a}) \cong \tau(N(\mathbf{a}))S_{i-1}$. Apply $\operatorname{Hom}(\ , S_t)$ and $\operatorname{Hom}(S_t,\ )$, respectively, we have the exact sequences

$$0 \longrightarrow \operatorname{Hom}(M(\mathbf{b}), S_t) \longrightarrow \operatorname{Hom}(M(\mathbf{c}), S_t) \longrightarrow \operatorname{Hom}(M(\mathbf{a}), S_t) \longrightarrow$$

and

$$0 \longrightarrow \operatorname{Hom}(S_t, M(\mathbf{a})) \longrightarrow \operatorname{Hom}(S_t, M(\mathbf{c})) \longrightarrow \operatorname{Hom}(S_t, M(\mathbf{b})) \longrightarrow$$

so we get that $\operatorname{Hom}(M(\mathbf{c}), S_t) \neq 0$ if and only if $t = i$ and $\operatorname{Hom}(S_t, M(\mathbf{c}), S_t) \neq 0$ if and only if $t = i - 1$. This shows that *top* $M(\mathbf{c})$ is a direct sum of copies of S_i and $\operatorname{soc} M(\mathbf{c})$ is a direct sum of copies of S_{i-1} and hence $c_{s,t} = 0$ for all $s \not\equiv i \mod (n)$ or $t \not\equiv 0 \mod (n)$. $\square$

Lemma 2.2. Let

$$0 \longrightarrow M_{i,ln} \oplus mM_{i,n} \longrightarrow M \xrightarrow{g} M_{i,n} \longrightarrow 0$$

be a short exact sequence, then $M \cong M_{i,(l+1)n} \oplus mM_{i,n}$ or $M \cong M_{i,ln} \oplus (m+1)M_{i,n}$ or $M \cong M_{i,ln} \oplus (m-1)M_{i,n} \oplus M_{i,2n}$.

Proof. By the last lemma $M \cong \bigoplus a_t M_{i,tn}$. Let $g = (g_{t,v})$ with $g_{t,v}$ a map from the vth copy of $M_{i,tn}$ to $M_{i,n}$, then $g_{t,v}$ is either epimorphism or zero since $M_{i,n}$ has dimensional vector $(1, 1, \cdots, 1)$ and all $M_{i,tn}$ have the same top S_i. Let t_0 be the minimal among t's with the property that $g_{t,v} \neq 0$ for some v and assume that $g_{t_0,v_0} \neq 0$. Then if $g_{t,v} \neq 0$, we see easily that $M_{i,t_0 n}$ is a factor module of $M_{i,tn}$ and $g_{t,v}$ factors through g_{t_0,v_0}, with $g_{t,v} = g_{t_0,v_0} h_{t,v}$ for some epmorphism $h_{t,v} : M_{i,tn}^{(v)} \longrightarrow M_{i,t_0 n}^{(v_0)}$. Let $h_{t,v} = 0$ if $g_{t,v} = 0$ and let $h = (h_{t,v})$. Then $h : M \longrightarrow M_{i,t_0 n}$ is an splitting epmorphism since h_{t_0,v_0} is an isomorphism. Let h' be such that $h'h = id_{M_{i,t_0 n}}$, then we have that $M = \operatorname{Im} h' + \ker h$ and in this case $g(\ker h) = gh(\ker h) = 0$. This proves that $\ker g = \ker g|_{\operatorname{Im} h} \oplus \ker h \cong \mathbf{r}^n M_{i,t_0 n} \oplus \bigoplus (a_t - \delta_{t,t_0}) M_{i,tn} \cong M_{i,(t_0-1)n} \oplus \bigoplus (a_t - \delta_{t,t_0}) M_{i,tn} \cong M_{i,ln} \oplus m M_{i,n}$, where $\delta_{s,t}$ is the Kronecker δ, with $\delta_{s,t} = 1$ whenever $s = t$ and 0 otherwise. So we have that $t_0 = 1, 2$ or $l+1$ and $M \cong M_{i,ln} \oplus m M_{i,n}$, or $M_{i,ln} \oplus (m-1) M_{i,n} \oplus M_{i,2n}$ or $M_{i,(l+1)n} \oplus m M_{i,n}$.

This proves the lemma. $\square$

Lemma 2.3. Let

$$0 \longrightarrow M_{i,ln} \xrightarrow{f} M \longrightarrow m M_{i,n} \longrightarrow 0$$

be a short exact sequence, then $M \cong M_{i,ln} \oplus m M_{i,n}$ or $M \cong M_{i,(l+1)n} \oplus (m-1) M_{i,n}$.

Proof. Using induction on m. The case $m = 1$ follows from lemma 1.1 of [1]. Assume that $m \geq 2$ and assume the Lemma for $m - 1$. Consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_{i,ln} \arrow[r, "f"] \arrow[d] & M' \arrow[r] \arrow[d] & (m-1)M_{i,n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{i,ln} \arrow[r, "f"] & M \arrow[r] \arrow[d] & mM_{i,n} \arrow[r] \arrow[d] & 0. \\
 & 0 \arrow[r] & M_{i,n} \arrow[r, "\mathrm{id}"] \arrow[d] & M_{i,n} \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r, equal] & 0 &
\end{tikzcd}$$

By the inductive assumption, we have that $M' \cong M_{i,ln} \oplus (m-1)M_{i,n}$ or $M' \cong M_{i,(l+1)n} \oplus (m-2)M_{i,n}$. So we have the short exact sequence $0 \longrightarrow M_{i,ln} \oplus (m-1) \xrightarrow{g} M \longrightarrow M_{i,n} \longrightarrow 0$ or $0 \longrightarrow M_{i,(l+1)n} \oplus (m-2)M_{i,n} \xrightarrow{h} M \longrightarrow M_{i,n} \longrightarrow 0$.

By the last Lemma, we get that $M \cong M_{i,(l+1)n} \oplus (m-1)M_{i,n}$ or $M \cong M_{i,ln} \oplus mM_{i,n}$ or $M \cong M_{i,ln} \oplus (m-2)M_{i,n} \oplus M_{i,2n}$. The third case is impossible since $(M \cong M_{i,ln} \oplus (m-2)M_{i,n} \oplus M_{i,2n})/M_{i,ln}$ will always has a direct summand of length equal to two, which is a contradiction. The rest is obvious.

