# m-Dependent Random Fields with Analytic Cumulant-Generating Function

F. Götze[*] L. Heinrich[†] C. Hipp[‡]

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**Abstract**

We consider m-dependent random fields of bounded random vectors and investigate the analyticity of the cumulant generating function of sums of these random vectors. Using the Kirkwood-Salsburg equations we derive upper bounds for the cumulant generating function and prove its analyticity in a neighbourhood of zero, where the normalized bounds and the neighbourhood are independent of the number of terms in the sum. The results are applied to spatial statistics in Poisson cluster processes and Boolean models and yield probabilities of large deviations as well as Berry-Esseen results for these statistics.

*Keywords and phrases*: m-dependent random field, cumulant-generating function, Kirkwood-Salsburg equations, Poisson cluster process, Boolean model. *1991 Mathematics Subject*

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## 1. Introduction and Summary

For $j$ in the lattice $\mathbb{Z}_+^d \triangleq \{1, 2, \ldots\}^d$ let $\xi_j$ be a random variable on a common probability space $[\Omega, \mathfrak{A}, \mathbb{P}]$ taking values in some measurable space $[\mathbb{X}_j, \mathfrak{X}_j]$. We shall assume throughout that the variables $\xi_j,\ j \in \mathbb{Z}_+^d$, are independent. For fixed integer $m \geq 0$ define

$$U_j \triangleq \times_{s=1}^d \{j_s, \ldots, j_s + m\} \quad \text{for} \quad j = (j_1, \ldots, j_d) \in \mathbb{Z}_+^d$$

[*]Universität Bielefeld, Fakultät für Mathematik, D-33501 Bielefeld, Germany
[†]TU Bergakademie Freiberg, Institut für Stochastik, Bernhard-von-Cotta-Str. 2, D-09596 Freiberg, Germany
[‡]Universität Karlsruhe, Lehrstuhl für Versicherungswissenschaft, Kronenstr. 34, D-76133 Karlsruhe, Germany

and

$$V_n^{(m)} \triangleq \times_{s=1}^{d}\{1,2,\ldots,n_s+m\} \quad \text{for} \quad n=(n_1,\ldots,n_d)\in\mathbb{Z}_+^d \quad , \quad V_n \triangleq V_n^{(0)}.$$

For $j \in \mathbb{Z}_+^d$ let $f_j$ be $\mathbb{R}^p$-valued and measurable on the product space $\times_{i\in U_j}\mathbb{X}_i$; the mappings $f_j$ are called *window functions* in [4]). We consider the $m-$dependent random field

$$X_j \triangleq f_j(\xi_i,\ i\in U_j), \quad j\in\mathbb{Z}_+^d \tag{1.1}$$

and the sum

$$S_n \triangleq \sum_{j\in V_n}(X_j - \mathbb{E}X_j) \tag{1.2}$$

It is the purpose of this paper to prove analyticity of the cumulant generating function (cgf)

$$\gamma_n(z) \triangleq \log\mathbb{E}\exp\{(z,S_n)\} \quad , \quad z\in\mathbb{C}^p \quad ,$$

in a neighbourhood $b(o,\delta)$ of the origin $o$, where $\delta$ is independent of $n$. This is done under the assumption that the random variables $X_j$ are uniformly bounded,

$$\|X_j\| \le c \quad \text{for} \quad j\in\mathbb{Z}_+^d \quad . \tag{1.3}$$

Here and throughout we use the following notations for complex $p-$vectors $y,z\in\mathbb{C}^p$ and $r>0$ :

$$(y,z) \triangleq \sum_{s=1}^{p} y_s\bar{z}_s$$

$$\|z\| \triangleq \sqrt{(z,z)}$$

$$b(y,r) \triangleq \{z\in\mathbb{C}^p : \|z-y\|\le r\}$$

(the same notation is used in Euclidean spaces and for any dimension $>1$) . The symbols $c, c_0, c_1, \ldots$ denote generic constants not depending on $n$. For finite sets $V$ we write $|V|$ for the number of elements in $V$. We shall still use the same notation for the modulus of real or complex numbers.

In recent years results on Berry-Esseen bounds and Edgeworth expansions have been obtained for sums of the form (1.2). These results state that under additional assumptions on the field $X_j$ (including conditional Cramér conditions) the remainder terms in these limit theorems have the same order as those in the corresponding theorems for independent $X_j$ , see [3, 4, 5, 8, 9]. We mention that analogous estimates and expansions for arbitrary $d-$dimensional $m-$dependent random fields ($d\ge 2$) are still unknown or less precise in cases for which the representation (1.1) is not possible.

In the particular case of $m-$dependent sequences (i.e. $d=1$) the moment generating function of the sum admits a product representation in some neighbourhood of zero (see

[7]). For the univariate case (i.e. $p = 1$) this yields analyticity of the cgf in the circle $b(0,\delta)$, $\quad \delta = c_1/(m+1)$, and the bound

$$|\gamma_n(z)| \le c_2\, n\, (m+1)|z|^2$$

provided the Bernstein condition

$$\max_{1\le j\le n} \mathbb{E}|X_j|^k \le c_0^k\, k!$$

is satisfied for $k \ge 2$. Using Cauchy's integral formula this can be equivalently expressed by the estimates

$$|\mathrm{cum}_k(S_n)| \le c_2\, c_1^{-(k-2)}\, n\, (m+1)^{k-1} k! \quad , \tag{1.4}$$

where $\mathrm{cum}_k(X)$ denotes the $k-$th order cumulant of $X$.

Estimates of type (1.4) or the analyticity of the cgf are essential for the asymptotic behaviour of probabilities of large deviations (in the sense of H. Cramér) , see e.g. [17], [7] and Sect. 3 below.

In contrast to the case $d = 1$ the best estimates for $d \ge 2$ and for general $m-$dependent random fields of univariate random variables seem to be

$$|\mathrm{cum}_k(S_n)| \le c_3^k(d)\, |V_n|\, (m+1)^{d(k-1)}(k!)^{3/2} \quad \text{for} \quad k \ge 2 \quad , \tag{1.5}$$

see [11] [1]

For $m-$dependent random fields admitting the representation (1.1) Theorem 1 improves the estimates (1.5) by removing the exponent 3/2 on the right hand side.

