A SPECIAL CLASS OF 2-GENERATOR KLEINIAN GROUPS

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Abstract. In this article we continue investigation of a special class of 2-generator Kleinian groups in which neither of the generators nor their commutator is strictly loxodromic. We give necessary and sufficient conditions for discreteness of non-elementary and non-Fuchsian groups of this class generated by two hyperbolic elements.

Introduction

Let $\Gamma = \langle A, B\rangle$ be a two-generator subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We study the following question: when Γ is discrete?

The same question for two-generator subgroups of $\mathrm{PSL}(2,\mathbb{R})$ has a problematic history in the mathematical literature. There were a lot of papers dealing with this problem, see [Br2], [DJ], [Gi1] – [Gi3], [GiM], [J], [KIR], [Kn], [LU], [Mt], [P1] – [P3], [PR], [R1] – [R5]. Rosenberger [R5] completed the classification of all two-generator Fuchsian groups and presented an algebraic algorithm for determining whether or not a two-generator subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is discrete. Another, geometric approach to this problem was started by Matelski [Mt] and developed by Gilman and Maskit [GiM].

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As for $\mathrm{PSL}(2,\mathbb{C})$, the problem is far from being completely solved. The remarkable Jørgensen's inequality [J] yields a necessary condition for discreteness of 2-generator non-elementary groups.

Necessary and sufficient discreteness conditions for subgroups of $\mathrm{PSL}(2,\mathbb{C})$ which are generated by two elliptic elements with mutually orthogonal skew axes were given in [K1], [K2]. Thus, some class of 2-generator Kleinian groups was first completely described. Maskit [M2] investigated another specific class of 2-generator Kleinian groups, where one of generators has exactly two fixed points in $\overline{\mathbb{C}}$ and the other maps one of these fixed points to the other. There were two open questions in [M2]. The answer to one of them is given in [K3]. There are discrete groups belonging to both classes studied in [K2] and [M2], however, neither of these classes contains the other.

We study a larger class $\mathcal{G}$ of 2-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$, where neither of the generators nor their commutator is strictly loxodromic. This class contains all 2-generator Fuchsian groups as well as all Kleinian groups described in [K2] and [M2]. There are a lot of non-elementary discrete groups which are not Fuchsian in $\mathcal{G}$. Most of such groups have been detailed in [K4], see also [K5]. Moreover in [K4] a description of these groups in terms of generators and relations is given, and all orbifolds that are uniformized by the groups are also presented.

Some examples of discrete two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$ have been constructed in [HMV].

In this paper we give necessary and sufficient conditions for discreteness of non-elementary and non-Fuchsian groups of class $\mathcal{G}$ generated by two hyperbolic elements (Theorem 4).

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$\mathrm{PSL}(2,\mathbb{C})$ and Lobachevsky space

As is known, $\mathrm{PSL}(2,\mathbb{C})$ acts as the full group of orientation preserving isometries of 3-dimensional Lobachevsky space $\mathbb{H}^3$.

Consider 3-dimensional Lobachevsky space realized as the upper half-space

$$\mathbb{H}^3 = \{\ (z,t) \mid z \in \mathbb{C},\ t > 0\}$$

with the hyperbolic metric

$$ds^2 = (|dz|^2 + dt^2)/t^2.$$

In this model, hyperbolic lines are Euclidean semicircles and rays orthogonal to the boundary $\partial\mathbb{H}^3 = \overline{\mathbb{C}}$ of Lobachevsky space; and hyperbolic planes are Euclidean semispheres and half-planes, orthogonal to $\overline{\mathbb{C}}$.

Two hyperbolic planes either are intersecting, or parallel, or they are disjoint. Two hyperbolic lines either lie in a hyperbolic plane (and then they are intersecting, parallel or disjoint), or they are skew.

In the next section we will need the following three propositions.

Proposition 1. *Let σ and ρ be mutually orthogonal planes, and let g be the line by which they intersect. Let h be a straight line which lies in ρ and is orthogonal to g. Then h is orthogonal to σ.*

Proposition 2. *If a plane σ is orthogonal to a straight line h lying in another hyperbolic plane ρ, then σ and ρ are mutually orthogonal.*

Propositions 1 and 2 are very simple and occur not only in hyperbolic geometry, but also in Euclidean and spherical ones.

