Volumes of the Fibonacci manifolds

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Abstract

This paper is devoted to the study of the compact hyperbolic 3-manifolds uniformized by the Fibonacci groups. It is shown that their volumes are equal to volumes of non-compact hyperbolic 3-manifolds arising as complements of some well-known knots. All these volumes are described in terms of the Lobachevsky function.

1. Introduction.

We consider the three dimensional compact orienatable hypebolic manifolds connected with the Fibonacci groups. The Fibonacci groups

$$F(2,m) = < x_1, x_2, ..., x_m : x_i x_{i+1} = x_{i+2},\ i \bmod m >$$

were introduced by J. Conway [2]. First of all the following question was connected with these groups: whether $F(2,m)$ are infinite groups or not. From [3], [6], [10], [8] and [19] it is known that the group $F(2,m)$ is finite if and only if $m = 1, 2, 3, 4, 5, 7$. The algebraic generalizations of $F(2,m)$ were considered in [5].

A new stage in the investigation of the Fibonacci groups is connected with a remakable work of Helling, Kim and Mennicke [8]. They showed that the group $F(2,2n)$, $n \geq 4$, is isomorphic to a discontinuous co-compact subgroup of

$PSL_2(\mathbb{C})$, the full group of orientation-preserving isometries of the hyperbolic space $\mathbb{H}^3$. Moreover the group $F(2,6)$ is isomorphic to a 3-dimentional affine group.

We will call hyperbolic manifolds $M_n = \mathbb{H}^3/F(2,2n)$, $n \geq 4$, uniformized by Fibonacci groups to be **Fibonacci manifolds**.

In [9] it is shown that manifolds M_n are n–fold cyclic covers of the S^3, branched over the figure-eight knot. We note that M_n are isometric to hyperbolic manifolds, discribed by Hempel [7].

We continue the investigation of the algebraical, topological and arithmetical properties of Fibonacci manifolds. We consider the properties of the volumes of the Fibonacci manifolds.

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2. Hyperbolic volumes. Thurston–Jørgensen theorem.

In this section we recall some properties of the volumes of the hyperbolic manifolds. An n-dimensional hyperbolic manifold can be defined as a quotient space $M^n = \mathbb{H}^n/\Gamma$, where Γ is a discrete isometry group of the Lobachevsky space $\mathbb{H}^n$ acting without fixed points. The concepts of hyperbolic area and hyperbolic volume in $\mathbb{H}^2$ and $\mathbb{H}^3$ are carried over naturally to M^2 and M^3. We are interested in the set $\mathcal{M}^n$, $n = 2,\, 3$, of n-dimensional orientable hyperbolic manifolds of the finite volume.

Consider the volume function $v_n : \mathcal{M}^n \to \mathbb{R}$, $n = 2, 3$ which makes correspondence between a manifold $M^n \in \mathcal{M}^n$ and its volume $vol(M^n)$. It is interesting to remark that the volume functions v_2 and v_3 have essentially different properties.

The two-dimensional case is described by the Gauss–Bonnet theorem. If M^2 is a hyperbolic surface of genus g with k points removed, then

$$vol(M^2) = 2\pi\,(2g - 2 + k).$$

Therefore the values of v_2 are positive numbers from the set $2\pi\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, and we have the following picture for the values of v_2 (see Figure 1):

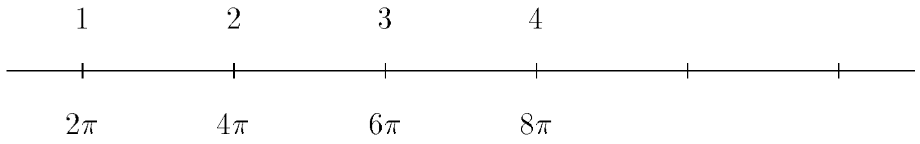

Figure 1. *Values of* v_2.

Given $v_0 = 2\pi n_0$, $n_0 \in \mathbb{N}$, we have only a finite number of non-homeomorphic surfaces M^2 with $vol(M^2) = v_0$. All of them satisfy the following condition:

$$2g - 2 + k = n_0$$

In particular, for each even $n_0 \in \mathbb{N}$, there are compact and non-compact surfaces with the same area $v_0 = 2\pi n_0$.

In the three-dimensional situation we have the following remarkable theorem of Thurston and Jørgensen: *the set of volumes of three-dimensional hyperbolic manifolds form a well-ordered subset of the real line of type* ω^ω. Therefore we have the following picture for v_3 (see Figure 2), where we show some known values:

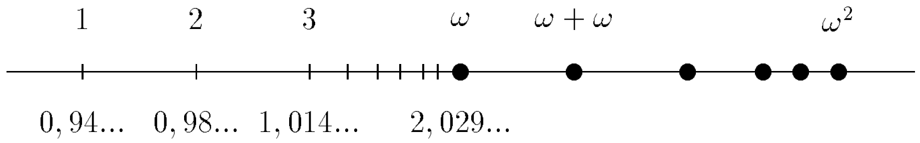

Figure 2. *Values of* v_3.

In particular, it follows from the Thurston–Jørgensen theorem that there exists a three-dimensional hyperbolic manifold with the smallest volume. The minimal known value of volumes of non-compact manifolds (which corresponds to the first limit ordinal number) is equal to the double volume of the regular ideal tetrahedron and it is the volume of the complement of the figure-eight knot. The minimal known value of volumes of compact manifolds is the volume of the Weeks–Matveev–Fomenko manifold [11], [22].