This proves the lemma. □

Lemma 2.4.

$$F^{M_{i,(l+1)n} \oplus (m-1)M_{i,n}}_{mM_{i.n}, M_{i,ln}} = 1$$

and

$$F^{M_{i,ln} \oplus mM_{i,n}}_{mM_{i.n}, M_{i,ln}} = q^m.$$

Proof. Similar to the calculations in appendix. □

Let

$$
\begin{aligned}
L_1 &= (l-2)M_{i,n} \oplus S_i \oplus S_{i-1} \oplus M_{i,n-1} \oplus M_{i+1,n-1} \\
L_2 &= (l-1)M_{i,n} \oplus S_{i-1} \oplus M_{i,n-1} \\
L_3 &= (l-1)M_{i,n} \oplus S_i \oplus M_{i+1,n-1} \\
N_1 &= (l-2)M_{i,n} \oplus M_{i,n-1} \oplus M_{i+1,n-1} \\
N_2 &= (l-2)M_{i,n} \oplus S_{i-1} \oplus M_{i,n-1} \oplus M_{i+1,n-1} \\
N_3 &= (l-2)M_{i,n} \oplus S_i \oplus M_{i,n-1} \oplus M_{i+1,n-1} \\
N_4 &= (l-2)M_{i,n} \oplus S_i \oplus M_{i,n-1} \\
N_5 &= S_{i-1} \oplus M_{i+1,n-1};
\end{aligned}
$$

and let

$$
\begin{aligned}
u_i &= u_{[S_i]} \\
u_{i,j} &= u_{[M_{i,j}]}, \\
u_{ss,t} &= u_{[t\bigoplus_{i=1}^n S_i]} \\
u_{t,i,j} &= u_{[tM_{i,j}]}, \\
v_{i,l,s} &= u_{[M_{i,(l-s)n} \oplus sM_{i,n}]} \\
x_{i,t} &= u_{[N_t]} \\
y_{i,t} &= u_{[L_t]}
\end{aligned}
$$

Proposition 2.5. If $F^L_{M,N} \neq 0$ for $(M,N) = (S_i, N_2)$ or, (N_3, S_{i-1}), or (N_4, N_5) or

$$F^L_{S_i,N_1,S_{i-1}} \neq 0.$$

Then L is isomorphism $lM_{i,n}$ or L_t for $t = 1,2,3$ and we have have the following tabulate for the coefficients of $u_{[L]}$.

	$u_{l,i,n}$	$y_{i,1}$	$y_{i,2}$	$y_{i,3}$
$u_i x_{i,1} u_{i-1}$	$(\frac{q^l-1}{q-1})(\frac{q^{l-1}-1}{q-1})$	$\frac{q^{l-1}-1}{q-1}q^{l-1}$	$\frac{q^{l-1}-1}{q-1}q^{l-1}$	$q^{2(l-1)}$
$x_{i,4}x_{i,5}$	$(\frac{q^l-1}{q-1})(\frac{q^{l-1}-1}{q-1})$	$\frac{q^{l-1}-1}{q-1}q^{l-1}$	$\frac{q^{l-1}-1}{q-1}q^{l-1}$	q^{2l-3}
$u_i x_{i,2}$	0	0	$\frac{q^{l-1}-1}{q-1}$	q^{l-1}
$x_{i,3}u_{i-1}$	0	$\frac{q^{l-1}-1}{q-1}$	0	q^{l-1}

Proof. Using the results in [1] and [2], we prove for the case N_4 and N_5, and the calculation for the case N_4 and N_5 will be given in the appendix. The rest follows similarly from the Lemma 1.2, 1.3 of [1] and Proposition 2.5 of [2]. Let

$$0 \longrightarrow N_5 \longrightarrow L \longrightarrow N_4 \longrightarrow 0$$

be a short exact sequence, consider the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M_{i+1,n-1} \arrow[r, "\mathrm{id}"] \arrow[d] & M_{i+1,n-1} \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & N_5 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & N_4 \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & S_{i-1} \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & N_4 \arrow[r] & 0 \\
 & 0 \arrow[r] & 0 & &
\end{tikzcd}$$

and we have that L' may be isomorphic to $N_4 \oplus S_{i-1}$ or $(l-1)M_{i,n} \oplus S_i$. Now using the pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_{i+1,n-1} \arrow[r] \arrow[d, "id"'] & L_2 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{i+1,n-1} \arrow[r] & L \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & S_1 \arrow[r, "\mathrm{id}"] \arrow[d] & S_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] & 0 &
\end{tikzcd}$$

with $L_1 \cong (l-2)M_{i,n} \oplus M_{i+1,n-1} \oplus S_{i-1}$, or $L_1 \cong (l-1)M_{i,n}$, so we have that $\mathrm{Ext}(L_1, M_{i+1,n-1}) = 0$ and thut $L_2 \cong L_1 \oplus M_{i+1,n-1}$. So it follows from Lemma 1.2 of [1] that $L \cong lM_{i,n}$ or L_t for $t = 1, 2, 3$. $\square$

Lemma 2.6. The matrix

$$\begin{pmatrix} (\frac{q^l-1}{q-1})(\frac{q^{l-1}-1}{q-1}) & \frac{q^{l-1}-1}{q-1}q^{l-1} & \frac{q^{l-1}-1}{q-1}q^{l-1} & q^{2(l-1)} \\ (\frac{q^l-1}{q-1})(\frac{q^{l-1}-1}{q-1}) & \frac{q^{l-1}-1}{q-1}q^{l-1} & \frac{q^{l-1}-1}{q-1}q^{l-1} & q^{2l-3} \\ 0 & 0 & \frac{q^{l-1}-1}{q-1} & q^{l-1} \\ 0 & \frac{q^{l-1}-1}{q-1} & 0 & q^{l-1} \end{pmatrix}$$

is not degenerate for $q \neq 0$ and q not a root of $(X^l-1)(X^{l-1})$.

