# Semigroups containing proximal linear maps

by

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## I. Introduction

A linear automorphism of a finite dimensional real vector space is called *proximal* if it has a unique eigenvalue – counting multiplicities – of maximal modulus. Proximal elements play an important role in the study of the dynamics of linear maps because of their simple structure: The induced map on projective space has a unique attracting fixed point whose basin consists of the whole space except for a hyperplane. This contraction property played an essential role in Tits' proof of what is now called the Tits alternative [Ti], in Furstenberg's work on boundary theory [F], in the paper of the two last named authors on maximal subgroups [MS], in the work of Guivar'ch and coauthors on random walks on groups (cf. [GR]). The authors were motivated to do this work by their hope for yet another application of proximality, namely in their work in progress on the Auslander conjecture. For work on this conjecture see [A, Mi, D, DG, FG, GK, GrM, Ma1-3, Me, To, S].

An efficient tool to find proximal elements is the following corollary of the main algebraic result of [GM]: Given a subgroup $G$ of $GL(V)$ which contains a proximal element. Then so does every Zariski dense subsemigroup $H$ of $G$, provided $V$ – considered as a $G$–module – is strongly irreducible (for a definition see the paragraph preceding 3.12). Here we show that under these hypotheses $H$ in fact provides for a rich supply of proximal elements: There is a finite subset $M$ of $H$ such that for every $g \in GL(V)$ at least one of the elements $\gamma g$, $\gamma \in M$, is proximal. We obtain such a subset $M$ with at most $(\dim V)^2$ elements. We show the main result (theorem 4.1) for a quantitative version of proximality, called $(r, \varepsilon)$–proximality. For another richness result see 4.10.

In section 5 of the paper we extend our result in two directions, namely for direct sums of strongly irreducible representations and for other types of Lyapunov filtrations. The latter means that instead of requiring that $G$ contains an element with exactly one eigenvalue of maximal modulus we look at the minimum number $n_1$ of eigenvalues of maximal modulus among elements of $G$. Then the same minimum is attained in the Zariski dense subsemigroup $H$. Similarly for the eigenvalues with maximum and next to maximum modulus, etc.

In the final section we give necessary and sufficient conditions when the image of a reductive group under an irreducible representation $\rho$ contains a proximal linear map. The conditions are phrased in terms of the highest weight of $\rho$.

As an application (actually of 4.1 and results of [GR]) we obtain (see 6.8) that for the group $G(\mathbb{R})$ of real points of a connected semi–simple algebraic group $G$ over $\mathbb{R}$ every Zariski dense subsemigroup $H$ contains $\mathbb{R}$–regular elements, where an element $g \in G(\mathbb{R})$ is called $\mathbb{R}$–regular if the number of eigenvalues – counting multiplicities – of modulus $1$ of $\mathrm{Ad}(g)$ is minimum possible among elements $g \in G(\mathbb{R})$. This was known before for sub*groups* $H$ ([BL], [P]). We also obtain a corresponding finiteness result.

The proofs of the finiteness results 4.1 and 5.14 make essential use of the notion of quasiprojective transformation due to Furstenberg [F] and the methods of the algebraic part of [GM]. The relevant notions and results are given in section 3. The proof of 4.1 is contained in section 4. There the simple structure of proximal linear maps is expounded, technically speaking in the form of "contractions" and "contractive sequences" on projective space. In section 5 we give the generalization to several irreducible summands and more general Lyapunov filtrations.

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## II. Proximality

A linear automorphism $g$ of a finite dimensional real vector space $V$ is called *proximal* if $g$ has a unique eigenvalue $\lambda = \lambda(g)$ of maximal absolute value — hence $\lambda$ is real — and the corresponding weight space

$$V_\lambda = \{v \in V;\ (g - \lambda E)^n v = 0 \quad \text{for some } n\}$$

is one–dimensional.

Fix a norm $\|\cdot\|$ on $V$ and let $\rho$ be the corresponding metric on $V$. If $g$ is proximal with eigenvalue $\lambda = \lambda(g)$ of maximal absolute value set $V^+ = V^+(g) = V_\lambda$ and let $V^< = V^<(g)$ be the unique $g$–invariant hyperplane complementary to $V^+$. Put

$$r(g) = \frac{\| g \mid V^+ \|}{\| g \mid V^< \|} = \frac{|\lambda(g)|}{\|g \mid V^<\|}$$

and

$$\varepsilon(g) = \rho\left(x^+, V^<\right)$$

where $x^+$ is any of the two vectors of $V^+$ of norm $1$. Let us call an element $g \in GL(V)$ $(r,\varepsilon)$–proximal, $r > 1$, $\varepsilon > 0$, if $g$ is proximal and $r(g) \geq r$ and $\varepsilon(g) \geq \varepsilon$. So $g \in GL(V)$ is $(r,\varepsilon)$–proximal iff there is a decomposition $V = V^+ \oplus V^<$ into $g$–invariant subspaces such that

1) $\dim V^+ = 1$

2) $\| g \mid V^+ \| \geq r \, \| g \mid V^< \|$ and

3) $\rho\,(x^+, V^<) \geq \varepsilon$ for $x^+ \in V^+$ of norm 1.

Note that "$g$ proximal" does not imply $r(g) \geq 1$. But if $g$ is proximal then $\sqrt[n]{r(g^n)}$ converges to $\lambda / \max\{|\mu|,\ \mu$ eigenvalue of $g \mid V^<\} > 1$, in particular $\lim\limits_{n\to\infty} r\,(g^n) = \infty$.

We take the following criterion for proximality from [Ti]. For $g \in GL(V)$ put

$$g_1(x) = \frac{g(x)}{\|g(x)\|} \qquad \text{for} \quad x \neq 0\ .$$

So $g_1$ maps the "sphere" $S = \{x \in V \ ; \ \|x\| = 1\}$ to itself. For a metric space $(X, \rho)$ define the *norm of a map* $f : X \longrightarrow X$ *with respect to* $\rho$ by

$$\|f\|_\rho = \sup_{x \neq y \text{ in } X} \frac{\rho(f(x), f(y))}{\rho(x, y)}$$

$(= 0$ if card $X \leq 1)$ .

**2.1 Lemma** *Let $g \in GL(V)$ and let $K$ be a compact subset of $S$ such that $g_1(K) \subset \mathring{K}$ and $\| g_1|K \| < 1$ . Here $\mathring{K}$ denotes the interior of $K$ with respect to the topology of $S$ . Then $g$ is proximal, $g_1$ has a unique fixed point in $K$ , namely the point in $V^+(g) \cap \mathring{K}$ and $V^<(g)$ does not intersect $K$ .*

The **proof** of [Ti 3.8 (ii)] for projective space works for the sphere, too.

## III. Quasiprojective transformations

In this section we shall recall the notion of quasiprojective transformation due to Furstenberg [F] and in particular the results on contractions from [GM].

It will be convenient to use different metrics on projective space. So we shall first of all fix a class of metrics.

Two metrics $\rho$ and $\rho'$ on a set $X$ are called *equivalent* if there is a positive constant $c \leq 1$ such that

$$c\,\rho(x, y) \leq \rho'(x, y) \leq c^{-1}\rho(x, y)$$

for every pair $x, y$ of points of $X$ . Let $X$ be a connected differentiable manifold with a Riemannian tensor $g$ . Then there is an induced metric $\rho = \rho_{x,g}$ , namely $\rho(x, y)$ is the infimum of the arc lengths with respect to $g$ of piecewise differentiable curves from $x$ to $y$ . The following facts are easily proved.

**3.1** *If $g$ and $g'$ are two Riemannian tensors on $X$ then $\rho_{X,g'} \mid K$ and $\rho_{X,g} \mid K$ are equivalent for any compact subset $K$ of $X$ .*

**3.2** *If* $U$ *is an open connected neighbourhood of a compact subset* $K$ *of* $X$ *then* $\rho_{X,g} \mid K$ *and* $\rho_{U,g} \mid K$ *are equivalent.*

It follows that

**3.3** *On a compact differentiable manifold* $X$ *there is a unique equivalence class of metrics* $\rho$ *containing the metrics induced by Riemannian tensors. This class is characterized by the following property: If* $\varphi$ *is a diffeomorphism from an open subset* $U$ *of* $X$ *to a finite dimensional normed vector space* $(V, \| \cdot \|)$ *then for any compact subset* $K$ *of* $U$ *the metric* $\rho'(x,y) = \|\varphi(x) - \varphi(y)\|$ *on* $K$ *is equivalent to* $\rho \mid K$

We consider the projective space $P = \mathbb{P}V$ associated with a finite dimensional real vector space $V$ and endow it with some metric of the distinguished class of metrics equivalent to the Riemannian metrics. We call such a metric *admissible.* A particularly pleasant admissible metric is the *angle metric* on $P$ with

$$\rho([x],[y]) = \arccos \frac{|(x,y)|}{\|x\| \cdot \|y\|}$$

for $x, y$ nonzero in $V$ supposing we are given a positive inner product $(\ ,\ )$ on $V$ with associated norm $\|x\| = (x,x)^{1/2}$ .

**3.4** A map $b : P \longrightarrow P$ is called *quasiprojective* if there is a sequence of projective transformations of $P$ converging pointwise to $b$ . Let $M_1(b)$ be the closure of the set of points, where $b$ is discontinuous and let $M_0(b)$ be the $b$–image of the set of continuity points of $b$ .

