A Note on Comparison Theorems of Splittings and Multisplittings of Hermitian Positive Definite Matrices

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Abstract. We discuss iterative methods for the solution of the linear system $Ax = b$, which are based on a single splitting or a multisplitting of A. In order to compare different methods, it is common to compare the spectral radius of the iterative matrix. For M–matrices A and weak regular splittings there exist well-known comparison theorems. Here, we give a comparison theorem for splittings of Hermitian positive definite matrices. Furthermore, we establish a comparison theorem for multisplittings of a Hermitian positive definite matrix.

§1 Introduction and Definitions

If we consider linear systems of equations $Ax = b$, $A \in \mathbb{C}^{n,n}, x, b \in \mathbb{C}^n$, that occur in the numerical analysis, in many cases the matrix $A = [a_{ij}]$ is an M–matrix, i.e. $A \in \mathbb{R}^{n,n}$, $a_{ij} \le 0$ for $i \ne j$, A^{-1} exists and is nonnegative, or A is a Hermitian positive definite matrix. Stimulated by O. Taussky in 1958 [T], a lot of common properties of these two classes of matrices were found, as for instance, determinant inequalities for principal submatrices of A. Here, we consider iterative methods for approximating the solution of the linear system $Ax = b$. We establish results for Hermitian positive definite matrices, which corresponds with well-known results for M–matrices.

One class of iterative methods is based on a *splitting* (M, N) of A, i.e. $A = M - N$, M nonsingular. Then, the iteration has the form

$$x^{i+1} = M^{-1}Nx^i + M^{-1}b \quad i = 0, 1, 2, \ldots \tag{1.1}$$

It is well-known that (1.1) converges for all x_0 if and only if $\rho(M^{-1}N) < 1$ [V], [BP]. Here, $\rho(M^{-1}N)$ denotes the spectral radius of $M^{-1}N$.

A splitting (M, N) of A is called *regular* if $M^{-1} \ge 0$ and $N \ge 0$, and *weak regular* if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$. Furthermore, Ortega [O] called a splitting *P-regular* if $M^H + N$ is Hermitian positive definite. Here, we use the notation $M \ge 0$ for a matrix to be nonnegative and $M \succ 0$ ($M \succeq 0$) for a matrix to be Hermitian positive (semi-) definite. For both sets of splittings, there exist well-known convergence theorems for the sequence (1.1):

Theorem 1.1 (Varga [V]) *Let $A \in \mathbb{R}^{n,n}$ and (M, N) be a weak regular or a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.*

Theorem 1.2 (Householder - John [H],[J]) *Let $A \in \mathbb{C}^{n,n}$ be Hermitian and let (M, N) be a P-regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A \succ 0$.*

In order to get a parallel algorithm for the sequence (1.1), O'Leary and White introduced in [OW] the concept of *multisplittings*. They called $(M_k, N_k, E_k)_{k=1}^m$ a multisplitting of A if for each k (M_k, N_k) is a single splitting of A, and E_k is a nonnegative diagonal matrix with $\sum_{k=1}^m E_k = I$. Here, I denotes the identity matrix. Then, the iteration to solve $Ax = b$ becomes

$$x^{i+1} = \sum_{k=1}^{m} E_k(M_k^{-1}N_k x^i + M_k^{-1}b). \tag{1.2}$$

Using the matrices

$$H = \sum_{k=1}^{m} E_k M_k^{-1} N_k, \quad G = \sum_{k=1}^{m} E_k M_k^{-1}, \tag{1.3}$$

we can express, if G is nonsingular, the multisplitting $(M_k, N_k, E_k)_{k=1}^m$ as a single splitting $(G^{-1}, G^{-1}H)$ of A. Multisplittings have been discussed in several recent papers, for example the papers of Neumann and Plemmons [NP], White [W1], [W2], Elsner [E] and Elsner, Bru and Neumann [BEN]. O'Leary and White established in [OW] the following theorem:

Theorem 1.3 *i) If, for $k = 1, \ldots, m$, (M_k, N_k) is a weak regular splitting of a matrix A satisfying $A^{-1} \geq 0$, then (1.2) is convergent.*
ii) If, for $k = 1, \ldots, m$ (M_k, N_k) is a P-regular splitting of a symmetric positive definite matrix A and $E_k = \alpha_k I$, then (1.2) is convergent.
iii) If, for $k = 1, \ldots, m$, $||(M_k, N_k)||_\infty < 1$, then (1.2) is convergent. Here, $||\cdot||_\infty$ is the maximum row sum matrix norm.