Proof. This follows from the fact that the determinant of the matrix is

$$q^{2l-3}(q^l-1)(\frac{q^{l-1}-1}{q-1})^3.$$

$\square$

We have immediatelly

Proposition 2.7.

$$\frac{(q^l-1)(q^{l-1}-1)}{(q-1)^2}u_{i,l,n} = \frac{q-1}{2q-1}u_i x_{i,1} u_{i-1} + \frac{q}{2q-1}x_{i,4}x_{i,5} - q^{l-1}(u_i x_{i,2} + x_{i,3}u_{i-1}).$$

Lemma 2.8. $u_{[M_{i,ln}]}$ is in the subalgebra of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ generated by

$$u_i, u_{i-1}, x_{i,1} \cdots x_{i,5}, u_{i,n}, \cdots, u_{i,(l-1)n}, u_{2,i,n}, \cdots u_{l-1,i,n}$$

for $l \geq 3$

Proof. By lemma 2.3, we have that $u_{i,ln} = u_{i,n}u_{i,(l-1)n} - qv_{i,l,1}$, and $v_{i,l,s-1} = u_{s,i,n}u_{i,(l-s)n} - q^s v_{i,l,s}$ for $s = 2, \cdots, l-2$ and $v_{i,l,l-2} = u_{l-1,i,n} - \frac{q^l-1}{q-1}u_{l,i,n}$. This shows that $u_{i,ln} \in < u_{i,n}, \cdots, u_{i,(l-1)n}, u_{2,i,n}, \cdots u_{l-1,i,n}, u_{l,i,n} >$, but by the last Lemma, $u_{i,l,n} \in < u_i, u_{i-1}, x_{i,1}, \cdots, x_{i,5} >$, so we have that

$$u_{i,ln} \in < u_i, u_{i-1}, x_{i,1} \cdots x_{i,5}, u_{i,n}, \cdots, u_{i,(l-1)n}, u_{2,i,n}, \cdots u_{l-1,i,n} >$$

This completes the proof. $\square$

Remark 2.9. 1. From the proof of the lemma, we see that the element $u_{i,ln}$ can be express explicitly as

$$\begin{aligned}\tfrac{q^{l-1}-1}{q-1}u_{i,ln} &= \sum(-1)^{t-1}q^{\frac{t(t-1)}{2}}\tfrac{q^{l-1}-1}{q-1}u_{t,i,n}u_{i,tn}\\ &+(-1)^{l}q^{\frac{(t-1)(t-2)}{2}}(\tfrac{q-1}{2q-1}u_ix_{i,1}u_{i-1}\\ &+\tfrac{q}{2q-1}x_{i,4}x_{i,5}-q^{l-1}(u_ix_{i,2}+x_{i,3}u_{i-1}))\end{aligned}$$

So lemma 2.8 also valid when $q=0$.

2. The results are also valid in the twisted case, since in lemma 3.5 and 3.3 only a constant is different in our arguement and the twisted case in each row of the matrices.

3. Generating sets for $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$

Theorem 5.1 of [1] gives us two generating set for $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$, namely,

$$\{u_1,\cdots,u_n,u_{ss,1}\cdots,u_{ss,t},\cdots\}$$

and

$$\{u_{i,j}|1\leq i\leq n, 1\leq j\leq\infty\}.$$

Now we prove the following Theorem.

Theorem 3.1. $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is generated by $\{u_1,\cdots,u_n,u_{ss,1}\}$.

Proof. Let $<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$ be the subalgebra of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ generated by $\{u_1,\cdots,u_n,u_{ss,1}\}$. We prove that $u_{i,j}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$ for all i,j, by induction on j. Clearly, we have that $u_{i,j}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$ for any $j<n$, since $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is a graded ring with the grading $\mathcal{H}=\sum_{\mathbf{m}\in\mathbf{Z}^n}H_{\mathbf{m}}$ with $H_{\mathbf{m}}=\sum_{\mathbf{dim}M=\mathbf{m}}\mathbf{Q}u_M$ and we see that $u_{i,n}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$. Assume that $u_{i,t}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$ for all $t<j$. If $j\neq ln$ for any l, say $j=rn+d$ for some $1<d<n$, then similar to above, using the grading, we have that $u_{ss,h}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$ and thus $u_{i,j}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1},\cdots,u_{ss,r}>\subset<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$. Now let $j=ln$, then it follows from lemma 3.7 we have that $u_{i,ln}\in<\mathcal{C}_{\mathbf{Q}}(\mathcal{T}),u_{ss,1}>$.

This proves the theorem by induction. □

Corrollary 3.2. $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is generated by $\{u_1, \cdots, u_n, w_n\}$ for any w_n with $w_n \in H_{(1,1,\cdots,1)} \setminus \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$.

Proof. Since $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) =< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), u_{ss,1} >$ we have that $w = w_1 + w_2$ with $w_1 \in< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), u_{ss,1} >$ and $w_2 \in \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$, by comparing the dimensional vector of the basic elements occuring in the expression of w, we have that $w_1 = du_{ss,1}$ and $w_2 \in \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$, and $d \neq 0$ since $w \notin \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$. Thus $u_{ss,1} \in< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), w >$ and hence $< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), w >=< \mathcal{C}_{\mathbf{Q}}(\mathcal{T}), u_{ss,1} >= \mathcal{H}_{\mathbf{Q}}(\mathcal{T})$. □

Note that $u_{i,n} \notin \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ for all i, we have also

Corrollary 3.3. $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is generated by $\{u_1, \cdots, u_n, u_{i,n}\}$ for $i = 1, 2, \cdots, n$.

4. Central element in $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$

Now we consider a homogeneous element of dimensional vector $(1, 1, \cdots, 1)$ in the centre of $\mathcal{H}(\mathcal{T})$. Let

$$c = \sum (1-p)^{\tau(M_t)-1} u_{[M_t]}$$

where M_t range over all the modules with dimensional vector $(1, 1, \cdots, 1)$ and $\tau(M_t) = \sum\limits_{i,j} a_{i,j}(M_t)$. We prove that c is in $Z_{\mathcal{H}(\mathcal{T})}(\mathcal{C}(\mathcal{T}))$, the centralizer of $\mathcal{C}(\mathcal{T})$ and that the element c does not lie in $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$.

Lemma 4.1. $u_h c = c u_h$ for $h = 1, 2, \cdots, n$.