It is straightforward that for $h \in GL(V)$ we have

$$M_0(h\,b) = h\,M_0(b),\ \ M_0(b\,h) = M_0(b) \tag{3.5}$$

$$M_1(h\,b) = M_1(b),\ \ M_1(b\,h) = h^{-1}M_1(b) \tag{3.6}$$

A quasiprojective transformation $b$ is called a *contraction* if $M_0(b)$ consists of one point only.

For every linear concept we denote the corresponding projective concept by writing a $\mathbb{P}$ in front. E.g. for $g \in GL(V)$ the corresponding projective map is $\mathbb{P}g : \mathbb{P}V = P \longrightarrow P$ . For a linear subspace $W \neq 0$ of $V$ the corresponding subspace of

$P$ will be denoted $\mathbb{P}W$ . For a non–zero vector $x$ of $V$ let $\mathbb{P}x$ be the point $\mathbb{P}(\mathbb{R}x) \in P$ .

Let us call a sequence $s = (B_n)_{n\in\mathbb{N}}$ in $GL(V)$ *contractive* if

(C 1) $\quad \mathbb{P}\, B_n$ *converges pointwise on* $P$

(C 2) $\quad$ *there is a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}^*$ *such that* $(\alpha_n \cdot B_n)_{n\in\mathbb{N}}$ *converges to a linear map* $b_1$ *of rank* $1$ .

Note that C 2 does not imply C 1.

**3.7 Lemma** *If* $s = (B_n)_{n\in\mathbb{N}}$ *is a contractive sequence in* $GL(V)$ *then* $b = \lim \mathbb{P}\, B_n$ *is a contraction. Conversely, if for a sequence* $s = (B_n)_{n\in\mathbb{N}}$ *in* $GL(V)$ *the sequence* $(\mathbb{P}\, B_n)_{n\in\mathbb{N}}$ *converges pointwise on* $P$ *to a contraction then* $s$ *contains a contractive subsequence.*

**Proof** For a contractive sequence $(B_n)_{n\in\mathbb{N}}$ in $GL(V)$ the limit $b = \lim \mathbb{P}\, B_n$ is represented by $b_1$ for points not in $\ker b_1$ , i.e. $b\,(\mathbb{P}\, x) = \mathbb{P}\,(b_1 x)$ for $b_1 x \neq 0$ . Hence $b$ is continuous on the dense open subset $P \smallsetminus \mathbb{P} \ker b_1$ , hence $M_0(b) = \mathbb{P}\, \mathrm{image}\, b_1$ . Thus $b$ is a contraction.

Conversely, let $(B_n)_{n\in\mathbb{N}}$ be a sequence in $GL(V)$ such that $(\mathbb{P}\, B_n)_{n\in\mathbb{N}}$ converges to the contraction $b$ . Then, passing to a convergent subsequence of $B_n \cdot \|B_n\|^{-1}$ , we may assume that there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^*$ such that $(\alpha_n\, B_n)_{n\in\mathbb{N}}$ converges to a non–zero linear map $b_1$ . So $b(\mathbb{P}\, x) = \mathbb{P}\,(b_1 x)$ for $x \notin \ker b_1$ and $b$ is continuous on the dense open subset $P \smallsetminus \mathbb{P} \ker b_1$ , hence $b_1$ takes its image in the line corresponding to $M_0(b)$ for points $x \notin \ker b_1$ . It follows that rank $b_1 = 1$ .

For a contractive sequence $s$ put

$$M_0(s) = \mathbb{P}\, \mathrm{image}\ \ b_1$$

and

$$L_1(s) = \mathbb{P} \ker\ b_1 \ ,$$

with notations as above. We have seen in the proof of 3.7 that

$$M_0(s) = M_0(b)$$

for $b = \lim \mathbb{P}\, s$ , so $M_0(s)$ depends only on $s$ . To see that $L_1(s)$ is well–defined note that for the sequence $\alpha_n$ in $C2$ we have a constant $c \geq 1$ such that

$$c^{-1} < |\alpha_n| \; \|B_n\| \; < c \; .$$

Hence

$$L_1(s) = \left\{ v \in V \; ; \; \|B_n\, v\| \; \cdot \; \|B_n\|^{-1} \longrightarrow 0 \right\}$$

is independent of the sequence $(\alpha_n)_{n \in \mathbb{N}}$ . It follows that

**3.8** $b_1$ *in* $C2$ *is uniquely determined by* $s$ *up to a nonzero scalar,* since $\ker b_1$ and the line $\operatorname{image} b_1$ are unique.

We have seen above that $M_0(s)$ depends only on $b$ . This is not true for $L_1(s)$ , e.g. for $b$ sending all of $P$ to one point every hyperplane occurs as $L_1(s)$ for some contractive sequence with $\lim \mathbb{P}\, s = b$ .

Note that for a contractive sequence $s$ and $h \in GL$ we have

$$L_1(h\, s) = L_1(s) \;\text{ and }\; L_1(s\, h) = h^{-1} L(s) \; . \tag{3.9}$$

**3.10 Lemma** *For a contractive sequence* $s$ *the sequence* $\mathbb{P}\, s$ *of projective transformations converges uniformly on compact subsets of* $P \setminus \mathbb{P}L_1(s)$ *to the constant map with image* $M_0(b)$ *. For every admissible metric* $\rho$ *on* $P$ *we have*

$$\|\mathbb{P}B_n \mid K\|_\rho \longrightarrow 0$$

*on every compact subset* $K$ *of* $P \setminus \mathbb{P}L_1(s)$ *.*

**Proof** We have pointwise convergence of $\mathbb{P}s$ on $P \setminus \mathbb{P}L_1(s)$ to $M_0(b)$ . It thus suffices to prove the second claim. To show this choose coordinates as follows: Put $L_1 := L_1(s)$ , let $e \notin L_1$ . Then the projection $\pi : V \setminus \{0\} \longrightarrow P, \pi(x) = \mathbb{P}x$ , induces a map $\pi | e + L_1 \longrightarrow P \setminus \mathbb{P}L_1$ which is an affine coordinate system. Put $e' = b_1(e)$ and let $L'$ be a hyperplane complementary to $e'$ . We thus have an affine coordinate system $\pi : e' + L' \longrightarrow P \setminus \mathbb{P}L'$ in some neighbourhood of $M_0(b)$ . Decompose $\alpha_n\, B_n$ according to $\mathbb{R}\, e' \oplus L'$ :

$$\begin{aligned} \alpha_n\, B_n e &= \lambda_n\, e' + y_n \qquad\quad , \quad y_n \in L' \\ \alpha_n\, B_n x &= \mu_n(x) e' + B_n'(x), \quad x \in L, \;\; B_n'(x) \in L' \end{aligned}$$

Then $\lambda_n \longrightarrow 1, y_n \longrightarrow 0$ , since $\alpha_n B_n e \longrightarrow b_1 e = e'$ . And $\|\mu_n \mid L_1\| \longrightarrow 0$, $\|B'_n\| \longrightarrow 0$ . Hence in the coordinate systems above we can represent $\mathbb{P} B_n$ by

$$D_n(x) = \frac{y_n + B'_n(x)}{\lambda_n + \mu_n(x)}$$

which is defined on the complement of some hyperplane in $L_1$ and takes values in $L'$ . It is now easy to see that on any compact subset $K$ of $L_1$ for $n$ sufficiently large $D_n$ is defined on $L_1$ and $\|D_n\| \longrightarrow 0$ , which implies our claim.

**3.11 Remark** Note the difference between $M_1(b)$ and $L_1(s)$ . If $b$ sends $P$ to one point $M_0(b)$ then $M_1(b) = \emptyset$ but $L_1(s)$ may be any hyperplane – as noted above. It thus is not true that $\mathbb{P}s$ converges uniformly to the constant map with value $M_0(b)$ on the complement of $M_1(b)$ , because that would imply that $\mathbb{P}B_n(P)$ is contained in a given neighbourhood of $M_0(b)$ for $n$ sufficiently large which is of course impossible, since $\mathbb{P}B_n$ is a bijection of $P$ .

We have not discussed the question yet if contractions exist. At this point we need the main algebraic result of [GM].

Let $H$ be a subsemigroup of $GL(V)$ . Then $V$ — considered as an $H$–module — is called *strongly irreducible* if there is no finite union $\bigcup W_j$ of linear subspaces that is invariant with respect to $H$ other than $0$ and $V$ . Let $G$ be the algebraic closure of $H$ and let $G_0$ be the connected component of the identity in $G$ with respect to the Euclidean or the Zariski topology, the following statement is true for both. Then $V$ is strongly irreducible iff $V$ considered as a $G_0$–module is irreducible in the usual sense of the word, as is easy to see, cf. [GM, lemma 6.2].

For our subsemigroup $H$ of $GL(V)$ , let $\overline{H}$ be the set of quasiprojective transformations which are pointwise limits of sequences of projective transformations induced by elements of $H$ . The main algebraic result of [GM] is the following

**3.12 Theorem** *Let $H$ be a strongly irreducible subsemigroup of $GL(V)$ . If $\overline{G}$ contains a contraction, so does $\overline{H}$ .*

This theorem is obtained from [GM Theorem 6.3] in view of [GM lemma 6.5 and the remark following that lemma].

The two concepts of proximality and contraction are of course related, as follows.