**Theorem 1.** Assume that the field of random vectors defined by (1.1) satisfies condition (1.3). Then there exist real numbers $\delta, \varepsilon > 0$ depending on $c$ and $m$ such that the cgf $\gamma_n(z)$ is an analytic function in some circle $b(o,\delta)$, and in this circle we have the following estimate:

$$|\gamma_n(z)| \le \varepsilon\, |V_n|\, \|z\|^2 \quad \text{for} \quad z \in b(o,\delta) \quad . \tag{1.6}$$

An admissible pair of numbers $\delta, \varepsilon$ is given by $\delta_m = (8\,c\,(2m+1)^{2d})^{-1}$ and $\varepsilon_m = 56\,c^2\,(2m+1)^{3d}$.

The application of Cauchy's integral formula to the cgf $\gamma_n(z)$ leads immediately to the following

**Corollary 1.** Under the assumptions and with the notation of Theorem 1 we have

$$|\mathrm{cum}_k((t,S_n))| \le \varepsilon\, |V_n|\, \|t\|^k\, \delta^{-(k-2)}\, k! \quad \text{for} \quad k \ge 3\,,\ t \in \mathbb{R}^p \quad .$$

[1] In [11] the estimate (1.5) was actually proved for $d = 2$.

# 2. Proof of Theorem 1

Without loss of generality we may assume that $\mathbb{E}X_j = 0$. It suffices to prove Theorem 1 for $p = 1$; the general multivariate case follows using the linearity of the scalar product and Schwarz' inequality

$$|(z, X_j)| \leq \|z\| \|X_j\|.$$

For any non-empty $V \subseteq V_n^{(m)}$ put

$$\varphi(z; x_j, j \in V) \triangleq \frac{\mathbb{E}(\exp\{zS_n\} | \xi_j = x_j, j \in V)}{\mathbb{E}\exp\{zS_n\}} \quad , \quad x_j \in \mathbb{X}_j$$

for $z \in \mathbb{C}^1$ in some neighbourhood of zero in which the denominator does not vanish.

Logarithmic differentiation provides that

$$\frac{d}{dz}\gamma_n(z) = \sum_{j \in V_n} \mathbb{E}X_j \; \varphi(z; \xi_i, i \in U_j) \quad . \tag{2.1}$$

Denote by $l = l(V)$ the lexikographically smallest element of $V \subseteq V_n^{(m)}$ and introduce an additional random element $\xi_l^*$ being independent of the family $\xi_j, j \in \mathbb{Z}_+^d$, and having the same distribution as $\xi_l$. With the abbreviation $W_l^{(m)} \triangleq \cup_{j \in V_n : U_j \ni l} U_j$ we define a family of kernel functions $k_V$ , $\emptyset \neq V \subseteq V_n^{(m)}$, as follows:

$$\begin{aligned} & k_V(z; x, x_j, j \in W_l^{(m)} \cap V, \xi_j, j \in W_l^{(m)} \setminus V) \triangleq \\ & \quad \exp\left\{ z \sum_{j \in V_n : U_j \ni l} F_j(x, x_i, i \in U_j \cap V, \xi_i, i \in U_j \setminus V) \right\} - 1 \quad , \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} & F_j(x, x_i, i \in U_j \cap V, \xi_i \in U_j \setminus V) \triangleq \\ & \quad f_j(\xi_i, i \in U_j)\Big|_{\xi_i = x_i; i \in V} - f_j(\xi_i, i \in U_j)\Big|_{\xi_l = x, \xi_i = x_i; l \neq i \in V} . \end{aligned}$$

Using the definition of $\varphi(z; .)$ and the independence of the $\xi_j$'s we get

$$\varphi(z; x_l) = 1 + \mathbb{E}k_V(z, \xi_l^*, x_l, \xi_i, l \neq i \in W_l^{(m)}) \varphi(z; \xi_l^*, \xi_i, l \neq i \in W_l^{(m)})) \tag{2.3}$$

for $V = \{l\}$ and

$$\begin{aligned} \varphi(z; x_i, i \in V) \quad = \quad & \varphi(z; x_i, l \neq i \in V) + \mathbb{E}k_V(z, \xi_l^*, x_i, i \in W_l^{(m)} \cap V, \xi_i, i \in W_l^{(m)} \setminus V) \\ & \times \varphi(z; \xi_l^*, x_i, l \neq i \in V, \xi_i, i \in W_l^{(m)} \setminus V)) \end{aligned} \tag{2.4}$$

for $|V| \geq 2$. Here the $x's$ are fixed, and the expectation is taken with respect to all $\xi$'s.

In other words the functions $\varphi(z;.)$ satisfy a (finite) system of linear integral equations. In statistical mechanics these are the Kirkwood-Salsburg equations for the correlation functions of (infinite) particle systems , see [16] , Chapt. 4.
Let $\eta \in (0,1)$ (to be specified later) and let $\boldsymbol{\Phi}_\eta$ be the Banach space of all (finite) sequences of bounded complex-valued functions

$$\boldsymbol{\rho} \triangleq \{\rho_V \mid \times_{j\in V} \mathbb{X}_j \mapsto \mathbb{C}^1 : \emptyset \neq V \subseteq V_n^{(m)}\}$$

equipped with the norm

$$\|\boldsymbol{\rho}\|_\eta \triangleq \max\{\eta^{|V|}\|\rho_V\|_\infty : \emptyset \neq V \subseteq V_n^{(m)}\} \quad ,$$

where

$$\|\rho_V\|_\infty \triangleq \sup\{\|\rho_V(x_j, j \in V)\| : x_j \in \mathbb{X}_j, j \in V\} \quad .$$

(see D. Ruelle [16] ).