Moreover, White gives in [W2] some more convergence results for Hermitian positive definite matrices satisfying a special zero pattern, i.e. matrices in dissection form or in nested dissection form. But here, we consider general Hermitian positive definite matrices and general multisplittings.

In order to compare the asymptotic convergence rate of (1.1) and (1.2), it is common to compare $\rho(M^{-1}N)$ for different single splittings or $\rho(H)$ for different multisplittings. For M–matrices, or more general for monotone matrices, and weak regular splittings there exist well-known comparison theorems. A matrix A is called *monotone*, if A^{-1} exists and $A^{-1} \geq 0$. In this note, we give a comparison theorem for single and multisplittings of Hermitian positive definite matrices. A matrix A is *positive definite*, if for all $x \in \mathbb{C}^n, x \neq 0$, $Re(x^H Ax) > 0$. Here, $Re(z)$ denotes the real part of the complex number z. These theorems have the same structure as the well-known comparison theorems for monotone matrices or M–matrices. The difference between these theorems for monotone matrices and for Hermitian positive definite matrices is just the different partial ordering of the two classes of matrices. Finally, we mention that the results given here, are part of [N].

§2 Single Splittings

Comparison theorems for regular splittings of monotone matrices were first introduced by Varga [V] in 1960 followed by the work of Csordas and Varga [CV], Miller and Neumann [MN] and Elsner [E]. Here, we mention the theorems due to Varga and Elsner:

Theorem 2.1 *Let $A^{-1} > 0$ and let $(M_1, N_1), (M_2, N_2)$ be two regular splittings of A. If $0 \leq N_1 \leq N_2$, $N_1 \neq 0, N_2 - N_1 \neq 0$, then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1$.*

Theorem 2.2 *Let $A^{-1} \geq 0$ and $A = M_1 - N_1 = M_2 - N_2$ be weak regular splittings. In either of the following cases*
i) $N_1 \leq N_2$
ii) $M_1^{-1} \geq M_2^{-1}, \quad N_1 \geq 0$
iii) $M_1^{-1} \geq M_2^{-1}, \quad N_2 \geq 0$
the inequality $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$ holds.

Since A^{-1} is a nonnegative matrix, the partial ordering for matrices M and N of the splittings used above, is quite natural. It is the partial ordering of nonnegative matrices, which is defined using the ordering of $\mathbb{R}$ elementwise.

To develop comparison theorems for Hermitian positive definite matrices, we also need an ordering of these matrices. Here, we use the partial ordering introduced by Löwner [L]. Let $B, C \in \mathbb{C}^{n,n}$ Hermitian then, $B \succeq C$ if and only $B - C$ is positive semidefinite. Similar, $B \succ C$ if and only if $B - C$ is positive definite. Furthermore, we need the following lemma ([HJ], Corollary 7.7.4):

Lemma 2.3 *Let $A, B \in \mathbb{C}^{n,n}$ with $A \succ 0$ and $B \succ 0$. Let $\lambda_k(A), \lambda_k(B)$ be the eigenvalues of A and B, then*
(i) $A \succeq B \succ 0$ if and only if $0 \prec A^{-1} \preceq B^{-1}$,
(ii) if $A \succeq B$, then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, \ldots, n$, if the respective eigenvalues $\lambda_k(A), \lambda_k(B)$ of A and B are arranged in the same (increasing or decreasing) order.

Lemma 2.3 remains true if we replace all weak inequalities by sharp inequalities. We are now ready to state our main result of this note.