Proposition 3. *Let σ_1 and σ_2 be disjoint planes in $\mathbb{H}^3$, and let ρ be a plane which is orthogonal to both of them. Then ρ passes through the common perpendicular to σ_1 and σ_2.*

Indeed, the lines of intersection of σ_1 and σ_2 with ρ are disjoint and, consequently, have their common perpendicular (lying in ρ) which is orthogonal to both σ_1 and σ_2 by Proposition 1.

For $A \in \mathrm{PSL}(2,\mathbb{C})$, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ be one of the two matrices which represent A. Then, as usually, $\operatorname{tr}^2 A \equiv (a+d)^2$.

We need a classification of non-trivial elements $A \in \mathrm{PSL}(2,\mathbb{C})$.

A	$\operatorname{tr}^2 A$	geometrical sense
elliptic	$\in [0,4)$	rotation
parabolic	$= 4$	limit rotation
hyperbolic	$\in (4,+\infty)$	translation
strictly loxodromic	$\notin [0,+\infty)$	rotation + translation with a common axis

An elliptic, parabolic or hyperbolic element, in contrast to a strictly loxodromic one, can be presented as the product of the reflections in two hyperbolic planes which are intersecting (the line of intersection is called the axis of the elliptic element; this is the fixed point set under the action of this element), parallel (the common point of the planes, which lies on $\overline{\mathbb{C}}$, is the unique fixed point of the parabolic element) or disjoint respectively (in the last case, the common perpendicular to the planes is the unique invariant line under the action of the hyperbolic element and is called the axis of this element).

An elliptic element is determined by its axis, the angle and direction of rotation about this axis. An elliptic element of finite order is called *primitive* (or *geometrically primitive*), if it is conjugate to a rotation of the form $F(z) = \exp(2\pi i/p)z$, $p \in \mathbb{N}$, and non-primitive otherwise. Non-primitive elliptic element of order p is conjugate to a rotation $F(z) = \exp(2\pi iq/p)z$, where p, $q \in \mathbb{N}$ are relatively prime, $q > 1$ and $q/p < 1/2$.

Further information on Kleinian groups and hyperbolic geometry can be found in [A], [Br1], [F], [GeM], [KAG], [M1], [Mi], [S], [Th], [V], [ZVC].

A class for which the problem could be solved

We return to our problem: when $\Gamma = \langle A, B\rangle < \mathrm{PSL}(2,\mathbb{C})$ is discrete? Consider the following class:

$$\mathcal{G} = \big\{\Gamma = \langle A, B\rangle \mid A,\ B \text{ and } [A,B] = ABA^{-1}B^{-1} \text{ are not strictly loxodromic}\big\}.$$

We have the following cases:

	A	B	$[A,B]$
1.	elliptic	elliptic	
2.	elliptic	parabolic	
3.	parabolic	parabolic	is not strictly loxodromic
4.	hyperbolic	parabolic	for all the cases
5.	elliptic	hyperbolic	
6.	hyperbolic	hyperbolic	

We will study each of these cases separately.

Case 1. Both elements A and B are elliptic.

The condition that $[A, B]$ is not strictly loxodromic is equivalent to the condition that the axes of A and B either lie on a hyperbolic plane, or they are mutually orthogonal skew lines.

Thus, if $\Gamma \in \mathcal{G}$ and has two elliptic generators A and B, then either

1 (a) the axes of A and B lie in a hyperbolic plane, or

1 (b) the axes of A and B are mutually orthogonal skew lines.

Consider 1 (a). If the axes of A and B intersect (or are parallel), then Γ is elementary. Moreover, any hyperbolic sphere (respectively horosphere) centered at the point of their intersection (lying in $\mathbb{H}^3$ or on $\overline{\mathbb{C}}$ respectively) is invariant under the action of A and B. If the axes of A and B are disjoint, then there is a plane orthogonal to both of them (this plane passes through the common perpendicular to the axes of A and B); this plane is invariant under the action of A and B, so Γ is discrete if and only if Γ is Fuchsian. Thus, in case 1 (a) Γ is a group of isometries of 2-dimensional sphere, Euclidean plane or Lobachevsky plane, and criteria for discreteness of such groups are known.