In order to simplify the notation we shall drop the subscript $V$ at $\rho$ whenever $V$ can be identified from the arguments of $\rho$. We are now in a position to define a linear bounded operator $\boldsymbol{K}(z) : \boldsymbol{\Phi}_\eta \mapsto \boldsymbol{\Phi}_\eta$ by the following system of integral operators:

$$(\boldsymbol{K}(z)\boldsymbol{\rho})(x_j, j \in V) \triangleq \begin{cases} \mathbb{E}k_V(z; \xi_l^*, x_l, \xi_i, l \neq i \in W_l^{(m)})\rho(\xi_l^*, \xi_i, l \neq i \in W_l^{(m)}) & \text{if} \quad V = \{l\} \\ \rho(x_i, l \neq i \in V) + \mathbb{E}k_V(z; \xi_l^*, x_i, i \in W_l^{(m)} \cap V, \xi_i, i \in W_l^{(m)} \setminus V) & \\ \qquad \times \rho(\xi_l^*, x_i, l \neq i \in V, \xi_i, i \in W_l^{(m)} \setminus V) & \text{if} \quad |V| \geq 2 \end{cases}$$

Thus the equations (2.3) and (2.4) can be formally rewritten as a linear equation in $\boldsymbol{\Phi}_\eta$:

$$\boldsymbol{\varphi}(z) = \boldsymbol{\alpha} + \boldsymbol{K}(z)\,\boldsymbol{\varphi}(z) \quad , \tag{2.5}$$

where

$$\boldsymbol{\varphi}(z) \triangleq \{\varphi(z;.) \mid \times_{j\in V} \mathbb{X}_j \mapsto \mathbb{C}^1 : \emptyset \neq V \subseteq V_n^{(m)}\}$$

and

$$\boldsymbol{\alpha} \triangleq \{\alpha_V : \emptyset \neq V \subseteq V_n^{(m)}\}$$

with $\alpha_V = 1$ if $|V| = 1$ and $\alpha_V = 0$ if $|V| \geq 2$.

Next we shall verify that $\boldsymbol{K}(z)$ is an entire analytical operator-valued function (as defined in [2]), i.e. the Taylor expansion

$$\boldsymbol{K}(z) = \sum_{r\geq 0} \frac{z^r}{r!}\boldsymbol{K}^{(r)} \quad , \quad \boldsymbol{K}^{(r)} \in \mathbb{L}(\boldsymbol{\Phi}_\eta) \quad ,$$

converges absolutely for $z \in \mathbb{C}^1$ with respect to the norm $\|.\|$ in the Banach space of linear bounded operators $\mathbb{L}(\mathbf{\Phi}_\eta)$. In view of (2.2) the operators $\boldsymbol{K}^{(r)}$ , $r \geq 0$ , are defined by the kernel functions $k_V^{(r)}$ , $\emptyset \neq V \subseteq V_n^{(m)}$ , $r \geq 1$ ,

$$k_V^{(r)}(x, x_i, i \in W_l^{(m)} \cap V, \xi_i, i \in W_l^{(m)} \setminus V) \triangleq \left( \sum_{j \in V_n : U_j \ni l} F_j(x, x_i, i \in U_j \cap V, \xi_i, i \in U_j \setminus V) \right)^r$$

so that in analogy to the definition of $\boldsymbol{K}(z)$

$$(\boldsymbol{K}^{(r)}\boldsymbol{\rho})(x_i, i \in V) \triangleq \mathbb{E} k_V^{(r)}(\xi_l^*, x_i, i \in W_l^{(m)} \cap V, \xi_i, i \in W_l^{(m)} \setminus V)\rho(\xi_l^*, x_i, l \neq i \in V, \xi_i, i \in W_l^{(m)} \setminus V)$$

for $r \geq 1$ , $V \neq \emptyset$ and

$$(\boldsymbol{K}^{(0)}\boldsymbol{\rho})(x_i, i \in V) \triangleq \begin{cases} \rho(x_i, l \neq i \in V) & \text{if} \quad |V| \geq 2 \\ 0 & \text{if} \quad |V| = 1 \end{cases} .$$

Since, by (1.3) and $|\{j \in V_n : U_j \ni l\}| \leq (m+1)^d$ , the kernel functions $k_V^{(r)}$ , $r \geq 1$ , are bounded by $\left(2c(m+1)^d\right)^r$ it follows that

$$\|(\boldsymbol{K}^{(r)}\boldsymbol{\rho})_V\|_\infty \leq \|\rho_{W_l^{(m)}}\|_\infty \left(2c(m+1)^d\right)^r .$$

Using the inequality $|V \cup W_l^{(m)}| - vertV| \leq |W_l^{(m)}| - 1$ we get

$$\eta^{|V|}\|\boldsymbol{K}^{(r)}\boldsymbol{\rho}\|_\infty \leq \eta^{|V|-|W_l^{(m)}|} \left(2c(m+1)^d\right)^r \|\boldsymbol{\rho}\|_\eta$$

and finally, by $|W_l^{(m)}| \leq (2m+1)^d$,

$$\|\boldsymbol{K}^{(r)}\| \leq \eta^{-(2m+1)^d+1} \left(2c(m+1)^d\right)^r \quad \text{for} \quad r \geq 1 .$$

Together with

$$\|\boldsymbol{K}^{(0)}\| \leq \eta$$

we have

$$\|\boldsymbol{K}(z)\| \leq \eta \left(1 + (e^{2c|z|(m+1)^d} - 1)\eta^{-(2m+1)^d}\right) \quad \text{for all} \quad z \in \mathbb{C}^1 \tag{2.6}$$

proving the analyticity of $\boldsymbol{K}(z) \in \mathbb{L}(\mathbf{\Phi}_\eta)$ in the whole complex plane.
Now let us specify $\eta \in (0,1)$ by

$$\eta \triangleq \frac{(2m+1)^d}{1+(2m+1)^d} \quad , \quad m \geq 0 .$$

Using the inequality $x \le e^x - 1 \le xe^x$ for $x \ge 0$ we obtain that the right side of (2.6) is bounded by

$$\frac{(2m+1)^d}{1+(2m+1)^d}\left(1+\frac{e^{10/9}}{6(2m+1)^d}\right) < 1 - \frac{1}{5(2m+1)^d}$$

for $m \ge 1$ and $z \in \mathbb{C}^1$ satisfying $|z| \le \delta_m = 1/(8\,c\,(2m+1)^{2d})$. In case $m = 0$ we obtain from (2.6) that

$$\|\boldsymbol{K}(z)\| \le \eta - 1 + e^{2\,c\,|z|} < 0.79 \quad \text{for} \quad |z| \le 1/8\,c\,.$$

