Tutte's 5-flow conjecture for graphs of nonorientable genus 5

Eckhard Steffen *

Abstract

We develop four constructions for nowhere-zero 5-flows of 3-regular graphs which satisfy special structural conditions. Using these constructions we show a minimal counter-example to Tutte's 5-flow conjecture is of order ≥ 44 and therefore every bridgeless graph of nonorientable genus ≤ 5 has a nowhere-zero 5-flow.

One of the structural properties is formulated in terms of the structure of the multigraph $G(\mathcal{F})$ obtained from a given 3-regular graph G by contracting the circles of a 2-factor $\mathcal{F}$ in G.

1 Introduction

All terminology and notation stated not explicitely follows that of Bondy and Murty [1]. Let G be a graph, $\phi : EG \to \{0, 1, \cdots, k-1\}$ be a function and D be an *orientation* of G, i.e. an assignment of a direction to each edge. We define for each vertex $v \in VG$,

$$\phi(v) = \sum_{uv \in D} \phi(uv) + \sum_{vu \in D} -\phi(vu) \tag{1}$$

to be the *weight of v under ϕ* and call (D, ϕ) a *k-weighting* of G. The *deficit set* $\Delta_{(D,\phi)}$ of (D, ϕ) is the set of vertices of G, where $\phi(v) \neq 0$. If $\Delta_{(D,\phi)} = \emptyset$, the k-weighting (D, ϕ) is called a *k-flow* of G. If, in addition, $\phi(e) \neq 0$ for each $e \in EG$ then (D, ϕ) is a *nowhere-zero k-flow.* In 1954 Tutte [6] conjectured the following,

5-Flow Conjecture: *Every bridgeless graph has a nowhere-zero 5-flow.*

There are several results which approximate this conjecture. The best one is due to Seymour [4]. He proved

Theorem 1.1 *Every bridgeless graph has a nowhere-zero 6-flow.*

*Fakultät für Mathematik, SFB 343, Postfach 100131, 33501 Bielefeld, Germany

The next theorem relates edge 3-colorings to nowhere-zero 4-flows for 3-regular graphs, see e.g. [2]. It shows the 5-flow conjecture is best possible.

Theorem 1.2 *A 3-regular graph G has a nowhere-zero 4-flow iff it is edge 3-colorable.*

A bridgeless 3-regular graph, which is not edge 3-colorable is a *snark*, e.g. the Petersen graph is a snark. To give an indirect proof of the 5-flow conjecture, the structure of a possible minimal counter-example has been the subject of many investigations. Celmins proved the following result, see e.g. [2].

Theorem 1.3 *Every minimal counter-example to the 5-flow conjecture is a cyclically 5-edge-connected snark of girth at least 7.*

Steinberg [5] showed the 5-flow conjecture is true for the projective plane. Möller, Carstens and Brinkmann [3] improved this result by proving

Theorem 1.4 *Every 3-regular graph of orientable genus ≤ 2 or non-orientable genus ≤ 4 has a nowhere-zero 5-flow.*

They checked all 3-regular graphs of order ≤ 28 and with girth ≥ 7 with a computer and applied the following lemma.

Lemma 1.1 *Let G be a 3-regular graph embedded in a surface S of minimal genus γ, such that there is no face having a boundary of length < 7. Then the following holds:*
(a) If S is orientable then $|VG| \leq 28(\gamma - 1)$.
(b) If S is non-orientable then $|VG| \leq 14(\gamma - 2)$.

However the lower bound of the order of a minimal counter-example can be easily deduced from the girth requirement and Theorem 1.5, which was proved by Jaeger [2]. It is a consequence of his more general considerations on "nearly F_4-graphs".

Theorem 1.5 *Every bridgeless 3-regular graph with a 2-factor having 0 or 2 odd components has a nowhere-zero 5-flow.*

Since the basic problem is difficult, one direction of research is to prove special results, see Jaeger [2], Section 10. In this paper we pursue this approach and extend the result of Möller, Carstens and Brinkmann [3]. We show

Theorem 1.6 *Every 3-regular graph of orientable genus ≤ 2 or non-orientable genus ≤ 5 has a nowhere-zero 5-flow.*

We specify four constructions for nowhere-zero 5-flows on 3-regular graphs. Using these constructions and Theorems 1.3 and 1.5 we theoretically prove

Lemma 1.2 *Every minimal counter-example to the 5-flow conjecture is of order* ≥ 44.

Furthermore we describe an infinite class of cubic graphs which have 2-factors satisfying special structural conditions. These structural properties guarantee the existence of a nowhere-zero 5-flow.

In the remainder of this text we use graph instead of 3-regular graph and with 3-coloring we mean edge 3-coloring.

2 Constructions

Let (D_1, ϕ_1) and (D_2, ϕ_2) be a k_1- and a k_2-weighting of a graph G, respectively. The *superposition* (D, ϕ) of (D_2, ϕ_2) to (D_1, ϕ_1) is defined as follows. Let

$$D := D_1|_{EG\setminus\{e|\phi_2(e)>\phi_1(e)\}} \cup D_2|_{EG\setminus\{e|\phi_1(e)\geq\phi_2(e)\}},$$

and for $e \in EG$ we define

$$\phi(e) := \begin{cases} \phi_1(e)+\phi_2(e) & \text{if} \quad e \in D_1 \cap D_2 \\ |\phi_1(e)-\phi_2(e)| & \text{if} \quad e \notin D_1 \cap D_2 \end{cases}$$

Clearly, (D, ϕ) is a k-weigthing of G and $k \leq k_1 + k_2$. If (D_1, ϕ_1) and (D_2, ϕ_2) are both flows of G, then (D, ϕ) is a k-flow of G.

In each construction we have to construct a nowhere-zero 4-flow from a given coloring of the graph. We describe a construction, which we will always use.

Remark 2.1 Let $c : EG \to \{1, 2, 3\}$ be a coloring of a graph G. Let $\{a, b, c\} = \{1, 2, 3\}$. We define an (a, b)-circle to be a circle that edges are alternately colored with colors a and b. The set of (a, b)-circles is a 2-factor $\mathcal{F}$ of G, and $\{e|c(e) = c\}$ is the complementary 1-factor $\overline{\mathcal{F}}$.

Choose orientations D_1 and D_2 such that the (3,2)- and the (3,1)-circles are directed circles, respectively. We define two integer functions ϕ_1 and ϕ_2 on the edge set of G as follows:

$\phi_1(e) := 2$, if e is an edge of a (3,2)-circle, $\phi_1(e) := 0$ otherwise, and

$\phi_2(e) := 1$, if e is an edge of a (3,1)-circle, $\phi_2(e) := 0$ otherwise.

Then the superposition of (D_2, ϕ_2) to (D_1, ϕ_1) is a nowhere-zero 4-flow of G. Note that the (former) (3,2)-circles are still directed circles.