Theorem 2.4 *Let $A \succ 0$ and let $(M_1, N_1), (M_2, N_2)$ be two splittings of A. If $0 \preceq N_1 \preceq N_2$ then,*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1. \tag{2.4}$$

If $0 \preceq N_1 \prec N_2$ then,

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1. \tag{2.5}$$

Proof: Since $0 \preceq N_1 \preceq N_2$, $A \prec 0$ and $M_i = A + N_i$, $i = 1, 2$, we obtain

$$0 \prec M_1 \preceq M_2.$$

Hence, both splittings are P-regular splittings and $\rho(M_i^{-1}N_i) < 1$ for $i = 1, 2$. Furthermore, the matrices $M_i^{-1}N_i$ are similar to the Hermitian positive semidefinite matrices $M_i^{-\frac{1}{2}}N_iM_i^{-\frac{1}{2}}$. Thus, $\sigma(M_i^{-1}N_i) \subset \mathbb{R}_0^+$ $(i = 1, 2)$, where $\sigma(M_i^{-1}N_i)$ denotes the spectrum of $M_i^{-1}N_i$ and $\mathbb{R}_0^+$ denotes the nonnegative real numbers. Similarly, $\sigma(M_i^{-1}A) \subset \mathbb{R}_0^+$. Thus, we have

$$1 > \rho(M_i^{-1}N_i) = \rho(I - M_i^{-1}A) = 1 - \lambda_{min}(M_i^{-1}A), \quad \text{for } i = 1, 2,$$

where $\lambda_{min}(M_i^{-1}A)$ is the smallest eigenvalue of $M_i^{-1}A$. Since $N_1 \preceq N_2$, we have $M_1 \preceq M_2$ and with Lemma 2.1 (i), $M_1^{-1} \succeq M_2^{-1}$. Therefore,

$$A^{\frac{1}{2}}M_2^{-1}A^{\frac{1}{2}} \preceq A^{\frac{1}{2}}M_1^{-1}A^{\frac{1}{2}}.$$

We obtain with Lemma 2.3 (ii)

$$\begin{aligned} \lambda_{min}(A^{\frac{1}{2}}M_1^{-1}A^{\frac{1}{2}}) &\geq \lambda_{min}(A^{\frac{1}{2}}M_2^{-1}A^{\frac{1}{2}}) \\ \lambda_{min}(M_1^{-1}A) &\geq \lambda_{min}(M_2^{-1}A) \end{aligned}$$

Thus, $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$. Anagously, we obtain (2.5). $\square$

Remarks: (i) Theorem 2.4 can also be shown by a proof which is similar to that of Theorem 2.1 given by Varga in [V]. With the conditions of Theorem 2.4, we have

$$\rho(M_i^{-1}N_i) = \frac{\rho(A^{-1}N_i)}{1 + \rho(A^{-1}N_i)} < 1.$$

Since $N_i \succeq 0$, $\rho(M_i^{-1}N_i)$ and $\rho(A^{-1}N_i)$ are eigenvalues of $M_i^{-1}N_i$ and $A^{-1}N_i$ respectively.
(ii) With the assumption of Theorem 2.4 we have $N_1 \preceq N_2 \Leftrightarrow M_1 \preceq M_2 \Leftrightarrow M_1^{-1} \succeq M_2^{-1}$.

Theorem 2.4 for Hermitian positive definite matrices and P-regular splittings, and Theorem 2.1 of Varga for monotone matrices and regular splittings, have the same structure related to the partial ordering used of the classes of matrices. The difference between these theorems is the different ordering of nonnegative and Hermitian positive definite matrices.

Theorem 2.1 essentially requires $A^{-1} > 0$, $M_i^{-1} \geq 0$ and $N_i \geq 0$, furthermore $0 \leq N_1 \leq N_2$. While in Theorem 2.4 we assume $A \succ 0$, $N_i \succeq 0$, and $0 \preceq N_1 \preceq N_2$, which lead to $A^{-1} \succ 0$, $M_i^{-1} \succ 0$, $N_i \succeq 0$, and $0 \preceq N_1 \preceq N_2$.