Hence,

$$\|\boldsymbol{K}(z)\| \le 1 - \frac{1}{5(2m+1)^d} < 1 \quad \text{for all} \quad m \ge 0 \quad \text{and} \quad |z| \le \delta_m$$

so that the inverse operator $(\boldsymbol{I} - \boldsymbol{K}(z))^{-1}$ belongs to $\mathbb{L}(\boldsymbol{\Phi}_\eta)$ and is analytical with

$$\|(\boldsymbol{I} - \boldsymbol{K}(z))^{-1}\| \le 5(2m+1)^d \quad \text{for} \quad |z| \le_m \delta\,. \tag{2.7}$$

Therefore (2.5) has a unique solution in $\boldsymbol{\Phi}_\eta$:

$$\boldsymbol{\varphi}(z) = (\boldsymbol{I} - \boldsymbol{K}(z))^{-1}\boldsymbol{\alpha} \quad \text{for} \quad |z| \le \delta_m\,. \tag{2.8}$$

The representation (2.8) and the inequality (2.7) yield the analyticity of $\varphi(z; x_j, j \in V)$ for each choice of $x_j, j \in V$ in $b(o, \delta_m)$. In particular, we obtain with $\|\boldsymbol{\alpha}\|_\eta = \eta$

$$\begin{aligned} \|\varphi(z;.)U_j\|_\infty &\le \eta^{-|U_j|}\|\boldsymbol{\varphi}(z)\|_\eta \\ &\le \left(1+\frac{1}{(2m+1)^d}\right)^{(m+1)^d-1} \|(\boldsymbol{I} - \boldsymbol{K}(z))^{-1}\| \\ &\le 7(2m+1)^d \end{aligned}$$

uniformly for all $|z| \le \delta_m$ , $j \in V_n$ , $d \ge 2$ and $m \ge 0$. In view of (1.3) and (2.1) it is readily checked that $\frac{d}{dz}\gamma_n(z)$ is an analytical function in $b(o, \delta_m)$ satisfying

$$\left|\frac{d}{dz}\gamma_n(z)\right| \le 7\;c\;(2m+1)^d|V_n| \quad \text{for all} \quad |z| \le \delta_m \quad .$$

Further, since $\frac{d}{dz}\gamma_n(z) = 0$ for $z = 0$ by assumption, applying Schwarz's lemma gives

$$\left|\frac{d}{dz}\gamma_n(z)\right| \le |z|\frac{7\;c}{\delta_m}\;(2m+1)^d|V_n| \quad \text{for} \quad |z| \le \delta_m \quad .$$

Finally the assertion of Theorem 1 follows by integrating (2.1) and using the inequality

$$|\gamma_n(z)| = \left|\int_0^z \left(\frac{d}{du}\gamma_n(u)\right) du\right| \le |z| \sup_{u \in b(o,|z|)} \left|\frac{d}{du}\gamma_n(u)\right| \quad .$$

This completes the proof of Theorem 1 for $p = 1$ . $\square$

## 3. Applications to Large Deviations

In this section we formulate and prove an integral limit theorem for probabilities of large deviations for sums of bounded random variables connected in a $d-$ dimensional $m-$dependent random field. For a detailed dicussion and the historical background of large deviations (in the sense of H. Cramér) the reader is referred to [17]. The proof of the subsequent Theorem 2 rests on the estimate (1.6) of the cgf of $S_n$. In the particular case $d = 1$ this result is valid under the (less restrictive) Bernstein condition ensuring the analyticity of the cgf of $S_n$, see [7] and Sect.1 of this paper. Another approach to obtain limit theorems for large deviations in case of (Gibbsian) random fields is the 'method of cluster expansions' leading to estimates of the cumulants of $S_n$, see references in [1]. Under certain rather technical conditions imposed on the cgf of the sum $S_n$ S.Čepulenas [1] proved large deviation relations for a class of stationary Gibbsian random fields which are similar to the ones obtained in our following theorem.

**Theorem 2.** Let $X_j$ , $j \in V_n$ , be a field of $m-$dependent random variables as defined in (1.1) for $p = 1$ satisfying (1.3). Then in the interval $0 \leq x \leq \Delta_n/14H_n$ the following relations for large deviations of the sum $S_n$ (defined by (1.3)) hold:

$$\frac{\mathbb{P}(S_n \geq xB_n)}{1-\Phi(x)} \leq \exp\left\{\frac{x^3}{B_n}\sum_{k\geq 0}\lambda_{kn}(\frac{x}{B_n})^k\right\}\left(1+c_4\frac{H_n(1+x)}{\Delta_n}\right) \tag{3.1}$$

and

$$\frac{\mathbb{P}(S_n \geq xB_n)}{\Phi(-x)} \geq \exp\left\{\frac{x^3}{B_n}\sum_{k\geq 0}\lambda_{kn}(\frac{x}{B_n})^k\right\}\left(1-c_5\frac{H_n(1+x)}{\Delta_n}\right) \ , \tag{3.2}$$

where

$$B_n^2 = \mathrm{var}(S_n) \quad , \quad H_n = \varepsilon_m|V_n|/B_n^2 \quad , \quad \Delta_n = \delta_m B_n \quad , \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$$

with $\delta_m$ , $\varepsilon_m$ from Theorem 1 and

$$\lambda_{kn} = \frac{1}{(k+2)(k+3)}\sum_{l=1}^{k+1}(-1)^{l-1}\binom{k+l+1}{l}\sum_{\substack{k_1+\cdots+k_l=k+1\\ k_i\geq 1}}\prod_{i=1}^{l}\frac{\mathrm{cum}_{k_i+2}(S_n)}{B_n^2(k_i+1)!} \tag{3.3}$$

for $k = 0, 1, 2, \ldots$ .