In [11] Matveev and Fomenko conjectured the structure of the initial part of the volume sequence of compact manifolds. But the set from [11] does not contain the Meyerhoff–Neumann manifold [12], which corresponds to the third minimal known volume (Theorem 3).

In [20] Thurston constructed an example of two non-compact manifolds with different numbers of the cusps, but with the same volume. This volume corresponds to the limit ordinal of the set ω^ω. But it was unknown if there are compact manifolds which volumes correspond to limit ordinals. It will be shown (Theorem 1) that the compact Fibonacci manifolds have this property.

3. Fibonacci manifolds as branched coverings.

It was shown by Hilden, Lozano and Montesinos [9] that the Fibonacci manifold M_n has a representation as an n–fold cyclic covering of S^3 branched over the figure-eight knot (see Figure 3). It means that M_n is the n–fold covering of the orbifold $\mathcal{O}(n)$ with underlying space S^3 and the figure-eight knot with index n as its singular set.

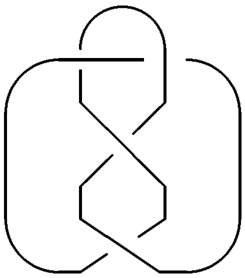

Figure 3. *Figure-eight knot.*

The orbifold $\mathcal{O}(n)$ has the rotation symmetry of order 2 which fixed-point set and the singular set of $\mathcal{O}(n)$ are disjoint. After factorization by this symmetry we obtain the orbifold $6_2^2(2,n)$ with underlying space S^3 and singular set the two-components link 6_2^2 in notation of [18] (see Figure 4) whose components have indices 2 and n respectively.

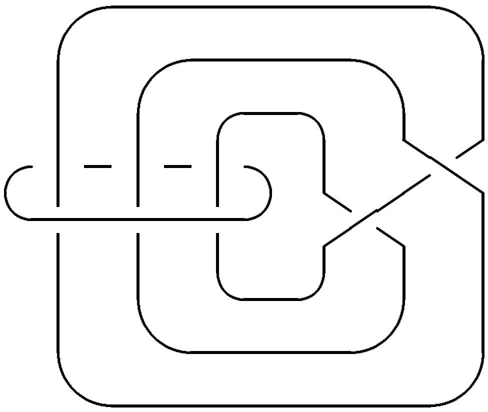

Figure 4. *The link* 6_2^2.

In the general case the orbifold $6_2^2(m,n)$, where $m,n \in \mathbb{N} \cup \{\infty\}$, is the orbifold with underlying space S^3 and singular set the two-component link 6_2^2 with indices m and n, and can be obtained by generalized Dehn surgery with parameters $(m,0)$ and $(n,0)$ on two components of the link 6_2^2. By index ∞ we denote a removed component and in this case we have a non-compact orbifold.

For the Fibonacci manifold M_n and the orbifolds $\mathcal{O}(n)$, $6_2^2(2,n)$ we have the following diagram of the covers (see Figure 5).

$$\begin{array}{c} M_n \\ \big\downarrow n \\ \mathcal{O}(n) \\ \big\downarrow 2 \\ 6_2^2(2,n) \end{array}$$

Figure 5. *The diagram of coverings.*

Therefore for hyperbolic volumes we have:

$$vol\,(M_n) \;=\; n\,vol\,(\mathcal{O}(n)) = 2\,n\,vol\,(6_2^2(2,n)) \tag{1}$$

Now we consider non-compact manifolds connected with the link 6_2^2. Denote by Th_n, $n \geq 2$, the closed 3-strings braid $\left(\sigma_1\sigma_2^{-1}\right)^n$. We note that members of this family Th_n are well-known. In particular, Th_2 is the figure-eight knot, Th_3 are the Borromean rings, Th_4 is the Turkey head knot 8_{18} and Th_5 is the knot 10_{123} according to the notation of [18]. It was shown in [20] that the manifold $S^3 \setminus Th_n$, $n \geq 2$ is hyperbolic and has a representation as an n–fold cyclic covering over the orbifold $6_2^2(n,\infty)$. In particular, for hyperbolic volumes we have:

$$vol\left(S^3 \setminus Th_n\right) = n\,vol\,(6_2^2(n,\infty)) \tag{2}$$

In Section 4 and Section 5 we calculate right-hand sides of (1) and (2).

4. Volumes of the compact orbifolds and manifolds.

In this section we calculate the volumes of the compact hyperbolic orbifolds in terms of the Lobachevsky function.

We remind the reader that an ideal tetrahedron T in $\mathbb{H}^3$ is described completely (up to isometry) by a single complex number z with $\operatorname{Im} z > 0$. We write $T = T_z$. In this case

$$\arg z, \qquad \arg\frac{z-1}{z}, \qquad \arg\frac{1}{1-z}$$

are dihedral angles of T_z (each of which occurs at two opposite edges).

It is well-known [13], [21], that for the hyperbolic volume of the ideal tetrahedron T_z we have following formula, essentially due to Lobachevsky,

$$vol\,(T_z) = \Lambda\,(\arg z) + \Lambda\left(\arg \frac{z-1}{z}\right) + \Lambda\left(\arg \frac{1}{1-z}\right),$$

where

$$\Lambda(x) = -\int_0^x \ln|\,2\,\sin\zeta\,\,|\,d\zeta$$

We recall some properties of $\Lambda(x)$:

$$\Lambda(-x) \,=\, -\Lambda(x), \qquad \Lambda(x+\pi) \,=\, \Lambda(x).$$

The following Lemma 1, Corollary 1 and 2 define the volumes of the hyperbolic orbifolds $\mathcal{O}(n)$, $6_2^2(2,n)$ and the hyperbolic manifolds M_n in terms of the Lobachevsky function $\Lambda(x)$.