**3.13 Lemma** *If $h$ is a proximal element of $GL(V)$ then for the semigroup $H = \{h^n; n \in \mathbb{N}\}$ every quasiprojective transformation of $\overline{H}$ not in $H$ is a contraction. Conversely, if $H$ is an irreducible subsemigroup of $GL(V)$ and $\overline{H}$ contains a contraction, then $H$ contains a proximal element.*

**Proof** If $h$ is proximal then the sequence $h^n \cdot \lambda\,(h^n)^{-1}$ converges to the projection onto $V^{\circ}(h)$ in the decomposition $V = V^{\circ}(h) \oplus V^{<}(h)$ . Hence, any subsequence $s$ of $(h^n)_{n\in\mathbb{N}}$ for which $\mathbb{P}s$ converges pointwise on $P$ is a contractive sequence, hence its limit is a contraction by 3.7. Conversely, if $\overline{H}$ contains a contraction then there is a contractive sequence in $H$ , by lemma 3.7. We may assume that $M_0(s) \notin L_1(s)$ , by passing from $s$ to $h\,s$ for some $h \in H$ by 3.5 and 3.9 and using the hypothesis that $H$ is irreducible on $V$ . It now follows from Tits' criterion 2.1 that every element $B_n$ of the contractive sequence $s$ is proximal for $n$ sufficiently large.

## IV. The finiteness result

In this section we prove our basic finiteness result. Consequences will be discussed in the following sections.

Fix a norm $\|\cdot\|$ on $V$ .

**4.1 Theorem** *Let $H$ be a strongly irreducible subsemigroup of $GL(V)$ . Suppose the algebraic closure $G$ of $H$ contains a proximal element. Then there is an $\varepsilon > 0$ and for every $r \geq 1$ there is a finite subset $M$ of $H$ such that for every $g \in GL(V)$ there is a $\gamma \in M$ such that $\gamma g$ is $(r, \varepsilon)$–proximal.*

If we let $X_{r,\varepsilon}$ be the set of $(r, \varepsilon)$–proximal elements of $GL(V)$ , we thus have

$$M \cdot X_{r,\varepsilon} = GL(V)$$

We shall check proximality using Tits' criterion (lemma 2.1). We thus need for every $g \in GL(V)$ a uniform bound for the norm of $\mathbb{P}g$ away from some set of singularities $\mathbb{P}W_1(g)$ . This set will be determined using the Cartan decomposition, as follows:

**4.2 Proposition** *We can associate with every* $g \in GL(V)$ *a hyperplane* $W_1(g)$ *of* $V$ *with the following property. For every* $\varepsilon > 0$ *there is a constant* $c = c(\varepsilon)$ *such that for every* $g \in GL(V)$ *for the map* $\mathbb{P}g$ *induced on* $P$ *we have*

$$\| \, \mathbb{P}\, g \mid P \smallsetminus U\,(\mathbb{P}W_1(g), \varepsilon) \, \|_\rho \leq c \, .$$

Here $U(A, \varepsilon) = \{y \in P \; ; \; \rho(A, y) < \varepsilon\}$ is the $\varepsilon$–neighbourhood of a subset $A$ of $P$ with respect to a given admissible metric $\rho$ on $P$ .

**Proof** Since the claim holds for every admissible metric $\rho$ if it holds for one we may assume that $V = \mathbb{R}^{n+1}$ and $\rho$ is the angle metric (see formula before 3.4) with respect to the standard Euclidean inner product on $\mathbb{R}^{n+1}$ . Let $g = k' \cdot p \cdot k$ be a Cartan decomposition of $g \in GL(V)$ , so $k, k' \in O_{n+1}(\mathbb{R})$ and $p$ is a diagonal matrix with positive entries $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} > 0$ . Put $W_1(g) = k^{-1}\langle e_2, \ldots, e_{n+1}\rangle_{\mathbb{R}}$ . To prove the claim we may assume that $\lambda_1 = 1$ by passing from $g$ to $\lambda_1^{-1} \cdot g$ . Since $k$ and $k'$ induce isometries of $P$ we may assume furthermore that $g = p$ and $W := W_1(g) = \langle e_2, \ldots, e_{n+1}\rangle_{\mathbb{R}}$ . Let $A$ be the affine plane $x_1 = 1$ in $\mathbb{R}^{n+1}$ . The map $g \mid A$ maps $A$ to itself and has norm $\leq 1$ with respect to the Euclidean norm of $\mathbb{R}^{n+1}$ , since $\lambda_2, \ldots, \lambda_n \leq 1$ . Let $\pi : A \longrightarrow P$ be the natural projection $\pi(x) = \mathbb{P}(\mathbb{R}x)$ . Then $\pi$ induces a diffeomorphism $A \xrightarrow{\sim} P \smallsetminus \mathbb{P}W$ , whose inverse we call $\pi^{-1}$ . We have

$$\mathbb{P}g = \pi \circ g|A \circ \pi^{-1},$$

hence for every compact ball $K = \{x \; ; \; \|x\| \leq R\}$ in $A$ the norm of $\mathbb{P}g$ is bounded on $\pi(K)$ by a constant depending only on $K$ , not on $g$ , by 3.3 and since $\| \, g \mid A \, \| \leq 1$ . This implies our claim since for every $\varepsilon$ there is such a $K$ with $\mathbb{P} \smallsetminus U(\mathbb{P}W, \varepsilon) \subset \pi(K)$ .

Let us first give a rough sketch of the proof of 4.1. We start by constructing a sufficiently rich finite set $S$ of contractive sequences, see lemma 4.4. Then for a given element $g \in G$ we find an $s' \in S$ such that $M_0(s')$ is $\varepsilon$–distant from $W_1(g)$ . We then find an $s \in S$ such that $gM_0(s')$ is $\varepsilon$–distant from $L_1(s)$ . Thus $\gamma_n$ in $s$ takes $gM_0(s')$ close to $M_0(s)$ . The elements of $S$ are so chosen that $M_0(s)$ is $\varepsilon$–distant from $L_1(s')$ for every pair $(s, s')$ of elements of $S$. Hence $\gamma'_m \gamma_n g$ takes $M_0(s')$ close to $M_0(s')$. Taking norms into account we obtain the theorem.

**4.3 Lemma** *Let $A. = (A_n)_{n\in\mathbb{N}}$ and $B. = (B_n)_{n\in\mathbb{N}}$ be sequences in $GL(V)$ for which $\mathbb{P}A_. = (\mathbb{P}A_n)_{n\in\mathbb{N}}$ and $\mathbb{P}B.$ converge pointwise on $P$ to $a$ and $b$, respectively.*

*a) Then there is a subsequence $C.$ of $(A_nB_n)_{n\in\mathbb{N}}$ for which $\mathbb{P}C.$ converges to $ab$.*

*b) If furthermore the sequences $(A_n{\cdot}\|A_n\|^{-1})_{n\in\mathbb{N}}$ and $(B_n{\cdot}\|B_n\|^{-1})_{n\in\mathbb{N}}$ converge to linear maps $a_0$ and $b_0$, and $a_0b_0 \neq 0$ – equivalently, $M_0(b) \not\subset L_1(A.)$ – then $(A_nB_n\|A_nB_n\|^{-1})$ converges to a non–zero multiple of $a_0b_0$. Hence*

$$0 \neq M_0(ab) \subset M_0(a)$$

*and*

$$P \neq L_1(C.) = \ker(a_0b_0).$$

*c) If $A.$ is contractive and $M_0(b) \not\subset L_1(A.)$ then there is a contractive subsequence $C.$ of $(A_nB_n)_{n\in\mathbb{N}}$. For every such $C.$ we have*

$$\lim \mathbb{P}C. = ab,$$
$$M_0(C.) = M_0(ab) = M_0(a)$$

*and*

$$L_1(C.) = \ker a_0b_0$$

*d) Similarly with the roles of $A.$ and $B.$ interchanged: If $B.$ is contractive and $M_0(b) \not\subset L_1(A.)$ then there is a contractive subsequence $C.$ of $(A_nB_n)_{n\in\mathbb{N}}$. For every such $C.$ we have*

$$\lim \mathbb{P}C. = ab,$$
$$M_0(C.) = M_0(b) \in M_0(a)$$

*and*

$$L_1(C.) = \ker(a_0b_0) = \ker b_0.$$

**Proof** Part a) is proved in the proof of [GM lemma 2.7].

b) Since the sequence $s = (A_n B_n \|A_n B_n\|^{-1})$ is contained in the compact metric space of operators of norm $\leq 1$, it suffices to prove that every convergent subsequence of $s$ has limit $a_0 b_0$. We thus may assume that $s$ converges. Evaluating the convergent sequences $A_n B_n \|A_n\|^{-1} \|B_n\|^{-1}$ and $s$ at a vector $\notin \ker(a_0 b_0)$, we see that $\|A_n B_n\| \cdot \|A_n\|^{-1} \cdot \|B_n\|^{-1}$ converges to a number $\alpha$ with $0 < \alpha < 1$, which proves that $s$ converges to $\alpha^{-1} a_0 b_0$. The other claims of b) follow easily.

c) is a consequence of a) and b).