Since the ordering of Hermitian positive definite matrices seems to be more restrictive and more difficult to verify, the assumptions of Theorem 2.4 also seems to be restrictive. However, the following examples show that we can not weaken them. Theorem 2.4 is not true, if we just assume that N_1 and N_2 are arbitrary Hermitian matrices or arbitrary positive definite matrices. Furthermore, similar assumptions to those in Theorem 2.2 by Elsner for the weak regular case, $N_2 \succ 0$ and $N_1 \preceq N_2$ (thus, $M_1^{-1} \succeq M_2^{-1}$), but not $N_1 \succ 0$, cannot replace the assumptions of Theorem 2.4.

Example 2.5 Let $A = M_1 - N_1 = M_2 - N_2$ with

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}, M_1 = \begin{bmatrix} 8 & -7 \\ -7 & 12 \end{bmatrix}, N_1 = \begin{bmatrix} 4 & -7 \\ -7 & 5 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 19 & -8 \\ -8 & 27 \end{bmatrix}, N_2 = \begin{bmatrix} 15 & -8 \\ -8 & 20 \end{bmatrix}.$$

Both splittings are P-regular, $N_2 \succ 0$ and $N_2 \succ N_1$ but N_1 is indefinite. We have $\rho(M_1^{-1}N_1) = 0.899 > 0.829 = \rho(M_2^{-1}N_2)$.

Example 2.6 Let $A = M_1 - N_1 = M_2 - N_2$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_1 = \begin{bmatrix} 5 & 7 \\ -7 & 4 \end{bmatrix}, N_1 = \begin{bmatrix} 4 & 7 \\ -7 & 3 \end{bmatrix},$$
$$M_2 = \begin{bmatrix} 12 & -2 \\ -5 & 9 \end{bmatrix}, N_2 = \begin{bmatrix} 11 & -2 \\ -5 & 8 \end{bmatrix}.$$

Both splittings are P-regular, N_1, N_2 and $N_2 - N_1$ are positive definite but not Hermitian. We have $\rho(M_1^{-1}N_1) = 0.940 > 0.920 = \rho(M_2^{-1}N_2)$.

In the following we give three examples of splittings with Hermitian positive definite matrices N. First, we mention the SSOR-method which is given by the splitting

$$\begin{aligned} A &= M - N, \\ M &= \frac{1}{\omega(2-\omega)}(D - \omega L)D^{-1}(D - \omega U), \\ N &= \frac{1}{\omega(2-\omega)}((1-\omega)D - \omega L)D^{-1}((1-\omega)D - \omega U). \end{aligned} \tag{2.6}$$

Here, we use the standard decomposition $A = D - L - U$, where D is the diagonal of A, $-L$ the lower triangular part of A and $-U$ the upper triangular part of A. For $A \succ 0$ and

$0 < \omega < 2$, we have $N \succeq 0$.

The second example is the incomplete Cholesky-factorization $A = LDL^T - R$, where L is a lower triangular, and D is a diagonal matrix. In the matrix L, zero elements may occur in arbitrary off-diagonal places, which can be chosen in advance. These places (i,j) will be given by the set

$$P \subseteq P(n) = \{(i,j) | 1 \leq i,j \leq n, i \neq j\},$$

and require both $(i,j) \in P$ and $(j,i) \in P$. Thus, a matrix $A \in C^{n,n}$ admits an incomplete Cholesky-factorization with respect to $P \subseteq P(n)$ if there exist matrices $L = [l_{ij}], D, R = [r_{ij}] \in \mathbb{C}^{n,n}$ such that $A = LDL^T - R$ with

$$\begin{aligned} l_{ij} &= 0 \qquad \text{if } (i,j) \in P, \\ r_{ij} &= 0 \qquad \text{if } (i,j) \notin P. \end{aligned}$$

For more details about incomplete Cholesky, or more general, the incomplete LDU-factorization see Meijerink and van der Vorst [MV] and Robert [R]. For M- and H-matrices the incomplete LDU-factorization exits for each $P \subseteq P(n)$. However, there are Hermitian positive definite matrices, which do not admit an incomplete Cholesky-factorization for all $P \subset P(n)$, e.g. in [R]. To avoid this, Gustafsson [G] and later on Robert [R] suggested the following technique:

The modulus of the zeroed elements $r_{ij}, (i,j) \in P$ are added to the diagonal elements of row i and row j of LDL^T. Thus, the row sum of A and the row sum of LDL^T are the same, but the diagonal of R is no longer zero. But, with this modification, each Hermitian positive definite matrix admits an incomplete factorization for each set $P \in P(n)$ [R].