Lemma 1. *For $n \geq 4$ the hyperbolic volume of the orbifold $\mathcal{O}(n)$ is equal to*

$$vol\,(\mathcal{O}(n)) = 2\,(\Lambda(\beta+\delta) + \Lambda(\beta-\delta)),$$

where $\delta = \frac{\pi}{n}$, $\quad \beta = \frac{1}{2}\,arccos\left(cos(2\delta) - \frac{1}{2}\right)$.

Proof. Consider the orbifold $\mathcal{O}(n)$ as the result of $(n,0)$-generalized Dehn surgery on the complement of the figure-eight knot. Similarly to [15] the orbifold $\mathcal{O}(n)$ can be obtained by completion of the non-complete hyperbolic structure on the union of two ideal tetrahedra T_z and T_w with complex parameters z and w which satisfy the equations:

$$\begin{cases} z\,w\,(z-1)\,(w-1) = 1, \\ (w\,(1-z))^n = 1, \\ \mathrm{Im}\,z > 0, \\ \mathrm{Im}\,w > 0. \end{cases} \tag{3}$$

From this we obtain the following equation for z:

$$z^2 \,+\, \left(2i\sin\frac{2\pi}{n} \,-\, 1\right)\,z \,+\, e^{-\frac{2\pi i}{n}} \,=\, 0$$

which has the solutions

$$z \,=\, \frac{1}{2} - i\,\sin\left(\frac{2\pi}{n}\right) \,\pm\, i\,\sqrt{1 - \left(\cos\left(\frac{2\pi}{n}\right) - \frac{1}{2}\right)^2}.$$

If we denote $\varphi = \frac{2\pi}{n}$, $n \geq 4$, then we get

$$-1 \,<\, \cos\varphi - \frac{1}{2} \,<\, 1,$$

and we choose ψ, $0 < \psi < \pi$, such that $\cos\psi = \cos\varphi - \frac{1}{2}$. Then

$$z = \frac{1}{2} + i\,(\pm\sin\psi - \sin\varphi)\,,$$

and as we need $\operatorname{Im} z > 0$, we choose the solution with the sign " + " :

$$z = \frac{1}{2} + i\,(\sin\psi - \sin\varphi) \tag{4}$$

and hence

$$w = \frac{\cos\varphi + i\sin\varphi}{\frac{1}{2} - i\,(\sin\psi - \sin\varphi)} \tag{5}$$

We remark that for $n \geq 5$ expressions (4) and (5) satisfy (3), but for $n = 4$ we get $\operatorname{Im} z < 0$ and the corresponding tetrahedron T_z is the "negative". It means that the volume of the orbifold is equal to the difference of volumes of T_w and T_z.

For finding the hyperbolic volume of the ideal tetrahedron T_z with complex parameter z we should know the values of the arguments of the following complex numbers :

$$\arg z, \qquad \arg\frac{z-1}{z}, \qquad \arg\frac{1}{1-z}.$$

Proposition 1. *With notation as above we have:*

$$\arg z = \ arg\frac{1}{1-z} = \frac{\pi - \varphi - \psi}{2},$$

and

$$\arg\frac{z-1}{z} \ = \varphi \ + \ \psi.$$

Proof. From (4) by direct computation we have:

$$\tan\left(\arg z\right) = \frac{\sin\psi - \sin\varphi}{\frac{1}{2}} = 2\ (\sin\psi - \sin\varphi) =$$

$$= 4\cos\frac{\psi+\varphi}{2}\,\sin\frac{\psi-\varphi}{2} = \frac{\cos\frac{\psi+\varphi}{2}\,\sin\frac{\psi-\varphi}{2}}{\frac{1}{4}} =$$

$$= \frac{\cos\frac{\psi+\varphi}{2}\,\sin\frac{\psi-\varphi}{2}}{\frac{1}{2}\,(\cos\psi - \cos\varphi)} = \frac{\cos\frac{\psi+\varphi}{2}\,\sin\frac{\psi-\varphi}{2}}{\sin\frac{\psi+\varphi}{2}\,\sin\frac{\psi-\varphi}{2}} =$$

$$= \cot\frac{\psi+\varphi}{2} \ = \ \tan\frac{\pi-\varphi-\psi}{2}.$$

Similarly, for the second complex paramater we see that

$$\frac{1}{1-z} = \frac{1}{\frac{1}{2} - i\ (\sin\psi - \sin\varphi)} = \frac{\frac{1}{2} + i\ (\sin\psi - \sin\varphi)}{\frac{1}{4} + (\sin\psi - \sin\varphi)^2}$$

and

$$\tan\left(\arg\frac{1}{1-z}\right) = \frac{\sin\psi - \sin\varphi}{\frac{1}{2}} = \tan\frac{\pi - \varphi - \psi}{2}.$$

Therefore

$$\arg z = \arg\frac{1}{1-z} = \frac{\pi - \varphi - \psi}{2}.$$

To prove the second statement of the Proposition we remark that

$$\arg z + \arg\frac{z-1}{z} + \arg\frac{1}{1-z} = 2\pi.$$

Hence

$$\arg\frac{z-1}{z} = \pi - (\pi - \varphi - \psi) = \varphi + \psi,$$

and it complete the proof.

From Proposition 1 we conclude that:

$$vol\,(T_z) = \Lambda\,(\varphi + \psi) + 2\,\Lambda\left(\frac{\pi - \varphi - \psi}{2}\right).$$

Now we consider the tetrahedron T_w with the complex parameter w.