*Proof of Theorem 2.* In view of Theorem 1 and the maximum principle for analytic functions we are led to

$$|\gamma_n(z/B_n)|\Big|_{|z|=\Delta_n} \leq H_n\Delta_n^2 \ .$$

Such an estimate of the cgf of $S_n/B_n$ enables us to apply a general lemma on large deviations for standardized random variables due to V.A. Statulevičius ( see e.g. [1, 17]). According to this lemma the $x$-values have to lie in the interval $[0, \delta\Delta_n/H_n]$ with

$$\delta < \frac{\delta^*}{2}(1+\frac{\delta^*}{H_n}), \tag{3.4}$$

where $\delta^*$ is the unique real solution of the equation $6\delta^* = (1 - \frac{\delta^*}{H_n})^3$. We take $\delta = 1/14$ here, and hence we have to check whether the corresponding $\delta^*$ satisfies (3.4). Combining the inequality $B_n^2 \leq (m+1)^d \sum_{j \in V_n} \mathbb{E} X_j^2$ with (1.3) yields $H_n \geq 56$ and hence

$$1/7 < \delta^* < 1/6.$$

A short calculation shows that $\frac{\delta^*}{2}(1 + \frac{\delta^*}{H_n}) = \delta^*(1 - \sqrt[3]{\frac{3\delta^*}{4}})$ is increasing in $\delta^*$ which implies that

$$\frac{\delta^*}{2}(1 + \frac{\delta^*}{H_n}) > \frac{1}{7}(1 - \sqrt[3]{\frac{3}{28}}) > \frac{1}{14} = \delta.$$

This completes the proof of Theorem 2. □

In addition to Theorem 2 we obtain the corresponding Berry-Esseen bound in the central limit theorem either directly from (3.1) and (3.2) or from (1.6) by applying Esseen's smoothing inequality. In either case, only standard arguments are needed to prove of the following Corollary 2. The details are left to the reader.

**Corollary 2.** Under the assumptions of Theorem 2 we have

$$\sup_{x \in \mathbb{R}^1} |\mathbb{P}(S_n < x B_n) - \Phi(x)| \leq c_6 \frac{H_n}{\Delta_n} = 448\ c_6\ (2m+1)^{5d} \frac{c^3 |V_n|}{B_n^3} \tag{3.5}$$

with some constant $c_6$ not depending on $n$ and $m$.

Note that in case of i.i.d. real-valued random variables $\xi_j$ , $j \in \mathbb{Z}^d$, and identical $f_j = f$ , $j \in V_n$, the estimate (3.5) was shown in [8].

## 4. Applications to Spatial Statistics

Boolean (or Poisson grain) models as defined in [13] represent an important class of random closed sets which have many applications in image analysis, stereology and spatial statistics, see [6, 19].

First of all we introduce some notation and recall necessary facts from random set and point process theory and give a rigorous definition of a Boolean model following the line in [12, 18, 19].

Let $\mathbb{F}$ denote the family of closed subsets of $\mathbb{R}^d$ and let $\sigma_f$ be the $\sigma-$field of subsets of $\mathbb{F}$ generated by the families

$$\mathbb{F}_K \triangleq \{F \in \mathbb{F} : F \cap K \neq \emptyset\} \quad , \quad K \in \mathbb{K} \quad ,$$

where $\mathbb{K}$ is the family of all non-void compact subsets of $\mathbb{R}^d$. A *random closed set* $Z$ in $\mathbb{R}^d$ is then defined to be an $(\mathfrak{A}, \sigma_f)-$ measurable mapping from $[\Omega, \mathfrak{A}, \mathbb{P}]$ into the measurable space $[\mathbb{F}, \sigma_f]$. The distribution of $Z$ (i.e. the probability measure induced by $Z$ on $(\mathbb{F}, \sigma_f)$) is entirely determined by the hitting functional $T_Z$ defined on $\mathbb{K}$ by

$$T_Z(K) \triangleq \mathbb{P}(Z \cap K \neq \emptyset) \quad , \quad K \in \mathbb{K} \cup \{\emptyset\} \quad , \quad \text{see [13].} \tag{4.1}$$

We now introduce the notion of an (independently) marked (Poisson) point process $\Psi$ on $\mathbb{R}^d$ with the particular measurable mark space $[\mathbb{K}, \mathfrak{B}(\mathbb{K})]$, where $\mathfrak{B}(\mathbb{K})$ designates the Borel $\sigma-$field corresponding to the Polish space $\mathbb{K}$ , see [12, 19]. Let $\mathbb{M}(\mathbb{K})$ be the set of all counting measures $\psi$ on $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{B}(\mathbb{K})$ satisfying $\psi(B \times \mathbb{F}) < \infty$ for all bounded $B \in \mathfrak{B}(\mathbb{R}^d)$ and let $\mathfrak{M}(\mathbb{K})$ denote the $\sigma-$field generated by the sets $\{\psi \in \mathbb{M}(\mathbb{K}) : \psi(B \times L) = j\}$ for $j \geq 0$, bounded $B \in \mathfrak{B}(\mathbb{R}^d)$ and $L \in \mathfrak{B}(\mathbb{K})$. It turns out that each locally finite counting measure $\psi \in \mathbb{M}(\mathbb{K})$ has a unique representation

$$\psi = \sum_{i \geq 1} \delta_{[x_i(\psi), z_i(\psi)]}$$

as sum of Dirac measures $\delta_{[x,z]}$ , $[x, z] \in \mathbb{R}^d \times \mathbb{K}$ , with respect to the at most countable family $\{[x_i(\psi), z_i(\psi)], i \geq 1\}$ of atoms of $\psi$ , where each atom is counted according to its multiplicity; the mappings $\psi \mapsto x_i(\psi)$ and $\psi \mapsto z_i(\psi)$ , $i \geq 1$ , are measurable, see [12]. In analogy to the definition of a random closed set a *marked point process* $\Psi$ is an $(\mathfrak{A}, \mathfrak{M}(\mathbb{K}))-$ measurable mapping from $[\Omega, \mathfrak{A}, \mathbb{P}]$ into $[\mathbb{M}(\mathbb{K}), \mathfrak{M}(\mathbb{K})]$.

For any given locally finite measure $\Lambda$ on $\mathfrak{B}(\mathbb{R}^d)$ and any probability measure $Q$ on $\mathfrak{B}(\mathbb{K})$ (called mark distribution) , an *independently marked Poisson process* $\Psi = \sum_{i \geq 1} \delta_{[X_i(\Psi), Z_i(\Psi)]}$ with *intensity measure* $\Lambda \times Q$ on $\mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{M}(\mathbb{K})$ (symbolically expressed by $\Psi \sim \Pi_{\Lambda,Q}$) is characterized by the following conditions:

(i) $\Psi(B_1 \times L_1), \cdots, \Psi(B_n \times L_n)$ are independent random variables for pairwise disjoint $B_1 \times L_1, \cdots, B_n \times L_n \in \mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{B}(\mathbb{K})$,

(ii) $\Psi(B \times L)$ is Poisson distributed with parameter $\Lambda(B)\, Q(L)$ , $B \in \mathfrak{B}(\mathbb{R}^d)$ , $L \in \mathfrak{B}(\mathbb{K})$ .