Now let us look at the matrix R of a modified incomplete LDL^T splitting $A = LDL^T - R$. We obtain that R is the sum of matrices of the form

$$\begin{bmatrix} 0 & & & & 0 \\ & |r_{ij}| & & r_{ij} & \\ & & 0 & & \\ & r_{ij} & & |r_{ij}| & \\ 0 & & & & 0 \end{bmatrix}.$$

Therefore, R is positive semidefinite.

In general, Theorem 2.4 is not applicable to these splittings, the SSOR–splittings as in (2.6) and the modified incomplete LDL^T splittings. For arbitrary matrices $A \succ 0$ the SSOR–splittings for different choices of ω need not satisfy the assumptions of Theorem 2.4. Moreover, the modified incomplete LDL^T splittings for different sets P need not satisfy the assumption of Theorem 2.4 in general. However, the next example describes such splittings.

The third example is the Jacobi-Overrelaxation-Method (JOR-method). The JOR-method is given by the splitting:

$$M = \frac{1}{\omega}D, \quad N = \frac{1-\omega}{\omega}D + L + U$$

Here, $N \succ 0$, if ω is small enough. With Theorem 2.4 we obtain the following theorem for the JOR-method:

Theorem 2.7 *Let $A \in \mathbb{C}^{n,n}$ with $A \succ 0$. Let $\omega_1, \omega_2 \in \mathbb{R}$ with $0 < \omega_1 \le \omega_2 < \frac{1}{1-\mu_{min}}$, where μ_{min} is the smallest eigenvalue of $D^{-1}(L+U)$. Then,*

$$\rho(H_{\omega_2}) \le \rho(H_{\omega_1}) < 1.$$

Here, $H_{\omega_i} = (1-\omega_i)I + \omega_i D^{-1}(L+U)$ is the JOR iterative matrix.

Proof: The JOR-method is based on the splitting:

$$A = (\frac{1}{\omega}D) - (\frac{1-\omega}{\omega}D + L + U).$$

We have

$$\tfrac{1-\omega}{\omega}D + L + U \succ 0$$

if and only if

$$T := \tfrac{1-\omega}{\omega}I + D^{\frac{1}{2}}D^{-1}(L+U)D^{-\frac{1}{2}} \succ 0.$$

Since the eigenvalues of T are $\frac{1-\omega}{\omega} + \mu_i$, where μ_i are the eigenvalues of $D^{-1}(L+U)$, we have

$$T \succ 0 \quad \text{if and only if} \quad 0 < \omega < \frac{1}{1-\mu_{min}}.$$

Furthermore, we have $\frac{1}{\omega_2}D \preceq \frac{1}{\omega_1}D$ and with Theorem 2.4 (Remark (ii)) we obtain

$$\rho(H_{\omega_2}) \le \rho(H_{\omega_1}) < 1. \quad \square$$

At the end of this section, we give two properties of P-regular splittings which we need in the next section. Notice, that in the definition of positive definiteness we use, the matrices need not to be Hermitian. However, the notation $A \succ 0$ implies both properties, Hermitian and positive definite.

Theorem 2.8 *Let $A \in \mathbb{C}^{n,n}$ be positive definite and let (M,N) be a P-regular splitting of A. Then M is positive definite.*

Proof: Since $M^H + N = M^H + M - A$ we have for all $x \in \mathbb{C}^n, x \neq 0$

$$0 < Re(x^H(M^H + N)x) = x^H(M^H + M)x - Re(x^H Ax) \leq x^H(M^H + M)x$$

Thus, $M^H + M$ and therefore M is positive definite. $\Box$

Theorem 2.9 *Let $A \succ 0$ and $A = M - N$ with N positive definite. Then, $\rho(M^{-1}N) < 1$ and the real parts of the eigenvalues of $M^{-1}N$ are positive.*