Proposition 2. *With notation as above we have:*

$$\arg w = \arg\frac{1}{1-w} = \frac{\pi - \psi + \varphi}{2},$$

and

$$\arg\frac{w-1}{w} = \psi - \varphi.$$

Proof. Using the Proposition 1, from (5) we obtain:

$$\arg w = \arg\frac{e^{i\varphi}}{1-z} = \varphi + \frac{\pi - \varphi - \psi}{2} = \frac{\pi - \psi + \varphi}{2}.$$

Similarly,

$$\frac{w-1}{w} = 1 - \frac{1}{w} = 1 - \frac{\frac{1}{2} - i\ (\sin\psi - \sin\varphi)}{\cos\varphi + i\ \sin\varphi} = \frac{\cos\varphi - \frac{1}{2} + i\ \sin\psi}{\cos\varphi + i\ \sin\varphi} =$$

$$= \frac{\cos\psi \,+\, i\,\sin\psi}{\cos\varphi \,+\, i\,\sin\varphi} \,=\, e^{i\,(\psi-\varphi)},$$

and

$$\arg\frac{w-1}{w} \,=\, \psi-\varphi.$$

Hence

$$\arg\frac{1}{1-w} \,=\, \pi \,-\, \arg w \,-\, \arg\frac{w-1}{w} \,=$$

$$= \pi \,-\, \frac{\pi-\psi+\varphi}{2} \,-\, (\psi-\varphi) \,=\, \frac{\pi-\psi+\varphi}{2},$$

and proof is complete.

From Proposition 2 we obtain

$$vol\,(T_w) \,=\, \Lambda\,(\psi-\varphi) \,+\, 2\,\Lambda\left(\frac{\pi+\varphi-\psi}{2}\right).$$

So the volume of the orbifold $\mathcal{O}(n)$ is:

$$vol\,(\mathcal{O}(n)) = vol\,(T_z) \,+\, vol\,(T_w) \,=$$

$$= 2\,\Lambda\left(\frac{\pi-\psi-\varphi}{2}\right) \,+\, \Lambda(\varphi+\psi) \,+\, 2\,\Lambda\left(\frac{\pi-\psi+\varphi}{2}\right) \,+\, \Lambda(\psi-\varphi) \,=$$

$$= 2\left(\Lambda\left(\frac{\psi+\varphi}{2}\right) + \Lambda\left(\frac{\psi-\varphi}{2}\right)\right).$$

For the last step we used the following property of the Lobachevsky function [21]:

$$2\,\Lambda(x) \,=\, \Lambda(2x) \,+\, 2\,\Lambda\left(\frac{\pi}{2}-x\right).$$

In order to get the statement of Lemma 1 we denote $\delta = \frac{\varphi}{2}$ and $\beta = \frac{\psi}{2}$, then $\delta = \frac{\pi}{n}$ and $\beta = \frac{1}{2}\,\arccos\left(\cos(2\delta)-\frac{1}{2}\right)$. Therefore,

$$vol\,(\mathcal{O}(n)) = 2\,\left(\Lambda(\beta+\delta) \,+\, \Lambda(\beta-\delta)\right).$$

The proof is complete.

According to the diagram of coverings in figure 5, from Lemma 1 we obtain

Corollary 1. *For $n \geq 4$ the hyperbolic volume of the Fibonacci manifold M_n is equal to*

$$vol\,(M_n) = 2\,n\,\left(\Lambda(\beta+\delta) \,+\, \Lambda(\beta-\delta)\right),$$

where $\delta = \frac{\pi}{n}$, $\beta = \frac{1}{2}\,\arccos\left(\cos(2\delta)-\frac{1}{2}\right)$.

Corollary 2. *For $n \geq 4$ the orbifold $6_2^2(2,n)$ is hyperbolic and*

$$vol\,(6_2^2(2,n)) = \Lambda(\beta+\delta) \, + \, \Lambda(\beta-\delta),$$

where $\delta = \frac{\pi}{n}$, $\beta = \frac{1}{2} \arccos\left(\cos(2\delta) - \frac{1}{2}\right)$.

For some n the arguments of the Lobachevsky function from Lemma 1 have the simple expressions.

Corollary 3. *For $n = 4$ we obtain*

$$vol\,(\mathcal{O}(4)) \; = \; \frac{3}{2} \, \Lambda\left(\frac{\pi}{3}\right).$$

Proof. For $n = 4$ we have $\delta = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$ and by Lemma 1 in this case

$$vol\,(\mathcal{O}(4)) \; = \; 2 \, \left(\Lambda\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \, + \, \Lambda\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \right) \; = \; 2 \, \left(\Lambda\left(\frac{7\pi}{12}\right) \, + \, \Lambda\left(\frac{\pi}{12}\right) \right).$$

We recall that the Lobachevsky function has the following property [13]:

$$\Lambda\,(m\,\theta) \; = \; m \sum_{k=0}^{m-1} \Lambda\left(\theta \, + \, \frac{k\pi}{m}\right). \tag{6}$$

From this for $m = 4$ and $\theta = \frac{\pi}{12}$ by direct computation we obtain:

$$\Lambda\left(\frac{\pi}{4}\right) \; = \; 4 \, \left(\Lambda\left(\frac{\pi}{12}\right) \, + \, \Lambda\left(\frac{\pi}{3}\right) \, + \, \Lambda\left(\frac{7\pi}{12}\right) \, - \, \Lambda\left(\frac{\pi}{6}\right) \right),$$

and for $m = 2$ and $\theta = \frac{\pi}{6}$ we get

$$2\,\Lambda\left(\frac{\pi}{6}\right) \; = \; 3\,\Lambda\left(\frac{\pi}{3}\right),$$

hence,

$$3\,\Lambda\left(\frac{\pi}{3}\right) \; = \; 4 \, \left(\Lambda\left(\frac{7\pi}{12}\right) \, + \, \Lambda\left(\frac{\pi}{12}\right) \right),$$

and using this we complete the proof of our Corollary.