In case 1 (b), Γ contains infinitely many strictly loxodromic elements with different axes (for example, the product AB), so it is non-elementary and non-Fuchsian. The necessary and sufficient conditions for discreteness of such groups are given by the following

Theorem 1 [K2]**.** *Let $A, B \in \mathrm{PSL}(2,\mathbb{C})$ be primitive elliptic elements of orders n and m respectively ($n \leq m$) with mutually orthogonal skew axes. Then*

(1) there exists an element $C \in \mathrm{PSL}(2,\mathbb{C})$ such that $C^2 = [A,B]$ and $(CB)^2 = 1$; and

(2) $\Gamma = \langle A, B\rangle$ is discrete if and only if one of the following conditions is satisfied:

(i) C is hyperbolic, parabolic or a primitive elliptic element;

(ii) $C = D^2$, where D is a primitive elliptic element of order p, p is odd, and

either $n = 2$, $p = m \geq 7$,
or $n = m = p \geq 7$.

Cases 2–4. A is elliptic, parabolic or hyperbolic, and B is parabolic.

The condition that $[A,B]$ is not strictly loxodromic has also a certain geometrical sense. Again, in these cases $\Gamma = \langle A, B\rangle$ is either elementary or Fuchsian or neither. In the last case we have the following

Theorem 2 [K4]**.** *Let $\Gamma = \langle A, B\rangle \in \mathcal{G}$ be non-elementary and non-Fuchsian, where A is hyperbolic, parabolic or primitive elliptic, and B is a parabolic element. Then*

(1) there exists an element $C \in \mathrm{PSL}(2,\mathbb{C})$ such that $C^2 = [A,B]$ and $(CB)^2 = 1$; and

(2) Γ is discrete if and only if C is hyperbolic, parabolic or a primitive elliptic element.

Case 5. A is elliptic and B is hyperbolic.

In this case, $\Gamma = \langle A, B\rangle \in \mathcal{G}$ if and only if either

5 (a) the axes of A and B lie in a hyperbolic plane, or

5 (b) the axes of A and B are mutually orthogonal skew lines.

However, this time in case 5 (b) Γ is discrete if and only if it is Fuchsian, because there is a hyperbolic plane (passing through the axis of B and orthogonal to the axis of A) which is invariant under the action of Γ.

In case 5 (a), the axes of A and B are either intersecting or parallel or disjoint. The question for the case of intersecting axes is still open. If the axes are parallel, then the group is not discrete, see [M1, p. 83]. If the axes are disjoint, then we have the following

Theorem 3 [K5]**.** *Let $A \in \mathrm{PSL}(2,\mathbb{C})$ be a primitive elliptic element of order n and $B \in \mathrm{PSL}(2,\mathbb{C})$ be a hyperbolic element, and let the axes of A and B be disjoint lines lying on a hyperbolic plane. Then*

(1) there exists an element $C \in \mathrm{PSL}(2,\mathbb{C})$ such that $C^2 = [A,B]$ and $(CB)^2 = 1$; and

(2) $\Gamma = \langle A, B\rangle$ is discrete if and only if one of the following conditions is satisfied:

(i) C is hyperbolic, parabolic or a primitive elliptic element;

(ii) $n = 2$, C is a non-primitive elliptic element of finite order, and $\langle CB, C\rangle$ is a Fuchsian group;

(iii) $C = D^2$, where D is a primitive elliptic element of order p, p is odd, and

either $n = p \geq 5$ and $X = CBAD^{-1}A$ is hyperbolic, parabolic or a primitive elliptic element of order $r \geq 4$,

or $n = 3$, $p = 5$ and $Y = CBA(D^{-1}A)^3$ is hyperbolic, parabolic or a primitive elliptic element of order $s \geq 3$.

Case 6. Both elements A and B are hyperbolic.

Here we give a detailed description of discrete groups arising in this case.

Again the condition that $[A,B]$ is not strictly loxodromic is equivalent to the condition that the axes of A and B either lie on a hyperbolic plane (and then this plane is invariant under the action of $\Gamma = \langle A, B\rangle$, i.e. Γ is discrete if and only if it is Fuchsian), or they are mutually orthogonal skew lines. In the last case, necessary and sufficient conditions for Γ to be discrete are given by the following

Theorem 4. *Let $A, B \in \mathrm{PSL}(2,\mathbb{C})$ be hyperbolic elements with mutually orthogonal skew axes. Then*

(1) there exists an element $C \in \mathrm{PSL}(2,\mathbb{C})$ such that $C^2 = [A,B]$ and $(CB)^2 = 1$; and

(2) $\Gamma = \langle A, B\rangle$ *is discrete if and only if* C *is hyperbolic, parabolic or a primitive elliptic element.*

Proof. To prove the theorem we will use the methods expounded in papers [K2] – [K5].