Proof. (A). Consider a module L of the middle term of a short exact sequence

$$0 \longrightarrow S_h \xrightarrow{\Phi} L \longrightarrow M \longrightarrow 0$$

for some module M of dimensional vector $(1, 1, \cdots, 1)$. Then $L = L_1 \oplus L_2$ and $\mathrm{Im}\phi = \mathrm{soc}L_1 = S_h$ and $M \simeq L_1/\mathrm{soc}L_1 \oplus L_2$, by Lemma 1. We have the following two cases:

(a)$L_1 = S_h$ and $L \simeq S_h \oplus M$ and we have that $F^{M\oplus S_h}_{M,S_h} = p+1$ if $a_{h,1} = 1$, $F^{M\oplus S_h}_{M,S_h} = p$ if $a_{h-l+1,l} = 1$ for some $l > 1$ and $F^{M\oplus S_h}_{M,S_h} = 1$ otherwise. On the other hand, if we have a short exact sequence

$$0 \longrightarrow S_h \longrightarrow L \longrightarrow N \longrightarrow 0$$

then $N \simeq M$ or $M = M_{h-l+1,l} \oplus M'$ and $N = M_{h-l+1,l-1} \oplus M' \oplus S_h = M''$. Thus we have that $a_{i,j}(M'') = a_{i,j}(M)$ if $(i,j) \neq (h-l+1,t)$ for $t \geq 1$ and $a_{h,1}(M) = 0$, $a_{h,1}(M'') = 1$ and $a_{h-l+1,l-1}(M'') = 1 = a_{h-l+1,l}(M)$. So we have that $\sum\limits_{i,j} a_{i,j}(M)+1 = \sum\limits_{i,j} a_{i,j}(M'')$ and $F^{M\oplus S_h}_{M'',S_h} = 1$ by Proposition 2.5 of [2].

If we have a short exact sequence

$$0 \longrightarrow N \longrightarrow M \oplus S_h \longrightarrow S_h \longrightarrow 0$$

then $N \simeq M$ or $M = M_{h,1} \oplus N'$ and $N = M_{h+1,l-1} \oplus N' \oplus S_h = N''$. Thus we have that $a_{i,j}(N'') = a_{i,j}(M)$ if $i \neq h, h+1$ for and $a_{h,1}(M) = 0$, $a_{h,1}(N'') = 1$ and $a_{h+1,l-1}(N'') = 1 = a_{h,l}(M)$. So we have that $\sum\limits_{i,j} a_{i,j}(M) + 1 = \sum\limits_{i,j} a_{i,j}(N'')$ and $F^{M\oplus S_h}_{S_h,N''} = 1$ by Proposition 2.5 of [2]. Therefore the coefficient of $u_{[M\oplus S_h]}$ in $cu_h - u_hc$ is

$$\begin{aligned}&(1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{M,S_h} - (1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{S_h,M}\\ &= (1-p)^{\sum a_{i,j}(M)-2} - (1-p)^{\sum a_{i,j}(M)-2} = 0\end{aligned}$$

if $a_{h,t} = 0 = a_{h-t+1,t}$ for all $t \geq 1$;and

$$\begin{aligned}&(1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{M,S_h} - (1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{S_h,M}\\ &= (1-p)^{\sum a_{i,j}(M)-2} \tfrac{p^2-1}{p-1} - (1-p)^{\sum a_{i,j}(M)-2} \tfrac{p^2-1}{p-1} = 0\end{aligned}$$

if $a_{h,1} = 1$;and

$$\begin{aligned}&(1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{M,S_h} + (1-p)^{\sum a_{i,j}(M'')-2} F^{M\oplus S_h}_{M,S_h} - (1-p)^{\sum a_{i,j}(M)-2} F^{M\oplus S_h}_{S_h,M}\\ &= (1-p)^{\sum a_{i,j}(M)-2} p + (1-p)^{\sum a_{i,j}(M)-1} - (1-p)^{\sum a_{i,j}(M)-2} = 0\end{aligned}$$

if $a_{h-t+1,t}$ for some $t \geq 2$;and

$$(1-p)^{\sum a_{i,j}(M)-2} F_{M,S_h}^{M\oplus S_h} - (1-p)^{\sum a_{i,j}(M)-2} F_{S_h,M}^{M\oplus S_h} - (1-p)^{\sum a_{i,j}(N")-2} F_{S_h,N"}^{M\oplus S_h}$$

$$= (1-p)^{\sum a_{i,j}(M)-2} - (1-p)^{\sum a_{i,j}(M)-2} p - (1-p)^{\sum a_{i,j}(M)-1} = 0$$

if $a_{h,t} = 1$ for some $t > 1$ So we prove that the coefficient of $u_{[M\oplus S_h]}$ in $cu_h - u_h c$ is 0.

(b)L_1 is not isomorphic to S_h. We have that $M = M_{h-l+1,l} \oplus L_2$ and $l > 1$. If there exists a short exact sequence

$$0 \longrightarrow S_h \xrightarrow{\psi} L \longrightarrow N \longrightarrow 0$$

and N is not isomorphic to M, then $L = L' \oplus L"$ and Imψ =socL'. Note that $(\mathbf{dim}L)_i = 1 + \delta_{i,h}$, thus L_1 is not isomorphic to $L'_1 = M_{h-l+1,l}$ and $L = L_1 \oplus L'_1 \oplus L'_2$. If $L'_1 \neq$Im$\psi \simeq S_h$, then we have that $(\mathbf{dim}L)_{h-1} \geq (\mathbf{dim}L'_1)_{h-1} + (\mathbf{dim}L_1)_{h-1} = 2$, a contradiction. This shows that L'_1 =Im$\psi \simeq S_h$, $L = L_1 \oplus S_h \oplus L'_2$, $M = S_h \oplus M_{h-l+1,l} \oplus L'_2$ and $N = M_{h-l+1,l} \oplus L'_2$. This reduces to the third case of (a). Thus the coefficient of $u_{[L]}$ in $cu_h - u_h c$ is zero.