Observe that for each of the (3,2)- and (3,1)-circles, we have two different ways to orient the edges to make it into a directed circle. The 4-flow defined above depends on how we orient the edges. In later application, we make use of this fact.

2.1 Construction I

Let $\mathcal{F}$ be a 2-factor of a graph G consisting of circles $C_1, \cdots C_n$, $n \in \mathbb{N}$. We define $G(\mathcal{F})$ to be the multigraph with vertex set $V(G(\mathcal{F})) = \{c_1, \cdots, c_n\}$, $n \in \mathbb{N}$ and between c_i and c_j there are as many edges as between C_i and C_j in G, $(i \leq j)$. That is, $G(\mathcal{F})$ is the multigraph obtained from G by contracting the circles of $\mathcal{F}$ in G.
Let $M = (VM, EM)$ be a multigraph with $\delta \geq 3$ and $U = \{u_1, \cdots, u_{2n}\} \subseteq VM$ $(n \in \mathbb{N})$ be the set of vertices with odd degree in M. We say M is *pairwise odd* iff there is a partition P of U into sets $P_j := \{u_{i_{2j-1}}, u_{i_{2j}}\}$ for $j = 1, 2, \cdots, n$ defined as follows: $P_j \in P :\Longleftrightarrow$ there is a circle C_j in M such that

i) $u_{i_{2j-1}} u_{i_{2j}} \in EC_j$, and

ii) $i \neq j \Longrightarrow C_i \cap C_j = \emptyset$, for all $i, j \leq n$.

Theorem 2.1 *Let G be a graph. If there is a 2-factor $\mathcal{F}$ of G such that $G(\mathcal{F})$ is pairwise odd, then G has a nowhere-zero 5-flow.*

Proof. Let $P = \{P_1, P_2, \cdots, P_m\}$ be a partition of the odd vertices of $G(\mathcal{F})$ into pairs, where $P_j = \{u_{i_{2j-1}}, u_{i_{2j}}\}$. Let $C_j = u_{i_{2j-1}}, u_{i_{2j}}, k_1^j, k_2^j, \cdots, k_{m_j}^j, u_{i_{2j-1}}$ be a circle in $G(\mathcal{F})$, where k_i^j's are vertices of even degree in $G(\mathcal{F})$. Recall that each even (odd) vertex of $G(\mathcal{F})$ corresponds to an even (odd) circle of G, and two vertices of $G(\mathcal{F})$ are adjacent if there is an edge joining the two corresponding circles. Therefore each C_j corresponds to a subgraph S_j of G as depicted in Figure 1, where K_i^j is the circle corresponding to k_i^j, and K_0^j, K_n^j are the circles corresponding to $u_{i_{2j-1}}, u_{i_{2j}}$, respectively.
Let $e_i := u_i u_i'$ be an edge between K_i^j and K_{i+1}^j, for $i = 0, 1, \cdots, n-1$, where $u_i \in VK_i^j$ and $u_i' \in VK_{i+1}^j$ and let v_i and v_i' be a neighbor of u_i in K_i^j and of u_i' in K_{i+1}^j, respectively. The edge between K_n^j and K_0^j is $e^j := w_j w_j'$, where $w_j \in VK_n^j$ and $w_j' \in VK_0^j$. In the following we write K_i instead of K_i^j for $i = 0, \cdots, n$.
Construction I:
STEP 1: We put in new vertices a_i and a_i' in $u_i v_i$ and $u_i' v_i'$, respectively and join them by an edge, for $i = 0, 1, \cdots, n-1$. We obtain a graph G' and a 2-factor $\mathcal{F}'$ of G' containing even circles only. Hence G' is 3-colorable.

STEP 2: We color the edges of $\overline{\mathcal{F}'}$ with color 1 and the edges of the circles of $\mathcal{F}'$ with colors 2 and 3 alternately, such that the edges $u_i a_i$ and $u_i' a_i'$ are of the same color. This is always possible, as we can color K_0 first, then K_1 and so on up to K_n. Note that each e_j is contained in a (3,1)- or in a (2,1)-circle of length 4.

STEP 3: We construct a nowhere-zero 4-flow (D, ϕ) on G' as described in Remark 2.1. We take the following under consideration. We arbitrarily choose directions for all the (3,2)- and (3,1)-circles, except for those (3,1)-circles of length 4 containing e_i

and $a_i a'_i$. The direction of these (3,1)-circles depends on the direction of the (3,1)-circle containing e^j. Without loss of generality we assume e^j is directed from w_j to w'_j. For each i such that e_i and $a_i a'_i$ are contained in a (3,1)-circle of length 4, we choose a direction of this circle so that the edge $a_i a'_i$ is directed from a'_i to a_i.

