On Geometric Structures of dihedral Θ-Orbifolds

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Abstract

This article studies the geometric structures of dihedral orbifolds with a knot and a bridge as it's singular set and the three-dimensional sphere as the underlying space. These knotted graphs are connected with certain families of links factorized by their involution. In some of our cases the bridge is a tunnel of the knot. The greater part of the orbifolds described have a hyperbolic structure. Most of the fundamental groups appearing are two generator groups and moreover, they are generalized triangle groups, whose explicit representations are given.

1. Introduction

The purpose of the present article is to investigate geometrical structures of orbifolds where the singular set is a knot or a two-component link with a bridge and underlying space S^3. Examples of this kind have already been studied in the literature, so Helling, Mennicke and Vinberg [10] considered the trefoil knot with a bridge and determined the geometric structure of the orbifold with such a singular set for all branching indices k, l, m greater or equal to two (see figure 1).

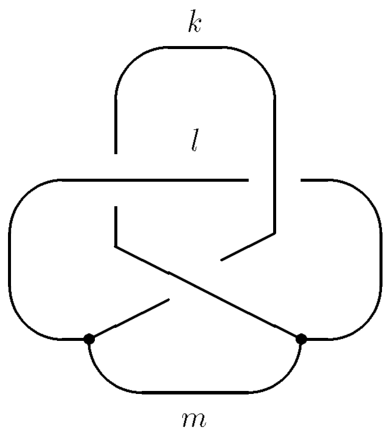

figure 1: *The trefoil with a bridge*

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Moreover the hyperbolic structures of the orbifolds with the Hopf-link or the 4_1^2-link with a bridge as singular set (see figure 2) were established by the first author [7].

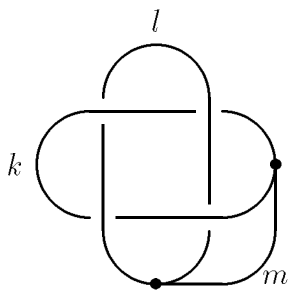

figure 2: *The double link 4_1^2 with a bridge*

In Theorems 1 and 2 we generalize these examples and construct families of hyperbolic dihedral orbifolds. Further examples of orbifolds similar to those above but having non-hyperbolic structure were obtained by Dunbar [6].

2. Embeddings of graphs in S^3 and orbifolds

It is well-known that if the singular set of a compact orbifold contains a graph (including also non-connected and multigraphs), then the order of any vertex is exactly three. Such a graph is called a cubic graph. Although cubic graphs are well-known as abstract graphs, they become rather complicate, as we are interested in their embeddings in S^3. Let us first consider two types of cubic multigraphs. It is obvious that there are only two abstract cubic graphs with two vertices (see figure 3).

figure 3: *Cubic graphs with two vertices and three edges*

We will call them **Θ-graph** and **"pince-nez"-graph** (as this name seems preferable to us than "handcuff"-graph the one chosen by Wolcott [14]). We remark that the singular set of the orbifolds from [10] (see figure 1) is isomorphic, as a graph, to the Θ-graph; and the singular set of the orbifold from [7] (see figure 2) is isomorphic, as a graph, to the "pince-nez"-graph.

The embeddings of the Θ-graph and the "pince-nez"-graph in the three dimensional sphere were already investigated in [14]; called **Θ-curves** and **"pince-nez"-curves**, respectively. Every Θ-curve has three edges, deleting any of these edges yields a knot. These three knots are called **the knots associated to the Θ-curve**. Obviously two Θ-curves having different sets of associated knots are topologically different. However, in general the converse is not true. This becomes clear from Kinoshita's example of a non-planar Θ-curve for which all associated knots are trivial. Thus, a Θ-curve cannot be completely described only by the set of associated knots (see examples in [14]).

As already known from the combinatoric point of view a cubic graph with exactly three vertices does not exist. We now consider a cubic graph with four vertices, namely, the complete graph K_4. Similar, its embeddings in S^3 are called **K_4-curves**. Such embeddings were studied in [15]. If we delete pairs of edges of the K_4-curve or triples of edges of the K_4-curve, we will get seven knots in S^3 altogether, associated to the K_4-curve. It was shown in [15] that, for any ordered 7-tuple of knot types, there exists a K_4-curve having these seven associated knots.

Similar to knots and links we can define **the projection** of the embedding of the graph, **the order of the projection** of a curve as the number of crossings of the projection, and **the order of the embedding** of the graph as the minimal order of all it's projections. The Θ-curves and the K_4-curves of small order where tabulated in [12].

Further we will be interested in the geometrical structures of orbifolds with underlying space S^3 and singular set as a Θ-curve or a K_4-curve. We will call such orbifolds **Θ-orbifolds** and **K_4-orbifolds** respectively.

According to [5] we will say that a vertex of the singular set of the orbifold (which is in fact the vertex of the cubic graph) is of tetrahedral, octahedral, icosahedral or dihedral type, corresponding to the type of the finite subgroup of $SO(3)$ connected with this point of the orbifold. The distances between vertices of some of such types were investigated in [5]. In the sequel we will be interested in orbifolds whose vertices are all of dihedral type. We call such orbifolds **dihedral** orbifolds. In the following sections dihedral Θ-orbifolds and dihedral K_4-orbifolds will be constructed.

3. Dehn surgery on some 3-component links

Consider the Borromean rings B (see figure 4), the link 6^3_2 (as all link numbers in the following this also refers to the list of Rolfsen [11]).