(B) Now assume that L is the middle term of a short exact sequence

$$0 \longrightarrow M^* \xrightarrow{\theta} L \longrightarrow S_h \longrightarrow 0$$

with $\mathbf{dim}M^* = (1, 1, \cdots, 1)$, then $F^L_{S_h,M^*} \neq 0$. Thus $L = W_1 \oplus W_2$ such top$W_1 = S_h$ and Imθ =rad$W_1 \oplus W_2$ by Lemma 1. If $W_1 = S_h$ and rad$W_1 = 0$, then we have a short exact sequence

$$0 \longrightarrow S_h \xrightarrow{\theta} L \longrightarrow W_2 \longrightarrow 0$$

and $\mathbf{dim}W_2 = (1, 1, \cdots, 1)$. This reduces to the fourth case in (a) of (A) if we take $M = W_2$ and $M^* = M"$. If $W_1 \simeq M_{h,t}$ for some $t > 1$, then $(\mathbf{dim}L)_i = (\mathbf{dim}W_1)_i + (\mathbf{dim}W_2)_i$, thus $(\mathbf{dim}W_2)_h = 1$ and $(\mathbf{dim}W_2)_{h+1} = 0$. This implies that S_h is a direct sum of the socle of W_2 and we have a short exact sequence

$$0 \longrightarrow S_h \xrightarrow{\theta} L \longrightarrow W_1 \oplus W_2/S_h \longrightarrow 0$$

and $\mathbf{dim} W_1 \oplus W_2/S_h = (1, 1, \cdots, 1)$, again we reduce to (A) if we take $M = W_1 \oplus W_2/S_h$:

So we prove that the coefficient of $u_{[L]}$ in $cu_h - u_h c$ is zero. And this completes the proof of the Lemma. $\square$

Remark Let ϵ be the Euler character of $\mathcal{T}$, and $x = (1, 1, \cdots, 1) \in \mathbf{Z}^n$, then it is easy to see that $\epsilon(x, y) = 0 = \epsilon(y, x)$ for any $y \in \mathbf{Z}^n$. The lemma remains true under the change of multiplication as defined by Ringel in [12] using Euler character.

Let $M = \bigoplus M_{i_t, j_t}$ be a module with dimensional vector $(1, 1, \cdots, 1)$. We have that $i_{t+1} \equiv i_t + j_t \mod (n)$ for $1 \leq t < r$, $i_r + j_r \equiv i_1 \mod (n)$ and $\sum j_t = n$. Let $j_r > 1$ and consider a product of the u_i's

$$w(M) = u_{i_r} u_{i_{r-1}} \cdots u_{i_1} u_{i_r+1} \cdots u_{i_1+j_1-1}.$$

we have that

Lemma 4.2. The coefficient of $u_{[M]}$ is 1 in $w(M)$, and if the coefficient of $u_{[N]}$ is not 0 in $w(M)$, then $\mathbf{a}(M) \prec_2 \mathbf{a}(N)$.

If $\tau(M) = \tau(N)$, then the coefficient $u_{[N]}$ in $w(M)$ is zero.

If $\tau(N) = l > m$, then the coefficient $u_{[N]}$ in $\sum\limits_{\tau(M)=m} w(M)$ is C_l^m.

Proof. The first assertion of the first statement is obvious. Assume that the coefficient of $u_[N]$ in $w(M)$ is non zero. Then N has a composition series of the type $S_{i_r}, S_{i_{r-1}}, \cdots S_{i_1}, S_{i_r+1}, S_{i_{r-1}+1}, \cdots S_{i_1+1} \cdots$. So by theorem C of [11], we see that $\mathbf{a}(M) \prec_2 \mathbf{a}(N)$.

Note that $\bigoplus_{t=1}^r S_{i_t}$ is a submodule of any mudule N with the coefficient of $u_{[N]}$ in $w(M)$ nonzero. Thus if $\tau(N) = \tau(M)$, then we have that $M \cong N$.

By [11], we see that if $\tau(N) > \tau(M)$, the N is obtaied from M by ' breaking' some of its direct summand into pieces. So if we consider the tops, those simples occuring in *top* M with the coefficient of $u_{[N]}$ in $w(M)$ nonzero can be regarded as chosen from those occuring in *top* N, so there are C_l^m types of tops for M. This proves the third statement. $\square$

Proposition 4.3. $c = \frac{(-q)^n}{1-q}u_{ss,1} + \sum_{t=1}^{n-1} \sum\limits_{\tau(M)=t\geq 2} \frac{((-1)^{l-1}-q)^{\tau(M)}}{1-q} w(M) + \sum\limits_{\tau(M)=1} w(M)$.

Proof. Note that $(1-q) - C_2^1 = -1-q$ and $\sum(1-q)^l - \sum_{j=2}^{l-1} C_l^j((-1)^{j-1} + (-q)^j) - C_l^1(1-q) = \sum_{j=0}^{l} C_l^j(-q)^j - \sum_{j=1}^{l-1} C_l^j(-q)^j + \sum_{j=1}^{l-1} C_l^j(-1)^j = (-q)^l - (-1)^l + (1-1)^l = (-q)^l + (-1)^{l-1}$ So we have that $c - \sum_{\tau(M)=2}^{t} \frac{(-q)^{\tau(M)}}{1-q} w(M) - \sum\limits_{\tau(M)=1} w(M) = \sum\limits_{\tau(M)>t} ((1-q)^{\tau(M)-1} - \sum_{r=2}^{t} C_{\tau(M)}^r \frac{(-q)^t}{1-q} - C_{\tau(M)}^1) w(M)$ for $t = 0, 1, \cdots, n-1$ and this completes the proof. $\square$

Since $u_{ss,1} \notin \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$, so we have the following.

Corrollary 4.4. $c \notin \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$.

Theorem 4.5. $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ is generated by $\{u_1, \cdots, u_n, u_{i,n}\}$ for $i = 1, 2, \cdots, n$.

Proof. This follows from the corollary 4.4 and corollary 3.2. $\square$

Corrollary 4.6. c is in the centre of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$.

Proof. This follows from lemma 4.1 and theorem 4.5. $\square$

So we can rewrite theorem 4.5 as follow.

Corrollary 4.7. $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) = \mathcal{C}_{\mathbf{Q}}(\mathcal{T})[c]$.

5. The proof of the main theorem

In this section we prove the c is an indeterminant over $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$.

Let $m \geq 2$ be an integer, we have studied the algebra $R(n,m) = R_k(n,m) = kQ_n/I^m$, where I^m is the ideal of kQ_n generated by the paths of length m in kQ_n and their Hall algebras $\mathcal{H}(n,m)$. Note that for a given m, the category of the finite $R(n,m)$ modules is equivalent to the full subcategory of $\mathcal{T}$ consisting of the modules of Loewy length $\leq m$. In the following, we using the same notation for a basic element

$u_{[M]}$ in $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ and in $\mathcal{H}_Q(n,m)$ is M is a $R(n,m)$ module. By the observation in section 5 of [1], we have the following result.