**4.4 Lemma** *Suppose a subsemigroup $H$ of $GL(V)$ is Zariski–connected and acts irreducibly on $V$. If $H$ contains a contractive sequence then there is a finite set $S$ of contractive sequences $s_1, \ldots, s_t$ in $H$ with the following properties:*

*1) The points $M_0(s_j)$ are distinct.*

*2) $M_0(s_i) \notin L_1(s_j)$ for $1 \leq i, j \leq t$.*

*3) $\bigcap L_1(s_j) = \emptyset$*

*4) The lines corresponding to $M_0(s_j)$, $j = 1, \ldots, t$, span $V$.*

*In fact we find such a set $S$ of cardinality* $\dim V$.

**Proof** We prove by induction on $m \leq \dim V$ that there are $m$ contractive sequences $s_1, \ldots, s_m$ in $H$ such that 1) and 2) hold for $i, j \leq m$,

$3_m$) $\operatorname{codim} \bigcap_{j=1}^{m} L_1(s_j) = m$ and

$4_m$) $\dim \bigvee_{j=1}^{m} M_0(s_j) = m - 1$.

The case $m = 0$ is trivial. So suppose we have $s_1, \ldots, s_m$ satisfying these conditions, $m < \dim V$. We know that there is a contractive sequence $s$ in $H$. There is an element $\gamma \in H$ such that

$$M_0(\gamma s) = \gamma M_0(s) \notin \bigvee_{j=1}^{m} M_0(s_j) \tag{4.5}$$

since $V$ is an irreducible $H$–module. Similarly for every $j \leq m$ there is a $\gamma \in H$ such that

$$M_0(\gamma s) = \gamma M_0(s) \notin L_1(s_j) \qquad (\ 4.6_j\ )$$

Every one of the conditions $(4.5)$ and $(4.6)_j$ defines a Zariski–open subset of $\gamma' s$ in $H$ , hence there is a $\gamma \in H$ satisfying all of them. A similar argument using 3.9 shows that there is an element $\beta \in H$ such that $s_1, \ldots, s_m, \gamma s \beta$ satisfy $1_{m+1}$ through $4_{m+1}$ .

**4.7 Proof of theorem 4.1** We work our way along the following sketch of proof. Let $S = \{s_1, \ldots, s_t\}$ be a sequence as in the preceding lemma. Fix $g \in GL(V)$ . For $\varepsilon$ sufficiently small there is an $s_i \in S$ such that $M_0(s_i)$ is $\varepsilon$–distant from $W_1(g)$ . Then there is an $s_j \in S$ such that $gM_0(s_i)$ is $\varepsilon$–distant from $L_1(s_j)$ . Now apply an element $\gamma_j$ in the contractive sequence $s_j$ which takes $gM_0(s_i)$ close to $M_0(s_j)$ . Since $M_0(s_j)$ is $\varepsilon$–distant from $L_1(s_i)$ a $\gamma_i$ sufficiently close to $s_i$ takes $\gamma_j g M_0(s_i)$ close to $M_0(s_i)$ . Finally $g$ takes $\gamma_i \gamma_j g M_0(s_i)$ back close to $gM_0(s_i)$ . It then follows that the set of maps $M = \{\gamma_i \gamma_j\}$ has the desired properties.

To get notations straight, look at the following picture:

$$\overset{\bullet}{gM_0(s_i)} \ \overset{\gamma_j}{\curvearrowright}\ \overset{\bullet}{M_0(s_j)} \ \overset{\gamma_i}{\curvearrowright}\ \overset{\bullet}{M_0(s_i)} \ \overset{g}{\curvearrowright}\ \overset{\bullet}{gM_0(s_i)}$$

Here $a \overset{\beta}{\curvearrowright} b$ means that $\beta$ maps $a$ close to $b$ .

So let $S = \{s_1, \ldots, s_t\}$ be as in the preceding lemma. Fix an admissible metric $\rho$ on $P$ . Then $\varepsilon_1 = \inf_{x \in P} \sum_{j=1} \rho(x, L_1(s_j))$ is positive by 3), hence for every point $x \in P$ we have $\rho(x, L_1(s_j)) > \frac{\varepsilon_1}{2t}$ for some $j$ . Similarly, for every hyperplane $L$ in $P$ we find a number $\varepsilon_2$ such that $\rho(L, M_0(s_j)) > \frac{\varepsilon_2}{2t}$ for some $j$ , using 4). So there is a number $\varepsilon > 0$ with the following properties. For every point $x \in P$ we have

$$\rho(x, L_1(s_j)) > 2\varepsilon \text{ for some } j. \qquad \text{(a)}$$

For every hyperplane $L$ in $P$ we have

$$\rho(L, M_0(s_j)) > 2\varepsilon \qquad \text{for some } j. \tag{b}$$

$$\rho(M_0(s_i), L_1(s_j)) > 2\varepsilon \quad \text{for } 1 \leq i, j \leq t. \tag{c}$$

So for a given $g \in GL(V)$ there are numbers $1 \leq i, j \leq t$ , such that

$$\rho(M_0(s_i), W_1(g)) > 2\varepsilon \tag{d}$$

and

$$\rho(gM_0(s_i), L_1(s_j)) > 2\varepsilon. \tag{e}$$

Hence

$$\|g \mid U(M_0(s_i), \varepsilon)\|_\rho \leq c \tag{f}$$

where $c = c(\varepsilon) \geq 1$ is the constant of proposition 4.2, hence

$$g(U(M_0(s_i), \varepsilon c^{-1}) \subset U(gM_0(s_i), \varepsilon). \tag{g}$$

Now choose for every $s_k \in S$ an element $\gamma_k$ of the sequence $s_k$ such that

$$\gamma_k(P \setminus U(L_1(s_k), \varepsilon)) \subset U(M_0(s_k), \tfrac{1}{2}\varepsilon c^{-1}) \tag{h}$$

and

$$\|\gamma_k \mid P \setminus U(L_1(s_k), \varepsilon)\|_\rho \leq \delta \tag{i}$$

where $\delta > 0$ will be fixed later on. Then

$$\gamma_j U(gM_0(s_i), \varepsilon) \subset U(M_0(s_j), \tfrac{1}{2}\varepsilon c^{-1}) \tag{j}$$

and

$$\gamma_i \gamma_j U(gM_0(s_i), \varepsilon c^{-1}) \subset U(M_0(s_i), \tfrac{1}{2}\varepsilon c^{-1}) \tag{k}$$

by (e), $(\mathrm{h})_j$ , (c) and $(\mathrm{h})_i$ , hence

$$g\gamma_i\gamma_j U(gM_0(s_i),\varepsilon) \subset U(gM_0(s_i),\tfrac{\varepsilon}{2}) \tag{l}$$

by (f) and

$$\|g\gamma_i\gamma_j \mid U(gM_0(s_i),\varepsilon)\|_\rho \leq c\cdot\delta^2 \tag{m}$$

by (i) and (f). Hence if we choose $\delta$ sufficiently small depending on $r$ the element $g\gamma_i\gamma_j$ is $(r,\varepsilon)$–proximal by Tits' criterion 2.1.

**4.8 Addendum** *Note that – independent of $r$ – a set $M$ of $(\dim V)^2$ elements suffices.*

Since in 4.4 we can take $t \leq \dim V$ and in the proof above we found $\#M = t^2$ . One cannot hope that a set $M$ with fewer than $(\dim V)^2$ elements will work since the condition that " $\gamma g$ is proximal" corresponds to " $gV^+(\gamma)$ is not near $V^<(\gamma)$ " and " $gV^+(\gamma) \in V^<(\gamma)$ " is a codimension one condition on $g$ .

The following sharpening of the theorem may be useful (see [PR] lemma 3.7 ff.)

**4.9 Addendum** *In the theorem we can replace " $\gamma g$ is $(r,\varepsilon)$ -proximal" by " $\gamma^n g$ is $(r,\varepsilon)$ -proximal for every natural number $n$ ".*

**Proof** The map $\gamma = \gamma_i\gamma_j$ constructed in the proof sends $U := U(M_0(s_i),\frac{1}{2}\varepsilon c^{-1})$ to itself by $(e)$ and $(h)$ and has norm $\|\gamma \mid U\|_\rho \leq \delta^2$ by $(e)$ and $(i)$ , hence we have for $n\in\mathbb{N}$ the inclusion $\gamma^n V\subset U$ , where $V := U(gM_0(s_i),\varepsilon c^{-1})$ , as in $(k)$ , and hence $g\gamma^n V \subset U$ as in $(l)$ and $\|g\gamma^n \mid V\|_\rho \leq c.\delta^{2n}$ as in $(m)$ .

Recall the following result of the same sort: If there is one proximal element, there are many.

**4.10 Theorem** [Ti; Ma 4, Appendix B] *Let $F$ be a subgroup of $GL(W)$ which is connected in the Zariski topology. Suppose $W$ is irreducible as a $F$–module. If $F$ contains a proximal element the set $X = \{x\in F; x \text{ and } x^{-1} \text{ are proximal}\}$ is Zariski dense in $F$ .*

We here obtain an analogous result for semigroups.

**4.11 Corollary** *Let $H$ be a subsemigroup of $GL(V)$ which is connected with respect to the Zariski topology. Suppose $V$ is irreducible as an $H$–module. If $H$ contains a proximal element – hence if the Zariski closure of $H$ contains a proximal element – then the set of proximal elements of $H$ is Zariski dense.*

**Proof** Let $P$ be the set of $(r,\varepsilon)$–proximal elements of $H$, where $(r,\varepsilon)$ is as in theorem 4.1. For every $h \in H$ there is an element $\gamma$ of the finite subset $M$ of $H$ of theorem 4.1 such that $\gamma h \in P$. So $H$ is contained in the finite union $\gamma^{-1}P$, $\gamma \in M$. Let $P$ be contained in an algebraic set $A$. Then $H \subset \bigcup_{\gamma \in M} \gamma^{-1}A$ hence $H$ is contained in $\gamma^{-1}A$ for one $\gamma \in M$ since $H$ is Zariski connected. But $\gamma H \subset A$ implies that for the Zariski closure $G$ of $H$ – a group – we have $G = \gamma G \subset A$, q.e.d.