We have $\Lambda(.) = \lambda\nu(.)$ ($\nu(.) = d-$dimensional Lebesgue measure) iff the (unmarked) Poisson process $\Psi((\,.\,) \times \mathbb{K})$ is stationary with intensity $0 < \lambda < \infty$.

An equivalent more constructive definition runs as follows: Let there be given an unmarked Poisson process $\Psi^* = \sum_{i \geq 1} \delta_{X_i}$ on $\mathbb{R}^d$ with intensity measure $\Lambda$ and a sequence of i.i.d. random compact sets $\{Z_i, i \geq 0\}$ (called grains) being independent of $\Psi^*$ with common distribution $Q(.) = \mathbb{P}(Z_0 \in (.))$ , where $Z_0$ is called the typical grain, see [19]. Remember that $Q(.)$ is completely determined by $T_{Z_0}(K) = Q(\mathbb{F}_K)$ , $K \in \mathbb{K}$. It is easily verified that the marked point process $\sum_{i \geq 1} \delta_{[X_i, Z_i]}$ satisfies (i) and (ii) and so its distribution coincides with $\Pi_{\Lambda,Q}$.

With the above preliminaries and the set operation $A \oplus B = \{x + y : x \in A, y \in B\}$ we are now ready to define the term Boolean model.

Let $\Psi = \sum_{i \geq 1} \delta_{[X_i(\Psi), Z_i(\Psi)]} \sim \Pi_{\Lambda,Q}$ an independently marked Poisson process on $\mathbb{R}^d$ with mark space $\mathbb{K}$ satisfying

$$\int_{\mathbb{R}^d} T_{Z_0}(K - x)\Lambda(dx) < \infty \quad \text{for} \quad K \in \mathbb{K}\,. \tag{4.2}$$

Then the $\mathbb{P}$-a.s. closed set

$$Z(\Psi) = \bigcup_{i \geq 1} (Z_i(\Psi) \oplus X_i(\Psi)) \tag{4.3}$$

is called the *Boolean model* derived from $\Psi$.

For notational simplicity the argument $\Psi$ on both sides of (4.3) is mostly omitted. In the particular case of a discrete typical grain $Z_0 = \{\zeta_1, \ldots, \zeta_N\}$ , where $\zeta_1, \zeta_2, \ldots$ are (not necessarily independent) random vectors in $\mathbb{R}^d$ and $\mathbb{P}(N \in \mathbb{Z}_+^1 \cup \{0\}) = 1$, the corresponding Boolean model is usually described by the random point process $\Psi_c = \sum_{i \geq 1} \sum_{\zeta \in Z_i} \delta_{\zeta + X_i}$ (with i.i.d. copies $Z_i$ , $i \geq 1$ , of $Z_0$ ) called *Poisson cluster process* with typical cluster $Z_0$ . Under mild conditions , e.g. $\mathbb{E}N < \infty$, $\Psi_c$ belongs $\mathbb{P}$-a.s. to the measurable space $[\mathbb{M}, \mathfrak{M}]$ of unmarked locally finite counting measures on $\mathbb{R}^d$. Well-studied examples of Poisson cluster processes are Neyman - Scott , Gauss - Poisson and self-exciting processes, see [12].
Note that for a stationary Boolean model (that is, $\Lambda = \lambda\nu(.)$) the "existence condition" (4.2) is equivalent to

$$\mathbb{E}\;\nu(Z_0 \oplus K) < \infty \quad \text{for} \quad K \in \mathbb{K}\,. \tag{4.4}$$

Under this condition the hitting functional (4.1) of a stationary Boolean model $Z$ takes on the simple form

$$T_Z(K) = 1 \; - \; \exp\{-\lambda\mathbb{E}\;\nu(Z_0 \oplus (-K))\} \quad \text{for} \quad K \in \mathbb{K}\,. \tag{4.5}$$

In the remaining part of this section we shall demonstrate the applicability of the results of the preceding sections to some statistical estimators for characteristics of stationary Boolean models and Poisson cluster processes, respectively. We thus obtain results on probabilities of large deviations as well as Berry-Esseen bounds for these estimators. For this purpose the following assumptions are needed:

(A) $Z$ is observed in a rectangular sampling region $A_n = [0, n_1) \times \cdots \times [0, n_d)$ which expands unboundedly in each direction, i.e. $\min_{1 \leq s \leq d} n_s \longrightarrow \infty$ or briefly $n \longrightarrow \infty$ ,

(B) the typical grain (or cluster) $Z_0$ is $\mathbb{P}$-a.s. bounded, say $\mathbb{P}(Z_0 \subset b(o, \rho)) = 1$ for some $0 < \rho < \infty$.

In order to avoid troubles with boundary - effects in calculating the subsequent estimators we always assume that additional information on $Z$ is available in a larger region, say $A_n \oplus b(o, r^*)$ for some $r^* > 0$. ¿From the view point of asymptotic statistics this means no restriction of generality.