Proposition 5.1. For each $m \geq 2$, there is an embedding of abelian group

$$\Psi : \mathcal{H}_Q(n,m) \hookrightarrow \mathcal{H}_{\mathbf{Q}}(\mathcal{T})$$

such that $\Psi(u_{i,j}) = u_{i,j}$ for all (i,j) with $j \leq m$ and

$$\Psi(u_{[M]}u_{[N]}) = \sum_{L:ll(L)\leq m} F^L_{M,N} u_{[L]}$$

where $ll(L)$ denote the Loewy length of L.

We remark that the Hall algebra $\mathcal{H}_{\mathbf{Q}}(n,m) \cong \mathcal{H}_{\mathbf{Q}}(\mathcal{T})/J$ where J is the ideal of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ generated by the basic elements corresponing to the finite module of $\mathcal{T}$ of Loewy length $> m$, and the image of composition algebra $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ is the composition subalgebra $\mathcal{C}_Q(n,m)$ of $\mathcal{H}_Q(n,m)$. Using the same notations for the elements in $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ and their images in $\mathcal{H}_Q(n,m)$. So we also have that $\mathcal{H}_Q(n,m) = \mathcal{C}_Q(n,m)[c]$.

It is know that there is an $(((\mathbf{Z})^+)^{nm}, \prec_2)$ filtration on $\mathcal{H}_Q(n,m)$ with the related graded ring $\overline{\mathcal{H}_Q(n,m)}$ a (finite) iterated skew ring with zero derivations. Let $GK\dim\Lambda$ be the Gelfand-Kirilov dimension of an algebra Λ. So we see from Proposition 3.5 of [4], we see that $\overline{\mathcal{H}_Q(n,m)}$ has finite Gelfand-Kirilov dimension, thus $\mathcal{H}_Q(n,m)$ also has finite Gelfand-Kirilov dimension. We have the following:

Theorem 5.2. $GK\dim\mathcal{H}_Q(n,m) = GK\dim\mathcal{C}_Q(n,m) + 1$.

Proof. Clearly $GK\dim\mathcal{H}_Q(n,m) \leq GK\dim\mathcal{C}_Q(n,m)+1$ since $\mathcal{H}_Q(n,m) = \mathcal{C}_Q(n,m)[c]$ is a homomorphic image of the polynomial ring $\mathcal{C}_Q(n,m)[X]$. We prove that

$$GK\dim\mathcal{H}_Q(n,m) \geq GK\dim\mathcal{C}_Q(n,m) + 1$$

Let $V_{h,1}$ be the $\mathbf{Q}$ vector spaces with basis $\{u_1, \cdots, u_n, u_{n,n}\}$ and $V_{c,t}$ be the $\mathbf{Q}$ vector spaces with basis $\{u_1, \cdots, u_n\}$. $V_{h,1}$ and $V_{c,1}$ are the generating subspaces of $\mathcal{H}_Q(n,m)$ and $\mathcal{C}_Q(n,m)$, respectively. Let $V_{h,t} = \sum_{u=0}^t V^u_{h,1}$ and $V_{c,t} = \sum_{u=0}^t V^u_{c,1}$. Let $\dim V_{h,t} =$

$d_{h,t}$ and $\dim V_{c,t} = d_{c,t}$. Then we have that $GK \dim \mathcal{H}_Q(n,m) = \overline{lim}\ log_t d_{h,t}$ and $GK \dim \mathcal{C}_Q(n,m) = \overline{lim}\ log_t d_{c,t}$.

By [11] theorem B and Proposition 1 we see that $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ has a basis corresponding to the elements in the subset $\{\mathbf{a} = (a_{i,j}) \in ((\mathbf{Z})^+)^{n\infty} | a_{n,jn} = 0\}$ of $((\mathbf{Z})^+)^{n\infty}$, while $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ has a basis corresponding to the elements in $((\mathbf{Z})^+)^{n\infty}$. So this basis of $\mathcal{H}_{\mathbf{Q}}(\mathcal{T})$ can be obtained from the corresponding basis of $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ together with those by adding the parts of (n, jn) type for all j with $jn \leq m$ to the given basis of $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$.

Clearly $\mathcal{C}_Q(n,m)$ has basis a corresponding to the elements in the subset $\{\mathbf{a} = (a_{i,j}) \in ((\mathbf{Z})^+)^{nm} | a_{n,rn} = 0\}$ of $((\mathbf{Z})^+)^{n\infty}$, while $\mathcal{H}_Q(n,m)$ has a basis corresponding to the elements in $((\mathbf{Z})^+)^{nm}$. So this basis of $\mathcal{H}_Q(n,m)$ can be obtained from the corresponding basis of $\mathcal{C}_Q(n,m)$ together with those by adding the parts of (n, rn) type for all $r \leq [\frac{m}{n}]$ to the given basis of $\mathcal{C}_Q(n,m)$. By comparing the correspond elements of the same dimension vector, we have that

$$d_{h,tn} = \sum_{u=0}^{t} \delta(u) d_{c,t}$$

where $1 \leq \delta(u) \leq p(u)$ and $p(u)$ is the number of the partitions of the number u. So we have that $d_{h,tn} = \sum_{u=0}^{t} \delta(u) d_{c,t} \geq \sum_{u=0}^{t} d_{c,t}$

Now assume that $GK \dim \mathcal{C}_Q(n,m) = \gamma$, then by Lemma 2.1 of [4], we have that $d_{c,v} \leq v^{\gamma-\epsilon}$ for all $v \geq L$ for some L. So we have that

$$d_{h,tn} \geq \sum_{u=0}^{t} d_{c,tn} = \sum_{u=0}^{L-1} d_{c,t} + \sum_{u=L}^{t} d_{c,t} \geq \sum_{v=l}^{t} v^{\gamma-\epsilon} + C(L) = \frac{1}{\gamma+1-\epsilon} t^{\gamma+1-\epsilon} + C'(L)$$

where $C(L)$ is a constant and $C'(L) = o(t)$.