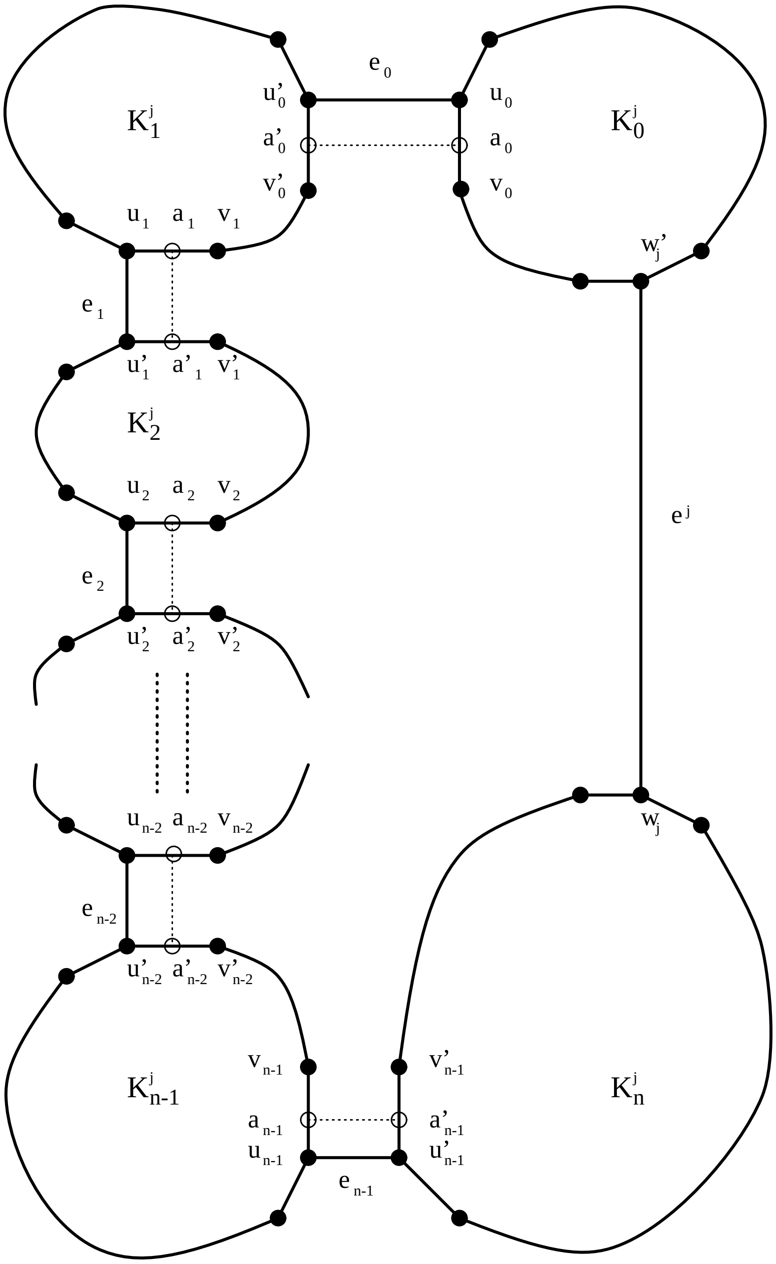

Figure 1: Substructure S_j.

STEP 4: If for some $i \in \{1, \cdots, n-1\}$ e_i and $a_i a'_i$ are not contained in a (3,1)-circle of length 4, then they are contained in a (2,1)-circle O_i of length 4. Let this be the case for $i_1, \cdots, i_m$. We choose an orientation $\tilde{D}$ such that for each $j \in \{1, \cdots, m\}$ O_{i_j} is a directed circle and $a_{i_j}a'_{i_j} \notin D \cap \tilde{D}$. We define $\tilde{\phi}(e) = 1$, if $e \in \bigcup_{j=1}^{m} EO_{i_j}$ and $\tilde{\phi}(e) = 0$, otherwise. The superposition of $(\tilde{D}, \tilde{\phi})$ to (D, ϕ) is a 4-flow (D', ϕ') on G with $\phi'(a_i a'_i) = 0$, $\phi'(e_i) \in \{0, 2\}$, and $\{\phi'(u_i a_i), \phi'(u'_i a'_i)\} \subseteq \{1, 3\}$. Notice, K_i and K_{i+1} are still directed circles.

STEP 5: We reconstruct G from G' by removing the additional vertices a_i, a'_i for $i = 0, 1, \cdots, n-1$. We transfer (D', ϕ') to G, to obtain a 4-weighting $(\hat{D}, \hat{\phi})$, defined as follows. The orientation $\hat{D}$ should be identical to D' on
$EG \setminus (\{v_i u_i | 0 \leq i \leq n-1\} \cup \{v'_i u'_i | 0 \leq i \leq n-1\})$ and for $0 \leq i \leq n-1$ we define $v_i u_i \in \hat{D}$ if $v_i a_i \in D'$ and $u_i v_i \in \hat{D}$ otherwise, and $v'_i u'_i \in \hat{D}$ if $v'_i a'_i \in D'$ and $u'_i v'_i \in \hat{D}$, otherwise.
For $e \in EG$ we define
$\hat{\phi}(e) := \phi'(e)$, if $e \in EG \setminus (\{v_i u_i | 0 \leq i \leq n-1\} \cup \{v'_i u'_i | 0 \leq i \leq n-1\})$, and
$\hat{\phi}(v_i u_i) := \phi'(v_i a_i)$, $\hat{\phi}(v'_i u'_i) := \phi'(v'_i a'_i)$ for $i = 0, \cdots, n-1$.
Then holds $\Delta_{(\hat{D}, \hat{\phi})} \subseteq \{u_i, u'_i | i = 0, \cdots, n-1\}$ and for each $1 \leq i \leq n-1$ there is a directed path p_i from u'_{i-1} to u_i, containing edges of K_i only. Further there is a directed path p_n from u'_{n-1} to u_0, consisting of e^j and of edges from K_n and K_0.

STEP 6: Case 1: $\Delta_{(\hat{D}, \hat{\phi})} = \emptyset$. We distinguish two cases.
If $\hat{\phi}(e_i) \neq 0$, for all $i \in \{0, \cdots, n-1\}$, then $(\hat{D}, \hat{\phi})$ is a nowhere-zero 4-flow of G.
If $\hat{\phi}(e_i) = 0$ for an $i \in \{0, \cdots, n-1\}$ we define

$$D^* := \hat{D}|_{EG \setminus \{e_i | 0 \leq i \leq n-1\}} \cup \{u_i u'_i | 0 \leq i \leq n-1\}, \text{ and}$$

$$\phi^*(e) = 1 \text{ if } e \in \bigcup_{i=1}^{n} Ep_i \cup \{e_i | 0 \leq i \leq n-1\} \text{ and } \phi^*(e) = 0, \text{ otherwise.}$$

The superposition of (D^*, ϕ^*) to $(\hat{D}, \hat{\phi})$ is a nowhere-zero 5-flow of G.

Case 2: $\Delta_{(\hat{D}, \hat{\phi})} \neq \emptyset$.
There is a smallest number $h \geq 0$ such that $\hat{\phi}(u'_h) \neq 0$. By the construction we have $\hat{\phi}(u'_h) = 1$. Let $k > h$ be the smallest number such that $\hat{\phi}(u_k) \neq 0$. Clearly $\hat{\phi}(u_k) = -1$, and for each $h < i < k$ holds $\hat{\phi}(e_i) \in \{0, 2\}$. We define

$$\phi^k(e) := 1 \text{ if } e \in \bigcup_{i=h+1}^{k} Ep_i \cup \{u_i u'_i | h < i < k\} \text{ and } \phi^k(e) = 0, \text{ otherwise.}$$

The superposition of (D^*, ϕ^k) to $(\hat{D}, \hat{\phi})$ is a 5-weighting (D_k, ϕ_k) with $\Delta_{(D_k, \phi_k)} \subset \Delta_{(\hat{D}, \hat{\phi})}$. Notice, for $1 \leq i \leq n$ the path p_i is still a directed path.
We repeat this procedure, starting from u'_k, where $\phi_k(u'_k) = 1$.

If there is a smallest $l > k$ such that $\hat{\phi}(u_l) \neq 0$, we proceed with k instead of h as described above.
If there is no such l it holds $\Delta_{(D_k,\phi_k)} = \{u'_k, u_h\}$ and $\phi_k(u_h) = -1$. We define

$$\phi^0(e) := 1 \text{ if } e \in \bigcup_{i=k+1}^{n} Ep_i \cup \bigcup_{i=1}^{h} Ep_i \cup \{u_i u'_i | 0 \leq i < h \vee k < i < n\} \text{ and}$$

$$\phi^0(e) = 0, \text{ otherwise.}$$

The superposition of (D^*, ϕ^0) to (D_k, ϕ_k) is a nowhere-zero 5-flow of G.
Since all substructures in G, which are induced by the circles in $G(\mathcal{F})$, are disjoint, we eventually get a nowhere-zero 5-flow of G.