## V. The Lyapunov filtration

In this section we extend our finiteness result for proximal maps to other types of Lyapunov filtrations and to direct sums of strongly irreducible representations.

Let $(V, \|\cdot\|)$ be a normed vector space over a local field $k$ of finite dimension and let $g$ be a linear endomorphism of $V$. For every vector $v \in V$ define the *Lyapunov exponent*

$$\alpha_g(v) := \limsup_{n\to\infty} \; n^{-1} \log \|g^n(v)\|. \tag{5.1}$$

In the terminology of [GM] this is the "Lyapunov index of the sequence $g^n, n \in \mathbb{N}$" in case $k = \mathbb{R}$. The Lyapunov exponent is independent of the norm chosen. It has the following formal properties of an

**5.2 "Ultrametric norm over the trivially valued field $k$"** *For $v, w \in V$ and $\lambda \neq 0$ in $k$ we have*

$$\alpha_g(\lambda v) = \alpha_g(v), \tag{a}$$

$$\alpha_g(v + w) \leq \max(\alpha_g(v), \alpha_g(w)), \tag{b}$$

$$\alpha_g(0) = -\infty. \tag{c}$$

**5.3** For $r \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ define

$$V_{\leq r} = \{v \in V; \alpha_g(v) \leq r\}$$

and

$$V_{<r} = \{v \in V; \alpha_g(v) < r\}.$$

These are vector subspaces of $V$ by the ultrametric properties of $\alpha_g$. The *Lyapunov filtration* of $V$ with respect to $g$ associates with every $r \in \overline{\mathbb{R}}$ the subspace $V_{\leq r}$ of $V$.

For $\lambda \in k$ let

$$V_\lambda = \{v \in V; (g - \lambda I)^n v = 0 \text{ for } n \gg 0\}$$

be the *weight space* of $g$ corresponding to $\lambda \in k$. Then the Jordan normal form implies

**5.4** *For $0 \neq v \in V_\lambda$ we have*

$$\alpha_g(v) = \log|\lambda|$$

*where $|\cdot|$ is the valuation of $k$. In fact, $n^{-1} \log \|g^n(v)\|$ converges to $\log|\lambda|$.*

Let $k \subset K$ be a finite field extension of local fields. Then in the situation above, i.e. for a $k$–endomorphism of the $k$–vector space $V$, consider the $k$–linear extension $g_K$ of $g$ to $V \otimes_k K$. Then

$$(V \otimes K)_{\leq r} = V_{\leq r} \otimes K \tag{5.5}$$

**Proof** For a vector in $V \otimes K$ the Lyapunov exponent of $g_K$ is the same as if we regard $g_K$ as a $k$–linear map, restricting scalars. Now choose a basis $e_1, \dots, e_m$ of $K$ over $k$ and take for $V \otimes K = \oplus_i V \otimes e_i$ the $k$–norm $\|\Sigma v_i \otimes e_i\| = \max_i \|v_i\|$ to see (5.5).

Combining 5.4 and 5.5 we get an algebraic description of the Lyapunov filtration.

**5.6 Corollary** *Let $K \supset k$ be a finite extension of local fields such that $K$ contains all the eigenvalues of $g \in End(V)$ . Then*

$$(V \otimes K)_{\leq r} = \bigoplus_{\log|\lambda| \leq r} (V \otimes K)_\lambda\,,$$
$$V_{\leq r} = (V \otimes K)_{\leq r} \cap V$$

*and $(V \otimes K)_{\leq r} = V_{\leq r} \otimes K$ . In particular, $V_{<r} \neq V_{\leq r}$ iff $r = \log|\lambda|$ for some eigenvalue $\lambda$ of $g$ .*

We can improve on the last remark. The Lyapunov filtration comes in fact from a graduation, as follows (cf. [Ma 4, II. 1]).

**5.7 Proposition** *For every $r \in \overline{\mathbb{R}}$ there is a unique $g$–invariant subspace $V_{=r}$ of $V_{\leq r}$ complementary to $V_{<r}$ . We have*

$$V_{=r} = V \cap \left( \bigoplus_{\log|\lambda| = r} (V \otimes K)_\lambda \right)$$

*in the situation of 5.6*

**Proof** $V_{\leq -\infty} = \{v \in V; g^n v = 0 \text{ for some } n \gg 0\}$ by 5.6. Here $n = \dim V$ suffices. Let $W = g^n V$ . Then $g$ induces an invertible map on $W$ . If follows easily that $W$ is the unique $g$–invariant complementary subspace of $V_{\leq -\infty}$ . We thus may assume that $g \in \mathrm{Aut}(V)$ . Then

$$V_{=r}(g) := V_{\leq r}(g) \cap V_{\leq -r}(g^{-1})$$

is the desired space, since this is true if our field contains all the eigenvalues of $g$, by 5.4, and follows in general by the descent result 5.6. Then the proof implies that

$$V_{=r}(g) = \{v \in V; \alpha_g(v) = r \text{ and } \alpha_{g^{-1}}(v) = -r\} \cup \{0\}. \tag{5.8}$$

for $g \in GL(V)$. The indexed collection of subspaces $V_{=r}$, $r \in \mathbb{R}$, is called the *Lyapunov graduation* of $V$ with respect to $g$ and $\dim V_{=r}$ is called the *multiplicity of the exponent* $r$. We have

$$V_{\leq r} = \bigoplus_{s \leq r} V_{=s}. \tag{5.9}$$

Let us order the different values of $\alpha_g$ on $V$

$$\alpha_0 = -\infty < \alpha_1 < \alpha_2 < \cdots < \alpha_t$$

and let

$$k_0, k_1, k_2, \ldots, k_t$$

be the corresponding multiplicities. Then $g$ is invertible iff $k_0 = 0$.

**5.10 Remark** *The linear automorphism $g \in GL(V)$ is proximal iff the multiplicity $k_t$ of the maximal Lyapunov exponent is $1$.*

**5.11. Corollary** *For $g \in GL(V)$ the induced map $\wedge^m g$ on the exterior powers $\wedge^m V$ is proximal iff $m$ is one of the numbers $k_r$, $k_r + k_{r-1}, \ldots, k_r + \cdots + k_1$.*

**Proof** Write the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $g$ in some local overfield $K$ of $k$ – every eigenvalue as often as its multiplicity indicates – in the following way: First the $k_1$ eigenvalues of modulus $\exp \alpha_1$, then the $k_2$ eigenvalues of modulus $\exp \alpha_2, \ldots$. The eigenvalues of $\wedge^m g$ are $\lambda_{i_1} \ldots \lambda_{i_m}$, $i_1 < \cdots < i_m$. There is a unique eigenvalue – counting multiplicities – of maximal modulus iff $m$ is one of the numbers $k_r$, $k_r + k_{r-1}, \ldots$.

**5.12** We define the *Lyapunov type* $L(g) = (e_0 < e_1 < \cdots < e_t)$ of $g \in GL(V)$ as the sequence of dimensions of the spaces $V_{\leq r}$, $r \in \mathbb{R}$. So $L(g) = (k_0, k_0 +$

$k_1, k_0+k_1+k_2, \ldots, k_0+k_1+\cdots+k_t = n)$ where $k_i$ is the multiplicity of the $i$–th Lyapunov exponent $\alpha_i$ and $n = \dim V$ . So $g \in GL(V)$ is proximal iff $n-1 \in L(g)$ . The following statements for $g \in GL(V)$ are equivalent: $e \in L(g)$ ; $\wedge^{n-e} g$ is proximal; if we order the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $g$ in some local overfield $K$ of $k$ according to their modulus then $|\lambda_e| < |\lambda_{e+1}|$ .

**5.13** As with proximality we need a quantitative version of the Lyapunov type. Fix a norm on $\wedge^m V$ for every $m = 0, \ldots, n$ . Let $\tau = (e_0, e_1, \ldots, e_s)$ be a *type of flag* i.e. an increasing sequence of integers $e_i$ with $0 \leq e_i \leq n$ . Define $P(\tau)$ as the set of $g \in GL(V)$ such that $L(g) \subset \tau$ and for $r > 1$ and $\varepsilon > 0$ define $P(\tau, r, \varepsilon)$ as the set of $g \in GL(V)$ such that $\wedge^m g$ is $(r, \varepsilon)$–proximal for every $m = n - e$ , $e \in \tau$ . If $\tau$ consists of one element only, say $\tau = (e)$ , then we write $P(e)$ for $P(\tau)$ and $P(e, r, \varepsilon)$ for $P(\tau, r, \varepsilon)$ .

Here is now our generalization of the finiteness result. It concerns the Lyapunov type, whereas the earlier result was concerned with proximality only.