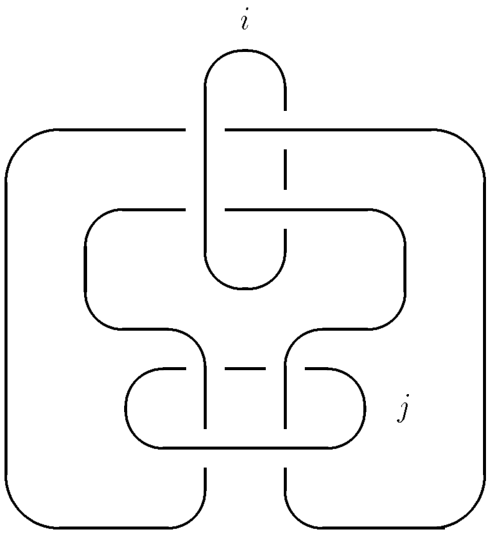

figure 4: *The Borromean rings* 6_2^3

According to [11] by $(1,i)$ and $(-1,j)$ Dehn surgeries on two of the three components of B, respectively, we get an alternating knot, denoted by B_j^i, with $2i+2j$ crossings for any positive integer i,j. So we have for example that B_1^1 is the knot 4_1, the so called figure–eight knot (see figure 5).

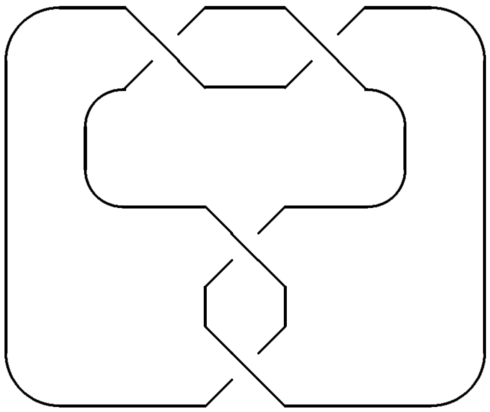

figure 5: *The knot* B_1^1

Of course in this family B_j^i and B_i^j represent the same knot type. The first members of this family, those with less than eleven crossings are the following: $B_1^1 = 4_1$, $B_2^1 = B_1^2 = 6_1$, $B_3^1 = B_1^3 = 8_1$, $B_4^1 = B_1^4 = 10_1$, $B_2^2 = 8_3$ and $B_2^3 = B_3^2 = 10_3$.
The knots B_j^i of this family have two obvious topological orientation-preserving symmetries τ and σ of order two as shown in the figure 6, where i and j denote the number of half-twists.

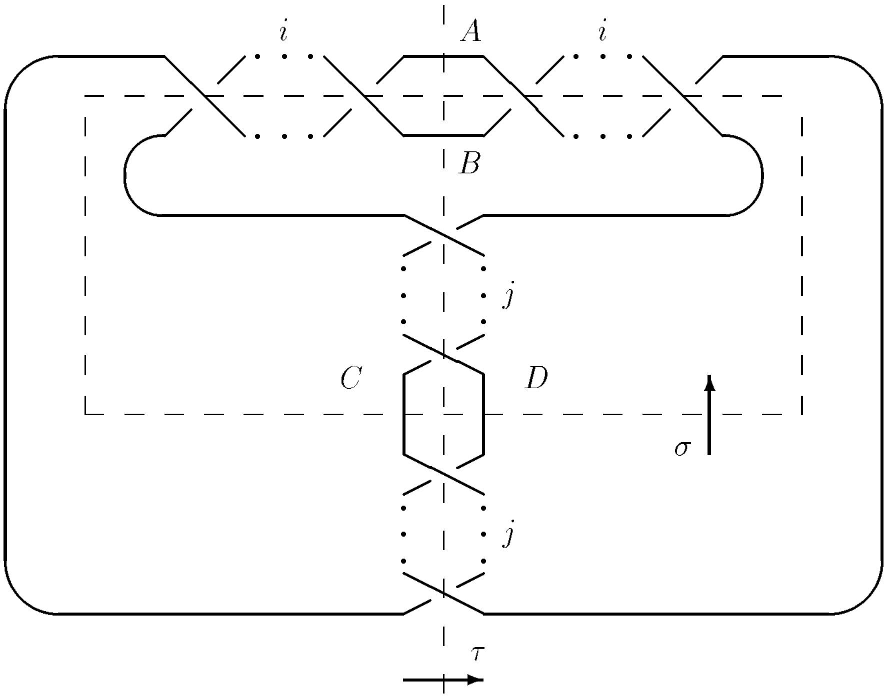

figure 6: *The symmetries τ and σ of the knot B^i_j*

Both symmetries τ and σ have exactly two fixed points on the knot, A,B and C,D, respectively. It is well-known that the complement in S^3 of the Borromean rings B has the hyperbolic structure [13]. Therefore by the hyperbolic Dehn surgery theorem [13], almost all knot complements of our family B^i_j are hyperbolic. Moreover all of them are hyperbolic for $i \geq 1$, $j \geq 1$, $i+j \leq 5$ (see [1]). By Mostov's rigidity theorem both symmetries τ and σ are also isometries of the corresponding hyperbolic structure on the knot complement of B^i_j in S^3.

Since for any hyperbolic knot there exist such n that for all $k \geq n$ the orbifolds with this knot as its singular set and k as its branching index are hyperbolic, we consider now such hyperbolic orbifolds $B^i_j(k)$ corresponding to the knots of our family. In this way, for example, holds, that $B^1_1(k)$ is hyperbolic for $k \geq 4$ and that the other $B^i_j(k)$, $i \geq 1$, $j \geq 1$, $i+j \leq 5$, are hyperbolic for $k \geq 3$ as one may verify using the direct calculations of SnapPea [1].