□

Given a multigraph M with $\delta \geq 3$, let $\mathcal{G}(M)$ be the set of 3-regular graphs obtained from M by expanding the vertices of M to circles. As an immediate consequence of the theorem above we get

Corollary 2.1 *Let M be a pairwise odd multigraph with $\delta \geq 3$, then each $G \in \mathcal{G}(M)$ has a nowhere-zero 5-flow.*

2.2 Three Further Constructions

The motivation for the development of constructions of nowhere-zero 5-flows on cubic graphs is the proof of Lemma 1.2. Unfortunately Construction I is not strong enough for this task. That is, we have to consider graphs G and a 2-factor $\mathcal{F}$ of G such that $G(\mathcal{F})$ is not pairwise odd. Clearly, if there is a substructure like S_j, see Figure 1, in G, Construction I is locally applicable on this substructure. The following constructions are for special cases. For example, if the odd circles of $G(\mathcal{F})$ can be paired such that condition i) of the definition of pairwise odd is satisfied for $G(\mathcal{F})$, but condition ii) is not.
The idea behind these constructions is the same as for Construction I. So we use the steps $1, \cdots, 6$ as given for Construction I for the description of these constructions. We will leave out those steps in the description which are not necessary for a construction. We have to require special conditions in steps of the constructions, especially in step 2 and 3. These conditions cannot be formulated in terms of multigraphs very well. So we avoid this formulation and underline the special requirements in the following descriptions.

2.2.1 Construction II

Let G be a graph and $\mathcal{F}$ be a 2-factor of G. Suppose that two odd circles U_1 and U_2 of $\mathcal{F}$ are connected by exactly one edge and by a path of length three. This path contains exactly one edge of a third circle $C \in \mathcal{F}$, different to U_1 and U_2.
Construction II yields a (local) nowhere-zero 5-flow without using any edge of C. The advantage is, C could be an odd circle or it could be used for the construction of a (local) nowhere-zero 5-flow with respect to two other odd circles, using Construction I.
Let u_1, v, w, u_2 be the vertices of the path of length 3 , and let $e' = w_1w_2$ be the edge between U_1 and U_2 and $u_i, w_i \in VU_i$, and let v_i be the neighbor of u_i in U_i, for $i = 1, 2$.

STEP 1: We put in new vertices a_1 and a_2 in u_1v_1 and u_2v_2, respectively and join them by an edge. We obtain a 3-colorable graph G' and a 2-factor $\mathcal{F}'$ of G'.

STEP 2: We color the edges of $\overline{\mathcal{F}'}$ with color 1 and the edges of the circles of $\mathcal{F}$ with colors 2 and 3 alternately, such that u_1a_1 and u_2a_2 are colored with color 3. This is always possible.
Requirement: vw is colorable with color 3.
Then vw, u_1a_1, u_2a_2 are in a (3,1)-circle C^* of length 6.

STEP 3: We construct a nowhere-zero 4-flow (D, ϕ) on G' as usual. Suppose, with no loss of generality, e' is directed from w_1 to w_2, then we direct C^*, such that a_1a_2 is directed from a_1 to a_2.

STEP 5: We reconstruct G from G' by removing the vertices a_1 and a_2. Analogously to step 5 of Construction I, we extend (D, ϕ) to a 4-weighting $(\hat{D}, \hat{\phi})$ of G. It holds $\Delta_{(\hat{D},\hat{\phi})} = \{u_1, u_2\}$.

STEP 6: It holds $\hat{\phi}(u_1) = 1$ and $\hat{\phi}(u_2) = -1$. There is a directed path p from u_1 to u_2 consisting of e' and of edges of $EU_1 \cup EU_2$. We define $\phi'(e) := 1$ if $e \in Ep$ and $\phi'(e) = 0$, otherwise. The superposition of $(\hat{D}, \phi')$ to $(\hat{D}, \hat{\phi})$ is a nowhere-zero 5-flow of G.

$\square$

2.2.2 Construction III

Construction III is applicable to four odd circles of an 2-factor $\mathcal{F}$ of G which form a 'star'. Let U, U_1, U_2, U_3 be four odd circles.
If there are two nonadjacent edges v_1v_2 and w_1w_2 in U such that v_i and w_i are adjacent to a vertex of U_i, $i = 1, 2$, and between U_3 and U are more than $\frac{1}{2}|VU_3|$ edges, then G has a nowhere-zero 5-flow.
Since $|VU|$ is odd, there is a path p of odd length from v_1v_2 to w_1w_2 in U. Choose $x \in \{v | v \in VU \wedge \exists w \in VU_3 : vw \in EG\}$, such that the distance between x and v_1v_2

is minimal in $U - Ep$. Let y be the neighbor of x in U_3 and let x' and y' be the neighbor of x in U and of y in U_3, respectively.

STEP 1: For $i = 1, 2$, let w_i be adjacent to u_i in U_i, and let u_i' be a neighbor of u_i in U_i. We put in new vertices a_1 and a_2 in u_1u_1' and u_2u_2', respectively and two new vertices b and c in xx' and yy', respectively, and add the edges a_1a_2 and bc. We obtain a graph G' with 2-factor $\mathcal{F}'$ containing even circles only.

STEP 2: We color the edges of $\overline{\mathcal{F}'}$ with color 1. The edges of the circles are colored with colors 2 and 3 alternately such that $v_1v_2, w_1w_2, a_1u_1, a_2u_2$ are colored with color 3 and xb and yc are of the same color. This is always possible and w_1w_2, a_1u_1, a_2u_2 are in a common (3,1)-circle C of length 6.

STEP 3: We construct a nowhere-zero 4-flow (D', ϕ') on G' as described in Remark 2.1, where we take the following under consideration. Choose D_1 such that there is directed path of odd length (in U) from v_1v_2 to w_1w_2 and the direction of C depends on the direction of the (3,1)-circle containing v_1v_2.
<u>Requirement:</u> Choose D_2 such that $v_1v_2 \in D_1 \cap D_2$ or $v_2v_1 \in D_1 \cap D_2$. (If $\mathcal{F}$ contains only these four odd circles this is always possible. A consequence of this requirement is $\phi'(v_1v_2) = 3$.)
Suppose, with no loss of generality, v_1v_2 is directed from v_1 to v_2. Then direct C such that a_1a_2 is directed from a_1 to a_2. If bc and xy are in (3,1)-circle of length 4 direct this circle such that bc is directed from c to b.

STEP 4: If bc and xy are in (2,1)-circle of length 4 proceed as in step 4 of Construction I to obtain a 4-flow (D', ϕ') of G with $\phi'(bc) = 0$.