Proof: Let $x \in \mathbb{C}^n$ an eigenvector of $M^{-1}N$ and λ the related eigenvalue, that is $M^{-1}Nx = \lambda x$ or $Nx = \lambda Mx$. With $A = M - N$ we have

$$x^H Nx = \lambda x^H Mx = \lambda x^H Ax + \lambda x^H Nx.$$

Since N is positive definite, we have $\lambda \neq 0$ and since A is nonsingular we have $\lambda \neq 1$ thus,

$$x^H Nx = \frac{\lambda}{1-\lambda} x^H Ax.$$

Together with the complex conjugate of this equation we get

$$x^H(N^H + N)x = (\frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}})x^H Ax.$$

Thus, since $x^H(N^H + N)x > 0$ and $x^H Ax > 0$

$$\frac{\lambda}{1-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}} = \frac{\lambda - 2\lambda\bar{\lambda} + \bar{\lambda}}{(1-\lambda)(1-\bar{\lambda})} = \frac{2Re(\lambda) - 2|\lambda|^2}{|1-\lambda|^2} > 0.$$

Therefore, $Re(\lambda) > |\lambda|^2 > 0$ and so $|\lambda| < 1$. $\Box$

§3 Multisplittings

Comparison theorems for multisplittings of monotone matrices are given by Neumann and Plemmons in [NP] and Elsner in [E]. Elsner proved the following theorem:

Theorem 3.1 *Let $A^{-1} \geq 0$ and let $(M_k, N_k, E_k)_{k=1}^m$ be a multisplitting of A, where each splitting (M_k, N_k) is a weak regular splittings of A. Furthermore, let $\tilde{M}_1, \tilde{M}_2$ be such that*

$$\tilde{M}_1 \leq M_k \leq \tilde{M}_2 \quad k = 1, \ldots, m$$

As in (1.3), let

$$H = \sum_{k=1}^{m} E_k M_k^{-1} N_k.$$

If $A = \tilde{M}_1 - \tilde{N}_1$ is a regular splitting then,

$$\rho(\tilde{M}_1^{-1}\tilde{N}_1) \le \rho(H).$$

If $A = \tilde{M}_2 - \tilde{N}_2$ is a regular splitting then,

$$\rho(H) \le \rho(\tilde{M}_2^{-1}\tilde{N}_2).$$

In order to get a comparison theorem for Hermitian positive definite matrices, we need the following two theorems. Henceforth, we assume that the weighting matrices E_k of a multisplitting have the form $E_k = \alpha_k I$ where $\alpha_k > 0$ and $\sum_{k=1}^m \alpha_k = 1$.

Theorem 3.2 *Let $A \succ 0$ and let $(M_k, N_k, E_k)_{k=1}^m$ be a multisplitting of A, where for each k, (M_k, N_k) is a P-regular splitting and $E_k = \alpha_k I$. Let G and H be as in* (1.3)*. Then, G is nonsingular and $(G^{-1}, G^{-1}H)$ is a P-regular splitting of A.*

Proof: With Theorem 2.9 each M_k is positive definite. Thus, G is also positive definite and in particular nonsingular. Since $GA = I - H$, we have $A = G^{-1} - G^{-1}H$. Therefore, it suffices to show that $G^{-H} + G^{-1}H \succ 0$. But

$$G^{-H} + G^{-1}H \succ 0$$

if and only if

$$G^H(G^{-H} + G^{-1}H)G = G + G^H - G^H AG \succ 0.$$

Now, we use essentially the same proof as O'Leary and White ([OW] Theorem 1).