1. The first step is to construct a group of reflections Γ^* in which Γ has finite index. In this case, the groups Γ and Γ^* are discrete or non-discrete simultaneously, therefore the question about discreteness of Γ is equivalent to the same question for Γ^*.

Let a and b be mutually orthogonal skew axes of hyperbolic elements A and B respectively, and let e be their common perpendicular. Denote by E the half-turn about e. There are two half-turns E_a and E_b (with their axes e_a and e_b) such that

$$A = E_aE, \qquad B = E_bE.$$

We have

$$[A, B] = (E_aE)(E_bE)(EE_a)(EE_b) = (E_aEE_b)^2 = (AE_b)^2. \tag{1}$$

It is clear that e_a and e lie on a hyperbolic plane, are disjoint, and a is their common perpendicular. The same statement is true for the three axes e_b, e, and b.

Let α, β, γ, δ be hyperbolic planes defined as follows: α contains e_a and is orthogonal to a, β contains e_b and is orthogonal to b, γ contains e_a, e and a, and δ contains e_b, e and b. Note that the pairs of planes α and γ, γ and δ, δ and β intersect at the angle of $\pi/2$, α and δ are disjoint (a is their common perpendicular), β and γ are also disjoint (b is their common perpendicular). The planes α and β are either disjoint or parallel or intersecting.

Let $\mathcal{P}_0$ denote the convex polyhedron (of infinite volume) bounded by α, β, γ and δ with the edges e_a, e, e_b, and $c = \alpha \cap \beta$, if this intersection is not empty (see Figure 1*). If we denote by R_α, R_β, R_γ, R_δ the reflections in the faces α, β, γ, δ respectively, then $A = R_\alpha R_\delta$, $E_b = R_\delta R_\beta$ and from (1) it follows that

$$[A, B] = (R_\alpha R_\beta)^2.$$

Denote by C the product of reflections R_β and R_α,

$$C = R_\alpha R_\beta.$$

Then C is the very square root of $[A, B]$ we need. Indeed, the product $CB = (R_\alpha R_\beta)(R_\beta R_\gamma) = R_\alpha R_\gamma = E_a$ is a half-turn. (Note that the other orientation preserving square root C' of non-parabolic commutator $[A, B]$ does not satisfy the condition that $C'B$ is a half-turn.) Let

$$\widetilde{\Gamma} = \langle A, B, E\rangle \qquad \text{and} \qquad \Gamma^* = \langle A, B, E, R_\delta\rangle = \langle R_\alpha, R_\beta, R_\gamma, R_\delta\rangle.$$

*All figures are done irrespective of any model of Lobachevsky space.

It is easy to see that either $\widetilde{\Gamma} = \Gamma$ (if $E \in \Gamma$), or Γ is a subgroup of $\widetilde{\Gamma}$ of index 2 (since $EAE^{-1} = A^{-1}$, $EBE^{-1} = B^{-1}$ and so $\widetilde{\Gamma} = \Gamma \cup \Gamma E$). Moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of Γ^* of index 2. Thus Γ, $\widetilde{\Gamma}$ and Γ^* are discrete or non-discrete simultaneously.

2. Assume that $C = R_\alpha R_\beta$ is hyperbolic, parabolic or a primitive elliptic element of order p (this means that α and β are disjoint, parallel or intersecting at the angle of π/p respectively). Then $\mathcal{P}_0$, together with the reflections R_α, R_β, R_γ, R_δ in its faces, satisfies the hypotheses of Poincaré's polyhedron theorem, and so $\Gamma^* = \langle R_\alpha, R_\beta, R_\gamma, R_\delta \rangle$ is discrete.