Corollary 4. *For $n = 6$ we obtain*

$$vol\,(\mathcal{O}(6)) \; = \; \frac{8}{3} \, \Lambda\left(\frac{\pi}{4}\right).$$

Proof. For $n = 6$ we have $\delta = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$ and from Lemma 1 we get

$$vol\,(\mathcal{O}(6)) \;=\; 2\,\left(\Lambda\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\,+\,\Lambda\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right) \;=\; 2\,\left(\Lambda\left(\frac{5\pi}{12}\right)\,+\,\Lambda\left(\frac{\pi}{12}\right)\right).$$

For $m = 3$ and $\theta = \frac{\pi}{12}$ from (6) by direct computation we obtain:

$$4\,\Lambda\left(\frac{\pi}{4}\right) \;=\; 3\,\left(\Lambda\left(\frac{5\pi}{12}\right)\,+\,\Lambda\left(\frac{\pi}{12}\right)\right),$$

and using this we get the statement of our Corollary.

Similarly, by the same computation we get

Corollary 5. *For $n = 10$ we obtain*

$$vol\,(\mathcal{O}(10)) \;=\; 2\,\left(\Lambda\left(\frac{3\pi}{10}\right)\,+\,\Lambda\left(\frac{\pi}{10}\right)\right).$$

5. Volumes of the non-compact orbifolds and manifolds.

For the computation of the volume of the manifold $S^3 \setminus Th_n$ we need the following lemma.

Lemma 2. *For $n \geq 2$ the orbifold $6_2^2(n, \infty)$ is hyperbolic and*

$$vol\,(6_2^2(n,\infty)) \;=\; 4\,\left(\Lambda(\alpha+\gamma)+\Lambda(\alpha-\gamma)\right),$$

where $\gamma = \frac{\pi}{2n}$, $\alpha = \frac{1}{2}\,arccos\,(\,cos(2\gamma) - \frac{1}{2})$.

Proof. We recall that the link 6_2^2 can be represented as in Figure 6, where a and τ denote corresponding elements of the $\pi_1(S^3 \setminus 6_2^2)$.

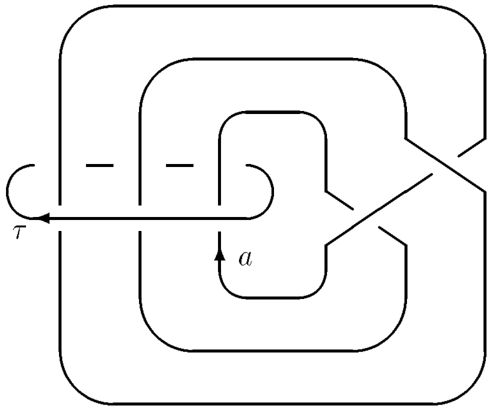

Figure 6. *The link* 6^2_2.

Using the Wirtinger algorithm [4] we obtain the following representation for $\pi_1(S^3 \setminus 6^2_2)$:

$$< a, \tau \mid \left(\tau a^{-1} \tau a \tau^{-1} a \tau^{-1}\right) \left(\tau^2 a^{-1} \tau a \tau^{-1} a \tau^{-2}\right) \left(\tau a^{-1} \tau a \tau^{-1} a \tau^{-1}\right)^{-1} = a > .$$

By choose of new generators u and r such that $a = u^{-1} r^{-1}$ and $\tau = r^{-1}$ this representation is equivalent to

$$\begin{aligned} \pi_1(S^3 \setminus 6^2_2) &=< u, r \mid \ (u\, r^{-1}\, u^{-2})\, (r^{-1}\, u\, r^{-1}\, u^{-2}\, r)\, (u\, r^{-1}\, u^{-2})^{-1} = u^{-1}\, r^{-1} > \\ &=< u, r \mid \ r^{-1}\, (u^2\, r\, u^{-1}\, r\, u^2)^{-1}\, r\, (u^2\, r\, u^{-1}\, r\, u^2) = 1 > . \end{aligned} \tag{7}$$

Now we construct the fundamental domain for this group. Let us consider four ideal regular tetrahedra (see Figure 7).

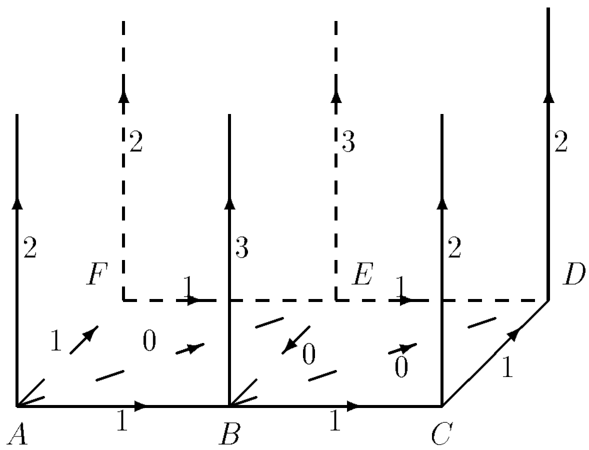

Figure 7. *The fundamental domain.*

Let us denote the ideal vertices of a polyhedron under consideration by A, B, C, D, E, F, and ∞. Let u, v, t, r be isometries of the hyperbolic space $\mathbb{H}^3$ which identify the following pairs of faces of this polyhedron:

$$\begin{array}{llcl}
u: & \mathrm{A\ B\ E} & \rightarrow & \mathrm{E\ D\ B} \\
v: & \mathrm{A\ E\ F} & \rightarrow & \mathrm{B\ D\ C} \\
t: & \mathrm{A\ F}\,\infty & \rightarrow & \mathrm{C\ D}\,\infty \\
r: & \mathrm{A\ B\ C}\,\infty & \rightarrow & \mathrm{E\ F\ D}\,\infty
\end{array}$$