Obviously these hyperbolic orbifolds $B^i_j(k)$ have still the same hyperbolic orientation-preserving isometries τ and σ of order two. Factorizing the hyperbolic orbifold $B^i_j(k)$ by the hyperbolic isometry τ we obtain a hyperbolic orbifold still with underlying space S^3 and the singular set as pictured in the following figure 7, where i and $4j$ as usual denote numbers of half-twists.

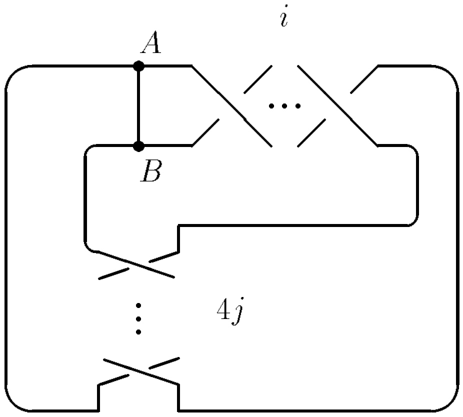

figure 7: *The singular set of* $B_j^i(k)/\tau$

The singular set in figure 7 is isomorphic, as an abstract graph, to a cubic graph with two vertices having no loops. Therefore the $B_j^i(k)/\tau$ are dihedral Θ-orbifolds according to the notation introduced above.
Let us consider a second 3–component link, the link 8_9^3, (see figure 8) and repeat the considerations above for this case.

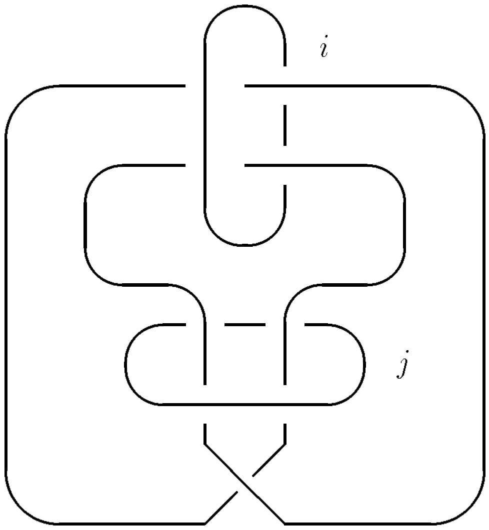

figure 8: *The link* 8_9^3

According to [1] the complement of this link is also hyperbolic, furthermore its hyperbolic volume is equal to the hyperbolic volume of the Borromean rings complement. Performing again Dehn surgeries on two of the components with the parameters $(1, i)$ and $(-1, j)$, where $i \geq 1$, $j \geq 0$, yields a second family of knots. Let us denote these knots by U_j^i in the same manner as the knots B_j^i. The knots U_j^i are again alternating knots having a projection of $2i + 2j + 1$ crossings. For the first members of this family with less than eleven crossings we have: $U_0^1 = 3_1$, $U_0^2 = 5_1$, $U_0^3 = 7_1$, $U_0^4 = 9_1$, $U_1^1 = 5_2$, $U_1^2 = 7_3$, $U_1^3 = 9_3$, $U_2^1 = 7_2$, $U_2^2 = 9_4$, $U_3^1 = 9_2$. Repeating the analogous considerations as for the knots B_j^i for the knots U_j^i and yields again dihedral

Θ–orbifolds $U_j^i(k)/\tau$. It will be convenient to join these families in one and consider the knots

$$H_j^i = \begin{cases} B_{j/2}^i & , \text{ for even } j, \\ U_{(j-1)/2}^i & , \text{ for odd } j \end{cases}$$

By construction each knot H_j^i is alternating with $2i+j$ crossings. The knots H_j^i with at most ten crossings are the following:

$H_1^1 = 3_1$	$H_1^2 = 5_1$	$H_1^3 = 7_1$	$H_1^4 = 9_1$
$H_2^1 = 4_1$	$H_2^2 = 6_1$	$H_2^3 = 8_1$	$H_2^4 = 10_1$
$H_3^1 = 5_2$	$H_3^2 = 7_3$	$H_3^3 = 9_3$	
$H_4^1 = 6_1$	$H_4^2 = 8_3$	$H_4^3 = 10_3$	
$H_5^1 = 7_2$	$H_5^2 = 9_4$		
$H_6^1 = 8_1$	$H_6^2 = 10_3$		
$H_7^1 = 9_2$			
$H_8^1 = 10_1$			

Among these knots H_1^i are the torus knots of order $2i+1$. All the other knots are hyperbolic according to [1]. In the following paragraph we study the dihedral Θ-orbifolds $H_j^i(k)/\tau$.

4. Dihedral Θ–orbifolds

The projection of a Θ-curve divides the plane in some regions. If we delete one of the edges of a Θ-curve, then we get a projection of one of the associated knots, which also divides the plane in some regions. If our deleted edge lies without intersections in one of such regions, then we call this edge a **bridge of the projection** and say that the Θ-curve is the **knot with the bridge**. Obviously the knot is the respective associated knot.

A bridge of the projection lies on the boundaries of two regions. Let e_1 and e_2 be the number of edges of each of these regions, respectively. In this case we say that our bridge is an m-**bridge** where $m = \min(e_1, e_2) - 2$. We denote a Θ-curve of the knot K with an m-bridge by $\Theta_m(K)$. Of course this notation does not uniquely determine the Θ-curve. Examples of the knot 5_1 with a 1-bridge and a 2-bridge are shown in figure 9.