3. Assume that $C = R_\alpha R_\beta$ is neither hyperbolic nor parabolic nor a primitive elliptic element. We have two possibilities: either C is an elliptic element of infinite order; or C is a non-primitive elliptic element of finite order. In the first case, Γ^* and Γ are not discrete. In the second case we need further investigations. To complete the proof of the theorem we must obtain the following

Lemma. *Let $A, B \in \mathrm{PSL}(2, \mathbb{C})$ be hyperbolic elements with mutually orthogonal skew axes, and let C as above be a non-primitive elliptic element of finite order. Then Γ is not discrete.*

Proof. We will prove it by contradiction. Suppose that Γ is discrete, and C is a non-primitive elliptic element, that is, the planes α and β intersect at an angle of $q(\pi/p)$, where p is the order of C, $q > 1$ and $q/p < 1/2$. Then Γ^* contains reflections in $q-1$ planes which pass through c and divide the corresponding dihedral angle of $\mathcal{P}_0$ into q smaller angles of π/p.

Now we need an additional construction.

The lines e_a and c are disjoint since they lie on disjoint planes γ and β. Denote by ε the plane which passes through the common perpendicular to e_a and c and is orthogonal to α. Then ε is orthogonal to e_a and e (Proposition 1), and consequently, to both β and γ (Proposition 2). Therefore ε passes through the common perpendicular to β and γ, i.e. through b (Proposition 3). Analogous argumant shows that there is a plane η which is orthogonal to α, β and δ and passes through a (Figure 2). Denote by $\mathcal{P}_c$ the compact polyhedron bounded by α, β, γ, δ, ε and η.

Recall that Γ^* contains the reflections R_α, R_β, R_γ and R_δ (and does not contain the reflections R_ε and R_η in ε and η); moreover, Γ^* contains reflections in $q-1$ planes ($q - 1 \geq 1$) passing through c and intersecting the interior of $\mathcal{P}_c$. We call these $q-1$ planes the *division planes (passing through c)*.

Consider the planes ε and η. Set $\Gamma^*_\varepsilon = \langle R_\alpha, R_\beta, R_\gamma \rangle$, $\Gamma^*_\eta = \langle R_\alpha, R_\beta, R_\delta \rangle$. It is easy to see that Γ^*_ε keeps the plane ε invariant, and Γ^*_η keeps the plane η invariant (since α, β and γ are orthogonal to ε, and α, β and δ are orthogonal to η). The division planes passing through c are orthogonal to both ε and η; and the reflections in them belong to both Γ^*_ε and Γ^*_η.

Both Γ^*_ε and Γ^*_η are subgroups of Γ^* which is discrete by assumption), consequently they are also discrete.

For convenience denote $\alpha_\varepsilon = \alpha \cap \varepsilon$, $\beta_\varepsilon = \beta \cap \varepsilon$, $\gamma_\varepsilon = \gamma \cap \varepsilon$, $\alpha_\eta = \alpha \cap \eta$, $\beta_\eta = \beta \cap \eta$, $\delta_\eta = \delta \cap \eta$ (Figure 3). We concentrate on ε, keeping in mind that analogous conclusions are valid also for η. Let ζ be the division plane nearest to α. A priori, ζ and γ are either disjoint or parallel or intersecting. In the last case, since Γ^*_ε is discrete, the triangle $\triangle_\varepsilon$ formed by α_ε, γ_ε and $\zeta_\varepsilon = \zeta \cap \varepsilon$ must be in Knapp's list, see [Kn]. Thus, we have the possibilities for $\triangle_\varepsilon$ presented at Figure 4 (where we write p instead of π/p).

A priori, there are infinitely many possibilities for the plane ε subdivided by the planes of reflections from Γ^*_ε. First of all, we consider the cases satisfying the following condition:

(*C1*) The number of division planes passing through c must be minimal for each admissible triangle $\triangle_\varepsilon$.

All possible subdivisions satisfying (*C1*) are represented at Figure 5. We will assume that case (i), Figure 5, contains subcases of non-compact triangle $\triangle_\varepsilon$ ($r = \infty$) and of disjoint planes ζ and γ. Note that two is the minimal number of division planes passing through c in case (ii), otherwise, β_ε intersects γ_ε.

We will denote the edges (and their lengths) and faces of $\mathcal{P}_c$ by $\overline{a} = a \cap \mathcal{P}_c$, $\overline{b} = b \cap \mathcal{P}_c$, $\overline{\gamma}_\varepsilon = \gamma_\varepsilon \cap \mathcal{P}_c$, ... , $\overline{\varepsilon} = \varepsilon \cap \mathcal{P}_c$, ... ¿From Figure 5 it is easy to see that

(*C2*) $\overline{\gamma}_\varepsilon$ has at most one point of intersection with the division planes.