Let Γ be a group generated by u, v, t, and r. By the Poincaré theorem we have the following list of relations for Γ:

$$\begin{array}{lc}
0: & u^2 = v \\
1: & u = r\,v\,t^{-1}\,v\,r \\
2: & t\,r\,t^{-1}\,r^{-1} = 1 \\
3: & r\,r^{-1} = 1
\end{array}$$

There are two classes of equivalence of ideal vertices which link diagrams are represented in Figure 8 and Figure 9.

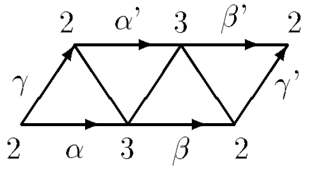

Figure 8.

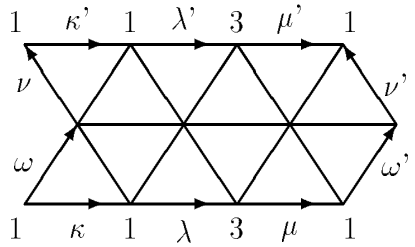

Figure 9.

All triangles in Figure 8 and 9 are regular Euclidean. The edges are paired by Euclidean isometries. So both cusps have complete hyperbolic structure and the following group

$$\begin{aligned}
\Gamma &= \langle\ u,\ v,\ t,\ r \mid u^2 = v,\ u = r\,v\,t^{-1}\,v\,r,\ t\,r\,t^{-1}\,r^{-1} = 1\ \rangle \\
&= \langle\ u,\ t,\ r \mid u = r\,u^2\,t^{-1}\,u^2\,r,\ t\,r\,t^{-1}\,r^{-1} = 1\ \rangle \\
&= \langle\ u,\ r \mid (u^2\,r\,u^{-1}\,r\,u^2)\ r\ (u^2\,r\,u^{-1}\,r\,u^2)^{-1}\ r^{-1} = 1\ \rangle
\end{aligned}$$

has the polyhedron in Figure 7 as its fundamental set in $\mathbb{H}^3$. But this represention is equal to the representation (7) and hence groups $\pi_1(S^3 \setminus 6_2^2)$ and Γ are isomorphic.

Now we are interested in the study of the orbifold $6_2^2(n, \infty)$ which is the result of a $(n, 0)$-generalized Dehn surgery on one of the cusps of the hyperbolic manifold $S^3 \setminus 6_2^2$. It means that the orbifold $6_2^2(n, \infty)$ can be obtained by completion of the non-complete hyperbolic structure on the union of four ideal tetrahedra in figure 7 with complex paramateries z_1, z_2, z_3, z_4 satisfying the some system of algebraic equations. To find this system we consider the link structures on both cusps of the manifold $S^3 \setminus 6_2^2$. For this we denote:

$$z' = \frac{z - 1}{z}, \qquad z'' = \frac{1}{1 - z}.$$

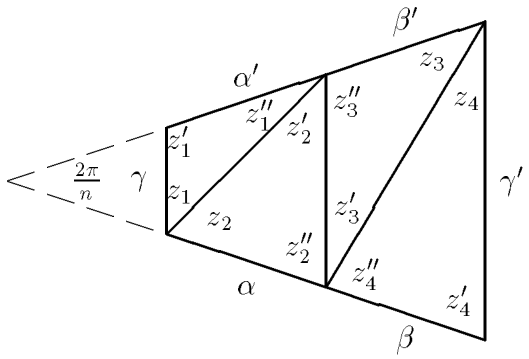

Figure 10. *The $(n, 0)$-generalized surgery on the cusp of $S^3 \setminus 6_2^2$.*

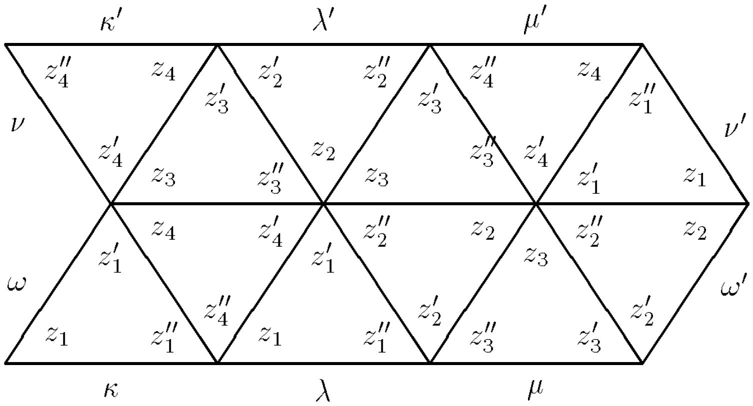

Figure 11. *The complete cusp of $S^3 \setminus 6_2^2$.*

From Figures 10 and 11 we get the following equations:

$$\begin{cases} (z_1 - 1)\, z_2\, z_3\, (z_4 - 1) \;=\; 1 \\ z_1\, (z_2 - 1)\, (z_3 - 1)\, z_4 \;=\; 1 \\ (z_2\, (1 - z_1))^n \;=\; 1 \\ z_3\, (1 - z_1) \;=\; 1 \end{cases} \tag{8}$$