$$\begin{aligned}
& G + G^H - G^H AG \\
= \;& \sum_{k=1}^m E_k M_k^{-1} + M_k^{-H} E_k - \sum_{k,j=1}^m M_k^{-H} E_k A E_j M_j^{-1} \\
= \;& \sum_{k=1}^m M_k^{-H}(M_k^H E_k + E_k M_k - E_k A E_k) M_k^{-1} - \sum_{\substack{k,j=1 \\ k\neq j}}^m M_k^{-H} E_k A E_j M_j^{-1} \\
= \;& \sum_{k=1}^m M_k^{-H}(M_k^H E_k + E_k A + E_k N_k - E_k A E_k) M_k^{-1} - \sum_{\substack{k,j=1 \\ k\neq j}}^m M_k^{-H} E_k A E_j M_j^{-1} \\
= \;& \sum_{k=1}^m M_k^{-H}(M_k^H E_k + E_k N_k + \sum_{\substack{j=1 \\ k\neq j}}^m E_k A E_j) M_k^{-1} - \sum_{\substack{k,j=1 \\ k\neq j}}^m M_k^{-H} E_k A E_j M_j^{-1} \\
= \;& \sum_{k=1}^m \alpha_k M_k^{-H}(M_k^H + N_k) M_k^{-1} + \sum_{\substack{k,j=1 \\ k\neq j}}^m \alpha_j \alpha_k M_k^{-H} A (M_k^{-1} - M_j^{-1}).
\end{aligned}$$

Since each (M_k, N_k) is a P-regular splitting, $M_k^H + N_k$ is positive definite. Thus, the first sum is positive definite. The second sum may be written as

$$
\begin{aligned}
& \sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_j \alpha_k M_k^{-H} A (M_k^{-1} - M_j^{-1}) \\
= & \sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_j \alpha_k (M_k^{-H} A M_k^{-1} - M_k^{-H} A M_j^{-1}) \\
= & \frac{1}{2} \sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_j \alpha_k (M_k^{-H} A M_k^{-1} - M_k^{-H} A M_j^{-1} - M_j^{-H} A M_k^{-1} + M_j^{-H} A M_j^{-1}) \\
= & \frac{1}{2} \sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_j \alpha_k (M_k^{-H} - M_j^{-H}) A (M_k^{-1} - M_j^{-1}).
\end{aligned}
$$

Since all the matrices $M_k^{-1} - M_j^{-1}$ may be singular, this sum is positive semidefinite. Together we obtain, $G + G^H - G^H A G$ is positive definite. □

Theorem 3.2 states that a multisplitting $(M_k, N_k, E_k)_{k=1}^m$, which consists of P-regular splittings and weighting matrices $E_k = \alpha_k I$, induces again a P-regular splitting. Therefore, we have $\rho(H) < 1$. For multisplittings with Hermitian positive definite matrices N_k the induced single splitting $(G^{-1}, G^{-1}H)$ is of the same type, i.e. $G^{-1}H \succ 0$.

Theorem 3.3 *Let $A \succ 0$ and let $(M_k, N_k, E_k)_{k=1}^m$ be a multisplitting of A, where each N_k is positive definite and $E_k = \alpha_k I$. Let G, H be as in* (1.3). *Then,*
i) the real parts of the eigenvalues of H are positive,
ii) if $N_k \succ 0$, i.e. the N_k are also Hermitian, then $G^{-1} \succ 0, G^{-1}H \succ 0$ and all eigenvalues of H are real and positive.

Proof: i) For $m = 1$, this is proved by Theorem 2.10. For $m > 1$ the multisplitting induces by Theorem 3.2 the single splitting $(G^{-1}, G^{-1}H)$. Thus, we have to show that $G^{-1}H$ is positive definite. However, $G^{-1}H = G^{-1} - A$ is positive definite if and only if $G^H(G^{-1} + G^{-H} - 2A)G$ is positive definite. But,

$$
\begin{aligned}
& G + G^H - 2G^H A G \\
= & \sum_{k=1}^{m} M_k^{-H} (M_k^H E_k + E_k M_k - 2E_k A E_k) M_k^{-1} - 2 \sum_{\substack{k,j=1\\k\neq j}}^{m} M_k^{-H} E_k A E_j M_j^{-1} \\
= & \sum_{k=1}^{m} M_k^{-H} ((A + N_k^H) E_k + 2E_k (A + N_k) - E_k A E_k) M_k^{-1} - 2 \sum_{\substack{k,j=1\\k\neq j}}^{m} M_k^{-H} E_k A E_j M_j^{-1} \\
= & \sum_{k=1}^{m} M_k^{-H} (\alpha_k (N_k^H + N_k) + 2\alpha_k A (I - \alpha_k I)) M_k^{-1} - 2 \sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_k \alpha_j M_k^{-H} A M_j^{-1}
\end{aligned}
$$