**5.14 Theorem** *Let $H$ be a subsemigroup of $GL(V)$ and let $\tau$ be a type of flag. Suppose $P(e)$ intersects $H$ for every $e \in \tau$ . If the representation of $H$ on $\wedge^{n-e} V$ is strongly irreducible for every $e \in \tau$ then $P(\tau)$ intersects $H$ . Moreover, there is an $\varepsilon > 0$ such that for every $r > 1$ there is a finite subset $M$ of $H$ such that $M.P(\tau, r, \varepsilon) = GL(V)$ .*

The proof is implied by the following results. The proof of the first claim follows from the following lemma applied to $\bigoplus_{e \in \tau} \wedge^{n-e} V$ in view of the remark following the definition of $(r, \varepsilon)$–proximality in section two.

**5.15 Lemma** *Suppose a representation $\rho : H \longrightarrow GL(V)$ decomposes into strongly irreducible subrepresentations $\rho = \rho_1 \oplus \cdots \oplus \rho_t$ . If $\rho_i(H)$ contains a proximal element for every $i = 1, \ldots, t$ , then there is an element $h \in H$ which is proximal for every $i = 1, \ldots, t$ .*

**Proof** For every $i$ we have the representation $\rho_i : H \longrightarrow GL(V_i)$ , the subsemigroup $H_i = \rho_i(H)$ of $GL(V_i)$ and the corresponding semigroup $\overline{H_i}$ of quasiprojective maps of $\mathbb{P}V_i$ . The set of all pointwise limits of $\rho(H)$ in $\overline{H}_1 \times \cdots \times \overline{H}_t$ forms a subsemigroup $\overline{H}$ of $\overline{H}_1 \times \cdots \times \overline{H}_t$ , the proof is as for the case of one factor [GM, Lemma 2.7]. For every $i$ there is an element $h_i \in H$ for which $\rho_i(h_i)$ is proximal. Hence there is a contractive sequence $\rho_i(s_i)$ on $V_i$ with $M_0^i := M_0(\rho_i(s_i))$ consisting of one point and $L_1^i \cap M_0^i = \emptyset$ where $L_1^i = L_1(\rho_i(s_i))$ . Now let $a = (a_1, \ldots, a_t) \in \overline{H}$ be such that $\Sigma \dim M_0(a_i)$ is minimal. We claim that this sum is zero. This implies the lemma as in 3.13.

If a linear automorphism $g$ is proximal, so are all its powers $g^n, n \geq 1$ . We may assume that $H$ is a subsemigroup of $GL(V)$ and hence assume that $H$ is Zariski–connected.

Suppose that $\Sigma \dim M_0(a_i)$ is positive, say $\dim M_0(a_1) > 0$ . We shall compose $a$ with an element $b = (b_1, \ldots, b_t) \in \overline{H}$ such that

$$M_0(a_i b_i) \subset M_0(a_i) \text{ for every } i$$

with strict inclusion for $i = 1$ . It suffices to find a $b \in \overline{H}$ such that $b_1$ is a contraction and

$$M_0(b_i) \not\subset L_1(a_i) \text{ for every } i,$$

by 4.3 b) and 4.3 d). There is a $b \in \overline{H}$ such that $b_1$ is a contraction, since $\rho_1(H)$ contains a proximal element, by 3.13. The subspace $L_1(a_i)$ is a proper subspace of $\mathbb{P}(V_i)$ . So by irreducibility of $\rho_i$ there is an element $h \in H$ such that $M_0(hb_i) = hM_0(b_i) \not\subset L_1(a_i)$ . This is a Zariski–open condition. So there is an $h \in H$ such that these conditions are satisfied for every $i = 1, \ldots, t$ , by Zariski–connectedness of $H$ . So $hb$ has all the required properties and we thus get a contradiction, proving that $\Sigma \dim M_0(b_i) = 0$ .

Under the assumption of 5.14, let $S$ be a finite set of sequences in $H$ such that $\rho_i(s)$ is contractive for every $i = 1, \ldots, t$ . Put $M_i^0(S) = \bigvee_{s \in S} M^0(\rho_i(s))$ and $L_i^1(S) = \bigcap_{s \in S} L_1(\rho_i(s))$ and let $\widetilde{M_i^0}(S)$ be the linear subspace of $V_i$ corresponding to $M_i^0(S)$ . We say that $S$ is in *general position* if the following condition holds for every $i = 1, \ldots, t$ .

(GP) *For every pair* $(S', S'')$ *of subsets of* $S$ *we have*

$$\dim \widetilde{M}_i^0(S') = \inf(\mathit{card}\, S', \dim V_i),$$
$$\mathit{codim}\, L_i^1(S'') = \inf(\mathit{card}\, S'', \dim V_i)$$

*and*

$$\dim(\widetilde{M}_i^0(S') + L_i^1(S'')) = \inf(\dim V_i, \dim \widetilde{M}_i^0(S') + \dim L_i^1(S'')).$$

**5.16 Lemma** *Suppose the hypotheses of 5.15 hold and* $H$ *is Zariski–connected. Then for every finite set* $S$ *as above in general position there is a sequence* $s$ *such that* $S \cup \{s\}$ *is in general position.*

**Proof** Recall that two subspaces $A$ and $B$ of a vector space $U$ are called *transversal*, denoted $A \pitchfork B$, if $A \cap B = 0$ or $A + B = U$. So the last condition of general position is equivalent to $\widetilde{M}_i^0(S') \pitchfork L_i^1(S'')$. Now there is a sequence $s$ in $H$ such that $\rho_i(s)$ is contractive for every $i = 1, \ldots, t$, by 5.15 and 3.13. For every pair $S'$, $S''$ of subsets of $S$ there is a $\gamma \in H$ such that

$$\widetilde{M}_i^0(\gamma s) = \gamma M_i^0(s) \pitchfork \widetilde{M}_i^0(S') + L_i^1(S''), \tag{a}$$

by irreducibility of $\rho_i$. So there is a $\gamma \in H$ such that (a) holds for every pair $S'$, $S''$ of subsets of $S$. Similarly, there is a $\beta \in H$ such that

$$L_i^1(s\beta) = \beta^{-1} L_i^1(s) \pitchfork \widetilde{M}_i^0(S') \cap L_i^1(S''), \tag{b}$$

and, if $\widetilde{M}_i^0(S') \cap L_i^1(S'') = 0$,

$$L_i^1(s\beta) = \beta^{-1} L_i^1(s) \pitchfork \widetilde{M}_i^0(S' \cup \{\gamma s\}) \cap L_i^1(S'') \tag{c}$$

It then follows that $S \cup \{\gamma s \beta\}$ is in general position, the case $(S' \cup \{\gamma s\beta\}, S'')$ by a), the case $(S', S'' \cup \{\gamma s \beta\})$ by b), and the case $(S' \cup \{\gamma s\beta\}, S'' \cup \{\gamma s\beta\})$ by a), b) and c).

We are now ready to prove our theorem.

**5.17 Theorem** *Let $H$ be a semigroup and let $\rho : H \to GL(V)$ be a representation. Suppose $\rho$ decomposes into strongly irreducible representations $\rho = \rho_1 \oplus \cdots \oplus \rho_t$ . Assume $H$ contains for every $i$ an element $h_i$ such that $\rho_i(h_i)$ is proximal. Then for a given norm on $V$ there is an $\varepsilon > 0$ and for every $r > 1$ a finite subset $M$ of $H$ such that for every $g = (g_1, \ldots, g_t) \in \prod GL(V_i)$ there is a $\gamma \in M$ such that $\rho_i(\gamma) \cdot g_i$ is proximal for every $i$ .*

**Proof** We just give a sketch of proof, since the details are very similar to those of 4.7. We may assume that $\rho(H)$ is Zariski–connected. Let $S$ be a set in general position of cardinality greater than $\sum_{i=1}^{t}(d_i - 1)$ , where $d_i = \dim V_i$ . For a given set $H_1, H_2, \ldots, H_t$ of hyperplanes $H_i$ in $V_i$ there are at most $d_i - 1$ elements $s \in S$ such that $M_i^0(s) \in H_i$ , hence there is an element $s \in S$ such that $M_i^0(s) \notin H_i$ for every $i$ . Similarly, for a given set $x_1, \ldots, x_t$ of points $x_i \in \mathbb{P}V_i$ there is an element $s \in S$ such that $\mathbb{P}L_i^1(s) \not\ni x_i$ for every $i$ . Now fix $g = (g_1, \ldots, g_t) \in \prod_i GL(V_i)$ . For $\varepsilon$ sufficiently small there is an $s \in S$ such that $M_i^0(s)$ is $\varepsilon$–distant from $W_1(g_i)$ for every $i$ . Then there is an $s' \in S$ such that $g_i M_i^0(s)$ is $\varepsilon$–distant from $L_i^1(s')$ . Now apply an element $\gamma'$ in the sequence $s'$ which takes $g_i M_i^0(s)$ close to $M_i^0(s')$ . In fact we want a compact subset $K_{i'}$ of the complement of $L_i^1(s')$ taken to a small neighbourhood of $M_i^0(s')$ and the norm of $\gamma'$ on $K_{i'}$ small. Now $M_i^0(s')$ is $\varepsilon$–distant from $L_i^1(s)$ , by the third condition of general position. So a properly chosen element $\gamma$ of the sequence $s$ takes a compact subset $K_i$ of the complement of $L_i^1(s)$ – and in particular $M_i^0(s')$ – close to $M_i^0(s)$ and has small norm on $K_i$ . Finally $g_i$ takes $\gamma\gamma' M_i^0(s)$ back close to $g_i M_i^0(s)$ and there is a universal bound for the norm of $g_i$ outside a neighbourhood of $W_1(g_i)$ for every $i$ . We thus prove our claim using Tits' criterion.