**Example 1:** Let $B \in \mathbb{K}$ be a star-shaped *structuring element* containing the origin $o$ ,e.g. $B = b(o,1)$ or $B = [0,1]^d$ or $B = s_x \stackrel{\triangle}{=} \{\alpha x : 0 \leq \alpha \leq 1\}$ for fixed vector $x$ with $\|x\| = 1$. see [19].
Providing $r_0\, B \subseteq b(o, r^*)$ the sequence of estimators

$$\hat{p}_n(r) \stackrel{\triangle}{=} \frac{\nu((Z \oplus (-rB)) \cap A_n)}{\nu(A_n)} \quad , \quad 0 \leq r \leq r_0\,, \tag{4.6}$$

turns out to be unbiased and strongly consistent for the probability

$$p(r) \stackrel{\triangle}{=} \mathbb{P}(Z \cap rB \neq \emptyset) = 1 - \exp\{-\lambda\mathbb{E}\;\nu(Z_0 \oplus (-rB))\} \quad .$$

It should be noted that $p(0) = \mathbb{P}(o \in Z)$ coincides with the so-called volume fraction $\mathbb{E}\ \nu(Z \cap [0,1]^d)$ - one of the basic characteristics of the random set $Z$. Moreover, (4.6) is suitable to establish an (asymptotically unbiased) estimate for the *contact distribution function* of $Z$ ,

$$H_B(r) \triangleq \mathbb{P}(Z \cap rB \neq \emptyset | o \in Z) = 1 - \frac{1-p(r)}{1-p(0)} = 1 - \exp\{-\lambda[\mathbb{E}\ \nu(Z_0 \oplus b(o,r)) - \mathbb{E}\ \nu(Z_0)]\} \ .$$

**Example 2:** Another important characteristic frequently used in statistical analysis of random closed sets is the *covariance* of $Z$ ,

$$c(x) \triangleq \mathbb{P}(o \in Z\ ,\ x \in Z) = p^2(0) + (1-p(0))[\exp\{\lambda \mathbb{E}\ \nu(Z_0 \cap (Z_0 - x))\} - 1] \quad \text{for} \quad x \in \mathbb{R}^d.$$

In analogy to Example 1 the sequence of estimators

$$\hat{c}_n(x) \triangleq \frac{\nu((Z \cap A_n) \cap (Z-x))}{\nu(A_n)} \quad , \quad 0 \leq \|x\| \leq r^* \quad , \tag{4.7}$$

for the covariance $c(x)$ can be shown to be unbiased and strongly consistent .

Next we consider a class of estimators for Palm distribution of certain point configurations of stationary Poisson cluster processes $\Psi_c$, see [10] and [14], pp. 102 - 104.

**Example 3:** Let us consider fixed sets $B_1, \ldots, B_k \in \mathbb{K}$ and non-negative integers $p_1, \ldots, p_k$ such that $b(o, r_*) \subseteq B_1 \cup \cdots \cup B_k \subseteq b(o, r^*)$ for some $0 < r_* < r^* < \infty$ . Then

$$\hat{p}_n \triangleq \frac{1}{\nu(A_n)} \sum_{x \in A_n : \Psi_c(\{x\}) > 0} \prod_{i=1}^{k} 1_{\{\psi \in \mathbb{M} : \psi(B_i + x) = p_i\}}(\Psi_c - \delta_x) \tag{4.8}$$

is a sequence of unbiased and strongly consistent estimators for

$$p \ \triangleq\ \lambda_c P_o^!(\{\psi \in \mathbb{M} : \psi(B_1) = p_1, \ldots, \psi(B_k) = p_k\}) \quad \text{with} \quad \lambda_c = \lambda \mathbb{E} N \quad ,$$

where $N$ is the number of cluster members in $Z_0$ and the reduced Palm distribution $P_o^!(.)$ is the probability of $\Psi_c - \delta_o \in (.)$ conditional on the null event that an atom of $\Psi_c$ is located at $o$, see [12] for details.

In the special case when $k = 1, p_1 = 0, B_1 = b(o,r)$ (4.8) makes it possible to estimate the *nearest-neighbour distance function* $D(r) \triangleq 1 - P_o^!(\{\psi \in \mathbb{M} : \psi(b(o,r)) = 0\})\ , 0 \leq r \leq r^*$ , see [10] for functional limit theorems of the empirical counterpart of this distribution function.

Finally we consider the empirical *second order moment function* of a hard-core point process which arises from a Poisson cluster process with bounded typical cluster by applying B. Matérn's (second) thinning procedure, see [19]. More precisely, let $\Psi_c = \sum_{i \geq 1} \sum_{\zeta \in Z_i} \delta_{\zeta + Z_i} = \sum_{i \geq 1} \delta_{Y_i}$ (after renumbering the atoms) be a stationary Poisson cluster process as in Example 3 independently marked by a sequence $U_i, i \geq 1,$ of independent random variables uniformly distributed on $(0,1)$. An atom $Y_i$ of $\Psi_c$ is deleted iff there is a further atom in $b(Y_i,R)$ with a mark smaller than $U_i$. The interpoint distances of the resulting stationary point process

$$\tilde{\Psi}_c \triangleq \sum_{i\geq 1} \delta_{Y_i} \prod_{j\neq i} \Big(1 - 1_{b(Y_i,R)}(Y_j) 1_{[0,U_i)}(U_j)\Big)$$

are always greater than $R$ .

**Example 4:** For any fixed $(R <)r \leq r^*$ the sequence of estimators

$$\hat{K}_n(r) \triangleq \frac{1}{\tilde{\lambda}_c \nu(A_n)} \sum_{x\in A_n : \tilde{\Psi}_c(\{x\})>0} (\tilde{\Psi}_c - \delta_x)(b(x,r)) \quad \text{with} \quad \tilde{\lambda}_c \triangleq \mathbb{E}\tilde{\Psi}_c([0,1)^d) \tag{4.9}$$

is unbiased and strongly consistent for the so-called second moment function of $\tilde{\Psi}_c$ given by

$$K(r) \triangleq \int_{\mathbb{M}} \psi(b(o,r)) \tilde{P}^!_o(d\psi) \quad ,$$

where $\tilde{P}^!_o$ denotes the Palm distribution of $\tilde{\Psi}_c$. The function $K(r)$ , $r \geq 0$ , is the main tool in the statistical analysis of stationary isotropic point patterns, see [19].