$$= \sum_{k=1}^{m} M_k^{-H}(\alpha_k(N_k^H + N_k) + 2\sum_{\substack{j=1\\k\neq j}}^{m} \alpha_k\alpha_j A)M_k^{-1} - 2\sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_k\alpha_j M_k^{-H} A M_j^{-1}$$

$$= \sum_{k=1}^{m} \alpha_k M_k^{-H}(N_k^H + N_k)M_k^{-1} + 2\sum_{\substack{k,j=1\\k\neq j}}^{m} \alpha_j\alpha_k M_k^{-H} A(M_k^{-1} - M_j^{-1}).$$

Following the second part of the proof of Theorem 3.2., we obtain the positive definiteness of $G + G^H - 2G^H AG$.

ii) With $N_k \succ 0$ and $A \succ 0$ we have $M_k \succ 0$ for $k = 1, \ldots, m$. Thus, $G^{-1} \succ 0$ and $G \succ 0$. Furthermore, $G^{-1}H = G^{-1} - A$ is Hermitian and the real parts of the eigenvalues are positive. Thus, $G^{-1}H \succ 0$. Further, $G^{-\frac{1}{2}}HG^{\frac{1}{2}} \succ 0$. Therefore, $G^{-\frac{1}{2}}HG^{\frac{1}{2}}$ is similar to H. So the eigenvalues of H are real and positive. $\square$

We are now ready to formulate and prove a comparison theorem for certain multisplittings of Hermitian positive definite matrices.

Theorem 3.4 *Let $A \succ 0$ and let $(M_k, N_k, E_k)_{k=1}^m$ be a multisplitting of A, where each N_k is positive definite and $E_k = \alpha_k I$. Furthermore, let $\tilde{M}_1, \tilde{M}_2 \succ 0$, such that*

$$\tilde{M}_1 \preceq M_k \preceq \tilde{M}_2 \quad k = 1, \ldots, m.$$

As in (1.3)*, let*

$$H = \sum_{k=1}^{m} E_k M_k^{-1} N_k.$$

If $\tilde{N}_1 = A - \tilde{M}_1 \succ 0$ then,

$$\rho(\tilde{M}_1^{-1}\tilde{N}_1) \leq \rho(H). \tag{3.1}$$

If $\tilde{N}_2 = A - \tilde{M}_2 \succ 0$ then,

$$\rho(H) \leq \rho(\tilde{M}_2^{-1}\tilde{N}_2). \tag{3.2}$$

Proof: With Theorem 3.2 the multisplitting induces the single splitting $(G^{-1}, G^{-1}H)$, where G, H are as in (1.3). With Theorem 3.3, $G^{-1} \succ 0$ and $G^{-1}H \succ 0$. Thus, we can apply Theorem 2.4 to $(G^{-1}, G^{-1}H), (\tilde{M}_1, \tilde{N}_1)$ and $(\tilde{M}_1, \tilde{N}_1)$. Since $\tilde{M}_1 \preceq M_k$, we have with Lemma 2.3 $\tilde{M}_1^{-1} \succeq M_k^{-1}$. With

$$\sum_{k=1}^{m} E_k = \sum_{k=1}^{m} \alpha_k I = I$$

we obtain

$$\sum_{k=1}^{m} E_k M_k^{-1} \preceq \tilde{M}_1^{-1}.$$

Again with Lemma 2.3

$$(\sum_{k=1}^{m} E_k M_k^{-1})^{-1} = G^{-1} \succeq \tilde{M}_1$$

and (3.1) follows with Theorem 2.4. Analogously we obtain (3.2). $\square$

Theorem 3.4 gives theoretical upper and lower bounds for a certain multisplitting and again, as in the case of single splittings, Theorem 3.4 has the same structure as the related Theorem 3.1 for monotone matrices.

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