STEP 5: We reconstruct G from G' by removing a_1, a_2, b and c. We extend (D', ϕ') to a 4-weighting $(\hat{D}, \hat{\phi})$ of G as usual. Then holds $\Delta_{(\hat{D},\hat{\phi})} \subseteq \{u_1, u_2, x, y\}$.

STEP 6: It holds $\hat{\phi}(u_1) = 1$ and $\hat{\phi}(u_2) = -1$, and there is a directed path p from u_1 to u_2, which contains v_1v_2 as the only edge of a circle of the 2-factor different to U_1 and U_2. We define $\phi(e) := 1$, if $e \in Ep$ and $\phi(e) = 0$, otherwise. The superposition of $(\hat{D}, \phi)$ to $(\hat{D}, \hat{\phi})$ is a 5-weighting (D_1, ϕ_1) on G with $\Delta_{(D_1,\phi_1)} \subseteq \{x, y\}$ and $\{v_1v_2\} = \{e|\phi_1(e) = 4\}$.
If $\phi_1(x) = \phi_1(y) = 0$ and $\phi_1(xy) \neq 0$, then (D_1, ϕ_1) is a nowhere-zero 5-flow of G.
If $\phi_1(x) = \phi_1(y) = 0$ and $\phi_1(xy) = 0$ we proceed as follows. Since $\phi_1(e) = 1$ for each edge between U_3 and U, and there are more than $\frac{1}{2}|VU_3|$ edges between U and U_3, there is an edge between these two circles, which is directed from U_3 to U and hence there is a directed path p from y to x which does not contain v_1v_2. Define $\phi_2(e) := 1$ if $e \in Ep \cup \{xy\}$ and $\phi(e) = 0$, otherwise. The superposition of $(D_1|_{EG\setminus\{xy\}} \cup \{xy\}, \phi_2)$ to (D_1, ϕ_1) is a nowhere-zero 5-flow of G.
If $\phi_1(x) = \phi_1(y) \neq 0$, then holds $\phi(x) = -1$, $\phi(y) = 1$ and $xy \in D_1$. We construct a nowhere-zero 5-flow of G analogously to the case above.

□

2.2.3 Construction IV

Construction IV is a variation of Construction III. Let U_1 and U_2 be two odd circles, which are not connected by an edge. Suppose there are two edges $e_1 = v_1v_2$, $e_2 = u_1u_2$, such that $e_1, e_2 \in \bigcup_{C\in\mathcal{F}} EC$ and v_1, u_1 and v_2, u_2 are adjacent to U_1 and to U_2, respectively. Let w_i be the vertex in U_i adjacent to v_i and x_i be a neighbor of w_i in U_i, for $i = 1, 2$.

STEP 1: For $i = 1, 2$ put in two new vertices a_i in w_ix_i and join them by an edge. We obtain a graph G' which is 3-colorable.

STEP 2: We color G' as usual where the following condition must be satisfied.
<u>Requirement 1:</u> The edges e_1 and e_2 are both colorable with color 3.
Then e_1, w_1a_1, w_2a_2 are in a $(3, 1)$-circle C of length 6.

STEP 3: Let w be the vertex in U_1 adjacent to u_1. We construct a nowhere-zero 4-flow (D, ϕ) on G' as usual, where the direction of C depends on the direction of the (3,1)-circle containing wu_1.
<u>Requirement 2:</u> $u_1u_2 \in D_1 \cap D_2$ or $u_2u_1 \in D_1 \cap D_2$. (This implies $\phi(u_1u_2) = 3$).
Let wu_1 be directed from w to u_1. We direct C such that a_1a_2 is directed from a_1 to a_2.

STEP 5: We reconstruct G from G' by removing a_1 and a_2. We extend (D, ϕ) to a 4-weighting $(\hat{D}, \hat{\phi})$ of G as usual. Then holds $\Delta_{(\hat{D},\hat{\phi})} = \{w_1, w_2\}$.

STEP 6: We have $\hat{\phi}(w_1) = 1$ and $\hat{\phi}(w_2) = -1$, and there is a directed path p from w_1 to w_2, which contains u_1u_2 as the only edge of a circle of the 2-factor different to U_1 and U_2. We define $\phi_1(e) = 1$ if $e \in Ep$ and $\phi_1(e) = 0$, otherwise. The superposition of $(\hat{D}, \phi_1)$ to $(\hat{D}, \hat{\phi})$ is a nowhere-zero 5-flow of G.

□

2.3 Proof of Lemma 1.2

Suppose there is a minimal counter-example G to the 5-flow conjecture of order ≤ 42. For two circles C_1 and C_2 of a 2-factor of G let $m(C_1, C_2)$ denote the number of edges connecting C_1 and C_2.

Case 1: There is a 2-factor $\mathcal{F}$ of G consisting of four odd circles A, B, C and D. By Theorem 1.3, each of the circles A, B, C, D has length ≥ 7. As $m(A, B) + m(A, C) + m(A, D) \geq 7$, at least one of $m(A, B), m(A, C), m(A, D)$ is greater or equal to 3. Without loss of generality we assume $m(A, B) \geq 3$. Similary one of $m(A, C), m(B, C), m(C, D)$ is greater than 2. If $m(C, D) \geq 3$ we use Construction I to get a nowhere-zero 5-flow of G. Thus we may assume $m(A, C) \geq 3$. Again one of $m(A, D), m(B, D), m(C, D)$ is greater than 2. The only case that Construction I can not be applied is that $m(B, D), m(C, D) \leq 1$. Therefore $m(A, D) \geq 3$ and

hence $m(B,C) \leq 1$.
If more than one circle of B, C, D have length ≥ 9 then $|VG| \geq 2 \cdot 9 + 7 + 2 \cdot 7 + 5 = 44$, a contradiction.
Suppose one of B, C, D has length 9, say B, and $|VC|, |VD| = 7$. For $X, Y \subseteq VG$ let X_Y be the set of vertices of X which are adjacent to vertices of Y, i.e. $X_Y := \{x \in X | \exists y \in Y : xy \in EG\}$. Then A_C and A_D must be independent sets, for otherwise G contains a circle of length ≤ 6. Furthermore, if $m(C,D) = 1$, then $A_C \cup A_D$ must be an independent set, for otherwise we can use Construction II to obtain a nowhere-zero 5-flow of G. Since $|A_C|, |A_D| \geq 5$ it follows $|VA| \geq 21$ and hence $|VG| \geq 23 + 21 = 44$, a contradiction. Therfore $m(C,D) = 0$.
If $m(B,C) = 0$ or $m(B,D) = 0$ it follows $|VG| \geq 44$. Thus $m(B,C), m(B,D) = 1$. If there is an edge x, y of A such that $x \in A_B$ and $y \in A_C \cup A_D$, then we again use Construction II. Thus no vertex of A_B is adjacent to a vertex of $A_C \cup A_D$. This implies $|VA \setminus (A_B \cup A_C \cup A_D)| \geq 2$. Therefore $|VG| \geq 2 \cdot 6 + 7 + 2 \cdot 7 + 9 + 2 = 44$, a contradiction.
Thus it holds $|VB|, |VC|, |VD| = 7$. If $m(B,C) = m(B,D) = m(C,D) = 1$, then, as above, we may conclude that $A_B \cup A_C \cup A_D$ is an independent set. As $|A_B|, |A_C|, |A_D| \geq 5$ we have $|VA| \geq 31$ and therefore $|VG| \geq 52$, a contradiction.
If $m(B,C) = m(B,D) = m(C,D) = 0$, then the four circles form a star, and we can use Construction III.
Thus without loss of generality we may assume that $m(B,C) = 1$ and $m(B,D) = 0$. Then $m(A,B) = 6$ and $A_B \cup A_C$ is an independent set of size 11. Therefore $|VA| \geq 23$ and $|VG| \geq 44$, a contradiction.