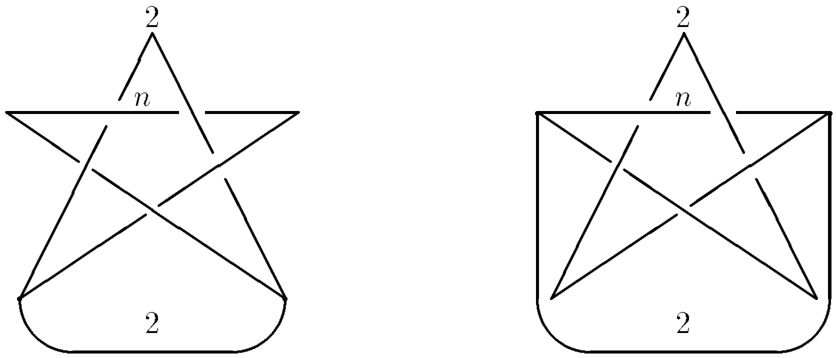

figure 9. *The singular sets of the orbifolds $\Theta_1^n(5_1)$ and $\Theta_2^n(5_1)$*

The corresponding dihedral Θ–orbifold with the underlying space S^3 and the singular set the Θ–curve consisting of a knot K with a m–bridge and branching indices $2, 2, n$ where the bridge has the index 2 will be denoted $\Theta_m^n(K)$. For the Θ–orbifolds, considered in the previous paragraph we have

Lemma 1. *Let $H_j^i(n)$ be the orbifold with underlying space S^3 and the knot H_j^i with the branching index n as it's singular set. Let τ be the orientation-preserving symmetry of order two as above. Then the factor-space $H_j^i(n) \,/\, \tau$ is the dihedral Θ–orbifold $\Theta_1^n(H_i^j)$ where bridge of H_i^j is exactly between the strings connecting i half-twists and $2j$ half-twists, i. e. with A and B as its endpoints (see figure 11).*

Proof. The knot H_j^i represented in figure 10, where as usual the labels i and j figure the numbers of half-twists, has the symmetry τ whose axis intersects H_j^i in two points A and B.

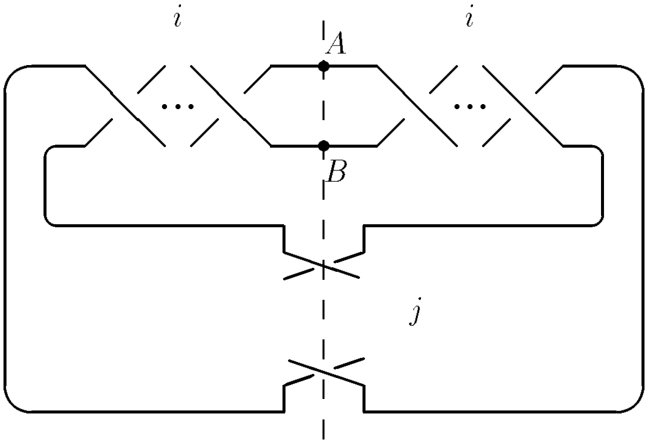

figure 10. *The knot H_j^i.*

By the factorization of $H_j^i(n)$ by τ, we get again S^3 from S^3 as the underlying space and from the singular set the knot with a 1-bridge at the endpoints A and B as represented in figure 11.

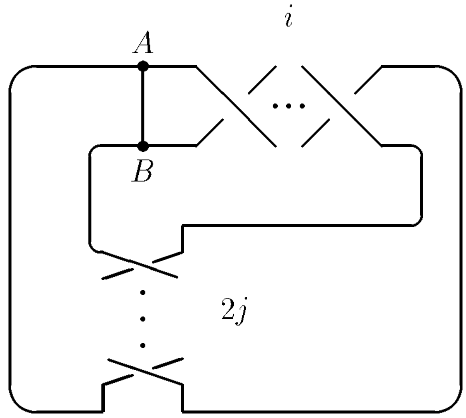

figure 11. *The factor space* H^i_j/τ.

It is not difficult to verify that, if we delete the edge AB from this Θ-curve, we get an alternating knot of order $2j+i$ of the same type as the knot H^j_i of our family. Moreover for the singular set of $H^i_j(n)/\tau$ we get the branching indices two on the bridge and on the edge without self crossings, and index n on the third edge. Therefore we get

$$H^i_j(n)/\tau = \Theta^n_1(H^j_i)$$

Moreover one sees that including the bridge in the considerations that it is as required (see figure 11).

In the cases where τ is a geometrical involution we have the following

Theorem 1. *The orbifold* $\Theta^n_1(H^j_i)$ *has the same geometric structure as the orbifold* $H^i_j(n)$.

By construction the knots H^i_j are results of Dehn surgeries on a hyperbolic 3–component link. Therefore they are hyperbolic for almost all i and j. Hence the orbifolds $H^i_j(n)$ are hyperbolic for almost all i, j and n. In particular one can verify by the program SnapPea [1] that all knots H^i_j with $2i+j \leq 10$, $j \neq 1$ are hyperbolic and the corresponding orbifolds $H^1_2(n)$ are hyperbolic for $n \geq 4$, the remaining fifteen $H^i_j(n)$ for $n \geq 3$.
Moreover we have the following

Corollary 1. Although the knots H^i_1, $i \geq 2$ are the torus knots of order $2i+1$ and therefore not hyperbolic, the dihedral orbifolds $\Theta^n_1(H^i_1)$ with the singular set the torus knot with a 1-bridge are hyperbolic for almost all n.