If we denote $\zeta = \frac{1}{1-z_1}$, then from (8) we get:

$$z_1 \;=\; \frac{\zeta - 1}{\zeta}, \quad z_2 \;=\; e^{\frac{2\pi i}{n}}\,\zeta, \quad z_3 \;=\; \zeta, \quad z_4 \;=\; 1 \;-\; \frac{1}{e^{\frac{2\pi i}{n}}\,\zeta}. \tag{9}$$

and the system (8) is reduced to the equation:

$$\left(e^{\frac{\pi i}{n}}\,\zeta \;+\; \frac{1}{e^{\frac{\pi i}{n}}\,\zeta} \;-\; \left(e^{\nu i} + e^{-\nu i} \right) \right)^2 \;=\; 1$$

where $\nu = \frac{\pi}{n}$. Let we choose θ such that $e^{\frac{\pi i}{n}}\,\zeta \;=\; e^{\theta i}$. Then

$$(2\,\cos\theta \;-\; 2\,\cos\nu)^2 \;=\; 1,$$

hence

$$\cos\theta \;=\; \cos\nu \;\pm\; \frac{1}{2}.$$

Since $\cos\theta \leq 1$, we choose the sign "–":

$$\cos\theta \;=\; \cos\nu \;-\; \frac{1}{2}.$$

We substitute $\zeta \;=\; e^{i\,(\theta-\nu)}$ in (9) and obtain:

$$z_1 \;=\; 1 \;-\; \frac{1}{e^{i(\theta-\nu)}}, \quad z_2 \;=\; e^{i(\theta+\nu)}, \quad z_3 \;=\; e^{i(\theta-\nu)}, \quad z_4 \;=\; 1 \;-\; \frac{1}{e^{i(\theta+\nu)}}.$$

By direct computation we get the following result.

Proposition 3. *With notation as above we have*

$$(i) \qquad \arg z_1 \;=\; \arg\frac{z_1 - 1}{z_1} \;=\; \frac{\pi - \theta + \nu}{2}, \qquad \arg\frac{1}{1 - z_1} \;=\; \theta \;-\; \nu;$$

$$(ii) \qquad \arg z_2 \;=\; \theta + \nu, \qquad \arg\frac{z_2 - 1}{z_2} \;=\; \arg\frac{1}{1 - z_2} \;=\; \frac{\pi - \theta - \nu}{2};$$

$$(iii) \qquad \arg z_3 \;=\; \theta - \nu, \qquad \arg\frac{z_3 - 1}{z_3} \;=\; \arg\frac{1}{1 - z_3} \;=\; \frac{\pi - \theta + \nu}{2};$$

$$(iv) \qquad \arg z_4 \;=\; \arg \frac{z_4 - 1}{z_4} \;=\; \frac{\pi - \theta - \nu}{2}, \qquad \arg \frac{1}{1 - z_4} \;=\; \theta \,+\, \nu.$$

As the tetrahedron in $\mathbb{H}^3$ is uniqualy defined by its dihedral angles, we see that $T_{z_1} = T_{z_3}$ and $T_{z_2} = T_{z_4}$. Therefore similarly to Lemma 1 we conclude:

$$vol\left(6_2^2\,(n,\infty)\right) \;=$$

$$= 2\left(\Lambda\,(\theta+\nu) + \Lambda\,(\theta-\nu) + 2\,\Lambda\left(\frac{\pi-\theta-\nu}{2}\right) + 2\,\Lambda\left(\frac{\pi-\theta+\nu}{2}\right)\right) =$$

$$= 4\left(\Lambda\left(\frac{\theta+\nu}{2}\right) + \Lambda\left(\frac{\theta-\nu}{2}\right)\right)$$

Remark that for $n = 2$ we again have the situation of “negative” tetrahedra. In this case the volume of the orbifold is equal to the difference of volumes of tetrahedra.

To finish the proof of Lemma 2 we denote $\gamma = \frac{\nu}{2}$, $\alpha = \frac{\theta}{2}$, then $\gamma = \frac{\pi}{2n}$ and $\alpha = \frac{1}{2}\,arccos\left(\cos(2\gamma) - \frac{1}{2}\right)$. Therefore

$$vol\left(6_2^2\,(n,\infty)\right) \;=\; 4\,\left(\Lambda\,(\alpha+\gamma) \,+\, \Lambda\,(\alpha-\gamma)\right).$$

Corollary 6. *For $n \geq 2$ the hyperbolic volume of the non-compact nanifold $S^3 \setminus Th_n$ is equal to*

$$vol\,(S^3 \setminus Th_n) \;=\; 4\,n\,\left(\Lambda(\alpha+\gamma) + \Lambda(\alpha-\gamma)\right),$$

where $\gamma = \frac{\pi}{2n}$, $\alpha = \frac{1}{2}\,arccos\,(\,cos(2\gamma) - \frac{1}{2})$.

6. Volumes of the Fibonacci manifolds.

The main result of this paper is the following theorem which is a consequence of Lemma 1 and Lemma 2. In fact, comparing Corollary 1 and Corollary 6 we conclude that the following theorem is true.

Theorem 1. *For $n \geq 2$ we have*

$$vol\,(M_{2n}) \;=\; vol\,\left(S^3 \setminus Th_n\right).$$

Therefore the volumes of the compact Fibonacci manifolds correspond to limit ordinals in the Thurston–Jørgensen theorem. In particular, the following assertions hold.