Case 2: There is a 2-factor of G consisting of four odd circles A, B, C, D and one even circle E.
Notice, girth$(G) \geq 7$ implies that two circles of length 7 are connected by at most 2 edges in G. In the following we often make use of this fact, without mention it explicitely.

Case 2.1: $|VA| = |VB| = |VC| = |VD| = 7$ and $|VE| \leq 14$.
There are at least 7 edges between the odd circles. Therefore none of $m(A,E)$, $m(B,E), m(C,E)$ and $m(D,E)$ is equal to 0. We now claim that $m(A,B) + m(C,D) \leq 2$. For otherwise, suppose $m(A,B) = 2$ and $m(C,D) \geq 1$. It is easy to see that we can apply Construction I. Analogously $m(A,C) + m(B,D) \leq 2$ and $m(A,D) + m(B,C) \leq 2$. Thus there are at most 6 edges between the odd circles, a contradiction.

Case 2.2: $|VA| = 9$, $|VB| = |VC| = |VD| = 7$ and $|VE| \leq 12$.
There are at least 9 and at most 11 edges between the odd circles, and between B, C and D are at most 6 edges. Therefore there are at least 3 edges between A and the three other odd circles.

If $m(A,X)=1$ for all $X\in\{B,C,D\}$ it follows $m(B,C)=m(B,D)=m(C,D)=2$ and therefore Construction I yields a nowhere-zero 5-flow of G.
Thus we may assume $m(A,B)\geq 2$.
(*) If Construction I is applicable on C,D or on C,D,E, we are done. Assume it is not applicable on C,D and on C,D,E then follows $m(C,D)\leq 1$ and $m(C,D)=0\vee m(C,E)=0\vee m(D,E)=0$.
Suppose $m(C,D)=1$. If $m(C,E)=0$, it holds $m(A,C)=4$ and $m(B,C)=2$, since $m(A,C)\leq 4$. There are at most 11 edges between the odd circles and therefore $m(A,E),m(B,E),m(D,E)\neq 0$. Thus if there is an edge between D and A or B, G has a nowhere-zero 5-flow, by applying Construction I in an appropriate way.
Hence $m(A,D)=m(B,D)=0$ and therefore $m(D,E)=6$. Since G is cyclically 5-edge connected it follows $|VE|=12$ and $m(A,B)=2$. But now Construction IV is applicable on B,D and E, and Construction I on A and C and G has a nowhere-zero 5-flow, a contradiction. If $m(D,E)=0$ we argue analogously.
Thus $m(C,D)=0$. Suppose $m(A,C)\geq 2$, then analogously follows $m(B,D)=0$ and therefore $m(A,D)\geq 1$.
If $m(A,D)\geq 2$ and therefore $m(B,C)=0$ it follows $m(A,E)=0$ and $|VE|=12$. It follows one of $m(B,E)+m(C,E),m(B,E)+m(D,E),m(C,E)+m(D,E)$, say $m(C,E)+m(D,E)$, is ≥ 8. Then Construction IV is applicable on C,D and E and Construction I on A and B.
Thus $m(A,D)=1$ and again follows $|VE|=12$, since $m(D,E)=6$ and E_D is an independent set. If $m(B,C)=0$ we argue as before. If $m(B,C)\geq 1$ it follows $m(A,E)\neq 0$ and therefore Construction II is applicable on A,D and E, and Construction I is applicable on B,C or on B,C and E.
Thus $m(A,C),m(A,D)\leq 1$ and therefore $m(A,E)\geq 1$, since $m(A,B)\leq 4$. From this and $m(B,C)+m(B,D)\leq 4$, we infer $m(A,C)=1$ or $m(A,D)=1$. Let $m(A,C)=1$. This implies $m(B,D)\geq 1$. Since we have to put in at least 9 edges it follows $m(A,B)=4,m(B,C)=2$ and $m(A,D),m(B,D)=1$. Thus Construction I is applicable on A,D,E and on B,C, a contradiction.

Case 2.3: $|VA|=|VB|=9$, $|VC|=|VD|=7$ and $|VE|\leq 10$.
There are at least 11 and at most 12 edges between the odd circles and it holds $m(C,E),m(D,E)\leq 5$.
If $m(A,C),m(A,D),m(B,C),m(B,D)\leq 1$ it follows $m(C,D)\geq 1$, since $m(A,B)\leq 6$. Since $m(C,D)\leq 2$ Construction I is applicable on A,B and on C,D or C,D,E. Thus we may assume, with no loss of generality, $m(A,C)\geq 2$ and therefore, analogously to Case 2.2 (*), $m(B,D)\leq 1$ and $m(B,D)=0\vee m(B,E)=0\vee m(D,E)=0$.
Suppose $m(B,D)=1$, then $m(B,E)=0$ or $m(D,E)=0$.
If $m(B,E)=0$ it follows $m(A,D)+m(B,D)+m(C,D)=1$ and $|VE|=10$ or $m(A,D)+m(B,D)+m(C,D)=2$ and $|VE|=8$. In both cases holds $m(D,E)\geq \frac{1}{2}|VE|$. But E_D is an independent set, a contradiction.

If $m(D,E) = 0$ it follows $m(A,D) \geq 4$. Since A_D is an independent set it follows $m(A,D) \leq 4$. Thus $m(A,D) = 4$ and therefore $m(C,D) = 2$. Since G is cyclically 5-edge connected we have $m(B,E) \leq 7$ and hence $m(A,B) \geq 1$ or $m(B,C) \geq 1$. It is easy to see Construction I is applicable in both cases, since there are at most 12 edges between the odd circles, a contradiction.
Thus $m(B,D) = 0$. Since $m(A,C) \leq 4$ and $m(C,D) \leq 2$ it follows $m(A,B) + m(B,C) \geq 5$. Further holds $m(B,C) \leq 4$ and therfore $m(A,B) \geq 1$.
If $m(A,B) = 1$ it follows $m(B,C) = 4$. Since there are at least 11 edges between the odd circles and $7 \leq m(A,C) + m(B,C) + m(C,D)$ it follows $m(A,D) \geq 2$. Construction I is applicable on A, D and B, C, a contradiction.
Thus $m(A,B) \geq 2$ and therefore $m(C,D) \leq 1$ and $m(A,D) \geq 1$. Since $|VA| = 9$ there are at least 2 edges between B, C, D, i.e. $m(B,C) + m(C,D) \geq 2$. But $m(A,D) \geq 1$ implies $m(C,D) = m(B,C) = 1$ and $m(C,E), m(D,E) \neq 0$ in each case. Hence Construction I is applicable on A, B and on C, D, E, a contradiction.