Corollary 2. Although the knots H^1_j, $j \geq 2$ are hyperbolic knots, the corresponding orbifolds $\Theta^n_1(H^1_j)$ may not be hyperbolic.

5. The geometrical structure on the complement of torus knots with a bridge

As already mentioned, in general, by the notation $\Theta_m(K)$ the Θ-curve is not uniquely determined, but it is not difficult to see that $\Theta_m(T)$ is unique for a torus knot T.

Let T_{2q+1} be the torus knot of the order $2q+1$, $q \geq 1$, defined by it's standard alternating diagram with two $(2q+1)$–gon regions, one of them the exterior, and $(2q+1)$ regions of two–gons. Let b be an m–bridge , $0 \leq m \leq q$ in the $(2q+1)$–gon region. By symmetry we may assume b to be the m–bridge in the exterior region. The Θ–curve with such a diagram we denote by $\Theta_m(T_{2q+1})$. Obviously the knots associated to $\Theta_m(T_{2q+1})$, $1 \leq m \leq q$, are the trivial knot, the torus knot T_{2q+1} and the torus knot T_r where $r = 2q+1-m$ for even m and $r = m$ for odd m. We remind that $\Theta_1(T_5)$ and $\Theta_2(T_5)$ are pictured in figure 9.

By $\Theta^n_m(T_{2q+1})$, $q \geq 2$, we denote the dihedral orbifold with the underlying space S^3 and the singular set the torus knot T_{2q+1} with an m–bridge and branching indices $2, 2, n$ where the associated trivial knot has the index 2.

To determine the geometrical structure of these orbifolds $\Theta^n_m(T_{2q+1})$, let us consider the 6-component non-alternating link on figure 12.

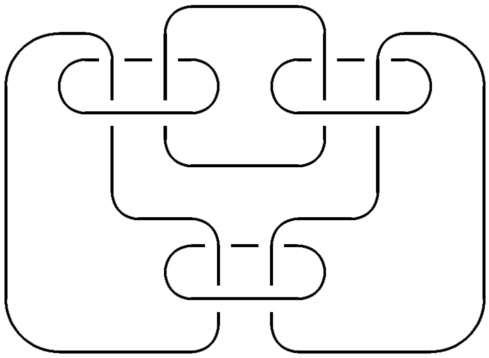

figure 12: *The 6–component link D_6*

This link is D_6 from the family of links considered in chapter 6 of [13], where it's hyperbolicity was shown. Dehn surgeries on three components of this link with the parameters $(1,i),(1,i),(1,j)$ yield a 3-component link, denoted by M^{2i}_{2j}, whose alternating diagram has $4i+2j$ crossings, see figure 13, where again $2i$ and $2j$ denote numbers of half-twists.

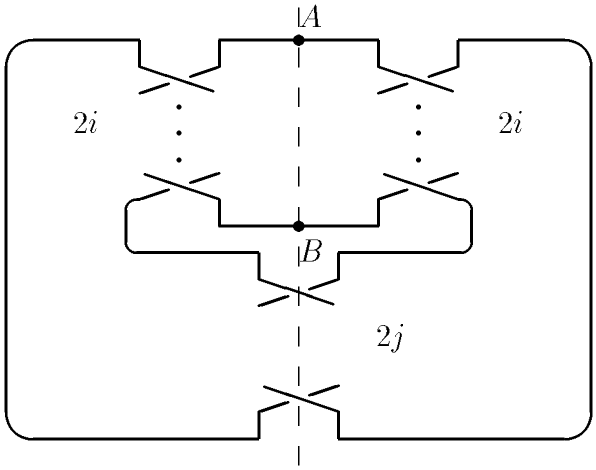

figure 13: *The link* M_{2j}^{2i}

Similar as for the family H_j^i, performing Dehn surgeries on an other 6-component link, we can generate the full family of the alternating links M_j^i with $2i+j$ crossings for any i, j, now not necessary even as above. M_j^i will be a knot if i is odd; a two–component link, if i is even and j odd; and a three–component link if i and j are both even. We remark that by construction $M_j^1 = H_j^1$ and for small values of i and j we have:

$M_1^1 = 3_1$	$M_1^2 = 5_1^2$	$M_1^3 = 7_4$	$M_1^4 = 9_4^2$
$M_2^1 = 4_1$	$M_2^2 = 6_1^3$	$M_2^3 = 8_5$	
$M_3^1 = 5_2$	$M_3^2 = 7_3^2$	$M_3^3 = 9_{35}$	
$M_4^1 = 6_1$	$M_4^2 = 8_1^3$	$M_4^3 = 10_{61}$	
$M_5^1 = 7_2$	$M_5^2 = 9_{10}^2$		
$M_6^1 = 8_1$			
$M_7^1 = 9_2$			
$M_8^1 = 10_1$			

All these knots and links, except of course M_1^1 are hyperbolic [1].
Let us denote by $M_j^i(n)$ the orbifold with underlying space S^3 and the singular set the link M_j^i with branching index n. Obviously also this orbifold has an orientation-preserving symmetry of order two (see figure 13). We denote this isometry by τ again.