Thus

$$\begin{aligned} GK \dim \mathcal{H}_Q(n,m) &= \overline{lim}\ log_t d_{h,t} \\ &= \overline{lim}\ log_t(\tfrac{1}{\gamma+1-\epsilon} t^{\gamma+1-\epsilon} + C'(L)) \\ &= \overline{lim}\ log_t \tfrac{1}{\gamma+1-\epsilon} t^{\gamma+1-\epsilon} = \gamma + 1 \end{aligned}$$

This completes the proof. $\square$

Theorem 5.3. c is an indeterminant over $\mathcal{C}_Q(n,m)$, and $\mathcal{H}_Q(n,m) \cong \mathcal{C}_Q[X]$.

Proof. Let $\mathcal{H}_Q(n,m) \cong \mathcal{C}_Q[X]/I$ for some ideal I of $\mathcal{C}_Q(n,m)[X]$, then by [1], $\mathcal{H}_Q(n,m)$ is a Noetherian ring without nonzero zero divisor, thus I is prime and $\mathcal{C}_Q(n,m)[X]$ has no nonzero zero divisor, since $\mathcal{C}_Q(n,m)$ is a subring of $\mathcal{H}_Q(n,m)$ and has no nonzero divisor. If $I \neq 0$, then by proposition 3.15 of [4], we have that $GK \dim \mathcal{C}_Q(n,m)+1 \geq GK \dim \mathcal{H}_Q(n, m+1$ and $GK \dim \mathcal{C}_Q(n,m) \geq GK \dim \mathcal{H}_Q$ a contradiction.

This proves that $I = 0$ and finishes the proof. □

Now we prove the main theorem of this paper

Theorem 5.4. *c is an indeterminant over $\mathcal{C}_{\mathbf{Q}}(\mathcal{T})$, and $\mathcal{H}_{\mathbf{Q}}(\mathcal{T}) \cong \mathcal{C}_{\mathbf{Q}}(\mathcal{T})[X]$.*

Proof. Assume that on the contrary, there is c is not an indeterminant, then there is an natural number l and there are elements $d_t \in \mathcal{C}_{\mathbf{Q}}(\mathcal{T})$ for $t = 0, 1, \cdots, l$ such that $d_0 + d_1 c + \cdots + d_l c^l = 0$. Let $d_t = \sum a_{t,s} u_{[N_s]}$ and $m = \sum\limits_{s; a_{t,s} \neq 0} l(N_s) + n$, then by proposition 6.1, we have that d_t, c and $d_0 + d_1 c + \cdots + d_l c^l$ are all contained in $\mathcal{H}_Q(n,m)$ and $d_t \in \mathcal{C}_Q(n,m)$, and we have that $d_0 + d_1 c + \cdots + d_l c^l = 0$, a contradiction to the fact that c is indeterminant over $\mathcal{H}_Q(n,m)$.

This completes the proof. □

6. Appendix

In [2], a formula to calculating $F^L_{N,M}$ with M semisimple (Proposition 2.5) and an inductive recipe to calculating $F^L_{N,M}$ for any M is given. Now we use the recipe and the formula give in [2] to calculation $F^L_{N_4,N_5}$. Now we consider the case $L = L_1$. Note that $\operatorname{soc} M_{i+1,n-1} \cong \operatorname{soc} M_{i,n} \cong S_{i-1}$, we have that

$$\begin{array}{ll} & F_{N_4,N_5}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ = & F_{S_i\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n},S_{i-1}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ + & F_{S_i\oplus M_{i,n-1}\oplus S_{i-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n},S_{i-1}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}} \\ + & F_{S_i\oplus 2M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-3)M_{i,n},S_{i-1}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-3)M_{i,n}} \\ - & F_{S_{i-1}\oplus M_{i+1,n-2},S_{i-1}}^{M_{i+1,n-2}\oplus 2S_{i-1}} F_{N_4,2S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ = & q^{l-1} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ + & q^{l-2} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}} \\ + & \frac{q^{l-2}-1}{q-1} F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus 2M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-3)M_{i,n}} \\ - & (q+1) F_{N_4,2S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \end{array}.$$

Note that $\operatorname{soc} M_{i+1,n-2} \cong \operatorname{soc} M_{i,n-1} \cong S_{1+n-2}$. Repeat the process for

$$F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}},$$

by a similar arguement as the case $lM_{i,n}$, we get

$$\begin{array}{ll} & F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ = & F_{N_4,S_{i-1}\oplus M_{i+1,n-3}}^{S_i\oplus M_{i,n-2}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ = & \cdots \\ = & F_{N_4,S_{i-1}\oplus S_{i+1}}^{S_i\oplus M_{i,2}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} \\ = & 0 \end{array}.$$

Similarly, we have that

$$F_{N_4,S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus 2M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-3)M_{i,n}} = 0$$

and

$$F_{N_4,2S_{i-1}\oplus M_{i+1,n-2}}^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}} = 0$$

On the other hand, we have that

$$
\begin{array}{rl}
& F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-2}} \\
= & F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{S_i\oplus M_{i,n-1}\oplus S_{i-1}\oplus M_{i+1,n-3}\oplus(l-2)M_{i,n},S_{i+n-2}} F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-3}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} \\
+ & F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}}_{S_i\oplus M_{i,n-2}\oplus S_{i-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n},S_{i+n-2}} F^{S_i\oplus S_{i-1}\oplus M_{i,n-2}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} \\
- & F^{S_{i+n-2}S_{i-1}\oplus M_{i+1,n-3}}_{S_{i-1}\oplus M_{i+1,n-3},S_{i+n-2}} F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus S_{i+n-2}\oplus M_{i+1,n-3}} \\
= & qF^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-3}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} + F^{S_i\oplus S_{i-1}\oplus M_{i,n-2}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} \\
= & \cdots \\
= & qF^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus S_{i+1}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus S_{i+1}} + F^{S_i\oplus S_{i-1}\oplus M_{i,2}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} \\
= & q\cdot q^{l-2}=q^{2l-3}
\end{array}
$$

where $F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus S_{i+n-2}\oplus M_{i+1,n-3}}=0$ since S_{i+n-3} is a summand of $\mathrm{soc}(S_{i-1}\oplus S_{i+n-2}\oplus M_{i+1,n-3})$ but not a summand of $\mathrm{soc}(S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n})$ and $F^{S_i\oplus S_{i-1}\oplus M_{i,2}\oplus M_{i+1,n-2}\oplus(l-2)M_{i,n}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}}=0$.

So we have that $F^{S_i\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i+1,n-1}\oplus(l-2)M_{i,n}}_{N_4,N_5}=q^{2l-3}$.