**5.18 Remark** The proof shows that there is such a set $M$ of cardinality $(1 + \sum_{i=1}^{t}(d_i - 1))^2$ .

**5.19 Remark** There is a similar theorem for a direct sum of representations $\rho_i$ and types $\tau_i$ of flags for every $i$ . The proof follows from 5.17.

## VI. Proximal elements in reductive groups

In this section we give necessary and sufficient conditions for when the image of a reductive group under an irreducible representation $\rho$ contains a proximal linear map. The conditions are given in terms of the highest weight of $\rho$ . At the beginning of this section we state in which sense we can restrict attention to irreducible representations. At the end of the section we apply our main finiteness result to obtain many $\mathbb{R}$ -regular elements.

**6.1** Let $H$ be a subsemigroup of $GL(V)$ . Suppose $H$ contains a proximal element $g$ with corresponding line $V^+$ and hyperplane $V^<$ . Let $W'$ be the maximal $H$-submodule of $V$ contained in $V^<$ . So $W' = \bigcap_{h\in H} hV^<$ . Also

$$W' = \{v \in V \ ; \ \alpha_g(v) < \text{ spectral radius of } g' \text{ for every } g' \text{ of the form}$$
$$g' = h\,g\,h^{-1}, h \in H\}.$$

Let $W$ be the smallest $H$-submodule of $V/W'$ containing the image of $V^+$ . *Then* $W_{\mathbb{C}}$ *is the unique minimal* $H$*-submodule of* $(V/W')_{\mathbb{C}}$ *. In particular, the corresponding representation* $\rho : H \longrightarrow GL(W)$ *is absolutely irreducible and the Zariski closure of* $\rho(H)$ *is a reductive subgroup of* $GL(W)$ .

**Proof** We may assume that $W' = 0$ . Then for every non–zero vector $v \in V_{\mathbb{C}}$ there is an element $h \in H$ such that $hv \notin (V^<)_{\mathbb{C}}$ . Hence $\frac{g^{2n}hv}{\|g^{2n}hv\|}$ converges for $n \to \infty$ to a non–zero vector of $(V^+)_{\mathbb{C}}$ , for some norm $\|.\|$ on $V_{\mathbb{C}}$ . So $(V^+)_{\mathbb{C}}$ is contained in every non–zero $H$-submodule of $V_{\mathbb{C}}$ .

If the Zariski closure of $H$ is reductive we obtain

**6.2 Proposition** *Let* $H$ *be a subsemigroup of* $GL(V)$ *whose Zariski closure* $\widetilde{H}$ *is reductive. Suppose* $\widetilde{H}$ *contains a proximal element with corresponding line* $V^+$ *and hyperplane* $V^<$ *. Let* $W$ *be the smallest* $H$ *submodule of* $V$ *containing* $V^+$ *and let* $W'$ *be the maximal* $H$*-submodule contained in* $V^<$ *. Then* $V = W \oplus W'$ *and* $W$ *is an absolutely irreducible* $H$*-submodule of* $V$ *.*

**Proof** In the claims we may suppose $H = \widetilde{H}$ , in which case everything follows from 6.1.

Let $G$ be a reductive linear algebraic group defined over $\mathbb{R}$ and let $\rho : G \longrightarrow GL(V)$ be an absolutely irreducible representation of $G$ defined over $\mathbb{R}$. We want to give criteria for when $\rho(G_{\mathbb{R}})$ contains a proximal element. Since the center of $G$ acts by homotheties on $V_{\mathbb{C}}$ it suffices to consider the case when $G$ is semisimple.

Recall the theory of *dominant weights* of representations of reductive groups. Let $G$ be a complex linear algebraic semi–simple group and let $B$ be a Borel subgroup of $G$ containing a maximal torus $T$ of $G$. Then for every irreducible representation $\rho : G \to GL(V)$ defined over $\mathbb{C}$ the restriction of $\rho$ to $T$ decomposes into weight spaces $V^{\chi}$, $\chi \in X^*(T) = \mathrm{Hom}(T, GL_1)$. Let $\Pi(\rho)$ be the set of weights of the representation $\rho$, i.e. $\Pi(\rho) = \{\chi \in X^*(T), V^{\chi} \neq 0\}$. Let $\leq$ be an order on $X^*(T)$ for which the roots of $T$ on $B$ are positive. There is a unique maximal weight $\lambda$ with respect to $\leq$ and the corresponding weight space $V^{\lambda}$ is one–dimensional. The line $V^{\lambda}$ can also be characterized as the unique line in $V$ stabilized by $B$. Let $\Phi$ be the root system of $G$ with respect to $T$ and let $\Phi^+$ be the set of roots of $T$ on $B$. Let $\Delta$ be the unique basis of the root system $\Phi$ contained in $\Phi^+$. Then every weight $\chi \in \Pi(\rho)$ is of the form

$$\chi = \lambda - \sum_{a \in \Delta} n_a \cdot a \quad \text{with} \quad n_a \geq 0.$$

Let us now turn to the real case. Suppose our group $G$, the vector space $V$ and the representation $\rho$ are defined over $\mathbb{R}$, that $T$ contains a maximal $\mathbb{R}$-split torus $S$ of $G$ and that $B$ is contained in a minimal parabolic $\mathbb{R}$-subgroup $P$ of $G$. Let ${}_{\mathbb{R}}\Delta$ be the basis of the root system ${}_{\mathbb{R}}\Phi$ of $S$ on $G$ which is contained in the set ${}_{\mathbb{R}}\Phi^+$ of roots of $S$ on $P$. Let $j : X^*(T) \longrightarrow X^*(S)$ be the restriction homomorphism and let $\Delta^{\circ} = \{a \in \Delta; j(a) = 0\}$. We sometimes write $\Pi(\lambda)$ instead of $\Pi(\rho)$.

**6.3 Theorem** *The following conditions are equivalent*

1) $\rho(G_{\mathbb{R}})$ *contains a proximal element.*

2) *The multiplicity of* $j(\lambda) = \lambda|S$ *is one.*

3) *There is no weight* $\chi \in \Pi(\lambda)$ *such that* $\lambda - \chi$ *is a linear combination of elements of* $\Delta^{\circ}$.

4) $\lambda$ *is orthogonal to* $\Delta^{\circ}$.

5) $\lambda$ *is fixed by the subgroup of the Weyl group* $W = W(T, G)$ *generated by the reflections* $r_a$ *corresponding to elements* $a \in \Delta^\circ$ .

6) $V^\lambda$ *is stabilized by* $P$ .

*In particular,* $\rho(G_\mathbb{R})$ *contains a proximal element if* $G$ *is* $\mathbb{R}$ *-split,* i.e. if $S = T$ , by 2).

The proof will occupy most of the remainder of this section.

Recall that every element $g$ of $GL(V)$ has a *multiplicative Jordan decomposition* $g = s \cdot u = u \cdot s$ which is uniquely determined by the conditions that $s$ is semisimple and $u$ is unipotent. Every semisimple element $s \in GL(V)$ has a *polar decomposition* $g = p \cdot k = k \cdot p$ uniquely determined by the following conditions: $p$ and $k$ are semisimple, the eigenvalues of $p$ are positive real and the eigenvalues of $k$ have modulus one. The following lemma is obvious.

**6.4 Lemma** *If* $g \in GL(V)$ *is proximal with corresponding line* $V^+$ *and hyperplane* $V^<$ *then so are its semisimple part* $s$ *and its polar part* $p$ *, with the same line and hyperplane. Furthermore, the moduli of the eigenvalues on* $V^+$ *and* $V^<$ *remain unchanged when passing from* $g$ *to* $s$ *to* $p$ .

**Proof of the theorem** 1) $\Longrightarrow$ 2) Suppose $g$ is a proximal element of $\rho(G_\mathbb{R})$ . We may suppose that $g$ is an element of the (Euclidean-) connected component $(\rho(G_\mathbb{R}))^\circ$ of $\rho(G_\mathbb{R})$ which is the same as $\rho((G_\mathbb{R})^\circ)$ , see [B] 7.4. It follows that the polar part of $g$ is also proximal which is of the form $\rho(p)$ for some polar element $p \in G_\mathbb{R}$ . Since any polar element is contained in a maximal $\mathbb{R}$ -split torus and maximal $\mathbb{R}$ -split tori are conjugate by elements of $G_\mathbb{R}$ we may assume that $p \in (S_\mathbb{R})^\circ$ . Using the action of the relative Weyl group ${}_\mathbb{R}W = N(S)/Z(S) = N(S)_\mathbb{R}/Z(S)_\mathbb{R}$ of $G$ over $\mathbb{R}$ we may assume that $p$ is actually in the Weyl chamber $C = \{s \in (S_\mathbb{R})^\circ ; b(s) \geq 1 \text{ for } b \in {}_\mathbb{R}\Delta\}$ . But

$${}_\mathbb{R}\Delta = j(\Delta \setminus \Delta^\circ)$$

by [BT] 6.3 and 6.8. So $a(p)$ is positive real for every $a \in X^*(T)$ and $\lambda(p) \geq \chi(p)$ for every $\chi \in \Pi(\lambda)$ . Since $\rho(p)$ is proximal we must have $\lambda(p) > \chi(p)$ for every $\chi \in \Pi(\lambda)$ other than $\lambda$ . In particular, the complex line $V^\lambda$ is the weight space for $j(\lambda) = \lambda|S$ in the decomposition of the complex vector space $V$ into weight spaces with respect to $S$ .