Lemma 2. *Let $M_j^i(n)$ be the orbifold with the underlying space S^3 and the link M_j^i with the branching index n as the singular set. Let τ be the orientation–preserving symmetry of order two as above. Then for odd i the factor space $M_j^i(n)/\tau$ is the dihedral Θ–orbifold $\Theta^n_{min(i,2j)}(T_{2j+i})$.*

Proof. On figure 13, where M_j^i for even i and j is presented, the symmetry τ has two fixed points A and B on the link itself. After the factorization by τ we get a link with an m–bridge, where $m = \min(i, 2j)$ (see figure 14).

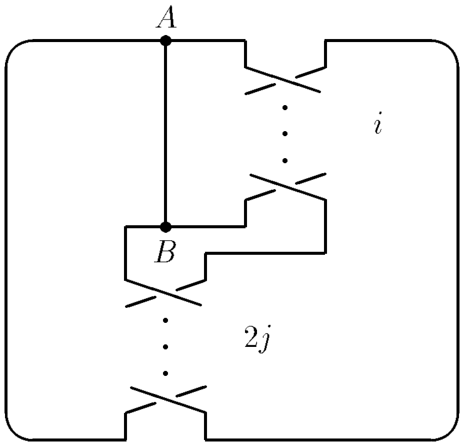

figure 14: *The singular set of $M^i_j(n)/\tau$*

To determine the link we remark that deleting the bridge AB gives an alternating diagram with two $(i+2j)$–gon regions and $i+2j$ two-gon regions. This is the diagram of a torus knot or link of two components with $i+2j$ crossings.

Remark that for even i we get an orbifold whose singular set consists of the plane Θ–curve linked with an S^1.

In the cases when τ is a geometrical involution, we get

Theorem 2. *For $q \geq 1$ the dihedral Θ-orbifold $\Theta^n_m(T_{2q+1})$, $m \leq q$ has the same geometrical structure as the orbifold $M^i_j(n)$, where $m = min(i, 2j)$ and $2j + i = 2q + 1$.*

As the orbifolds $M^i_j(n)$ are results of Dehn surgeries on a hyperbolic link, they are hyperbolic for almost all indices i, j, n. Therefore we have

Corollary 3. The dihedral Θ–orbifolds with the singular set a torus knot with a m–bridge are almost always hyperbolic.

6. Geometrical structures on K_4–curve complements

Geometrical structures of dihedral orbifolds whose singular set is as a graph isomorphic to a cubic graph with four vertices were studied for example in [9] there the singular set is as pictured in figure 15 with index n on one edge and index 2 on the others.

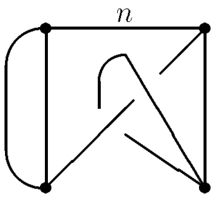

figure 15: *The singular set from [9]*

We will now consider dihedral orbifolds with a K_4–curve as its singular set.

As above let B be the Borromean rings and the knot B^i_j the result of $(1,i)$ and $(-1,j)$ Dehn surgeries on two components of B. As it was shown in paragraph 3, B^i_j has two orientation–preserving symmetries σ and τ of order two with the fixed points A, B and C, D, respectively. Therefore the orbifold $B^i_j(n)$ has the same symmetries and by factorization $B^i_j(n)/\tau/\sigma$ we obtain an orbifold with the singular set as shown in figure 16 where again $2i$ and $2j$ denote the number of half-twists.

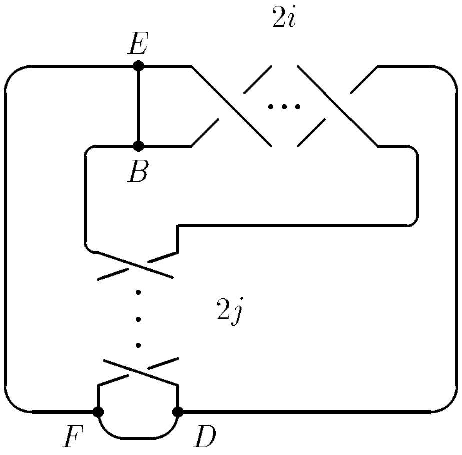

figure 16: *The singular set of* $B^1_1(n)/\tau/\sigma$

E and F are the intersection points of the axe of τ with the axe of σ. As an abstract graph this singular set is isomorphic to the full graph of four vertices K_4. It is possible to redraw this singular set as done for the case $i = j = 1$ in figure 17, where the edge BD has the branching index n and all others edges have index 2.

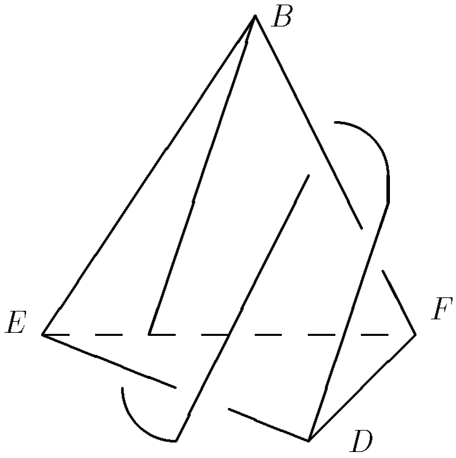

figure 17: *The singular set of* $B^1_1(n)/\tau/\sigma$

Among the seven knots associated with this K_4–curve are six trivial knots and the knot B^i_j, which arises when we delete the edges EB and FD.
As the knots B^i_j are all hyperbolic, we have

Theorem 3. *For almost all i, j, n the dihedral orbifolds $B^i_j(n)/\tau/\sigma$ with underlying space S^3 and the singular set the K_4-curve which is B^i_j-knot with two bridges and branching index n are hyperbolic.*

7. The fundamental groups of the orbifolds

We will use the well-known Wirtinger algorithm (see for example [3]) to obtain a representation of the fundamental group of the complement of the knot H^i_j. Let us consider the elements a, $b \in \pi_1(S^3 \setminus H^i_j)$ before and the elements a_k, $b_k \in \pi_1(S^3 \setminus H^i_j)$ after k-half-twists as on the figure 18. We will name such half-twists negative or positive as they correspond to Dehn surgery with negative or positive parameter according to our agreement.