We now show how to represent the estimators $\hat{p}_n(r)$ , $\hat{c}_n(x)$ , $\hat{p}_n$ and $\hat{K}_n(r)$ as a normalized sum of random variables of the form (1.1). To this end let

$$E_j \triangleq [-1,0)^d + j \quad \text{for} \quad j \in \mathbb{Z}^d \quad (\text{so that} \quad A_n = \cup_{j\in V_n} E_j \quad \text{and} \quad |V_n| = \nu(A_n))$$

and set

$$|V_n|\; \hat{p}_n(r) = \sum_{j\in V_n} X_j^{(1)} \quad \text{with} \quad X_j^{(1)} \triangleq \nu((Z \oplus (-rB)) \cap E_j)\, , \tag{4.10}$$

$$|V_n|\; \hat{c}_n(x) = \sum_{j\in V_n} X_j^{(2)} \quad \text{with} \quad X_j^{(2)} \triangleq \nu(Z \cap (Z - x) \cap E_j) \tag{4.11}$$

$$|V_n|\; \hat{p}_n = \sum_{j\in V_n} X_j^{(3)} \quad \text{with} \quad X_j^{(3)} \triangleq \sum_{x\in E_j : \Psi_c(\{x\})>0} \prod_{i=1}^{k} 1_{\{\psi\in\mathbb{M}:\psi(B_i+x)=p_i\}}(\Psi_c - \delta_x) \tag{4.12}$$

and

$$|V_n|\; \hat{K}_n(r) = \sum_{j\in V_n} X_j^{(4)} \quad \text{with} \quad X_j^{(4)} \triangleq \frac{1}{\tilde{\lambda}_c} \sum_{x\in E_j : \tilde{\Psi}_c(\{x\})>0} (\tilde{\Psi}_c - \delta_x)(b(x,r))\, . \tag{4.13}$$

Further introduce the set $\mathbb{M}_j(\mathbb{K})$ , $j \in \mathbb{Z}^d$ , consisting of those $\psi \in \mathbb{M}(\mathbb{K})$ with support $E_j \times \mathbb{K}$ endowed with the $\sigma-$ field $\mathfrak{M}_j(\mathbb{K})$ generated by sets of the form $\{\psi \in \mathbb{M}(\mathbb{K}) : \psi_j(B\times L) \triangleq \psi((E_j\cap B)\times L) = k\}$ for $k \geq 0$ , $B \in \mathfrak{B}(\mathbb{R}^d)$ and $L \in \mathfrak{B}(\mathbb{K})$ .

Following an idea of C. Preston [15] we can describe the above-defined independently marked Poisson process $\Psi \sim \Pi_{\Lambda,Q}$ as a lattice process by using the obvious fact (see [15], pp. 89/90) that the measurable space $[\mathbb{M}(\mathbb{K}), \mathfrak{M}(\mathbb{K})]$ is isomorphic by the mapping $\psi \mapsto (\psi_j)_{j\in\mathbb{Z}^d}$ to the product space $[\times_{j\in\mathbb{Z}^d}\mathbb{M}_j(\mathbb{K}), \otimes_{j\in\mathbb{Z}^d}\mathfrak{M}_j(\mathbb{K})]$ . By condition (i) the point processes $\Psi_j$ , $j \in \mathbb{Z}^d$ , are mutually independent - even i.i.d. in case $\Psi$ is stationary.

By virtue of condition (B) we get $Z_i \oplus X_i \oplus (-rB) \subset b(o, \rho + r^*)$ for $0 \le r \le r_0$ . Hence, the random closed set $(Z(\Psi) \oplus (-rB)) \cap E_j$ is completely determined by $\Psi$ restricted to the set $E_j \oplus b(o, \rho + r^*)$ or by the $(2q+1)^d$-tuple $\Psi_i, i \in U_j - \tau$, where

$$q \triangleq \mathrm{ent}(\rho + r^*) + 1 \quad \text{and} \quad \tau \triangleq (q, \ldots, q)$$

and $U_j$ is defined in Sect. 1 with $m = 2q$. Therefore there exists a real-valued $\otimes_{i\in U_j-\tau}\mathfrak{M}_j(\mathbb{K})$-measurable function $f_j^{(1)}$ on $\times_{i\in U_j-\tau}\mathbb{M}_j(\mathbb{K})$such that

$$X_j^{(1)} = \nu((Z \oplus (-rB)) \cap E_j) = f_j^{(1)}(\Psi_i, i \in U_j - \tau) \quad \text{with} \quad 0 \le f_j^{(1)} \le 1 \ , \ j \in V_n .$$

In quite the same manner we can treat $X_j^{(2)}$ and $X_j^{(3)}$. To get a bound of $X_j^{(3)}$ we argue as follows: Let $a \ge 1$ be the smallest integer satisfying $a \ge \sqrt{d}/r_*$, i.e. $[0, a^{-1})^d \subseteq b(y, r_*)$ for all $y \in [0, a^{-1})^d$. The conditions imposed on $B_1, \ldots, B_k$ imply that $X_j^{(3)} \le [a(p_1 + \cdots + p_k + 1)]^d$.

We conclude this section by summarizing the above results and remarks concerning the estimators (4.6) - (4.9) in

**Theorem 3.** Under (A) , (B) and the assumptions made in Example 1, 2, 3 and 4, Theorem 1 (and thus Theorem 2) is applicable to the following $m-$dependent random fields $\{X_j \ , \ j \in V_n\}$ :

$$\begin{aligned}
X_j &= X_j^{(1)} - p(r) \quad \text{for} \quad 0 \le r \le r_0 \quad \text{with} \quad c = 1 \\
X_j &= X_j^{(2)} - c(x) \quad \text{for} \quad 0 \le \|x\| \le r^* \quad \text{with} \quad c = 1 \\
X_j &= X_j^{(3)} - p \qquad \text{with} \quad c = [(\mathrm{ent}(\sqrt{d}/r_* + 1)(p_1 + \cdots + p_k + 1)]^d \\
X_j &= X_j^{(4)} - K(r) \quad \text{with} \quad c = N_0 N(r)/\tilde{\lambda}_c \quad \text{and} \\
& \qquad N_0 = \max\{n \ge 1 : \exists x_1, \ldots, x_n \in [0,1)^d : \|x_i - x_j\| > R, i \ne j\} \\
& \qquad N(r) = \max\{n \ge 1 : \exists x_1, \ldots, x_n \in b(o,r) : \|x_i\| > R, \|x_i - x_j\| > R, i \ne j\} \ ,
\end{aligned}$$

where $m = 2(\mathrm{ent}(\rho + r^*) + 1)$ and $X_j^{(1)}, X_j^{(2)}$ , $X_j^{(3)}$ and $X_j^{(4)}$ are defined by (4.10) , (4.11) , (4.12) and (4.13), respectively. This implies in particular that for the statistics (4.6) - (4.9) a Berry-Esseen bound and Cramér-type probabilities of large deviations hold.

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