2) $\Longrightarrow$ 3) Write $\chi \in \Pi(\lambda)$ in the form

$$\chi = \lambda - \sum_{a \in \Delta} n_a \cdot a.$$

Then $n_a$ are non–negative integers. The dominant weight has multiplicity $1$ . For the restriction $j(\chi)$ of $\chi$ to $S$ we thus have

$$j(\chi) = j(\lambda) = \sum_{a \in \Delta} n_a j(a).$$

Now $j(\Delta) \subset {}_{\mathbb{R}}\Delta \cup \{0\}$ , hence

$$j(\chi) = j(\lambda) - \sum_{b \in {}_{\mathbb{R}}\Delta} k(b) \cdot b, \tag{6.5}$$

where $k(b) = \sum_{j(a)=b} n(a)$ .

3) $\Longrightarrow$ 1) Again, let $C \subset (S_{\mathbb{R}})^\circ$ be the Weyl chamber $\{s \in (S_{\mathbb{R}})^\circ, b(s) \geq 1 \text{ for } b \in {}_{\mathbb{R}}\Delta\}$ . Then the computation above shows that for $s$ in the interior $\overset{\circ}{C}$ of $C$ , i.e. if $s \in (S_{\mathbb{R}})^\circ$ and $b(s) > 1$ for every $b \in {}_{\mathbb{R}}\Delta$ , we have $\chi(s) < \lambda(s)$ for $\chi \neq \lambda$ in $\Pi(\lambda)$ , provided $\lambda - \chi$ is not a linear combination of $\Delta^\circ$ . Hence $\rho(s)$ is proximal for $s \in \overset{\circ}{C}$ .

3) $\Longleftrightarrow$ 4) There is a criterion due to Satake for $\mathbb{R}$ and $\mathbb{C}$ and for arbitrary fields of characteristic zero to Borel–Tits (see [BT] 12.16 and the reference there) as to when for a given irreducible representation with dominant weight $\lambda$ a subset $\Theta$ of $\Delta$ is of the form $\Theta(q) = \{a \in \Delta; n_a \neq 0\}$ for some weight $q = \lambda - \sum_{a \in \Delta} n_a \cdot a$ , as follows: A subset $\Theta$ of $\Delta$ is of the form $\Theta(q)$ iff $\{\lambda\} \cup \Theta$ is connected. Here connectedness is defined using a graph defined like the Dynkin diagram. So there is a subset $\Theta$ of $\Delta^\circ$ which is of the form $\Theta(q)$ for some $q \in \Pi(\lambda)$ iff the pair $(\lambda, a)$ is an edge for at least one $a \in \Delta^\circ$ , i.e. iff $\lambda$ is not orthogonal to at least one $a \in \Delta^\circ$ . This shows the equivalence of 3) and 4) .

4) $\Longleftrightarrow$ 5) $\Longleftrightarrow$ 6) Let $W'$ be the subgroup of the Weyl group $W(T,G)$ in 5), i.e. generated by $r_a$ , $a \in \Delta^\circ$ . The equivalence of 4) and 5) is clear. The group $P_{\mathbb{C}}$ is the standard parabolic corresponding to the subset $\Delta^\circ$ of $\Delta$ , by [BT] 6.3. We thus have the Bruhat decomposition $P_{\mathbb{C}} = \bigcup_{w \in W'} BwB$ . Since $B$ stabilizes $V^\lambda$ this shows the equivalence of 5) and 6).

**6.6 Remark** The proof shows that if the group $\rho(G_{\mathbb{R}})$ contains a proximal map then every inner point of a Weyl chamber $C$ maps under $\rho$ to a proximal map. There are representations, though, for which also elements on walls of $C$ – hence singular elements – give proximal maps. More precisely, the proof yields the following corollary. For $s \in C$ define

$$\text{type}(s) = \{b \in {}_{\mathbb{R}}\Delta \ ; \ b(s) > 1\}$$

and

$$\text{cotype}(s) = \{b \in {}_{\mathbb{R}}\Delta \ ; \ b(s) = 1\}$$

and similarly

$$\text{type}_\Delta(s) = \{a \in \Delta \ ; \ a(s) > 1\}$$

and

$$\text{cotype}_\Delta(s) = \{a \in \Delta \ ; \ a(s) = 1\}.$$

We have by the results of [BT] cited in the proof of 1) $\Longrightarrow$ 2)

$$\text{type}_\Delta(s) = \{a \in \Delta \ ; \ j(a) \in \ \text{type } (s)\}$$

and

$$\text{cotype}_\Delta(s) = \{a \in \Delta \ ; \ j(a) \notin \ \text{type } (s)\},$$

where $j : X^*(T) \longrightarrow X^*(s)$ is the restriction homomorphism.

Suppose the hypothesis of the theorem. So we do not assume that $\rho(G_{\mathbb{R}})$ contains a proximal element. The theorem may be considered as the special case $\text{type}\,(s) = {}_{\mathbb{R}}\Delta$ , or equivalently $\text{cotype}_{\Delta}(s) = \Delta^0$ .

**6.7 Corollary** *Suppose $\rho$ fulfills the hypotheses of the theorem. Then for $s \in C$ the following conditions are equivalent.*

1) $\rho(s)$ *is proximal*

2) $\lambda$ *is fixed by the standard parabolic subgroup of $G$ of type $cotype_{\Delta}(S)$ .*

3) $\lambda$ *is fixed by the standard parabolic $\mathbb{R}$ -subgroup ${}_{\mathbb{R}}P_{\text{cotype}(s)}$ .*

4) $\lambda$ *is fixed by the centralizer $\mathcal{Z}_G(s)$ of $s$ in $G$ .*

5) $\lambda$ *is orthogonal to $cotype_{\Delta}(S)$ .*

6) *For every $\chi \neq \lambda$ in $\Pi(\lambda)$ we have $k(b) > 0$ for at least one $b \in type(s)$ .*

**Proof** The equivalence of 1) and 6) follows from 6.5. The equivalences of 2), 3) and 5) are proved as the corresponding equivalences of 3) through 6) in the theorem. Finally, the standard parabolic $P$ of $G$ whose type is $\text{cotype}_{\Delta}(s)$ is generated by $B$ and $\mathcal{Z}_G(s)$ , which shows 2) $\iff$ 4).

Recall from 6.4 that $\rho(g)$ is proximal iff $\rho(pol(g))$ is proximal, where $p = pol(g)$ is the polar part of $g$ . Since $p$ is conjugate in $G_{\mathbb{R}}$ to a unique element $s$ of $C$ , we can define

$$\text{type}\,(g) := \text{type}\,(p) := \text{type}\,(s),$$

and 6.7 gives a complete answer to the question when $\rho(g)$ is proximal.

The following theorem fits into the framework of the present chapter on representations of algebraic groups, but does not use any of its results. The proof uses some basic facts about $\mathbb{R}$ -regular elements from [PR], in particular a connection with proximality.

Let $G$ be a connected semi–simple algebraic group defined over $\mathbb{R}$ and let $G(\mathbb{R})$ be the group of $\mathbb{R}$–points of $G$ . Let $\text{Ad}$ be the adjoint representation of $G(\mathbb{R})$ on its Lie algebra $\mathfrak{g}$ . We assume that $G(\mathbb{R})$ is noncompact. An element $g \in G(\mathbb{R})$ is called $\mathbb{R}$*–regular* if the number of eigenvalues, counted with multiplicity, of modulus $1$ of $\text{Ad}\, g$ , is minimum possible. It is known that every $\mathbb{R}$–regular element is semisimple (see [PR]) and it is easy to see that an element of $SL_n(\mathbb{R})$ is $\mathbb{R}$–regular iff all its eigenvalues in the natural $n$–dimensional representation are real and distinct. The first two parts of the following theorem are also proved in [BL] and [P] for the case that $\Gamma$ is a subgroup.

**6.8 Theorem** *Let $\Gamma$ be a Zariski dense subsemigroup of $G(\mathbb{R})$ . Then $\Gamma$ contains an $\mathbb{R}$ –regular element. In fact, the set of $\mathbb{R}$–regular elements of $\Gamma$ is Zariski–dense. There is a finite subset $M$ of $\Gamma$ such that for every $g \in G(\mathbb{R})$ at least one of the elements $\gamma g, \gamma \in M$ , is $\mathbb{R}$ –regular.*

**Proof** Let $G(\mathbb{R}) = KAN$ be an Iwasawa decomposition of $G(\mathbb{R})$ , let $\mathfrak{n}$ be the Lie algebra of $N$ and let $k$ be its dimension. In the representation $\wedge^k \mathrm{Ad}$ of $G(\mathbb{R})$ on $\wedge^k \mathfrak{g}$ let $V$ be the smallest $G(\mathbb{R})$–submodule containing the one–dimensional subspace $\wedge^k \mathfrak{n}$ and let $\rho : G(\mathbb{R}) \to GL(V)$ be the corresponding representation. The representation $\rho$ is irreducible since $V$ is a highest weight module, see [PR 3.1]. An element $g \in G(\mathbb{R})$ is $\mathbb{R}$–regular iff $\rho(g)$ is proximal, by [PR Lemma 3.4]. So the theorem follows from theorem 4.1.

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