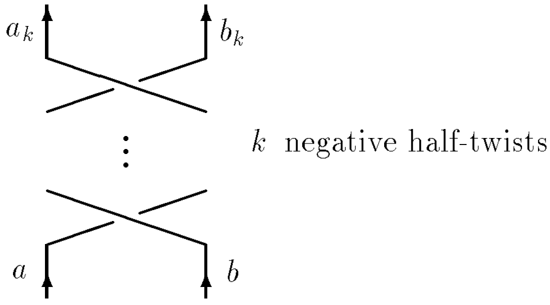

figure 18. Half-twists in the knot diagram.

Now a_k and b_k can be expressed as words in the elements a and b. More precisely by the Wirtinger algorithm we have the following recursive formulas:

$$a_{i+1} = b_i \text{ and } b_{i+1} = b_i^{-1} a_i b_i, \qquad 0 \leq i \leq k-1 \text{ where } a_0 = a,\ b_o = b.$$

Lemma 3. *For k half-twists we may express the paths a_k and b_k in terms of a and b. In particular*
(i) for negative half-twists $b_k = n_k(a,b)$, $a_k = n_{k-1}(a,b)$ where

$$n_k(a,b) = x_1^{-1} x_2^{-1} \dots x_k^{-1} x_{k+1} x_k \dots x_1,$$

with $x_i = \begin{cases} b, & i = 1 \bmod 2 \\ a, & i = 0 \bmod 2 \end{cases}$*,*
(ii) and in analogy for positive half-twists $a_k = p_k(a,b)$, $b_k = p_{k-1}(a,b)$ where

$$p_k(a,b) = y_1 y_2 \dots y_k y_{k+1} y_k^{-1} \dots y_1^{-1},$$

with $y_i = \begin{cases} a, & i = 1 \bmod 2 \\ b, & i = 0 \bmod 2 \end{cases}$.

Proof. We will prove this lemma by induction.
(i) Case $k = 1$ is trivial. Show that the case $k-1$ implies the case k:

$$\begin{aligned} b_k &= b_{k-1}^{-1} a_{k-1} b_{k-1} \\ &= b_{k-1}^{-1} b_{k-2} b_{k-1} \\ &= x_1^{-1} \dots x_k^{-1} x_{k-1} x_k \dots x_1 \\ &= x_1^{-1} \dots x_k^{-1} x_{k+1} x_k \dots x_1 = n_k(a,b) \end{aligned}$$

where we used in the last step that $x_{k+1} = x_{k-1}$.
(ii) The case of positive half-twists can be proved in the same way.

Using this lemma in view of the construction of our family of knots H_j^i we now obtain a representation of the fundamental group of the complement of the knot H_j^i. This group is two generated and has two relations, where one is a consequence of the other and therefore can be omited. That one of the relations is superfluent is not only well-known from the general theory as found for example in [4], but can also be deduced using Lemma 4 as in the proof of Theorem 6, where in fact exactly the equivalence of the two relations is shown. Remark that furthermore our generators are geometric.

Theorem 4. *We have the representation*

$$\pi_1(S^3 \setminus H_j^i) = < a, b \mid w_j^i(a,b) = 1 >,$$

where $w_j^i(a,b) = a^{-1} p_{2i-1}(n_j(a,b), b^{-1})$.

Using this theorem we get for example the following well-known representations. As the trefoil 3_1 is the knot H_1^1 in our notations $\pi_1(S^3 \setminus 3_1) = \pi_1(S^3 \setminus H_1^1)$ is the two generated group with the relation

$$\begin{aligned} 1 &= a^{-1}p_1(n_1(a,b), b^{-1}) \\ &= a^{-1}p_1(b^{-1}ab, b^{-1}) \\ &= a^{-1}b^{-1}abb^{-1}(b^{-1}ab)^{-1} \\ &= a^{-1}b^{-1}ab^{-1}a^{-1}b \end{aligned}$$

which is equivalent to

$$b^{-1}ab^{-1} = ab^{-1}a$$

Therefore, if we denote $c = b^{-1}$, we have:

$$\pi_1(S^3 \setminus 3_1) =< a, c \mid cac = aca >$$

which is the usual representation for example in [4, page 89].

Next consider the figure-eight knot 4_1 as our knot H_2^1, and therefore $\pi_1(4_1)$ is the two generated group with the relation $a^{-1}p_1(n_2(a,b), b^{-1}) = 1$. as above we have:

$$\begin{aligned} 1 &= a^{-1}p_1(n_2(a,b), b^{-1}) \\ &= a^{-1}p_1(b^{-1}a^{-1}bab, b^{-1}) \\ &= a^{-1}b^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}ab \end{aligned}$$

which is equivalent to

$$b^{-1}a^{-1}b^{-1}ab = a^{-1}b^{-1}aba$$

Therefore, if we denote $c = b^{-1}$, we have:

$$\pi_1(S^3 \setminus 4_1) =< a, c \mid ca^{-1}cac^{-1} = a^{-1}cac^{-1}a >$$

which is also the usual representation in [4, page 89].