Corollary 7. *The volume of the manifold M_4 is equal to the volume of the complement of the figure-eight knot.*

Corollary 8. *The volume of the manifold M_6 is equal to the volume of the complement of the Borromean rings.*

Many properties of hyperbolic manifolds are defined by arithmeticity or non-arithmeticity of their fundamental group [1]. As proven in [8] and [9] the manifold M_n is arithmetic for $n = 4, 5, 6, 8, 12$ and non-arithmetic otherwise. It is well-known [16] that Th_2, the figure-eight knot, is the only arithmetic knot. It is shown in [20] that Th_3, the Borromean rings, is also arithmetic.

Corollary 9. *We have the following table of manifolds of equal volume:*

n	M_{2n}	$S^3 \setminus Th_n$
2	*arithmetic*	*arithmetic*
3	*arithmetic*	*arithmetic*
4	*arithmetic*	*non-arithmetic*
5	*non-arithmetic*	*non-arithmetic*

Therefore there exist an arithmetic compact manifold and a non-arithmetic non-compact manifold which have the same volume. A. W. Reid kindly informed us about his unpublished results where he constructed compact arithmetic and non-compact arithmetic manifolds with the same volume by number-theoretical methods.

7. Fibonacci manifolds as two-fold coverings.

The following theorem gives one more relation between Fibonacci manifolds M_n and links Th_n.

Theorem 2. *For any $n \geq 2$ the manifold M_n can be represented as a two-fold covering of the S^3 branched over Th_n.*

Consider the orbifold $Th_n(2)$ with underlying space S^3 and singular set geven by the link Th_n with index 2. Then we have the following diagram of coverings (Figure 11).

$$\begin{tikzcd}[column sep=small]
 & M_n \arrow[dl, "2"'] \arrow[dr, "n"] & \\
Th_n(2) \arrow[dr, "n"'] & & \mathcal{O}(n) \arrow[dl, "2"] \\
 & 6_2^2(2,n) &
\end{tikzcd}$$

Figure 11. *The diagram of coverings.*

We recall that Meyerhoff and Neumann [12] have obtained a compact hyperbolic manifold $N = W(3,-2;6,-1)$ by means of Dehn surgery on the Whitehead link which volume approximately (up to 10^{-50}) equals to the volume of the regular ideal tetrahedron. They asked if the volumes are strictly equal and if the manifold N is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. As a consequence of the above results we have affirmative answers on this questions.

Theorem 3. *The Meyerhoff–Neumann manifold is arithmetic over the field* $\mathbb{Q}(\sqrt{-3})$ *and its volume equals to the volume of the regular ideal tetrahedron.*

For the demonstration of this theorem we remark that by the Montesinos theorem [14] any manifold obtained by a Dehn surgery on a strongly invertable link can be represented as a two-fold covering of S^3 branched over some link. This theorem also gives an algorithm for constructing such link. By Theorem 2 the manifold M_4 can be realized as a two-fold covering of the three-sphere branched over the knot Th_4 (see Figure 12).

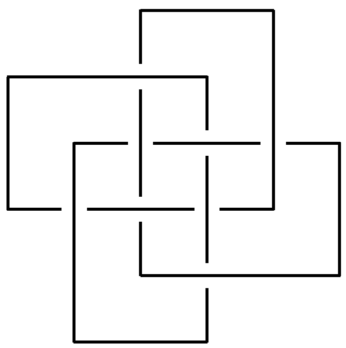

Figure 12. *The knot* $Th_4 = 8_{18}$.

The knot Th_4 has an evident rotational symmetry of order 2. After factorization by this symmetry one gets the two-component link Ω as shown in Figure 13.

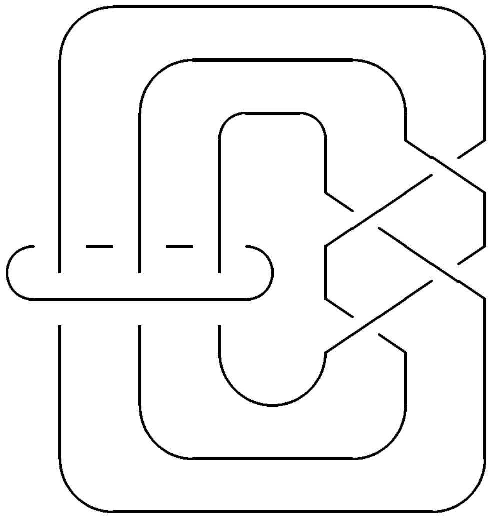

Figure 13. *The link Ω.*

We denote by $\Omega(2,2)$ the orbifold with underlying space S^3 and singular set consisting of the components of Ω labelled by 2. Using the Montesinos algorithm for N we get the following diagram of coverings (see Figure 14):

$$\begin{tikzcd}
 & & M_4 \arrow[d, "2"] \\
N \arrow[dr, "2"'] & & Th_4(2) \arrow[dl, "2"] \\
 & \Omega(2,2) &
\end{tikzcd}$$

Figure 14. *The diagram of coverings.*

So, manifolds N and M_4 are commensurable. Therefore the manifold N is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$, because it is true for M_4 [8]. Moreover by (1) and Corollary 3 we have: $vol(N) = \frac{1}{2} vol(M_4) =$ the volume of the ideal regular tetrahedron.

It is surprising that the manifold N does not appear in the list of Matveev and Fomenko. They conjectured that their list contains all hyperbolic manifolds whose volume not exceed 2,944414.

Corollary 10. *The Fomenko–Matveev conjecture about the structure of the initial part of volume sequence in the Thurston–Jorgensen theorem is not true.*

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