Thus, from (*C1*) it follows (*C2*). It is not difficult to see that (*C2*) holds even if (*C1*) does not occur.

We have that similar condition is necessarily realized for η:

(*C2′*) $\overline{\delta}_\eta$ has at most one point of intersection with the division planes.

At last, we show that (*C1*) follows from (*C2*) and (*C2′*). Indeed, if (*C1*) does not hold, then at least two division planes intersect $\overline{b}$ and, consequently, also $\overline{\delta}_\eta$, what contradicts (*C2′*). Thus, we conclude that the condition (*C1*) always holds and there are only the subdivisions of ε presented at Figure 5.

It is clear that Figure 5 gives also the full list of subdivisions of η by the planes of reflections of Γ^*_η.

Since the division planes passing through c are the same for ε and η, there are the following three possibilities for the pair $\{\varepsilon, \eta\}$:

$\langle 1\rangle$ {(i), (iii)} or {(iii), (i)};
$\langle 2\rangle$ {(ii), (ii)};
$\langle 3\rangle$ {(iii), (iii)}.

Note that (iv) is impossible (it could be combined only with (iii), but there $p = 7$).

To complete the proof of Lemma it remains to show that the situations $\langle 1\rangle$ – $\langle 3\rangle$ cannot occur.

In $\langle 3\rangle$, the division plane intersects e in a point, say x (Figure 6). We have two axes of elliptic elements of order 7 passing through x, what is impossible in a discrete group. (It is an easy corollary of the following fact. If some subgroup of a discrete group of isometries of $\mathbb{H}^3$ keeps a sphere invariant, then this subgroup is a discrete group of isometries of 2-dimensional sphere. For the subgroup generated by two elliptic elements of order 7 any sphere centered at x is invariant).

In the hyperbolic geometry a triangle is determined by its angles, therefore, in case $\langle 2\rangle$, $\overline{\alpha}_\varepsilon = \overline{\beta}_\eta$, and $\overline{\beta}_\varepsilon = \overline{\alpha}_\eta$ (see Figure 7). However, on the one hand, $\overline{\alpha}_\varepsilon > \overline{\beta}_\varepsilon$ (it is easy to see from Figure 5(ii) for the plane ε). On the other hand, α_ε is the common perpendicular to c and e_a, and, consequently, $\overline{\alpha}_\varepsilon < \overline{\alpha}_\eta$. We get a contradiction.

In case $\langle 1\rangle$ we can without loss of generality handle the pair $\{$(iii), (i)$\}$. Denote by ζ_γ the line of intersection of the division plane with γ. First of all, ζ_γ cannot intersect e (otherwise, there should be two intersecting axes of order 7 and of order 2 in the discrete group what is possible only if the angle of their intersection is equal to $\pi/2$, and in γ there is a triangle with two right angles). The lines ζ_γ and e cannot be parallel, because none of Euclidean groups contains elliptic element of order 7. The last possibility is that ζ_γ and e are disjoint. In this case, in the plane orthogonal to both ζ_γ and e we have a triangle with angles of $\pi/2$, $\pi/7$, φ. This triangle is an orthogonal section of a triangle prism for which ζ_γ and e are lateral edges. Thus, this triangle must be smaller than any other section of the prism. On the other hand, by discreteness of Γ^*, we must have $\varphi = \pi/n$ $(n \geq 3)$ or $\varphi = 2\pi/7$. However all such triangles are larger than the one hatched on Figure 8 (the hatched triangle is a part of the triangle with angles $\pi/2$, $\pi/7$, $\pi/3$). We arrived to a contradiction. The proof of Lemma (and Theorem 4) is complete.

Remarks.

1. It is interesting to compare this result with the 2-dimensional case. For Fuchsian groups as well the case of two hyperbolic generators is the most complicated one, but there criteria for discreteness are given in the form of an algorithm. The answer in the 3-dimensional case depends only on the commutator.

2. Surprisingly, there are no discrete groups with two hyperbolic generators for which the above mentioned square root C of their commutator is a non-primitive elliptic element (for example, compare this with the case of two elliptic generators).

3. To obtain a full classification for discrete groups of the class $\mathcal{G}$ it remains only to study the case in which A is primitive elliptic of order $n > 2$, B is hyperbolic, and the axes of A and B are intersecting but not mutually orthogonal lines.

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