According to Theorem 4 we obtain using [8] the fundamental group of the orbifold $H_j^i(n)$. We have

Theorem 5. *For any integer i, j, n*

$$\pi_1(H_j^i(n)) =< a, b \mid a^n = b^n = w_j^i(a,b) = 1 >$$

where $w_j^i(a,b) = a^{-1}p_{2i-1}(n_j(a,b), b^{-1})$.

Now we are going to obtain a representation of the fundamental group of the dihedral orbifold $\Theta_1^n(H_j^i)$. To accomplish this we need the following

Lemma 4. *For any $k \geq 1$ we have*

$$\text{i)} \qquad p_{k+1}(\alpha,\beta)\, p_k(\alpha,\beta) = \alpha\,\beta$$

$$\text{ii)} \qquad n_k(\alpha,\beta)\, n_{k+1}(\alpha,\beta) = \alpha\,\beta$$

Proof. By the definition of $p_k(\alpha,\beta)$ and as for any i we have $y_{i+2} = y_i$, hence

$$p_{k+1}(\alpha,\beta)\, p_k(\alpha,\beta) =$$

$$= y_1 ... y_{k+1} y_{k+2} y_{k+1}^{-1} ... y_1^{-1}\, y_1 ... y_k y_{k+1} y_k^{-1} ... y_1^{-1} =$$

$$= y_1 ... y_{k+2} y_k^{-1} ... y_1^{-1} = y_1\, y_2 = \alpha\,\beta$$

and in a similar way

$$n_k(\alpha,\beta)\, n_{k+1}(\alpha,\beta) =$$

$$= x_1^{-1} ... x_k^{-1} x_{k+1} x_k ... x_1 x_1^{-1} ... x_{k+1}^{-1} x_{k+2} x_{k+1} ... x_1 =$$

$$= x_1^{-1} ... x_k^{-1} x_{k+2} ... x_1 = x_2\, x_1 = \alpha\,\beta$$

Determining the fundamental group of the complement of $\Theta_1(H^i_j)$, which is the knot H^i_j with the bridge, we get

Theorem 6. *The bridge is a tunnel of the knot H^i_j.*

Proof. Let us denote elements $a, b, c \in \pi_1(S^3 \setminus \Theta_1(H^i_j))$ as on the figure 19.

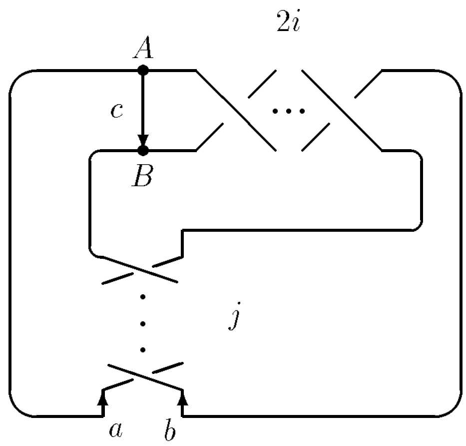

figure 19. *The knot H^i_j with the bridge*

By Lemma 3 we get the following relations in the points A and B:

$$\begin{array}{llll} A: & w_A & = & p_{2i-1}(n_j(a,b),b^{-1})\,a^{-1}\,c^{-1} & =1 \\ B: & w_B & = & p_{2i}(n_j(a,b),b^{-1})\,c\,n_{j-1}(a,b) & =1 \end{array}$$

To prove that these two relations are equivalent, we eliminate c from both equations. Then

$$p_{2i-1}(n_j(a,b),b^{-1})\,a^{-1} = p_{2i}^{-1}(n_j(a,b),b^{-1})\,n_{j-1}^{-1}(a,b)$$

and therefore

$$p_{2i}(n_j(a,b),b^{-1})\,p_{2i-1}(n_j(a,b),b^{-1}) = n_{j-1}^{-1}(a,b)\,a$$

Using Lemma 4 on the left hand side of the equation, we have

$$n_j(a,b)\,b^{-1} = n_{j-1}^{-1}(a,b)\,a$$

and so

$$n_{j-1}(a,b)\,n_j(a,b) = a\,b$$

which is true by Lemma 4.
Therefore

$$\pi_1(S^3 \setminus \Theta_1(H_j^i)) = < a,b,c \mid p_{2i-1}(n_j(a,b),b^{-1})\,a^{-1}\,c^{-1} = 1 >$$

Now using this last relation we may eliminate the generator c from this representation and hence obtain that $\pi_1(S^3 \setminus \Theta_1(H_j^i))$ is a free group of rank two. In other words our bridge AB is a tunnel of the knot H_j^i and the complement of the knot with the bridge is homeomorphic to the handlebody.

Similar considerations together with [8] yield the fundamental group of the orbifold $\Theta_1^n(H_j^i)$:

Theorem 6.

$$\pi_1(\Theta_1^n(H_j^i)) = < a,b \mid a^2 = b^n = (w_j^i(a,b))^2 = 1 >$$

where $w_j^i(a,b) = a^{-1}p_{2i-1}(n_j(a,b),b^{-1})$.

Note that $\pi_1(\Theta_1^n(H_j^i))$ is a generalized triangle group in the sense of [2]. Together with Theorem 1 and Corollary 1 now we have

Corollary 5. For almost all i,j,n the generalized triangle groups

$$< a,b \mid a^2 = b^n = (w_j^i(a,b))^2 = 1 >$$

where $w_j^i(a,b) = a^{-1}p_{2i-1}(n_j(a,b),b^{-1})$, has a faithful representation in $\mathrm{PSL}_2(\mathbb{C})$.

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