Case 2.4: $|VA| = |VB| = |VC| = 9$, $|VD| = 7$ and $|VE| = 8$.
There are exactly 13 edges between the odd circles and it holds $m(D,E) \leq 4$. Thus there are at least 3 edges between D and the other odd circles.
Suppose $m(A,D) \geq 2$. Analogously to Case 2.2 (*) we infer $m(B,C) \leq 1$ and $m(B,C) = 0 \vee m(B,E) = 0 \vee m(C,E) = 0$.
Suppose $m(B,C) = 1$, then $m(B,E) = 0$ or $m(C,E) = 0$. At this point there is difference between B and C, so we may assume $m(B,E) = 0$. It follows $m(A,B) \geq 4$, since $m(B,D) \leq 4$. Between C and the other odd circles are at least 3 edges. Thus $m(A,C) \geq 1$ and therefore $m(B,D) \geq 2$, since $|VA| = 9$. This implies $m(A,C) = m(C,D) = 1$. Now Construction I is applicable on B and D and on A, C and E.
Thus $m(B,C) = 0$. Since $m(C,E) \leq 4$ and $m(B,E) \leq 6$ it holds $m(A,B) \geq 2 \vee m(B,D) \geq 2$ and $m(A,C) \geq 2 \vee m(C,D) \geq 2$. If $m(A,B) \geq 2$ then follows $m(A,C) \geq 2$. Since $|VA| = 9$ there are 4 edges between B, C and D and therefore $m(B,D) \geq 2 \vee m(C,D) \geq 2$. Thus Construction I yields a nowhere-zero 5-flow. If $m(B,D) \geq 2$ we argue analogously.
Thus $m(A,D), m(B,D), m(C,D) \leq 1$. Since $m(D,E) \leq 4$ it follows $m(A,D) = m(B,D) = m(C,D) = 1$. For at least one of A, B, C, say A holds $m(A,E) \neq 0$. Thus Construction I is applicable on A, D and E and therefore $m(B,C) \leq 1$. But there are at least $13 - 4$ edges between A, B and C and all them must be incident to A, which is impossible.

Case 2.5: $|VA| = 11$, $|VB| = |VC| = |VD| = 7$ and $|VE| \leq 10$.
There are at least 11 and at most 12 edges between the odd circles and at least 5 edges between A and the three circles of length 7.
We may assume $m(A,B) \geq 2$, with no loss of generality. Analogously to Case 2.2 (*) we infer $m(C,D) \leq 1$ and $m(C,D) = 0 \vee m(C,E) = 0 \vee m(D,E) = 0$.

Suppose $m(C,D)=1$. It follows one of $m(C,E), m(D,E)$ is equal to 0. With no loss of generality assume $m(C,E)=0$. Then holds $m(A,C)\geq 4$ and therefore $m(B,D)\leq 1$. On the other side, $m(A,C)\leq 5$ and therefore $m(B,C)\geq 1$. Further, there are at least 2 edges between D and the other odd circles.
If $m(A,D)\geq 1$ it is easy to see that Construction I is applicable on A,D or on A,D,E. Since $m(A,C)+m(A,B)\geq \frac{1}{2}|VA|$ and A_C, A_B are independent sets Construction II is applicable on B,C and A.
Thus $m(A,D)=0$ and therefore $m(B,D)=1$ and $m(B,E)=5$. Thus $|VE|=10$, since E_B is an independent set. Since $m(B,C)\leq 2$ it follows $m(B,E)\neq 0$. Thus Construction I is applicable on A,C and on B,D,E. That is, G has a nowhere-zero 5-flow, a contradiction.
Thus $m(C,D)=0$. Since $|VB|=7$ and there at least 11 edges between the odd circles one of $m(A,C), m(A,D)$, say $m(A,C)$, is ≥ 2. It follows $m(B,D)\leq 1$ and hence $m(A,D)\geq 1$, since $m(D,E)\leq 5$.
Suppose $m(B,D)=1$. Either Construction I is applicable on B,D and E or $m(B,E)=0 \vee m(D,E)=0$. It holds $m(D,E)\neq 0$, for otherwise we have $m(A,D)=6$ and therefore $|VE|\geq 12$, since A_D is an independent set, a contradiction. If $m(B,E)=0$ it follows $m(B,C)\geq 1$. Since $m(B,C)\leq 2$, it follows $m(A,D)+m(A,B)\geq \frac{1}{2}|VA|$. Thus Construction II is applicable on B,C and A. Further Construction I is applicable on A,C,E, since $m(A,E), m(C,E)\neq 0$.
Thus $m(B,D)=0$ and hence $m(A,D)\geq 2$ and $m(B,C)\leq 1$. If $m(B,C)=1$ it follows $m(B,E), m(C,E)\neq 0$, since A_C and A_B are independent sets. Therefore Construction I is applicable on A,D and on B,C and E.
Thus $m(B,C)=0$ and therefore $m(A,E)=0$. Then Construction III is applicable on A,B,C,D.

Case 2.6: $|VA|=11$, $|VB|=9$, $|VC|=|VD|=7$ and $|VE|=8$.
There are 13 edges between the odd circles.
Suppose $m(A,C)\geq 2$, then analogously to Case 2.2 (*) we infer $m(B,D)\leq 1$ and $m(B,D)=0 \vee m(B,E)=0 \vee m(D,E)=0$.
Suppose $m(B,D)=1$ then $m(B,E)=0$ or $m(D,E)=0$.
Suppose $m(B,E)=0$. Then holds $m(A,B)\geq 4$ and therefore $m(C,D)\leq 1$. G is cyclically 5-edge connected. This implies $m(A,B)\leq 7$ and hence $m(B,C)\geq 1$. Since $m(D,E)\leq 4$ we have $m(A,D)\geq 1$.
If $m(A,D)=1$ it follows $m(C,D)=1$ and either Construction I is applicable on C,D and E, or $m(C,E)=0$, since $m(D,E)\neq 0$. But then follows $m(B,C)\geq 2$ and Construction I is applicable on B,C and on A,D and E.
Thus $m(A,D)\geq 2$ and hence $m(A,B)=7$, $m(B,C)=1$, $m(A,C)=2$ and therefore $m(A,D)=2$. But then Construction II is applicable on B,C and A, since A_B and A_C are independence sets. Construction I is applicable on A and D. Thus $m(B,E)\neq 0$ and therefore $m(D,E)=0$ (and $m(B,D)=1$).