Now let $L=lM_{i,n}$, using lemma 1.2 of [1] for $M=S_{i-t}$ for $t=1,2,\cdots$, we have the following

$$
\begin{array}{rcl}
F^{lM_{i,n}}_{N_4,N_5} & = & F^{lM_{i,n}}_{(l-1)M_{i,n}\oplus M_{i,n-1},S_{i-1}} F^{(l-1)M_{i,n}\oplus M_{i,n-1}}_{N_4,S_{i-1}\oplus M_{i+1,n-2}} \\
& - & F^{M_{i+1,n-2}\oplus 2S_{i-1}}_{S_{i-1}\oplus M_{i+1,n-2},S_{i-1}} F^{lM_{i,n}}_{N_4,2S_{i-1}\oplus M_{i+1,n-2}} \\
& = & \frac{q^l-1}{q-1}F^{(l-1)M_{i,n}\oplus M_{i,n-1}}_{N_4,S_{i-1}\oplus M_{i+1,n-2}} = \frac{q^l-1}{q-1}F^{(l-1)M_{i,n}\oplus M_{i,n-2}}_{N_4,S_{i-1}\oplus M_{i+1,n-3}} \\
& = & \cdots \\
& = & \frac{q^l-1}{q-1}F^{(l-1)M_{i,n}\oplus M_{i,2}}_{N_4,S_{i-1}\oplus S_{i+1}} = \frac{q^l-1}{q-1}\frac{q^{l-1}-1}{q-1}
\end{array}
$$

since $F_{S_{i-1}\oplus M_{i+1,n-t},S_{i+n-t}}\neq 0$ implies that $M\cong S_{i-1}\oplus M_{i+1,n-t+1}$ or $\cong S_{i-1}\oplus M_{i+1,n-t}\oplus S_{i+n-t}$ and $F^{(l-1)M_{i,n}\oplus M_{i,n-t+1}}_{N_4,S_{i+n-t}\oplus S_{i-1}\oplus M_{i+1,n-t}}=0$ since soc $M_{i+1,n-t}$ is not a summand of the socle of $(l-1)M_{i,n}\oplus M_{in-t}$ for all t, and for $t\geq 0$, $F^{(l-1)M_{i,n}\oplus M_{i,n-t+1}}_{M,S_{i+n-t}}\neq 0$ implies that $M\cong S_{i-1}\oplus M_{i+1,n-t}$. And we use Proposition 2.5 of [2] for the last step.

Now let $L=(l-1)M_{i,n}\oplus M_{i,n-1}\oplus S_{i-1}$. Similar to the above, we have.

$$
\begin{aligned}
&F_{M_4,M_5}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}}\\
&\quad= \sum_{M'} F_{M',S_{i-1}}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}} F_{M_4,S_{n-1}\oplus M_{i+1,n-1}}^{M'}\\
&\quad\quad - F_{S_{i-1}\oplus M_{i+1,n-2},S_{i-1}}^{2S_{i-1}\oplus M_{i+1,n-2}} F_{M_4,2S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}}\\
&\quad= q^{l-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus M_{i,n-1}}\\
&\quad\quad + \tfrac{q^{l-1}-1}{q-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-2}}^{(l-2)M_{i,n}\oplus S_{i-1}\oplus 2M_{i,n-1}}\\
&\quad\quad - (q+1)F_{M_4,2S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}}\\
&\quad= q^{l-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-3}}^{(l-1)M_{i,n}\oplus M_{i,n-2}}\\
&\quad\quad + \tfrac{q^{l-1}-1}{q-1}(q+1)F_{M_4,S_{i-1}\oplus M_{i+1,n-3}}^{(l-2)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i,n-2}}\\
&\quad\quad - (q+1)F_{M_4,2S_{i-1}\oplus M_{i+1,n-3}}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-2}}\\
&\quad= \cdots\\
&\quad= q^{l-1}F_{M_4,S_{i-1}\oplus S_{i+1}}^{(l-1)M_{i,n}\oplus M_{i,2}}\\
&\quad\quad + \tfrac{q^{l-1}-1}{q-1}(q+1)F_{M_4,S_{i-1}\oplus S_{i+1}}^{(l-2)M_{i,n}\oplus S_{i-1}\oplus M_{i,n-1}\oplus M_{i,2}}\\
&\quad\quad - (q+1)F_{M_4,2S_{i-1}\oplus S_{i+1}}^{(l-1)M_{i,n}\oplus S_{i-1}\oplus M_{i,2}}\\
&\quad= q^{l-1}\tfrac{q^{l-1}-1}{q-1} + \tfrac{q^{l-1}-1}{q-1}(q+1)q^{l-2} - (q+1)\tfrac{q^{l-1}-1}{q-1}q^{l-2}\\
&\quad= q^{l-1}\tfrac{q^{l-1}-1}{q-1}
\end{aligned}
$$

This proves the case $L=L_2$. Let $L=L_3$, we have that

$$
\begin{aligned}
F_{M_4,M_5}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-1}} &= \sum_{M'} F_{M',S_{i-1}}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-1}} F_{M_4,S_{n-1}\oplus M_{i+1,n-1}}^{M'}\\
&\quad - F_{S_{i-1}\oplus M_{i+1,n-2},S_{i-1}}^{2S_{i-1}\oplus M_{i+1,n-2}} F_{M_4,2S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-1}}\\
&= q^{l-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-2}}\\
&\quad + \tfrac{q^{l-1}-1}{q-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-2}}^{(l-2)M_{i,n}\oplus S_i} \oplus M_{i,n-1}\oplus M_{i+1,n-1}\\
&\quad - (q+1)F_{M_4,2S_{i-1}\oplus M_{i+1,n-2}}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-1}}\\
&= q^{l-1}F_{M_4,S_{i-1}\oplus M_{i+1,n-3}}^{(l-1)M_{i,n}\oplus S_i\oplus M_{i+1,n-3}}\\
&= \cdots\\
&= q^{l-1}F_{M_4,S_{i-1}\oplus S_{i+1}}^{(l-1)M_{i,n}\oplus S_i\oplus S_{i+1}}\\
&= q^{l-1}\tfrac{q^{l-1}-1}{q-1}
\end{aligned}
$$

Where by a similar arguement as in the first case, we have that the second and the third terms are zero.

This finishes the calculations.

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