Then holds $m(C,D) \geq 1$ and $m(A,D) \geq 4$ and therefore $m(B,C) \leq 1$. Since $m(B,E) \leq 6$ it follows $m(A,B) \geq 1$.
If $m(C,D) = 2$, Construction I is applicable on A, B and E or $m(A,E) = 0 \vee m(B,E) = 0$. In both cases there are more than 13 edges between the odd circles, a contradiction.
Thus $m(C,D) = 1$ and hence $m(A,D) = 5$. This implies $m(A,D) + m(A,C) \geq \frac{1}{2}|VA|$ and since A_C and A_C are independent sets, Construction II is applicable on C, D and A. It is easy to see that $m(A,B) \geq 2$ or $m(A,E), m(B,E) \neq 0$ holds. Therefore Construction I is applicable on A, B or on A, B and E.

Thus $m(B,D) = 0$. Since $m(B,E) \leq 6$ and $m(D,E) \leq 4$ it follows $m(A,B) \geq 2 \vee m(B,C) \geq 2$ and $m(A,D) \geq 2 \vee m(C,D) \geq 2$.
If $m(A,B) \geq 2$ then $m(C,D) \leq 1$ and therefore $m(A,D) \geq 2$ and $m(B,C) \leq 1$. Since there are 13 edges between the four odd circles, there are at least 2 edges between B, C and D and hence $m(B,C) = m(C,D) = 1$. Construction I is applicable on C, D and E or $m(A,C) = 5 \vee m(A,D) = 5$ In both cases the application of Construction II on C, D and A, and of Construction I on A, B yields a nowhere-zero 5-flow of G, a contradiction.
If $m(B,C) \geq 2$ it follows $m(A,D) \leq 1$ and therefore $m(C,D) \geq 2$ and $m(A,B) \leq 1$. But there are at least $13 - 7$ edges between A, B and D. Thus Construction I is always applicable.

Thus $m(A,C), m(A,D) \leq 1$ and therefore $m(A,B) \geq 2$, since $m(A,E) \leq 7$. Construction I is applicable on C, D and E or $m(C,D) \leq 1$ and $m(C,D) = 0 \vee m(C,E) = 0 \vee m(D,E) = 0$.
If $m(C,E) = 0$ or $m(D,E) = 0$ it follows $m(B,C) \geq 5$ or $m(B,D) \geq 5$. But $|VB| = 9$ and B_C and B_D are independent sets, a contradiction.
Thus $m(C,D) = 0$ and therefore $m(B,C), m(B,D) \geq 2$, since $m(C,E), m(D,E) \leq 4$. But there are 4 edges between A, C and D. Thus $m(A,C) \geq 2 \vee m(A,D) \geq 2$, and Construction I always yields a nowhere-zero 5-flow of G, a contradiction.

Case 2.7: $|VA| = 13$, $|VB| = |VC| = |VD| = 7$ and $|VE| = 8$.
There are exactly 13 edges between the odd circles. Since $m(B,C) + m(B,D) + m(C,D) \leq 6$, we may assume $m(A,B) \geq 2$, with no loss of generality. Analogously to Case 2.2 (*), we infer $m(C,D) \leq 1$ and $m(C,D) = 0 \vee m(C,E) = 0 \vee m(D,E) = 0$.
Let $m(C,D) = 1$. If $m(C,E) = 0$, then $m(A,C) \geq 4$ and therefore $m(B,D) \leq 1$. Since $m(D,E) \leq 4$ it follows $m(A,D) = m(B,D) = m(C,D) = 1$ or $m(A,D) \geq 2$.
If $m(A,D) = m(B,D) = m(C,D) = 1$, either Construction I is applicable on A, C and B, D, E, or $m(B,E) = 0$. Then holds $m(B,C) = 2$ and $m(A,E), m(A,D) \neq 0$ and Construction I is applicable on B, C and on A, D and E.
Thus $m(A,D) \geq 2$ and hence $m(B,C) \leq 1$ and $m(A,C) \geq 5$. Since A_C and A_D are independent sets, Construction II is applicable on C, D and A and Construction I is applicable on A and B.

If $m(D,E)=0$ we argue analogously.
Thus $m(C,D)=0$. Since $m(C,E), m(D,E) \leq 4$ it follows $m(A,C) \geq 2 \vee m(B,C) = 2$ and $m(A,D) \geq 2 \vee m(B,D) = 2$.
If $m(A,C) \geq 2$ it follows $m(A,D) \geq 2$. If there is an edge between B and C or D, say C, then either Construction I is applicable on A, D and on B, C and E or $m(B,E) = 0 \vee m(C,E) = 0$. But then holds $m(A,B) + m(A,C) \geq \frac{1}{2}|VA|$. Since A_B and A_C are independent sets, Construction II is applicable on B, C and A and Construction I is applicable on A, D, a contradiction. Thus $m(B,C)+m(B,D) = 0$. But now Construction III is applicable on the star A, B, C, D, a contradiction.
If $m(B,C) = 2$ it follows $m(B,D) = 2$. But there are at least 5 edges between A and C, D. Thus $m(A,C) \geq 2 \vee m(A,D) \geq 2$, a contradiction.

Case 3: There is a 2-factor of G consisting of six odd circles A, B, C, D, E and F. All these circles are of length 7, and each of them is connected to at least two other circles by two edges.
Consider the graph G^* obtained from G by contracting $A, \cdots, F$ to vertices $a, \cdots, f$ and connecting two vertices x and y by an edge if the circles X and Y are connected by two edges in G.
Case 3.1: If G^* is 2-regular, then it is a circle of length 6 or it consists of two triangles. In the first case G^* has an 1-factor. The application of Construction I on the pairs of circles corresponding to the vertices of the edges of the 1-factor of G^* yields a nowhere-zero 5-flow.
In the second case let a, b, c be the vertices of one triangle and d, e, f be the vertices of the other. Now we argue in G. Then in A there are two adjacent vertices u and w such that u is adjacent to one of E, D or F and w is adjacent to B or C. Let u be adjacent to E and w be adjacent to B. Furthermore there is an edge between B and E. Thus Construction II is applicable to B, E and A, and Construction I to A, C and to D, F.
Case 3.2: If there are two vertices of degree 3 in G^* then they are not adjacent, otherwise the removal of this edge leads to Case 3.1. It is easy to see all possible graphs have a 1-factor.
Case 3.3: If there are more than two vertices of degree 3 then at least two of them are adjacent. The removal of the edges between vertices of degree leads to Case 3.2 or 3.1.

$\square$

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